FUNCTION-THEORETIC TECHNIQUES FOR THE ELECTROMAGNETIC SCATTERING BY A RESISTIVE WEDGE

by

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To my loving parents,

Alice and Ivan

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LIST OF SYMBOLS

Chapter I:

$\bar{E}$ (also E, E_z, E^i, E^s, etc)	electric field intensity
$\bar{H}$ (also H, H_z, H^i, H^s, etc)	magnetic field intensity
$\bar{K}_e, \bar{K}_m$	electric and magnetic surface current densities
S (also S_+, S_-)	scattering surface
R	surface resistivity
U (also U_1, U_2, U^i, U^s, etc)	scalar electric field
V (also V_1, V_2, V^i, V^s, etc)	scalar magnetic field
Y	free space admittance
Z	free space impedance
$\varepsilon, \varepsilon_o$	complex permittivity
η	normalized impedance or resistivity
μ, μ_o	complex permeability
σ	conductivity
ω	angular frequency

Chapter II:

I^e, I^o	even and odd incident fields
O( )	order relation
$U^{es}_{1,2}, U^{os}_{1,2}$	even and odd scattered electric fields

$V^{es}_{1,2}, V^{os}_{1,2}$	even and odd scattered magnetic fields
(ρ, ϕ, z)	cylindrical coordinates
ϕ_o	angle of incidence
ψ	half-angle of the wedge

Chapter III:

$M(\alpha)$	multiplier function
$P(\alpha)$ (also $P_p(\alpha)$)	particular solution to a difference equation
$s(\alpha)$ (also $s_1(\alpha), s_2(\alpha)$, etc)	function related to the electric field
$S_1(\alpha), S_2(\alpha)$	functions defined by s_1, s_2
$t(\alpha)$ (also $t_1(\alpha), t_2(\alpha)$, etc)	function related to the magnetic field
$T_1(\alpha), T_2(\alpha)$	functions defined by t_1, t_2
α	complex variable
γ	contour in the complex α plane
$\chi(\alpha)$	general solution to a difference equation

Chapter IV:

$A(\nu), B(\nu)$ (also $A_{1,2}, B_{1,2}$, etc)	unknown coefficients for the transformed electric field
$C(\nu), D(\nu)$ (also $C_{1,2}, B_{1,2}$, etc)	unknown coefficients for the transformed magnetic field

c (also c_1, c_2, c_1^e, c_1^o, etc)	value of the scattered electric field at $\rho = 0$
d (also d_1, d_2, d_1^e, d_1^o, etc)	value of the scattered magnetic field at $\rho = 0$
$\tilde{f}(\nu) = K[f(\rho)]$	Kontorovich-Lebedev (K-L) transform of the function $f(\rho)$
$f(\rho) = K^{-1}[\tilde{f}(\nu)]$	inverse K-L transform of the function $\tilde{f}(\nu)$
$\hat{f}(\alpha)$	Maliuzhinets function representing $f(\rho)$ and related to $\hat{f}(\nu)$
$F(\nu), F'(\nu)$ (also F_e, F_o, F_e', F_o')	functions related to $A_1(\nu)$, $C_1(\nu)$, etc.
$f_e(\tau), f_o(\tau), f_e'(\tau), f_o'(\tau)$	functions related to F_e, F_o, F_e', F_o' respectively
$G(\nu), G'(\nu)$	functions related to $B_1(\nu), D_1(\nu)$
$g(\tau)$ (also g_e, g_o, g_e', g_o')	inhomogeneous term of a Fredholm integral equation
$H_\nu^{(1)}(k\rho)$	Hankel function of the first kind and order ν
$J_\nu(k\rho)$	Bessel function of the first kind and order ν
$K_\nu(-ik\rho)$	Modified Bessel function of the second kind and order ν

$K(\tau,\tau')$ (also K_e, K_o, K'_e, K'_o)	kernel of the Fredholm integral equation
$u(\rho,\phi), v(\rho,\phi)$ (also u_1, v_1, u_2, v_2, etc)	modified scattered electric and magnetic fields
$\tilde{u}(\nu,\phi), \tilde{v}(\nu,\phi)$ (also $\tilde{u}_1, \tilde{v}_1, \tilde{u}_2, \tilde{v}_2$, etc)	K-L transforms of u,v
$\tau = i\nu$	complex variable
ν	complex K-L transform variable

Chapter V:

$g(x,y)$	metric function for x,y of the vector space X
$K^n(\tau,\tau')$ (also K^n_e, K^n_o, etc)	nth iterated kernel of $K(\tau,\tau')$
T	linear operator on the Banach space X
T^n	nth iterated operator of T
T_e, T_o, T'_e, T'_o	Fredholm integral operators for f_e, f_o, f'_e, f'_o, respectively
T_e^{-1}	inverse of the operator T_e
X	normed linear vector space, or a Banach space
Z	space of complex numbers
$\|\|x\|\|$	norm of the element x
λ	Fredholm integral parameter related to η

CHAPTER I. INTRODUCTION

1.1 Previous Analyses of the Scattering from Wedges: A Review

The scattering of electromagnetic waves by wedge-shaped regions has been investigated by several authors, although few exact solutions exist [1-28]. For the case of a perfectly conducting wedge, the first solution is attributed to Sommerfeld (see Carslaw, [1]), and to MacDonald [2]. These works addressed the two-dimensional scalar problem, and were based on an extension of the method of images described by Carlslaw [3]. A special case, in which the included angle of the wedge is zero, was elegantly solved by Sommerfeld [4]. This solution for the half-plane has served as a model for investigating the effects of edge diffraction. Clemmow [5] developed a technique for generalizing these results to the case of obliquely incident plane wave excitation. Modal expansions for the solution can be found in [6], for example, while an expression for the general vector problem in the form of a dyadic Green's function is given by Tai [7].

In the event that the wedge is not perfectly conducting, an exact solution cannot be found. However, for the case in which the conductivity is large but finite, Leontovich [8] has developed an approximate boundary condition, known as an impedance boundary condition, which allows several otherwise intractable problems to be solved.

The major attribute of this approximation is that it replaces the fields within the conducting body with approximate surface currents via the boundary condition. With the interior field thus accounted for, the number of unknowns is halved, and only the exterior field need be determined. In particular, Senior [9] found a solution for the impedance half-plane under normal plane wave incidence, and later extended it to include oblique incidence [10]. This extension was related to Clemmow's technique by Williams [11]. The results are obtained through application of the Wiener-Hopf method (see [12]) to the unknown currents excited on the half-plane. For a wedge of arbitrary angle and differing face impedances, a general solution to the scalar problem with plane wave incidence was developed by Maliuzhinets in his doctoral thesis and subsequent works [13-15]. His method is a further generalization of the method of images, and involves an integral representation for the field along a Sommerfeld contour. Maliuzhinets replaces the integral equation for the unknown by an equivalent functional difference equation. Senior [16] and Williams [17], following similar methods, also arrived at the solution for the case in which the face impedances are the same.

Variations on the half-plane problem and its solution are given by Rawlins [18], Hurd [19], and Bucci and Franceschetti [20], which are essentially founded on either the Wiener-Hopf or Maliuzhinets' method.

Another function-theoretic technique for solving scattering problems in wedge-shaped regions was presented by Kontorovich and Lebedev [21] in the form of an integral transform bearing their names. A solution for the perfectly conducting half-plane was given as an example. In a recent paper [22], Jones derives sufficient conditions for the existence of the transform and a generalized inverse. In a major extension of their work, Lebedev and Skal'skaya [23] applied the transform to the impedance wedge and developed a closed form solution for a special set of wedge angles. Their method resulted in a functional difference equation, paralleling Maliuzhinets' technique, but the equation is of second order instead of first. In fact, Maliuzhinets [24] described the relationship between his method and the Kontorovich-Lebedev (K-L) transform via a Fourier transformation.

In addition to the impedance boundary condition, of which infinite conductivity and its dual are special cases, another approximation exists which accounts for the material properties of a scattering body via an equivalent boundary condition. In this case, the body is assumed to consist of a thin dielectric shell of a particular shape, and is approximated by an infinitely thin, partially transparent layer. Because it is most applicable to lossy dielectrics, this approximation is referred to as a resistive boundary condition, although due to the transparency of the sheet, and hence the existence of both interior and exterior fields, it is more exactly a transition condition, analogous to those applied at an interface between two dielectrics. Derivations of this condition can be found in [25,26]. Very few exact solutions for resistive bodies exist. Senior [27] and Anderson [28] give solutions for a resistive half-plane using a Wiener-Hopf analysis. An excellent review of the impedance and resistive boundary conditions for the half-plane is presented by Senior in [29], and a condensed version comparing the two is contained in Section 1.3 of this work.

Finally, it should be noted that the electromagnetic problems discussed above have analogs in other disciplines involving wave phenomena, especially in acoustics. The acoustic equivalents of the impedance and resistive boundary conditions in the context of Babinet's principle are pursued by Senior in [30].

1.2 Outline of the Scattering Problem

In order to further understand the nature of the scattering from partially penetrable objects, in particular those composed of resistive materials, an investigation into the scattering of an arbitrarily polarized electromagnetic plane wave normally incident on a wedge with resistive faces is presented here. The resistivity of the wedge is assumed to be a complex scalar constant, independent of position on the wedge. The unknown fields exist in two regions, the exterior of the wedge (Region 1), from which the plane wave is incident, and the interior of the wedge (Region 2), as shown in Fig. 1.1. The primary distinction between this problem and those discussed previously is the existence of a nonzero field in the interior region. This field severely complicates the task of finding a solution. In some respects, the problem closely resembles the scattering by a dielectric wedge, where an interior field is also present. Indeed, exact solutions for dielectric bodies are very few, and those which have been developed for special cases of the

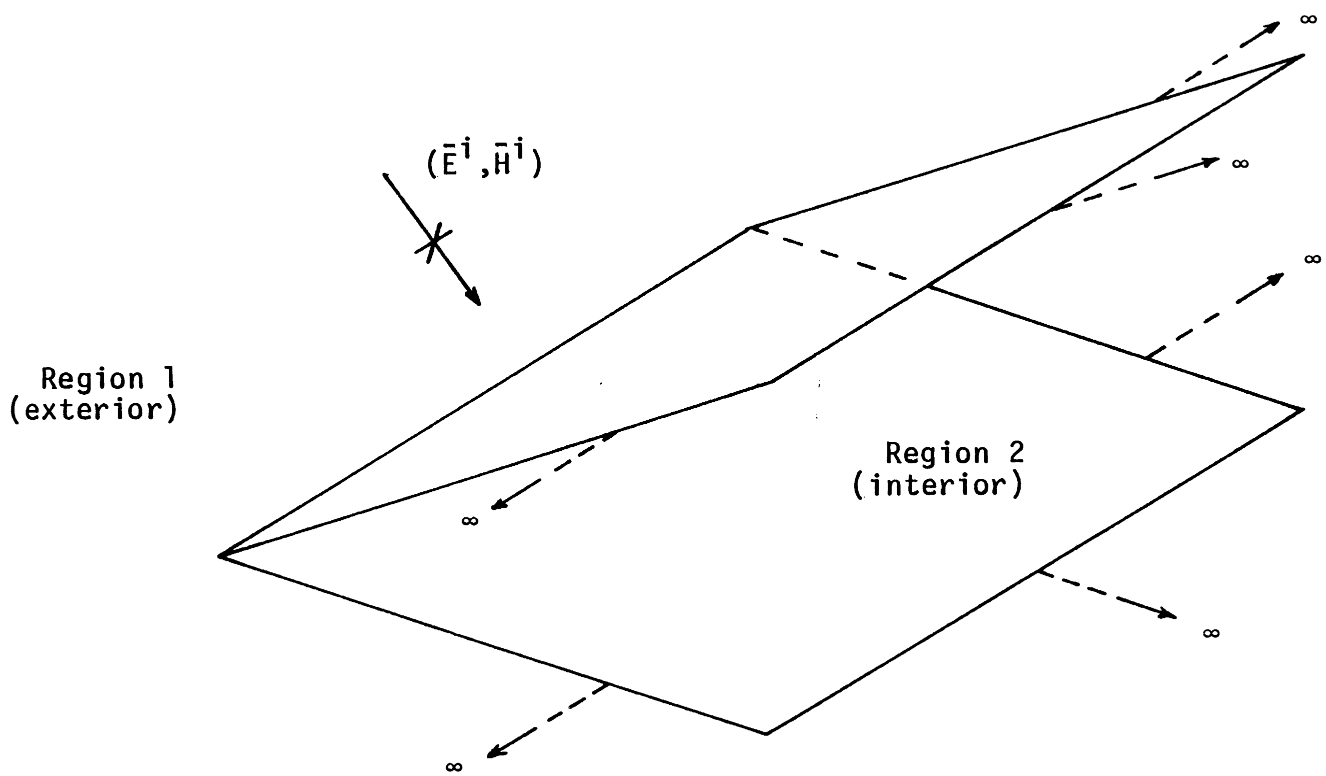


Figure 1.1: Three-Dimensional Geometry for the Scattering by a Resistive Wedge.

wedge [31-32], are either extremely complicated or somewhat in doubt [33].

Following a brief derivation and comparison of the impedance and resistive boundary conditions in Section 1.3, a formulation of the scattering problem to be considered is presented in Chapter II. By appropriately decomposing the incident field into two components, with either the E- or H-field vector parallel to the edge of the wedge, the solution can be constructed from a pair of scalar two-dimensional fields. The problem is further simplified by decomposing the scalar fields into components which are either symmetric or antisymmetric about a plane bisecting the wedge. Thus a total of four quantities are required to specify the general solution.

The approach taken in this work is to examine the feasibility of extending the function-theoretic techniques used in solving the impedance wedge scattering problem to the resistive wedge. While the Wiener-Hopf method has successfully been applied to the half-plane and other geometries (see for example, Carlson and Heins [34,35]) it is not appropriate for wedges of arbitrary angles, with the possible exception of the right-angle wedge. Senior discusses this shortcoming, as well as certain analogies between Maliuzhinets' method and the Wiener-Hopf technique in [16,29]. Therefore, the emphasis here will be on the methods of Maliuzhinets and the Kontorovich-Lebedev transform. It is shown that both methods lead to difference equations, as was the case for the impedance wedge. However, when applying the method of Maliuzhinets, the resulting difference equation for the resistive wedge is of third order, compared to the first order equation obtained for an impedance wedge. In general, a straightforward technique for solving this third order equation is not available [36]. A derivation of this equation and its consequences is given in Chapter III.

In Chapter IV, the relationship between the representation of Maliuzhinets and Kontorovich and Lebedev is presented [24]. The latter is applied to the resistive wedge formulation, and again a set of difference equations for the unknowns is generated. As before, the equations do not yield to the same methods of solution available for the impedance wedge. However, a novel technique for converting the difference equations to Fredholm integral equations of the second kind is developed. The kernels of the integrals are bounded and well behaved. This type of integral equation is well understood in the literature [37,38], and various means of finding exact or approximate solutions are available.

Such a method is developed in Chapter V for the Fredholm integral equations obtained from the K-L representation. From the theory of linear operators, the method of successive approximations is applied to obtain an iterative power series expansion for the unknowns. The expansion is in terms of the resistivity of the wedge, and converges uniformly for particular ranges of values of this parameter. Bounds for the regions of a convergence are given. The chapter is concluded with a discussion of how similar expansions obtained from the difference equations may not exist in certain cases.

A summary and discussion of the results follow in Chapter VI. Appendix A contains a set of conditions for the existence of the representations used by Maliuzhinets and Kontorovich and Lebedev, along with their corresponding inverses.

1.3 Discussion of the Impedance and Resistive Boundary Conditions

Before proceeding with an analysis of the scattering by a resistive wedge, it is appropriate to review the mathematical implications of the impedance and resistive boundary conditions.

In vector form, the impedance boundary condition on the surface S of a body is given by

$$\bar{E} - (\hat{n} \cdot \bar{E})\hat{n} = \eta Z \hat{n} \times \bar{H} \quad ; \quad \text{on } S \ , \tag{1.1}$$

where $(\bar{E},\bar{H})$ are the total fields in the region surrounding the body, assumed to be free space, Z is the intrinsic impedance of free space, and $\hat{n}$ is a unit vector normal to S and directed into the region containing the fields (see Fig. 1.2). The dimensionless parameter η is the surface impedance of the boundary normalized to free space. Physically speaking, if the body consists of a material with large refractive index, and hence large relative complex permittivity, η has the form [39]

$$\eta = \left\{ \frac{\mu_0}{\mu} \left[\frac{\varepsilon}{\varepsilon_0} + i \frac{\sigma}{\omega \varepsilon_0} \right] \right\}^{-1/2} \ , \tag{1.2}$$

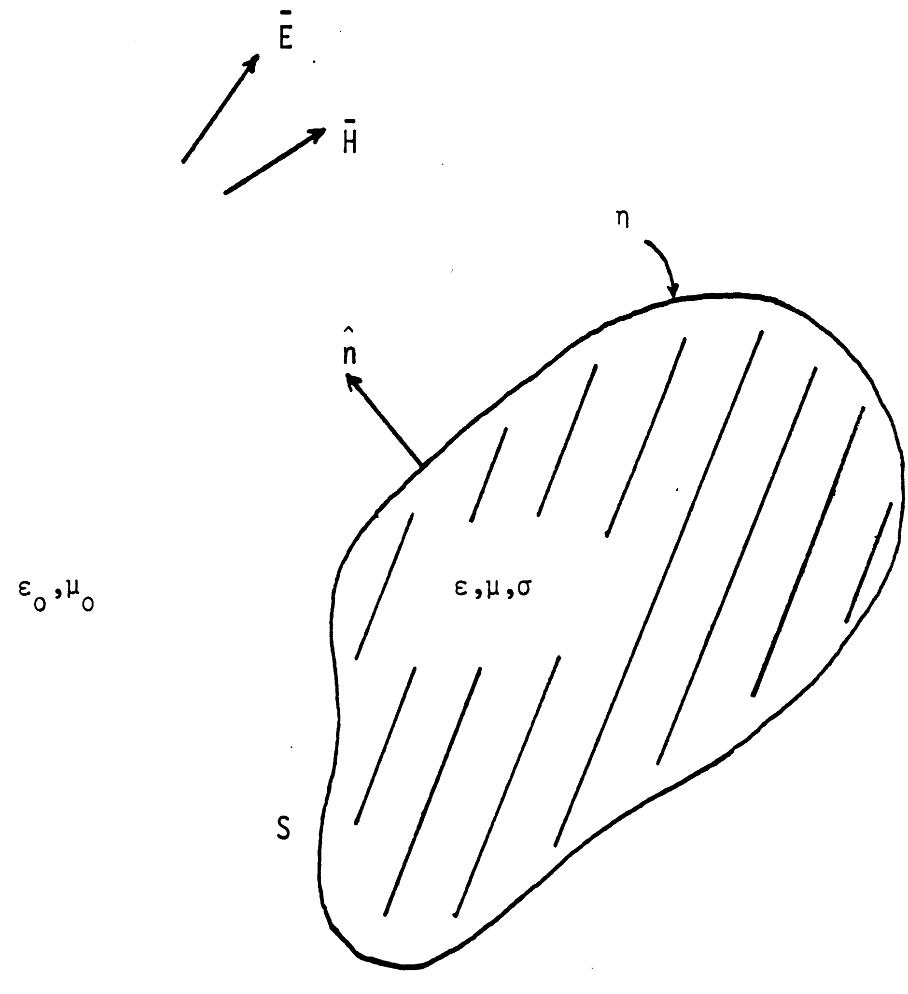


Fig. 1.2 Diagram for the Derivation of the Impedance Boundary Condition ($\sigma \to \infty$).

where ε, μ, σ are the permittivity, permeability, and conductivity of the body, respectively, and ε_0, μ_0 are the corresponding free space parameters. An $e^{-i\omega t}$ time dependence has been assumed and suppressed here and throughout this work with ω the angular frequency of the EM field. Note that as $\sigma \to \infty$, $\eta \to 0$ and (1.1) becomes the boundary condition for perfect conductivity, i.e.,

$$\hat{n} \times \bar{E} = 0 \quad ,$$

as expected. It was in this context that (1.1) was introduced [8] as an approximation to the boundary conditions at a body with large but finite conductivity. A discussion of the validity of the approximation can be found in [40]. The utility of (1.1) is that it reduces the problem of determining the fields both inside and outside the body to that of solving the exterior problem alone, subject to a boundary condition which describes the material properties of the body. All interior fields are identically zero. As discussed in Section 1.1, several scattering problems satisfying such a condition have been solved.

Regardless of its physical implications, mathematically (1.1) relates the tangential components of $\bar{E}$ and $\bar{H}$ via a parameter η. In terms of the equivalent electric and magnetic surface currents

$$\bar{K}_e = \hat{n} \times \bar{H} \tag{1.3a}$$

$$\bar{K}_m = -\hat{n} \times \bar{E} \quad , \tag{1.3b}$$

respectively, it follows that

$$\bar{K}_m = -\eta Z\hat{n} \times \bar{K}_e \quad . \tag{1.4}$$

An interesting duality transformation exists for the impedance boundary value problem. It can be stated via the following theorem [39]:

Theorem 1: If the electromagnetic field incident upon a body satisfying the boundary condition (1.1) is denoted by

$$(\bar{E}^i,\bar{H}^i) = (\bar{F},Y\bar{G}) \quad ,$$

and the scattered field by

$$(\bar{E}^s,\bar{H}^s) = (\bar{f}(\eta),Y\bar{g}(\eta)) \quad ,$$

where

$$(\bar{E},\bar{H}) = (\bar{E}^i + \bar{E}^s \ , \ \bar{H}^i + \bar{H}^s) \quad ,$$

then for an incident field

$$(\bar{E}^i,\bar{H}^i) = (-Z\bar{G},\bar{F}) \quad ,$$

the scattered field is

$$(\bar{E}^s,\bar{H}^s) = (-Z\bar{g}(1/\eta), \ \bar{f}(1/\eta)) \quad .$$

This is equivalent to the transformation $\bar{E} \rightarrow \bar{H}$, $Z\bar{H} \rightarrow -Y\bar{E}$, $\eta \rightarrow 1/\eta$, and can easily be derived by taking the cross product of (1.1) with $\hat{n}$, yielding

$$\bar{H} - (\hat{n} \cdot \bar{H})\hat{n} = -\frac{1}{\eta} Y\hat{n} \times \bar{E} \quad . \tag{1.5}$$

Equation (1.5) is simply a restatement of (1.1) under the prescribed transformation. By allowing $\eta \to \infty$, (or $1/\eta \to 0$), (1.5) reduces to

$$\hat{n} \times \bar{H} = 0 \quad ,$$

which is the boundary condition for a perfect (nonetheless fictitious) magnetic conductor.

Suppose instead that the material body in Fig. 1.2 is replaced by a thin shell of thickness δ, coincident with surface S. The shell is composed of the same material as the body, with parameters ε,μ,σ. A solution for the scattering from such a structure requires the determination of the fields in the three regions defined in Fig. 1.3. If, however, one again considers the situation when the complex permittivity becomes large while simultaneously allowing the thickness to decrease, such that their product remains constant, then the three region problem can be approximated by a two region problem with an appropriate set of boundary conditions on the surface S. A mathematical description of the limiting process is given in [25,26]. Referring to the geometry in Fig. 1.4, the approximate vector boundary conditions that now apply are

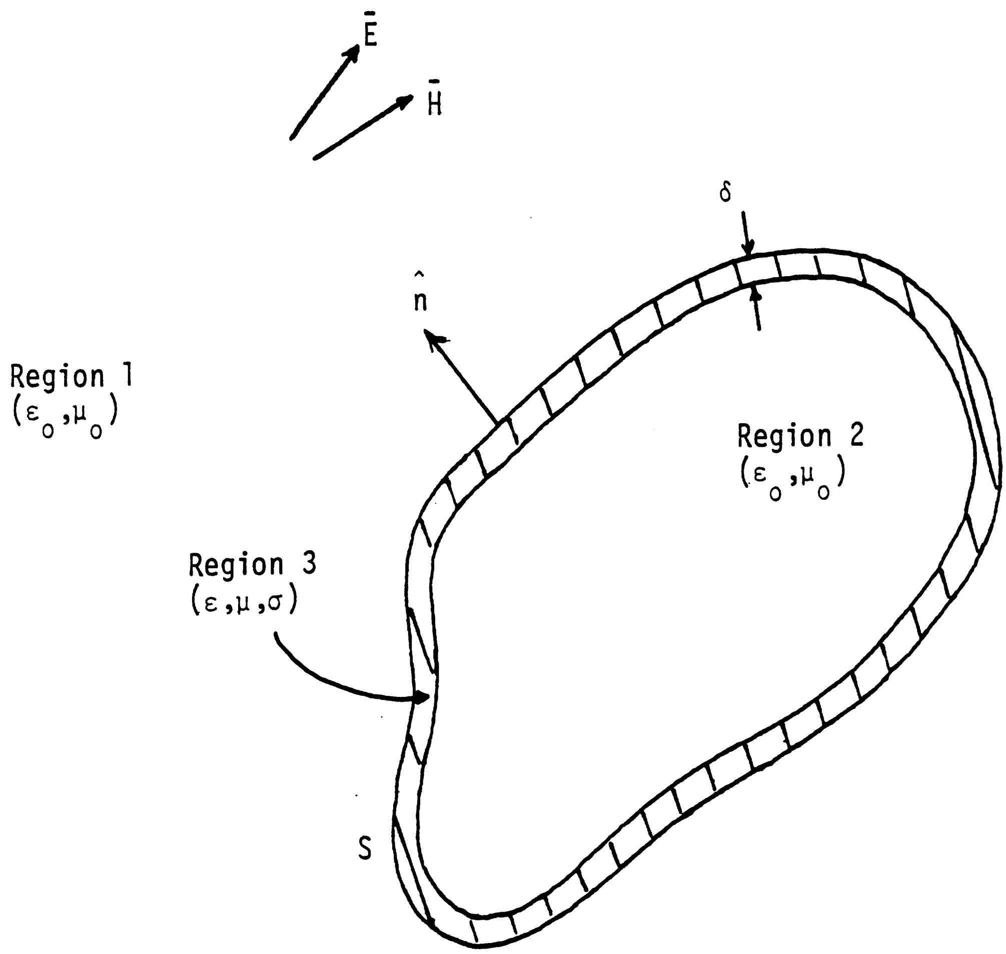


Fig. 1.3 Diagram for the Derivation of the Resistive Boundary Condition.

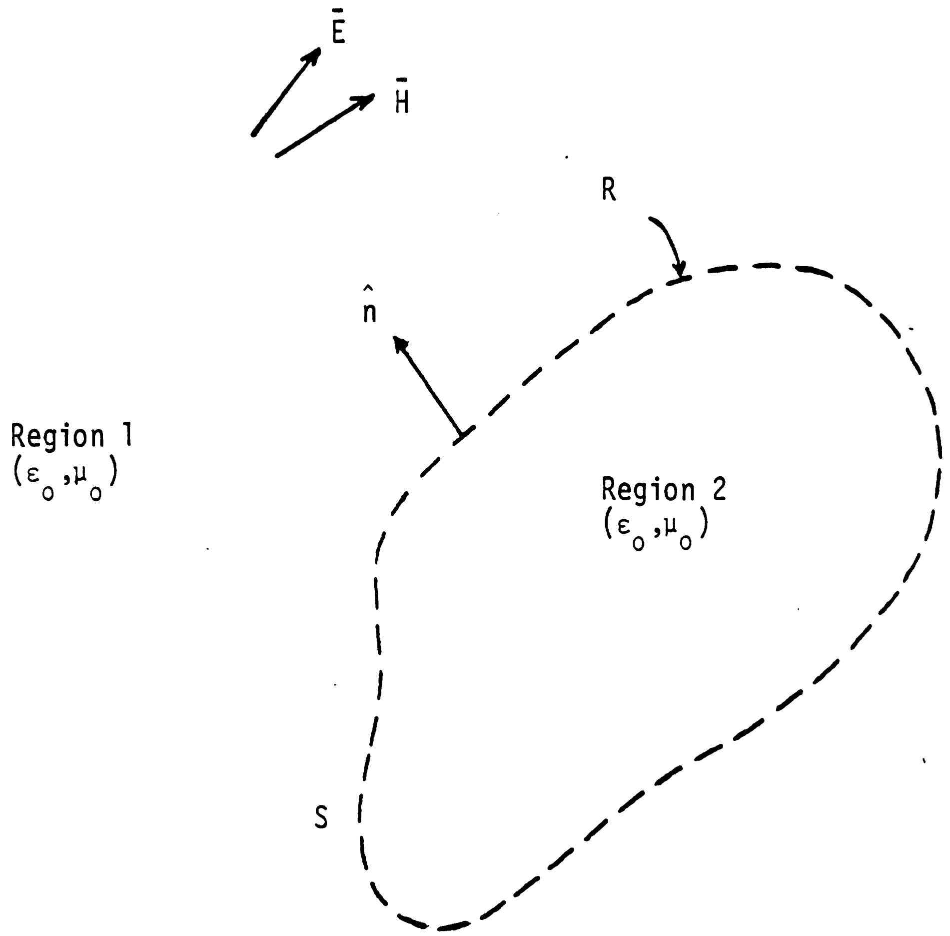


Fig. 1.4 Equivalent Geometry for Fig. 1.3 with the Scattering Body Replaced by a Resistive Surface.

$$\hat{n} \times [\bar{E}]_{-}^{+} = 0 \tag{1.6a}$$

$$\hat{n} \times (\hat{n} \times \bar{E}) = -R\hat{n} \times [\bar{H}]_{-}^{+} \tag{1.6b}$$

where the notation $[\ ]_{-}^{+}$ denotes the discontinuity in the value of a quantity across the surface S, or more exactly,

$$[f]_{-}^{+} = f_1 - f_2 \quad ; \quad \text{on } S \quad ,$$

with f an arbitrary function. The subscripts refer to the corresponding values in Regions 1 and 2 of Fig. 1.4. In terms of equivalent surface currents

$$\bar{K}_e = \hat{n} \times [\bar{H}]_{-}^{+} \tag{1.7a}$$

$$\bar{K}_m = -\hat{n} \times [\bar{E}]_{-}^{+} \quad , \tag{1.7b}$$

(1.6) becomes

$$\bar{K}_m = 0 \tag{1.8a}$$

$$\hat{n} \times (\hat{n} \times \bar{E}) = -R\bar{K}_e \quad . \tag{1.8b}$$

The parameter R, the resistivity of the layer, is given by

$$R = \frac{i}{\omega(\varepsilon - \varepsilon_0)\delta} \tag{1.9}$$

in the limit as described above. The resistivity R has units of ohms per square in the MKS system. When $\sigma >> \varepsilon$, R tends toward a purely real number, given by [26]

$$R = \frac{1}{\sigma\delta} . \qquad (1.10)$$

In general, for a passive material with $\sigma \geq 0$, $\varepsilon \geq \varepsilon_0$, R is a complex number lying in the first quadrant of the complex plane. From (1.7), a resistive surface is equivalent to an electric current layer whose strength is proportional to the tangential electric field at its surface. Since the tangential electric field is continuous across the layer (from (1.6a)), there are no magnetic currents. It is interesting to note that in the limit as $R \to 0$, (1.6) becomes a perfectly conducting boundary condition, while for $R \to \infty$, (1.6) can be written as

$$\hat{n} \times [\bar{E}]_-^+ = 0$$

$$\hat{n} \times [\bar{H}]_-^+ = 0 .$$

These continuity conditions are satisfied by the incident field alone, and hence there is no scattered field. Equivalently, the scattering body has ceased to exist.

As was done for the impedance boundary condition (1.1), it is convenient to normalize the resistivity via a dimensionless parameter

$$\eta = \frac{2R}{Z} . \qquad (1.11)$$

The factor of two is introduced because of an interesting result from half-plane diffraction [27]. The total electric current on a half plane with surface impedance η is identical to that on a resistive half-plane with R satisfying (1.11). However, this result cannot be generalized for arbitrary geometries.

Recalling the duality transformation presented in Theorem 1, its application to (1.6) yields

$$\hat{n} \times [\bar{H}]_{-}^{+} = 0 \tag{1.12a}$$

$$\hat{n} \times (\hat{n} \times \bar{H}) = R^{*}\hat{n} \times [\bar{E}]_{-}^{+} , \tag{1.12b}$$

where

$$R^{*} = \frac{Y}{2\eta} . \tag{1.13}$$

While (1.5) is simply a restatement of the impedance boundary condition under the prescribed duality transform, i.e., an impedance boundary condition is its own dual, examination of (1.12) indicates a similar analogy does not exist for the resistive boundary condition. Indeed, (1.12) describes a surface characterized by a jump discontinuity in the tangential electric field, but no discontinuity in the tangential magnetic field. Such a surface supports only a magnetic surface current $\bar{K}_m$, and has been referred to as a "conductive" sheet, with conductivity R* mhos per square [27]. A discussion of the duality relations for impedance and resistive boundary conditions is presented by Senior [41] in relation to Babinet's principle.

It is appropriate to further compare the impedance and resistive boundary conditions in the context of the problem considered here. A wedge with upper and lower faces S_+ and S_- respectively is illuminated by a plane electromagnetic wave incident normal to the edge of the wedge. By considering the cases in which the incident electric or magnetic field is parallel to the edge (E- or H-polarization respectively), as illustrated in Figs. 1.5a and 1.5b, the vector scattering problem can be reduced to two scalar, two-dimensional scattering problems.

In the event that S_+ and S_- satisfy impedance boundary conditions, (1.5) for E-polarization becomes

$$\frac{\partial U}{\partial n} + \frac{ik}{\eta} U = 0 \quad ; \qquad \text{on } S_+, S_- \tag{1.14a}$$

where $\bar{E} = \hat{u}U$, $\hat{u}$ being a unit vector parallel to the edge of the wedge. The normal derivative $\partial/\partial n$ is defined as

$$\frac{\partial}{\partial n} = \hat{n} \cdot \nabla$$

i.e., it is taken in the direction of the unit vector $\hat{n}$, which from Fig. 1.4 is directed toward Region 1 containing the incident field. The propagation constant k is defined as

$$k = \frac{\omega}{c} ,$$

where c is the velocity of light in vacuo.

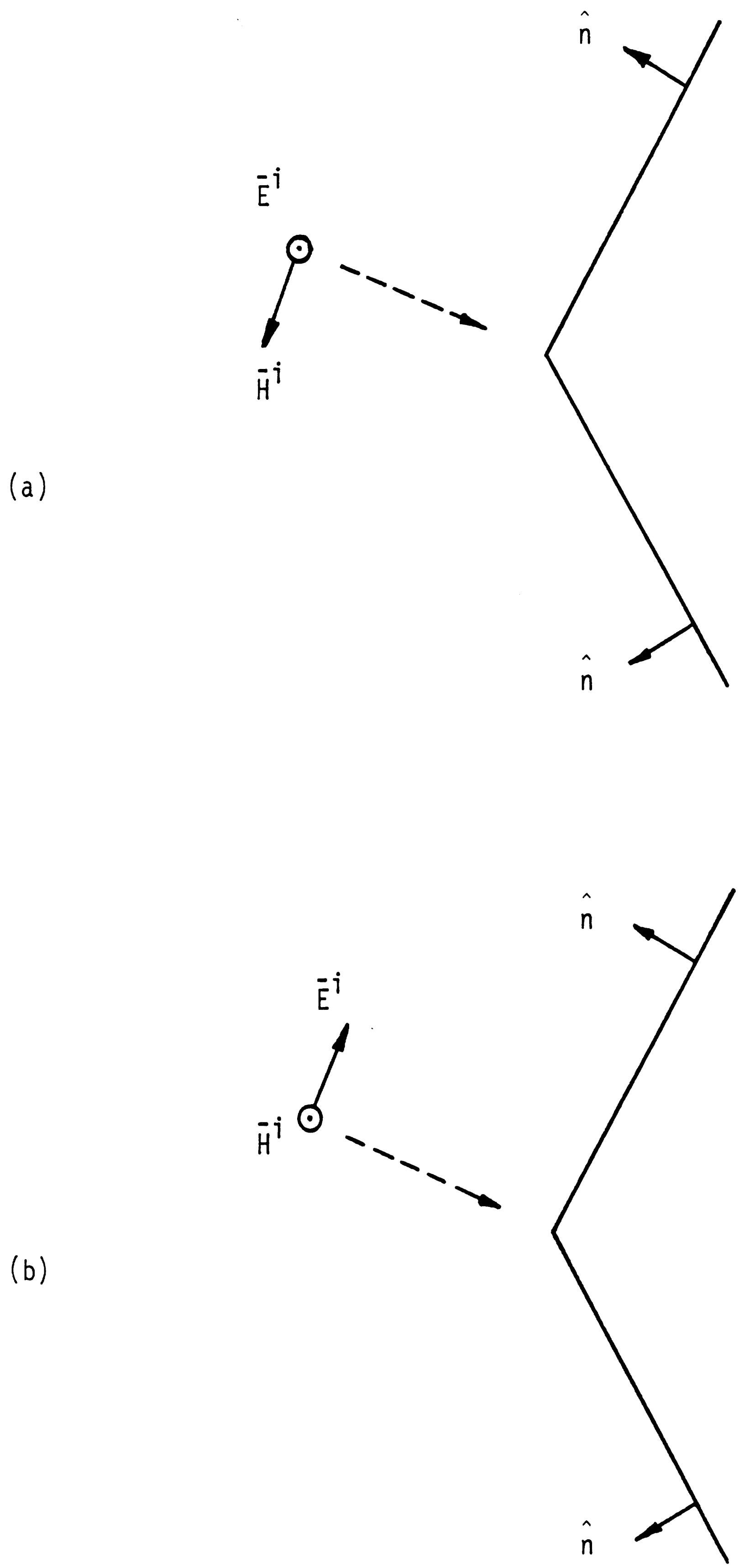


Fig. 1.5 Two-Dimensional Excitation of a Wedge by an (a) E-Polarized and (b) H-Polarized Plane Wave.

Similarly, for H-polarization, (1.5) becomes

$$\frac{\partial V}{\partial n} + ik\eta V = 0 \quad ; \quad \text{on } S_+, S_- \tag{1.14b}$$

where $\bar{H} = \hat{u}V$. Since it will be shown in Chapter II that both U and V satisfy the scalar Helmholtz equation, as well as similar edge and radiation conditions, the determination of a single quantity, U or V, is sufficient, since the duality transformation of Theorem 1 serves to specify the remaining unknown.

When the wedge satisfies resistive boundary conditions, (1.6) applies, which can be written as

$$\left[\frac{\partial U}{\partial n}\right]_-^+ + \frac{2ik}{\eta} U = 0 \tag{1.15a}$$

$$[U]_-^+ = 0 \quad ; \quad \text{on } S_+, S_- \tag{1.15b}$$

for an E-polarized incident field. For H-polarization the equivalent equations are

$$\frac{\partial V}{\partial n} + \frac{ik\eta}{2} [V]_-^+ = 0 \tag{1.16a}$$

$$\left[\frac{\partial V}{\partial n}\right]_-^+ = 0 \quad ; \quad \text{on } S_+, S_- \quad . \tag{1.16b}$$

Obviously, (1.15) and (1.16) are not duals of each other, and therefore the scattering problem for an arbitrary polarized incident plane wave requires the solution of two different scalar problems. Further complications arise from the existence of fields in both Regions 1 and 2, as implied by the bracketed quantities in (1.15) and (1.16).

For a more detailed comparison of the impedance and resistive, as well as "conductive" boundary conditions, the reader is referred to the works of Senior [27,29,30,41,42].

CHAPTER II. FORMULATION OF THE SCATTERING BY A RESISTIVE WEDGE

2.1 Statement of the Problem

Having provided a qualitative analysis of the resistive boundary condition, including its physical and mathematical implications, and having compared it to more conventional boundary conditions, a rigorous formulation of the electromagnetic scattering of a plane wave normally incident on a resistive wedge is presented.

The geometry under consideration is shown in Fig. 2.1. A wedge of included angle 2ψ composed of two resistive sheets S_+ and S_- has its vertex coincident with the z-axis of a cylindrical coordinate system (ρ,ϕ,z). The resistivity R of the sheets is a scalar constant. The azimuthal coordinate ϕ is chosen to take on the values $-\psi \leq \phi \leq 2\pi - \psi$. The upper face of the wedge lies in the plane $\phi = \psi$; the lower face in the plane $\phi = 2\pi - \psi$. The exterior region (Region 1) is defined by

$$(\rho,\phi,z) \in \text{Region 1} \Rightarrow \begin{cases} 0 \leq \rho < \infty \\ \psi \leq \phi \leq 2\pi - \psi \\ -\infty < z < \infty \end{cases} .$$

Similarly, the interior region (Region 2) is defined by

$$(\rho,\phi,z) \in \text{Region 2} \Rightarrow \begin{cases} 0 \leq \rho < \infty \\ -\psi \leq \phi \leq \psi \\ -\infty < z < \infty \end{cases} .$$

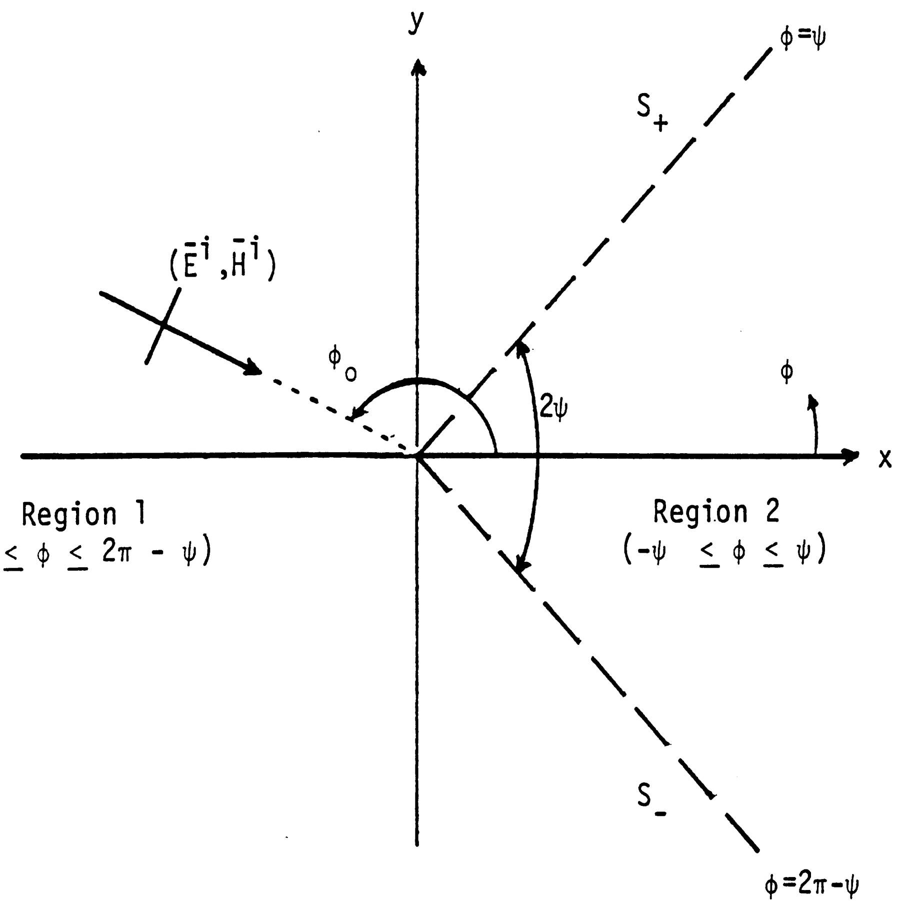


Fig. 2.1 Resistive Wedge of Included Angle 2ψ Illuminated by a Plane Wave at Normal Incidence.

Region 1 and 2 are assumed to be free space, with $\varepsilon = \varepsilon_0$, $\mu = \mu_0$. An E(H)- polarized electromagnetic plane wave with

$$\begin{bmatrix} E_z^i \\ H_z^i \end{bmatrix} = e^{-ik\rho\cos(\phi-\phi_0)} = \begin{bmatrix} U^i \\ V^i \end{bmatrix} \qquad (2.1)$$

is normally incident upon the wedge from an angle ϕ_0, which lies in Region 1 (see Fig. 2.1). Because of the symmetry of the geometry about the plane $y = 0$, it is sufficient to consider, without loss of generality, the following relation between the angle of incidence and the half angle of the wedge:

$$0 < \psi < \phi_0 \leq \pi \quad .$$

Since the entire problem is independent of the z coordinate, the two-dimensional problem in the plane $z = 0$ will be considered from this point on. As stated previously, a harmonic time variation of the form $e^{-i\omega t}$ is assumed and suppressed throughout.

Because the problem is two-dimensional, the total electromagnetic field $(\bar{E},\bar{H})$ can be determined from the two scalar quantities E_z and H_z. By assuming in turn the E- and H-polarized incident fields (2.1), (for which H_z and E_z are zero, respectively), the solution to the general problem is reduced to solving two scalar problems for a single unknown.

As implied in (2.1), the unknowns E_z,H_z will be denoted by

$$E_z = U = U^i + U^s \tag{2.2a}$$

$$H_z = V = V^i + V^s \tag{2.2b}$$

where the superscripts i and s denote the incident and scattered fields, respectively. The fields U and V are functions of the variables (ρ,ϕ), and as a result it is convenient to distinguish their values in Regions 1 and 2 via appropriate subscripts, viz,

$$U = U_1 = U_1^i + U_1^s \quad , \quad \text{for } (\rho,\phi) \in \text{Region 1} \tag{2.3a}$$

$$= U_2 = U_2^i + U_2^s \quad , \quad \text{for } (\rho,\phi) \in \text{Region 2} \quad , \tag{2.3b}$$

and similarly for V. As is traditionally the case, the incident field is assumed to permeate the entire space.

From Maxwell's curl equations in free space

$$\nabla \times \bar{E} = i\omega\mu_0 \bar{H} \quad ,$$

$$\nabla \times \bar{H} = -i\omega\varepsilon_0 \bar{E} \quad ,$$

the following results can be shown in two dimensions $(\partial/\partial z = 0)$:

<u>E-polarization</u>: $E_\rho = E_\phi = H_z = 0 \quad , \quad E_z = U$

$$ikZH_\rho = \frac{1}{\rho}\frac{\partial U}{\partial \phi} \tag{2.4a}$$

$$ikZH_\phi = -\frac{\partial U}{\partial \rho} \quad . \tag{2.4b}$$

<u>H-polarization</u>: $H_\rho = H_\phi = E_z = 0$, $H_z = V$

$$ikYE_\rho = -\frac{1}{\rho}\frac{\partial V}{\partial \phi} \tag{2.5a}$$

$$ikYE_\phi = \frac{\partial V}{\partial \rho} \ . \tag{2.5b}$$

In addition, it can be shown that both U and V satisfy the scalar Helmholtz equation

$$(\nabla^2 + k^2)\begin{bmatrix} U \\ V \end{bmatrix} = 0 \tag{2.4}$$

in Regions 1 and 2, where ∇^2 is the two-dimensional scalar Laplacian. Expressing the resistive boundary conditions (1.15) and (1.16) in polar coordinates (ρ,ϕ), the following conditions on the surfaces S_+ and S_- hold for E-polarization:

$$\begin{aligned}
&\frac{1}{\rho}\left[\frac{\partial U_1}{\partial \phi}\bigg|_{\phi=\psi} - \frac{\partial U_2}{\partial \phi}\bigg|_{\phi=\psi}\right] + \frac{2ik}{\eta} U_1\bigg|_{\phi=\psi} = 0 \\
&U_1\bigg|_{\phi=\psi} - U_2\bigg|_{\phi=\psi} = 0 \\
&\frac{1}{\rho}\left[\frac{\partial U_1}{\partial \phi}\bigg|_{\phi=2\pi-\psi} - \frac{\partial U_2}{\partial \phi}\bigg|_{\phi=-\psi}\right] - \frac{2ik}{\eta} U_1\bigg|_{\phi=2\pi-\psi} = 0 \\
&U_1\bigg|_{\phi=2\pi-\psi} - U_2\bigg|_{\phi=-\psi} = 0
\end{aligned} \tag{2.6}$$

Similarly, for H-polarization the conditions are

$$
\begin{aligned}
\frac{1}{\rho}\left.\frac{\partial V_1}{\partial \phi}\right|_{\phi=\psi} + \frac{ikn}{2}\left[\left. V_1\right|_{\phi=\psi} - \left. V_2\right|_{\phi=\psi}\right] &= 0 \\
\frac{1}{\rho}\left[\left.\frac{\partial V_1}{\partial \phi}\right|_{\phi=\psi} - \left.\frac{\partial V_2}{\partial \phi}\right|_{\phi=\psi}\right] &= 0 \\
\frac{1}{\rho}\left.\frac{\partial V_1}{\partial \phi}\right|_{\phi=2\pi-\psi} - \frac{ikn}{2}\left[\left. V_1\right|_{\phi=2\pi-\psi} - \left. V_2\right|_{\phi=-\psi}\right] &= 0 \\
\frac{1}{\rho}\left[\left.\frac{\partial V_1}{\partial \phi}\right|_{\phi=2\pi-\psi} - \left.\frac{\partial V_2}{\partial \phi}\right|_{\phi=-\psi}\right] &= 0
\end{aligned}
\qquad (2.7)
$$

Because the medium and the scattering body are both infinite in extent, the functions U and V are required to satisfy an additional constraint as $\rho \to \infty$ in the form of a Sommerfeld radiation condition. Specifically, by decomposing U and V into a geometrical optics component, obtained from ordinary ray theory, and a diffracted component, viz

$$U = U^g + U^d \qquad (2.8a)$$

$$V = V^g + V^d \qquad (2.8b)$$

each of which is a discontinuous function of ϕ, then the diffracted components satisfy*

---

*The reasoning behind this decomposition is discussed by Williams [11].

$$\lim_{\rho \to \infty} \rho^{1/2} \left[\frac{\partial}{\partial \rho} - ik \right] U^d_{1,2} = 0 \tag{2.9a}$$

$$\lim_{\rho \to \infty} \rho^{1/2} \left[\frac{\partial}{\partial \rho} - ik \right] V^d_{1,2} = 0 \tag{2.9b}$$

uniformly for all ϕ appropriate to Regions 1 and 2. Condition (2.9) is discussed in [6] and elsewhere; its vector analog, the Silver-Müller condition, can be found in several texts on electromagnetic theory, see Jones [43], for example. Roughly speaking (2.9) is equivalent to requiring that the diffracted field have the form of an outgoing cylindrical wave at infinity, decaying as the reciprocal of the square root of the distance from the line $z = 0$.

Finally, the geometrical singularity presented by the vertex of the wedge requires the specification of an additional condition governing the behavior of the fields in the vicinity of the edge. This physical constraint, known as the edge condition, is usually expressed by requiring that the stored electric and magnetic energy in any neighborhood of the edge be finite; that is

$$\lim_{V \to 0} \int_V (\varepsilon_0 |\bar{E}|^2 + \mu_0 |\bar{H}|^2) dv = 0 \quad . \tag{2.10}$$

Jones [43] derives a uniqueness theorem for finite dielectric bodies and infinite perfectly conducting bodies based on (2.9) and (2.10). He indicates, however, that these equations do not appear to be sufficient to insure uniqueness for infinite dielectric structures (such as a dielectric wedge) [44]. While the uniqueness of the resistive wedge problem has not been addressed in the literature, it will be assumed that a solution satisfying (2.1) through (2.10) provides the desired results. This assumption is legitimately questioned, since, as was pointed out in Chapter I, there exist similarities between the resistive and dielectric wedge scattering problems.

There are several physically intuitive means of expressing the edge condition [45,46] which are equivalent to (2.10) for a perfect conductor. These include zero induced charge or finite surface current and charge conditions. An overview of these works and a general discussion of the edge condition for perfect conductors can be found in Jones [43].

It is relatively easy to show for a straight edge that (2.10) is equivalent to requiring that no component of the fields be more singular than $\rho^{-\delta}$ in the neighborhood of the edge, where $\delta < 1$. More exactly,

$$\max\{f_i\} = O(\rho^{-\delta}) \quad ; \quad \rho \to 0 \quad , \tag{2.11}$$

where $\{f_i\}$ are the components of the electric and magnetic fields, and $O(\ )$ is the standard order relation, whereby

$$f = O(g) \quad ; \quad \rho \to 0$$
$$\Rightarrow \quad \exists \; A = \text{constant} \; : \; \lim_{\rho \to 0} \left(fg^{-1}\right) \leq A \quad .$$

Meixner and others [47,48] have investigated the possible values of the parameter δ for various wedge-like regions, including both perfectly conducting and dielectric wedges. By assuming series expansions in powers of ρ for the various field components, Meixner derives expressions for the series coefficients and admissible values of δ via Maxwell's equations and the boundary conditions. In particular, he shows that the lowest order (most singular) terms are equivalent to those obtained for the corresponding static field problem. An important consequence of this result is that the field components parallel to the edge (in this case, E_z and H_z) are finite for all geometries. Although some doubt has recently been expressed regarding the validity of Meixner's series expansion in the dynamic ($\omega \neq 0$) case [49], it appears that the lowest order terms are still the correct asymptotic forms.

Thus, for the resistive wedge, it is assumed that

$$U,V = O(\rho^{\delta}) \quad ; \quad \delta \geq 0 \quad , \quad \rho \to 0 \quad , \tag{2.12}$$

uniformly for all ϕ in each of Regions 1 and 2. Note that in the event that $\delta = 0$, i.e., U,V = constant at $\rho = 0$, equation (2.6) guarantees that the value of the constant is the same in Regions 1 and 2 for E-polarization. However, the same is not true for H-polarization (see (2.7)); there may be a discontinuity in the value of V at $\rho = 0$ in passing from Region 1 to Region 2. This observation is related to the fact that a resistive layer can support only an electric current, which is radially directed for H-polarization. In the event that this current is zero at the edge, then the discontinuity disappears and the value of V at $\rho = 0$ is the same in both regions.

In summary, the problem addressed in this work may be stated as follows. For the geometry of Fig. 2.1, illuminated by alternately E- and H-polarized plane waves of the form (2.1), two scalar unknowns U and V are sought which satisfy the Helmholtz equation (2.4), the boundary conditions (2.6) and (2.7), respectively, and the radiation and edge conditions (2.9) and (2.10), or equivalently, (2.12). The problem will be analyzed by applying the function-theoretic techniques of Maliuzhinets and Kontorovich and Lebedev. It is worthwhile to note that the more conventional method of mode matching, otherwise known as the method of separation of variables, is not applicable to resistive (or impedance) wedge scattering problems unless a resistivity (or impedance) which varies in a specified manner as a function of ρ is assumed (see [50], for example). Since a constant resistivity is assumed in this work, the mode matching method is not a valid approach to the problem.

Before proceeding with the analysis, a decomposition for the unknowns will be outlined which will prove instrumental in simplifying several of the resulting expressions, especially the boundary conditions (2.6) and (2.7).

2.2 The Method of Symmetric and Antisymmetric Components

It is a well known result that an arbitrary function $f(x)$ defined on an interval (a,b) can be decomposed into two functions which are respectively even (symmetric) and odd (antisymmetric) about the midpoint c of (a,b). Denoting these functions by f^e and f^o, and letting

$$f(x) = \frac{1}{2}[f^e(x) + f^o(x)] \quad , \tag{2.13}$$

then

$$\begin{aligned} f^e(x) &= f(x) + f(2c - x) \\ f^o(x) &= f(x) - f(2c - x) \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} f^e(x) &= f^e(2c - x) \\ f^o(x) &= -f^o(2c - x) \end{aligned} \quad . \tag{2.15}$$

Furthermore, if f is continuous and differentiable at $x = c$, then from (2.14) and (2.15) it can be shown that

$$\left.\frac{df^e}{dx}\right|_{x=c} = 0 \tag{2.16a}$$

$$\left.f^o\right|_{x=c} = 0 \quad . \tag{2.16b}$$

An important consequence of this result is that a knowledge of the single function $f(x)$ on (a,b) is equivalent to a knowledge of two functions, $f^e(x)$ and $f^o(x)$, on the half interval $(a,c]$.

The exterior and interior regions of the resistive wedge (Fig. 2.1) can be considered as intervals $[\psi, 2\pi-\psi]$ and $[-\psi,\psi]$ in the angular variable ϕ. Likewise, the unknowns U_1,V_1 and U_2,V_2 in these regions represent functions of ϕ on the appropriate intervals. Hence, the decomposition (2.13) can be applied to the unknowns in Regions 1 and 2 by writing

$$U_{1,2}(\rho,\phi) = \frac{1}{2}[U^e_{1,2}(\rho,\phi) + U^o_{1,2}(\rho,\phi)] \qquad (2.17a)$$

$$V_{1,2}(\rho,\phi) = \frac{1}{2}[V^e_{1,2}(\rho,\phi) + V^o_{1,2}(\rho,\phi)] \quad , \qquad (2.17b)$$

where, neglecting the ρ-dependence,

$$U_1^{e,o}(\phi) = U_1(\phi) \pm U_1(2\pi - \phi) \qquad (2.18a)$$

$$U_2^{e,o}(\phi) = U_2(\phi) \pm U_2(-\phi) \qquad (2.18b)$$

and similarly for V_1 and V_2. The plus sign corresponds to the symmetric (e) component, and the minus sign to the antisymmetric (o) component. From (2.15) it is obvious that U_1^e,V_1^e (U_1^o,V_1^o) are even (odd) about $\phi = \pi$; while U_2^e,V_2^e (U_2^o,V_2^o) are even (odd) about $\phi = 0$. Of greater significance is equation (2.16), from which it is determined that

$$\frac{\partial}{\partial\phi}(U_1^e, V_1^e)\Big|_{\phi=\pi} = 0 \qquad (2.19a)$$

$$(U_1^o, V_1^o)\Big|_{\phi=\pi} = 0 \qquad (2.19b)$$

and

$$\frac{\partial}{\partial\phi}(U_2^e, V_2^e)\Big|_{\phi=o} = 0 \qquad (2.20a)$$

$$(U_2^o, V_2^o)\Big|_{\phi=o} = 0 \quad . \qquad (2.20b)$$

It is beneficial to recall equation (2.3), in which U and V were expressed as the sum of an incident and scattered field. The same result holds for U^e, U^o, V^e, and V^o, i.e.,

$$\begin{bmatrix} U^e_{1,2} \\ \\ U^o_{1,2} \end{bmatrix} = \begin{bmatrix} U^{ei}_{1,2} + U^{es}_{1,2} \\ \\ U^{oi}_{1,2} + U^{os}_{1,2} \end{bmatrix} \qquad (2.21)$$

and likewise for $V^e_{1,2}$, $V^o_{1,2}$. It is hoped that the reader will forgive the rather cumbersome use of superscripts and subscripts which the author has employed. From (2.18) and (2.1) it is simple to show that

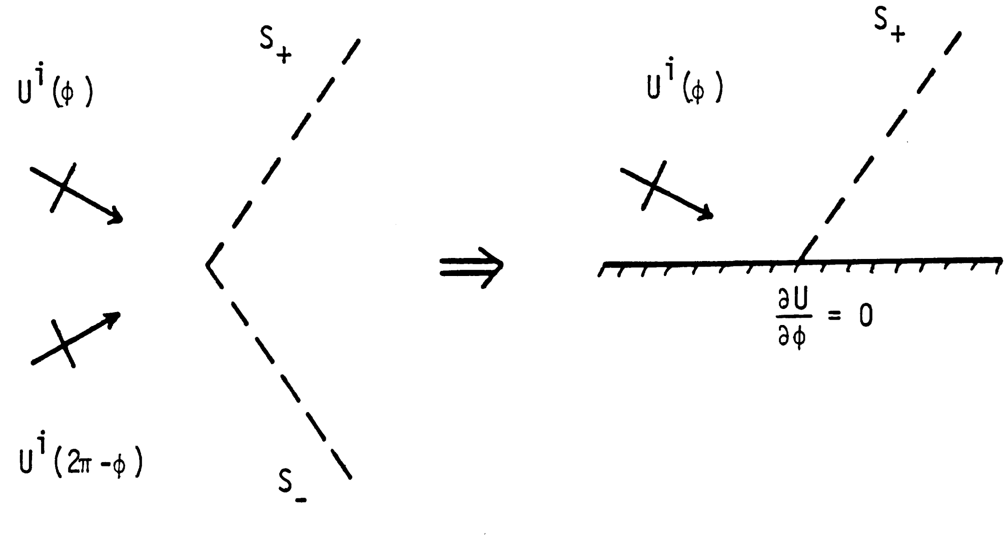
$$U_1^{ei} = U_2^{ei} = V_1^{ei} = V_2^{ei} = e^{ik\rho\cos(\phi-\phi_0)} + e^{ik\rho\cos(\phi+\phi_0)} \qquad (2.22a)$$

$$U_1^{oi} = U_2^{oi} = V_1^{oi} = V_2^{oi} = e^{ik\rho\cos(\phi-\phi_0)} - e^{ik\rho\cos(\phi+\phi_0)} \; . \qquad (2.22b)$$

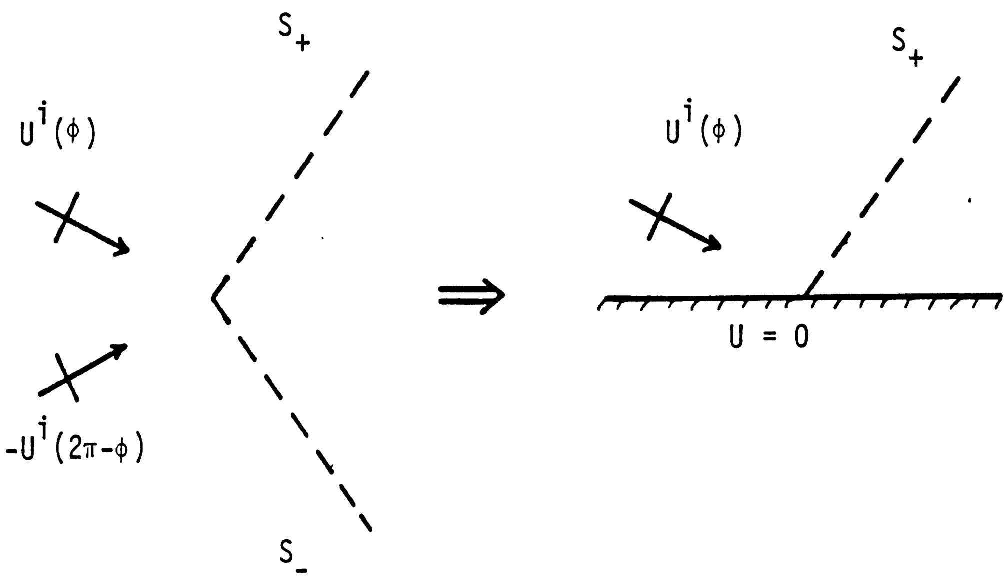
Therefore the notation for the various incident field components can be simplified by denoting the right-hand sides of equations (2.22a,b) by I^e and I^o, respectively. Equation (2.21) now has the less confusing form

$$\begin{bmatrix} U^e_{1,2} \\ U^o_{1,2} \\ V^e_{1,2} \\ V^o_{1,2} \end{bmatrix} = \begin{bmatrix} I^e + U^{es}_{1,2} \\ I^o + U^{os}_{1,2} \\ I^e + V^{es}_{1,2} \\ I^o + V^{os}_{1,2} \end{bmatrix} \; . \qquad (2.23)$$

From the symmetry of the geometry it is easy to show that the left-hand side of (2.23) satisfies the boundary conditions of the original problem (2.6) and (2.7), as well as the wave equation and radiation and edge conditions. Thus $(U^e_{1,2}, V^e_{1,2})$ are the E- and H-polarized solutions to the scattering problem with incident field I^e, while $(U^o_{1,2}, V^o_{1,2})$ are the E- and H-polarized solutions for the incident field I^o. Furthermore, by essentially replacing the boundary conditions on the lower wedge surface S_- by the conditions (2.19) and (2.20), it is sufficient to determine the unknowns in the upper half space $y \geq 0$ to completely solve the problem, as illustrated in Fig. 2.2.



(a)



(b)

Fig. 2.2: Symmetric (a) and Antisymmetric (b) Excitation of a Resistive Wedge and the Equivalent Half-Space Problems. Analogous Results hold for $V(\phi)$.

The new boundary conditions for $U^e_{1,2}$ become

$$\frac{1}{\rho}\left[\frac{\partial U_1^e}{\partial\phi} - \frac{\partial U_2^e}{\partial\phi}\right]\Bigg|_{\phi=\psi} + \frac{2ik}{\eta} U_1^e\Big|_{\phi=\psi} = 0 \tag{2.24a}$$

$$[U_1^e - U_2^e]\Big|_{\phi=\psi} = 0 \tag{2.24b}$$

$$\frac{\partial U_1^e}{\partial\phi}\Bigg|_{\phi=\pi} = 0 \tag{2.24c}$$

$$\frac{\partial U_2^e}{\partial\phi}\Bigg|_{\phi=0} = 0 \ , \tag{2.24d}$$

while for $V^e_{1,2}$ the following hold:

$$\frac{1}{\rho}\ \frac{\partial V_1^e}{\partial\phi}\Bigg|_{\phi=\psi} + \frac{ik\eta}{2}[V_1^e - V_2^e]\Big|_{\phi=\psi} = 0 \tag{2.25a}$$

$$\frac{1}{\rho}\left[\frac{\partial V_1^e}{\partial\phi} - \frac{\partial V_2^e}{\partial\phi}\right]\Bigg|_{\phi=\psi} = 0 \tag{2.25b}$$

$$\frac{\partial V_1^e}{\partial\phi}\Bigg|_{\phi=\pi} = 0 \tag{2.25c}$$

$$\frac{\partial V_2^e}{\partial\phi}\Bigg|_{\phi=0} = 0 \ . \tag{2.25d}$$

For $U^o_{1,2}, V^o_{1,2}$, equations (2.24c,d) and (2.25c,d) are respectively replaced by the primed equations

$$U^o_1\Big|_{\phi=\pi} = 0 \qquad (2.24c')$$

$$U^o_2\Big|_{\phi=o} = 0 \quad , \qquad (2.24d')$$

and

$$V^o_1\Big|_{\phi=\pi} = 0 \qquad (2.25c')$$

$$V^o_2\Big|_{\phi=o} = 0 \quad . \qquad (2.25d')$$

In the remaining equations the symmetric (e) components are simply replaced by the antisymmetric (o) components.

The most important consequence of the modified conditions (2.24 through 2.25) is the presence of only a single "mixed" transition equation for each unknown, i.e., equations (2.24a) and (2.25a). These conditions, which contain the only dependence on the resistivity η, are most responsible for the complexity of the problem. The method of symmetric and antisymmetric components has essentially eliminated this equation for the lower sheet. The implications of this result will become apparent in Chapters III and IV, in which the scattering problem will be analyzed via Maliuzhinets' method and the Kontorovich-Lebedev transform.

CHAPTER III. THE METHOD OF MALIUZHINETS

By far the most successful technique for solving boundary value problems in wedge-like regions was put forth by G. D. Maliuzhinets in his doctoral thesis [13] and subsequent publications [14,15]. His method, which fundamentally is a generalization of the method of images described by Carslaw [3], was able to provide the solution for the scattering from an impedance wedge with differing impedances on each of the faces [15]. Various aspects of Maliuzhinets' technique were inadvertently and independently described by Senior [16] and Williams [17] in solving similar problems.

The basis for Maliuzhinets' method is the ability to represent the total scalar fields (U,V) as a Sommerfeld integral of the form

$$\begin{bmatrix} U(\rho,\phi) \\ \\ V(\rho,\phi) \end{bmatrix} = \frac{1}{2\pi i} \int_{\gamma} e^{ik\rho\cos\alpha} \begin{bmatrix} s(\alpha - \phi) \\ \\ t(\alpha - \phi) \end{bmatrix} d\alpha \quad , \qquad (3.1)$$

where γ is a contour in the complex α-plane, consisting of two loops γ_1 and γ_2, symmetric about the point $\alpha = \pi$, as illustrated in Fig. 3.1. A contour integral of this type was first employed by Sommerfeld in his classic solution for the scattering of a plane wave by a perfectly conducting half-plane [1], and as a result (3.1) is often referred to as a Sommerfeld integral. Maliuzhinets has shown that (3.1) has

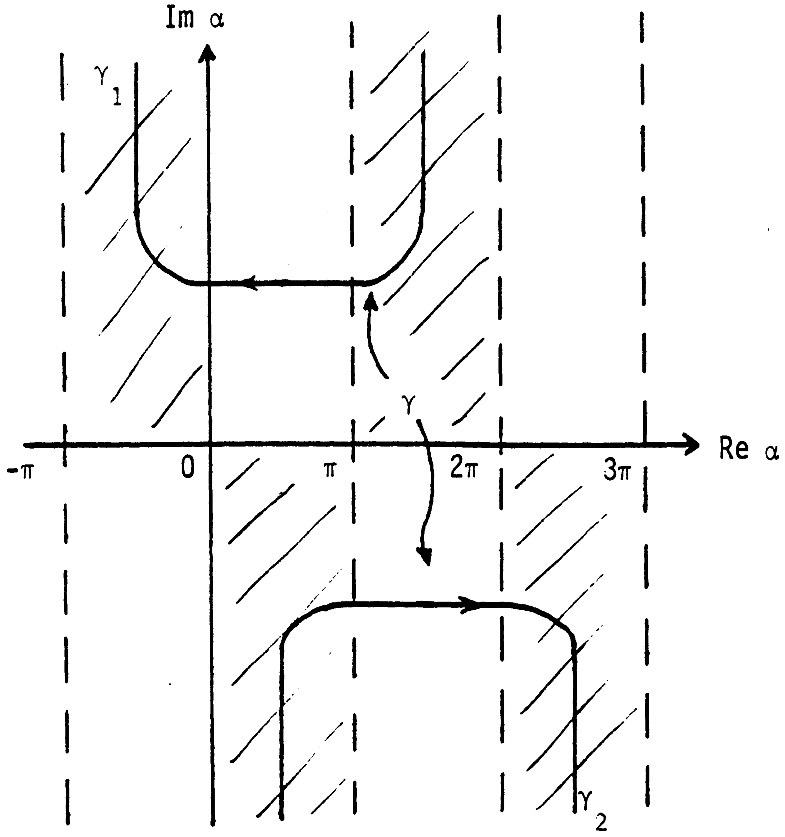


Fig. 3.1: The Contour of Integration γ in the Complex α-Plane for Use with the Maliuzhinets Method.

a unique inversion (s,t) within a particular set of functions, provided (U,V) satisfy certain boundedness conditions at $\rho = 0$ and $\rho \to \infty$. An outline of Maliuzhinets' uniqueness theorem is given in Appendix A.

A direct consequence of this uniqueness theorem is that a nonzero solution of the homogeneous equation

$$\int_{\gamma} f(\alpha) e^{ik\rho\cos\alpha} \, d\alpha = 0 \tag{3.2}$$

must be even about the point $\alpha = \pi$, i.e.,

$$f(\alpha) = f(2\pi - \alpha) \quad , \tag{3.3}$$

provided $f(\alpha) = O(e^{(1-a)|\mathrm{Im}\alpha|})$ as $|\mathrm{Im}\,\alpha| \to \infty$ within the loops γ_1 and γ_2, where $a > 0$. This result is also derived in Appendix A. The order relation above holds for functions of (ρ,ϕ) satisfying the radiation and edge conditions (2.9) and (2.10), respectively.

Recalling the representation (3.1) for (U,V), it is easy to show that indeed (U,V) satisfy the Helmholtz equation

$$(\nabla^2 + k^2) \begin{bmatrix} U \\ V \end{bmatrix} = 0 \quad . \tag{3.4}$$

In order to apply (3.1) to the boundary conditions (2.6) and (2.7) (note the (3.1) is a representation for the total fields (U,V)), it is appropriate to first consider the normal derivative

$$\frac{1}{\rho}\frac{\partial}{\partial\phi}\begin{bmatrix} U \\ V \end{bmatrix} = \frac{1}{2\pi i\rho}\int_{\gamma} e^{ik\rho\cos\alpha}\frac{\partial}{\partial\phi}\begin{bmatrix} s(\alpha-\phi) \\ t(\alpha-\phi) \end{bmatrix} d\alpha$$

$$= -\frac{1}{2\pi i\rho}\int_{\gamma} e^{ik\rho\cos\alpha}\frac{\partial}{\partial\alpha}\begin{bmatrix} s(\alpha-\phi) \\ t(\alpha-\phi) \end{bmatrix} d\alpha\ . \quad (3.4)$$

The interchange of the order of differentiation and integration is justified by the uniform convergence of the integral along γ provided (s,t) satisfy the order relation mentioned previously. Integration by parts leads to

$$\frac{1}{\rho}\frac{\partial}{\partial\phi}\begin{bmatrix} U \\ V \end{bmatrix} = -\frac{1}{2\pi i\rho}\, e^{ik\rho\cos\alpha}\left[\begin{matrix} s(\alpha-\phi) \\ t(\alpha-\phi) \end{matrix}\Bigg|_{\alpha\to\gamma_\infty}\right]$$

$$-\frac{k}{2\pi}\int_{\gamma}\sin\alpha\; e^{ik\rho\cos\alpha}\begin{bmatrix} s(\alpha-\phi) \\ t(\alpha-\phi) \end{bmatrix} d\alpha \quad (3.5)$$

where γ_∞ denotes the various endpoints of the contour γ as $|\mathrm{Im}\ \alpha| \to \infty$. From the behavior of (s,t) and the exponential (for $\rho > 0$) within the shaded portions of Fig. 3.1, it can be shown that the first term of equation (3.5) is zero, and thus

$$\frac{1}{\rho}\frac{\partial}{\partial\phi}\begin{bmatrix} U \\ V \end{bmatrix} = -\frac{k}{2\pi}\int_{\gamma}\sin\alpha\; e^{ik\rho\cos\alpha}\begin{bmatrix} s(\alpha-\phi) \\ t(\alpha-\phi) \end{bmatrix} d\alpha\ . \quad (3.6)$$

Substitution of (3.1) and (3.6) into the boundary conditions (2.6) and (2.7) results in the following equations for (s,t): for E-polarization

$$\int_{\gamma} \left[\sin\alpha \left(s_1(\alpha - \psi) - s_2(\alpha - \psi) \right) - \frac{2}{\eta} s_1(\alpha - \psi) \right] e^{ik\rho\cos\alpha}\, d\alpha = 0$$

$$\int_{\gamma} \left[s_1(\alpha - \psi) - s_2(\alpha - \psi) \right] e^{ik\rho\cos\alpha}\, d\alpha = 0$$

$$\int_{\gamma} \left[\sin\alpha \left(s_1(\alpha - 2\pi + \psi) - s_2(\alpha + \psi) \right) + \frac{2}{\eta} s_1(\alpha - 2\pi + \psi) \right] e^{ik\rho\cos\alpha}\, d\alpha = 0$$

$$\int_{\gamma} \left[s_1(\alpha - 2\pi + \psi) - s_2(\alpha + \psi) \right] e^{ik\rho\cos\alpha}\, d\alpha = 0 \tag{3.7}$$

and for H-polarization:

$$\int_{\gamma} \left[\sin\alpha\, t_1(\alpha - \psi) - \frac{\eta}{2} \left(t_1(\alpha - \psi) - t_2(\alpha - \psi) \right) \right] e^{ik\rho\cos\alpha}\, d\alpha = 0$$

$$\int_{\gamma} \sin\alpha \left[t_1(\alpha - \psi) - t_2(\alpha - \psi) \right] e^{ik\rho\cos\alpha}\, d\alpha = 0$$

$$\int_{\gamma} \left[\sin\alpha\, t_1(\alpha - 2\pi + \psi) + \frac{\eta}{2} \left(t_1(\alpha - 2\pi + \psi) - t_2(\alpha + \psi) \right) \right] e^{ik\rho\cos\alpha}\, d\alpha = 0$$

$$\int_{\gamma} \sin\alpha \left[t_1(\alpha - 2\pi + \psi) - t_2(\alpha + \psi) \right] e^{ik\rho\cos\alpha}\, d\alpha = 0 \tag{3.8}$$

In equations (3.7) and (3.8) the subscripts 1,2 on (s,t) correspond to the values of the functions in Regions 1,2, respectively.

Following Maliuzhinets' derivation for the impedance wedge [15], and by virtue of equations (3.2), (3.3), and the results of Appendix A, equations (3.7) and (3.8) are equivalent to the following set of functional equations:

E-polarization:

$$\eta \sin\alpha[s_1(\alpha - \psi) - s_2(\alpha - \psi)] - 2s_1(\alpha - \psi)$$

$$= -\eta \sin\alpha[s_1(2\pi - \alpha - \psi) - s_2(2\pi - \alpha - \psi)] - 2s_1(2\pi - \alpha - \psi) \quad (3.9a)$$

$$s_1(\alpha - \psi) - s_2(\alpha - \psi) = s_1(2\pi - \alpha - \psi) - s_2(2\pi - \alpha - \psi) \quad (3.9b)$$

$$\eta \sin\alpha[s_1(\alpha - 2\pi + \psi) - s_2(\alpha + \psi)] + 2s_1(\alpha - 2\pi + \psi)$$

$$= -\eta \sin\alpha[s_1(-\alpha + \psi) - s_2(2\pi - \alpha + \psi)] + 2s_1(-\alpha + \psi) \quad (3.9c)$$

$$s_1(\alpha - 2\pi + \psi) - s_2(\alpha + \psi) = s_2(-\alpha + \psi) - s_2(2\pi - \alpha + \psi) \quad (3.9d)$$

H-polarization:

$$2 \sin\alpha\, t_1(\alpha - \psi) - \eta[t_1(\alpha - \psi) - t_2(\alpha - \psi)]$$

$$= -2 \sin\alpha\, t_1(2\pi - \alpha - \psi) - \eta[t_1(2\pi - \alpha - \psi) - t_2(2\pi - \alpha - \psi)] \quad (3.10a)$$

$$t_1(\alpha - \psi) - t_2(\alpha - \psi) = -t_1(2\pi - \alpha - \psi) + t_2(2\pi - \alpha - \psi) \quad (3.10b)$$

$$2 \sin \alpha \, t_1(\alpha - 2\pi + \psi) + \eta[t_1(\alpha - 2\pi + \psi) - t_2(\alpha + \psi)]$$

$$= -2 \sin \alpha \, t_1(-\alpha + \psi) + \eta[t_1(-\alpha + \psi) - t_2(2\pi - \alpha + \psi)] \quad (3.10c)$$

$$t_1(\alpha - 2\pi + \psi) - t_2(\alpha + \psi) = -t_1(-\alpha + \psi) + t_2(2\pi - \alpha + \psi) \quad (3.10d)$$

Equations (3.9) and (3.10) can be put in simpler terms. Adding (3.9a) and (3.9b) and subtracting (3.9c) and (3.9d) results in two functional equations of similar form:

$$(1 - \eta \sin \alpha)S_1(\alpha) = (1 + \eta \sin \alpha)S_1(2\pi - \alpha) \quad (3.11a)$$

$$(1 - \eta \sin \alpha)S_2(\alpha) = (1 + \eta \sin \alpha)S_2(2\pi - \alpha) \;, \quad (3.11b)$$

where

$$S_1(\alpha) = s_1(\alpha - \psi) - s_2(2\pi - \alpha - \psi) \quad (3.12)$$

$$S_2(\alpha) = s_1(-\alpha + \psi) - s_2(\alpha + \psi) \;. \quad (3.12)$$

Analogously, equations (3.10a) through (3.10d) can be reduced to

$$(\eta - \sin \alpha)T_1(\alpha) = (\eta + \sin \alpha)T_1(2\pi - \alpha) \quad (3.13a)$$

$$(\eta - \sin \alpha)T_2(\alpha) = (\eta + \sin \alpha)T_2(2\pi - \alpha) \quad (3.13b)$$

where

$$T_1(\alpha) = t_1(\alpha - \psi) + t_2(2\pi - \alpha - \psi)$$

$$T_2(\alpha) = t_1(-\alpha + \psi) + t_2(\alpha + \psi) \quad . \tag{3.14}$$

Hence, equations (3.11) and (3.13) represent a set of coupled first-order functional equations for the unknowns s_1, s_2 and t_1, t_2 respectively. An important observation is that S_1, S_2, T_1, T_2 all satisfy identical equations, and therefore can differ at most by a multiplicative factor $M(\alpha)$ satisfying (see Chapter 2 of [36])

$$M(\alpha) = M(2\pi - \alpha) \quad . \tag{3.15}$$

Having determined S_1 and S_2, one can obtain an uncoupled set of first-order, inhomogeneous functional equations for s_1 and s_2, viz

$$s_1(\alpha - \psi) - s_1(\alpha - 2\pi + 3\psi) = S_1(\alpha) - S_2(2\pi - \alpha - 2\psi) \tag{3.16a}$$

$$s_2(\alpha - \psi) - s_2(\alpha - 2\pi + 3\psi) = S_2(\alpha - 2\pi + 2\psi) - S_1(2\pi - \alpha) \quad . \tag{3.16b}$$

A similar result holds for t_1 and t_2, i.e.,

$$t_1(\alpha - \psi) - t_1(\alpha - 2\pi + 3\psi) = T_1(\alpha) - T_2(2\pi - \alpha - 2\psi) \tag{3.17a}$$

$$t_2(\alpha - \psi) - t_2(\alpha - 2\pi + 3\psi) = -T_2(\alpha - 2\pi + 2\psi) + T_1(2\pi - \alpha) \tag{3.17b}$$

Once again there is a similarity among the functional equations (3.16) and (3.17) describing $s_{1,2}$ and $t_{1,2}$. Indeed, if $\chi(\alpha)$ is an arbitrary solution of the homogeneous equation

$$\chi(\alpha - \psi) - \chi(\alpha - 2\pi + 3\psi) = 0 \qquad (3.18)$$

then s_1, s_2, t_1, t_2 are all of the form

$$\begin{bmatrix} s_{1,2} \\ t_{1,2} \end{bmatrix} = \chi(\alpha) + P(\alpha) \quad , \qquad (3.19)$$

where $P(\alpha)$ is a solution of the appropriate inhomogeneous equation (3.16) or (3.17). That is to say, $s_{1,2}$ and $t_{1,2}$ are determined up to an additive function of period $2\pi - 4\psi$.

In order to uniquely determine $s_{1,2}, t_{1,2}$ additional constraints are required. One such constraint is the order relation previously imposed:

$$\begin{bmatrix} s_{1,2} \\ t_{1,2} \end{bmatrix} = O(e^{(1-a)|\mathrm{Im}\,\alpha|}) \quad ; \qquad |\mathrm{Im}\,\alpha| \to \infty \qquad (3.20)$$

within and on the loops γ_1 and γ_2. Furthermore, as is shown in Appendix A, for an incident field of the form

$$U^i = V^i = e^{-ik\rho\cos(\phi - \phi_0)} \quad , \qquad (3.21)$$

the functions

$$\begin{bmatrix} s_1(\alpha - \pi) \\ t_1(\alpha - \pi) \end{bmatrix} - (\alpha - 2\pi + \phi_0)^{-1} \tag{3.22a}$$

must be regular in the strip $\psi \leq \mathrm{Re}\,\alpha \leq 2\pi - \psi$, while

$$\begin{bmatrix} s_2(\alpha) \\ t_2(\alpha) \end{bmatrix} \tag{3.22b}$$

must be regular in the strip $\pi - \psi \leq \mathrm{Re}\,\alpha \leq \pi + \psi$. Conditions (3.20) and (3.22) serve to limit the choices of $\chi(\alpha)$ and $P(\alpha)$, as well as $M(\alpha)$ (see (3.15)) in determining the unknowns s_1, s_2, t_1, t_2.

In the event that it may be advantageous to preclude the derivation of the corresponding conditions on S_1, S_2, T_1, T_2, as well as to avoid the large number of arbitrary unknowns χ, P, M and so on, it is possible to eliminate s_2 and t_2 from equations (3.9) and (3.10) directly at the expense of arriving at a pair of considerably more complex functional equations for s_1 and t_1. After some rather tedious manipulation, one can obtain the following:

$$\sin(\alpha - 2\psi)\left\{[1 - \eta \sin\alpha]\, s_1(\alpha - \psi) - s_1(2\pi - \alpha - \psi)\right\}$$

$$= -\sin\alpha\left\{[1 + \eta \sin(\alpha - 2\psi)]\, s_1(\alpha - 2\pi - \psi) - s_1(-\alpha + 3\psi)\right\} \tag{3.23}$$

$$[\eta - \sin\alpha]t_1(\alpha - \psi) - \sin\alpha\ t_1(2\pi - \alpha - \psi)$$

$$= [\eta + \sin(\alpha - 2\psi)]t_1(\alpha - 2\pi - \psi) + \sin(\alpha - 2\psi)t_1(-\alpha + 3\psi) \ . \qquad (3.24)$$

Similar expressions can be derived for s_2 and t_2 , or they can be obtained directly from s_1 and t_1 , respectively.

It is evident that (3.23) and (3.24) are linear third-order functional equations of the unknowns s_1 and t_1. Theoretically, solutions to this type of equation exist, but there is no straightforward means of deriving them, except in certain special cases [36]. This is analogous to the task of finding solutions to general differential equations of order greater than one.

To further emphasize the complexity of equations (3.23) and (3.24), consider the corresponding functional equation developed by Maliuzhinets in solving the scattering by an impedance wedge, namely

$$(1 - \eta \sin\alpha)s(\alpha - \psi) = (1 + \eta \sin\alpha)s(\alpha - 2\pi + \psi) \ . \qquad (3.25)$$

Equation (3.25) is a linear first-order functional equation for $s(\alpha)$. Two important differences exist between (3.25) and either (3.23) or (3.24):

1. The functional equations for the resistive wedge (3.23) and (3.24) are of third order, while (3.25) for the impedance wedge is of first order. This difference in itself is sufficient to preclude the possibility of finding a solution.

2. Equation (3.25) is a special type of functional equation known as a difference equation, where the functional dependence of the argument of s is in the form of a finite increment. In (3.25), the increment is $2\pi - 2\psi$. However, (3.23) and (3.24) are not difference equations. This is evident from the fact that the arguments of s_1 and t_1 contain dependences on both $+\alpha$ and $-\alpha$, which cannot be expressed via a simple increment. Thus the author was unable to apply techniques appropriate to difference equations toward finding a solution.

In order to possibly obviate the difficulties discussed above, the method of symmetric and antisymmetric components outlined in Section 2.2 is considered. By writing

$$\begin{bmatrix} U^e \\ V^e \end{bmatrix} = \frac{1}{2\pi i} \int_\gamma e^{ik\rho\cos\alpha} \begin{bmatrix} s^e(\alpha - \phi) \\ t^e(\alpha - \phi) \end{bmatrix} d\alpha \qquad (3.26)$$

and similarly for U^o, V^o, and then applying the boundary conditions (2.24) and (2.25), four sets of functional equations are developed:

<u>E-polarization</u> (symmetric):

$$\eta \sin \alpha[s_1^e(\alpha - \psi) - s_2^e(\alpha - \psi)] - 2s_1^e(\alpha - \psi)$$

$$= -\eta \sin \alpha[s_1^e(2\pi - \alpha - \psi) - s_2^e(2\pi - \alpha - \psi)] - 2s_1^e(2\pi - \alpha - \psi) \quad (3.27a)$$

$$s_1^e(\alpha - \psi) - s_2^e(\alpha - \psi) = s_1^e(2\pi - \alpha - \psi) - s_2^e(2\pi - \alpha - \psi) \quad (3.27b)$$

$$s_1^e(\alpha - \pi) = - s_1^e(\pi - \alpha) \quad (3.27c)$$

$$s_2^e(\alpha) = -s_2^e(2\pi - \alpha) \quad (3.27d)$$

<u>E-polarization</u> (antisymmetric):

$$\eta \sin \alpha[s_1^o(\alpha - \psi) - s_2^o(\alpha - \psi)] - 2s_1^o(\alpha - \psi)$$

$$= -\eta \sin \alpha[s_1^o(2\pi - \alpha - \psi) - s_2^o(2\pi - \alpha - \psi)] - 2s_1^o(2\pi - \alpha - \psi) \quad (3.28a)$$

$$s_1^o(\alpha - \psi) - s_2^o(\alpha - \psi) = s_1^o(2\pi - \alpha - \psi) - s_2^o(2\pi - \alpha - \psi)$$

$$(3.28b)$$

$$s_1^o(\alpha - \pi) = s_1^o(\pi - \alpha) \quad (3.28c)$$

$$s_2^o(\alpha) = s_2^o(2\pi - \alpha) \quad (3.28d)$$

<u>H-polarization</u> (symmetric):

$$2 \sin\alpha\, t_1^e(\alpha - \psi) - \eta[t_1^e(\alpha - \psi) - t_2^e(\alpha - \psi)]$$

$$= -2 \sin\alpha\, t_1^e(2\pi - \alpha - \psi) - \eta[t_1^e(2\pi - \alpha - \psi) - t_2^e(2\pi - \alpha - \psi)] \quad (3.29a)$$

$$t_1^e(\alpha - \psi) - t_2^e(\alpha - \psi) = -t_1^e(2\pi - \alpha - \psi) + t_2^e(2\pi - \alpha - \psi) \quad (3.29b)$$

$$t_1^e(\alpha - \pi) = -t_1^e(\pi - \alpha) \quad (3.29c)$$

$$t_2^e(\alpha) = -t_2^e(2\pi - \alpha) \quad (3.29d)$$

<u>H-polarization</u> (antisymmetric):

$$2 \sin\alpha\, t_1^o(\alpha - \psi) - \eta[t_1^o(\alpha - \psi) - t_2^o(\alpha - \psi)]$$

$$= -2 \sin\alpha\, t_1^o(2\pi - \alpha - \psi) - \eta[t_1^o(2\pi - \alpha - \psi) - t_2^o(2\pi - \alpha - \psi)] \quad (3.30a)$$

$$t_1^o(\alpha - \psi) - t_2^o(\alpha - \psi) = -t_1^o(2\pi - \alpha - \psi) + t_2^o(2\pi - \alpha - \psi) \quad (3.30b)$$

$$t_1^o(\alpha - \pi) = t_1^o(\pi - \alpha) \quad (3.30c)$$

$$t_2^o(\alpha) = t_2^o(2\pi - \alpha) \quad (3.30d)$$

Again, the subscripts 1 and 2 refer to the values of the corresponding function in Regions 1 and 2. As noted in Section 2.2 when deriving the boundary conditions, the only difference between (3.9) and (3.27-28) or (3.10) and (3.29-30) is in the final two equations, which are considerably simpler in form. However, when the algebraic manipulations are performed which eliminate s_2^e, s_2^o, t_2^e, t_2^o as was done for equations (3.23) and (3.24), there appears to be little gain. The resulting equations are:

E-polarization:

$$\sin(\alpha - 2\psi)\left\{[1 - \eta \sin\alpha]\, s_1^e(\alpha - \psi) + s_1^e(\alpha - 2\pi + \psi)\right\}$$

$$= -\sin\alpha\left\{[1 + \eta \sin(\alpha - 2\psi)]\, s_1^e(\alpha - 2\pi - \psi) + s_1^e(\alpha - 3\psi)\right\} \qquad (3.32)$$

$$\sin(\alpha - 2\psi)\left\{[1 - \eta \sin\alpha]\, s_1^o(\alpha - \psi) - s_1^o(\alpha - 2\pi + \psi)\right\}$$

$$= -\sin\alpha\left\{[1 + \eta \sin(\alpha - 2\psi)]\, s_1^o(\alpha - 2\pi - \psi) - s_1^o(\alpha - 3\psi)\right\} \qquad (3.33)$$

H-polarization:

$$[\eta - \sin\alpha]\, t_1^e(\alpha - \psi) + \sin\alpha\, t_1^e(\alpha - 2\pi + \psi)$$

$$= [\eta + \sin(\alpha - 2\psi)]\, t_1^e(\alpha - 2\pi - \psi) - \sin(\alpha - 2\psi)\, t_1^e(\alpha - 3\psi) \qquad (3.34)$$

$$[\eta - \sin\alpha]\, t_1^o(\alpha - \psi) - \sin\alpha\, t_1^o(\alpha - 2\pi + \psi)$$

$$= [\eta + \sin(\alpha - 2\psi)]\, t_1^o(\alpha - 2\pi - \psi) + \sin(\alpha - 2\psi)\, t_1^o(\alpha - 3\psi) \qquad (3.35)$$

Similar expressions are available for $s_2^{e,o}, t_2^{e,o}$, or they can be derived directly from $s_1^{e,o}, t_1^{e,o}$ respectively. Comparison of (3.32-35) with (3.23-24) indicates that the decomposition into symmetric and antisymmetric components has done little to simplify the form of the functional equations. There has been no reduction in order; (3.32-35) are still third-order equations. If any improvement has been made, it is that (3.32-35) are now "difference" equations, in that the arguments of the unknowns are of the form $\alpha + \Delta_n$, $n = 0,1,2,3$, where Δ_n is a finite increment, and the increments are not uniform (i.e., $\Delta_{n+1} - \Delta_n \neq$ constant). It is perhaps in doubt whether any of the techniques available for solving difference equations are applicable to equations of this type.

Unique solutions for $s_1^{e,o}, t_1^{e,o}$ are guaranteed by requiring that they satisfy the order relation (3.20) with $a > 0$. The regularity condition (3.22a) must be modified to account for the presence of additional plane waves in the incident field (see equation (2.22)); the result is that

$$\begin{bmatrix} s_1^{e,o}(\alpha - \pi) \\ t_1^{e,o}(\alpha - \pi) \end{bmatrix} - (\alpha - 2\pi + \phi_0)^{-1} \mp (\alpha - \phi_0)^{-1} .$$

must be regular in the strip $\psi \leq \mathrm{Re}\,\alpha \leq 2\pi - \psi$. The minus (plus) sign corresponds to the symmetric (antisymmetric) components, respectively.

Equations (3.32-33) are the fundamental functional equations for the resistive wedge problem formulated via Maliuzhinets' method. The author has attempted to solve them through appropriate substitutions and/or factorizations, with no success. Methods described in texts on functional equations [36] are not applicable to third-order equations of this type. Though an exact solution has not been found, equations (3.32-33) are amenable to iterative techniques for generating approximate solutions. However, it is not within the scope of this work to pursue those methods here. Instead, an alternative formulation based upon the Kontorovich-Lebedev transform is presented in the next chapter.

CHAPTER IV. THE KONTOROVICH-LEBEDEV TRANSFORM

4.1 The Kontorovich-Lebedev (K-L) Transform and Its Relationship to the Maliuzhinets Representation

In 1938, two Russian authors, M. J. Kontorovich and N. N. Lebedev, put forth a cylindrical or radial transformation, along with the corresponding inverse, which is analogous to the LaPlace (or Fourier) transform in Cartesian or linear coordinates [21]. The transform, which now bears their names, found applications in boundary value and diffraction problems where the unknown functions are defined along the radial coordinate ρ of a cylindrical coordinate system (ρ,ϕ,z).

If the K-L transform of a function $f(\rho)$ is denoted by

$$\tilde{f}(\nu) = K[f(\rho)] \quad , \tag{4.1}$$

where ν is the transform variable, then, similarly, the inverse transform is written as

$$f(\rho) = K^{-1}[\tilde{f}(\nu)] \quad , \tag{4.2}$$

provided both $K[f(\rho)]$ and $K^{-1}[f(\nu)]$ are defined.

In [21], the transformation is presented in the form of a theorem providing sufficient conditions on a function $\tilde{f}(\nu)$ for the inversion (4.2) to exist. The theorem can be stated as follows:

Theorem 2:

Given a complex number $k = |k|e^{i\delta}$, $0 < \delta < \pi$, and $w(\nu)$ a function of the complex variable $\nu = \sigma + i\tau$, with $w(\nu)$ satisfying the following conditions:

1. $w(\nu)$ is regular (analytic) in the strip $|\mathrm{Re}\ \nu| < \beta$, $\beta > 0$,
2. $w(\nu)$ is an even function of ν, i.e., $w(\nu) = w(-\nu)$,
3. the integral

$$\int_{-\infty}^{\infty} |(\sigma + i\tau)w(\sigma + i\tau)| e^{\delta\tau + (\pi/2)(|\tau| - \tau)} \, d\tau < \infty$$

   for all $|\sigma| < \beta$, and

4. $|(\sigma + i\tau)w(\sigma + i\tau)| \; e^{\delta\tau + (\pi/2)(|\tau| - \tau)} \to 0$ as $|\tau| \to \infty$ uniformly for all $|\sigma| < \beta$.

Then supposing that

$$f(\rho) = -\frac{1}{2}\int_{-i\infty}^{i\infty} \mu w(\mu) e^{-i(\pi/2)\mu} J_\mu(k\rho)\, d\mu$$

where $\rho > 0$, it follows that

$$w(\nu) = \int_0^\infty f(\rho)\; e^{i(\pi/2)\nu}\; H_\nu^{(1)}(k\rho)\, \frac{d\rho}{\rho}$$

for all ν in the strip $|\mathrm{Re}\ \nu| < \beta$.

The function J_μ and $H_\nu^{(1)}$ are the Bessel function and Hankel function of the first kind, respectively.

From the results of this theorem, it is convenient to define the transformed function

$$\tilde{f}(\nu) = w(\nu)\, e^{-i(\pi/2)\nu} , \qquad (4.3)$$

whereby it follows that

$$\tilde{f}(\nu) = K[f(\rho)] = \int_0^\infty f(\rho) H_\nu^{(1)}(k\rho) \frac{d\rho}{\rho} , \qquad (4.4a)$$

$$f(\rho) = K^{-1}[\tilde{f}(\nu)] = -\frac{1}{2}\int_{-i\infty}^{i\infty} \nu \tilde{f}(\nu) J_\nu(k\rho) d\nu$$

$$= -\frac{i}{4}\int_{-i\infty}^{i\infty} \nu \sin \nu\pi \; \tilde{f}(\nu) e^{i\nu\pi} H_\nu^{(1)}(k\rho) d\nu , \qquad (4.4b)$$

the second integral being a result of the properties of J_ν and $H_\nu^{(1)}$.

It is important to point out that the transformation put forth in the manner of Theorem 2 must be used with caution. By assuming properties for $\tilde{f}(\nu)$ (i.e., $w(\nu)$), and then defining $f(\rho)$ via (4.4b), the theorem restricts the class of functions for which the transform (4.4a) exists. In practice, it is the properties of $f(\rho)$ which are known, and therefore it is necessary to show that $\tilde{f}(\nu)$ exists as defined by (4.4a), and that the expression on the right-hand side of (4.4b) does indeed return the function $f(\rho)$.

To this end, Jones, in a recent paper [22], derives sufficient conditions for $f(\rho)$ such that the transform integral and its inverse (4.4a,b) exist. In particular, Jones defines the inverse as

$$f(\rho) = K^{-1}[\tilde{f}(\nu)] = \lim_{\varepsilon \to 0^+} -\frac{1}{2}\int_{-i\infty}^{i\infty} e^{\varepsilon\nu^2} \nu\, \tilde{f}(\nu) J_\nu(k\rho) d\nu \quad (4.4c)$$

in order to guarantee its existence under the assumed properties of $f(\rho)$. An outline of Jones' theorem, and its relation to the properties of the functions of ρ expected for the resistive wedge, are given in Appendix A. Henceforth, all transformations and inverses in this chapter will be assumed to exist, either in the sense of (4.4b) or (4.4c), based upon the results of the appendix.

It is interesting to note that under certain conditions, there exists a relationship [24] between the K-L transform of $f(\rho)$ and the Maliuzhinets representation

$$f(\rho) = \frac{1}{2\pi i}\int_\gamma e^{ik\rho\cos\alpha}\, s(\alpha - \phi) d\alpha \quad , \qquad (4.5)$$

where the explicit dependence of $f(\rho)$ on ϕ has not been shown, and where γ is the contour of Fig. 3.1. Using the symmetry of the contour $\gamma = \gamma_1 + \gamma_2$, and letting

$$\hat{f}(\alpha) = \frac{1}{2}[s(\alpha - \phi) - s(2\pi - \alpha - \phi)] \quad , \qquad (4.6)$$

(4.5) becomes

$$f(\rho) = \frac{1}{2\pi i}\int_\gamma e^{ik\rho\cos\alpha}\, \hat{f}(\alpha) d\alpha \quad . \qquad (4.7)$$

As outlined in Appendix A, if $f(\rho) = O(\rho^{-1+a}\, e^{b\rho})$, where a,b > 0, then the integral equation (4.7) has a unique solution within the class of functions

$$\hat{f}(\alpha) = O(e^{(1-a)|\mathrm{Im}\,\alpha|}) \quad ; \qquad |\mathrm{Im}\,\alpha| \to \infty \quad ,$$

where $\hat{f}(\alpha)/\sin\alpha$ is analytic within and on γ, given by

$$\hat{f}(\alpha) = -\frac{ik\sin\alpha}{2}\int_0^\infty f(\rho)\, e^{-ik\rho\cos\alpha}\, d\rho \quad . \qquad (4.8)$$

Note from (4.6) that $\hat{f}(\alpha)$, as well as γ, are odd about $\alpha = \pi$, i.e.,

$$\hat{f}(\alpha) = -\hat{f}(2\pi - \alpha) \quad ,$$

implying from (4.7) that

$$f(\rho) = \frac{1}{\pi i}\int_{\gamma_1} e^{ik\rho\cos\alpha}\, \hat{f}(\alpha)d\alpha \quad . \qquad (4.9)$$

Maliuzhinets has shown [51] that in the event the constant $a \geq 1$, or equivalently, that $\hat{f}(\alpha)$ is bounded at the end points of γ, then

$$f(0) = 2i\hat{f}(i\infty) \quad . \qquad (4.10)$$

From Appendix A, the existence of the K-L transform of $f(\rho)$ requires $f(0)$ and hence $\hat{f}(i\infty)$ to be zero. With these restrictions on $\hat{f}(\alpha)$, it can be concluded that $\hat{f}(\alpha)$ is regular in the strip $\pi/2 - \varepsilon < \text{Re}\ \alpha < 3\pi/2 + \varepsilon$. Based on this fact, and defining the Fourier transform

$$g(\nu) = \frac{i}{\pi} \int_{-i\infty}^{i\infty} \hat{f}(\alpha - \pi) e^{i\nu\alpha}\, d\alpha \quad , \tag{4.11}$$

which is absolutely convergent in the strip $|\text{Re}\ \nu| < a - 1$, Maliuzhinets has shown [24] that

$$\tilde{f}(\nu) = \frac{2}{\nu} e^{-i(\pi/2)\nu} g(\nu) \quad . \tag{4.12}$$

Equations (4.11) and (4.12) thus establish a Fourier transform relationship between the K-L transform (4.4) and the Maliuzhinets representation (4.7) of the function $f(\rho)$.

4.2 Application of the K-L Transform to the Scattering by a Resistive Wedge

Recalling the geometry under consideration, the electromagnetic E- and H-polarized fields (U,V) scattered by a resistive wedge can be written

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} U^i + U^s \\ V^i + V^s \end{bmatrix} \quad , \tag{4.13}$$

where

$$U^i = V^i = e^{-ik\rho\cos(\phi-\phi_0)}$$

is a plane wave incident upon the wedge at an angle ϕ_0 from within Region 1 (see Fig. 4.1). The included angle of the wedge is 2ψ.

The fields (U,V) satisfy the Helmholtz equation

$$(\nabla^2 + k^2)\begin{bmatrix} U \\ V \end{bmatrix} = 0 \tag{4.14}$$

in Regions 1 and 2, as well as the boundary conditions (2.6), (2.7), the radiation condition (2.9), and the edge condition (2.12). For the time being is assumed that Im $k > 0$. Equation (4.14) is also satisfied by the scattered fields (U^s,V^s).

Throughout the remainder of this analysis, it is convenient to define the modified scattered fields (denoted by lower case letters)

$$u(\rho,\phi) = U^s(\rho,\phi) + ce^{ik\rho} \tag{4.15a}$$

$$v(\rho,\phi) = V^s(\rho,\phi) + de^{ik\rho} \tag{4.15b}$$

where

$$\begin{bmatrix} c \\ d \end{bmatrix} = -\begin{bmatrix} U^s(0,\phi) \\ V^s(0,\phi) \end{bmatrix} . \tag{4.16}$$

The quantities c,d are independent of ϕ within each of the Regions 1 and 2, which simply means that the scattered field is uniquely defined at the apex of the wedge within each region. As was discussed in Section 2.1 regarding the edge condition (2.12), it is possible for d to take on two values, one each in Regions 1

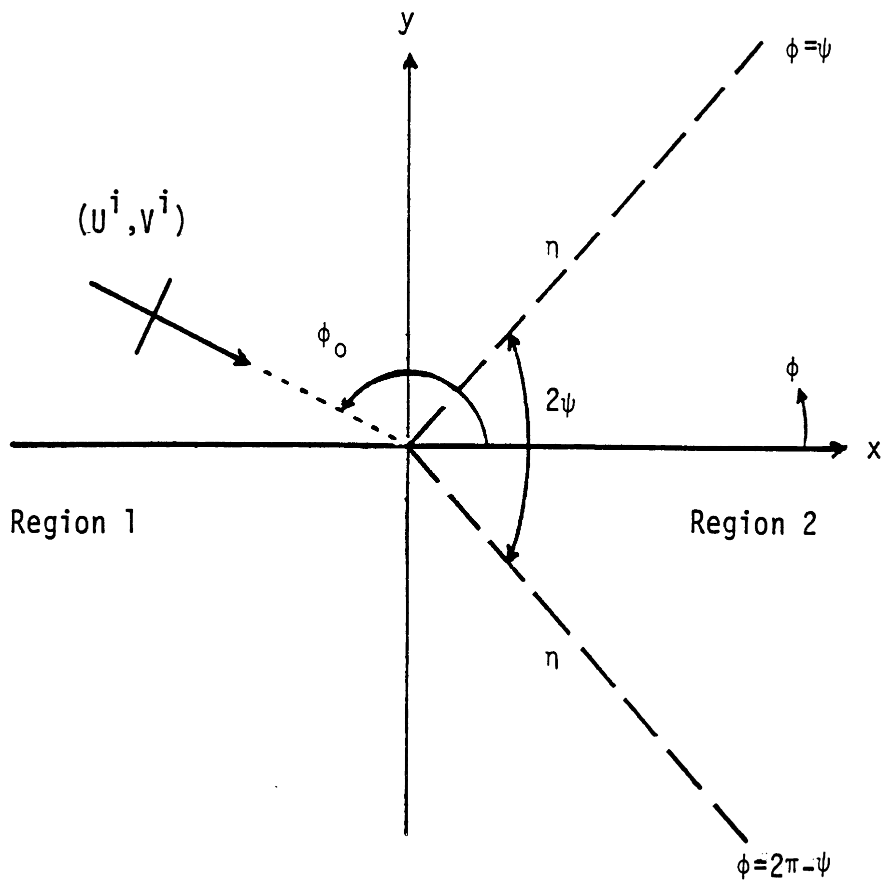


Fig. 4.1: Geometry for the Application of the Kontorovich-Lebedev Transform to the Scattering by a Resistive Wedge.

and 2, but due to the continuity of U across the boundary, the value of c is the same in both regions. From equation (4.16), it is evident that (u,v) are zero at $\rho = 0$.

Expressing (4.14) in cylindrical coordinates, and substituting (4.13) and (4.14), the following equations for (u,v) are obtained:

$$\left[\rho^2 \frac{\partial^2}{\partial\rho^2} + \frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial\phi^2} + k^2\rho^2\right]\begin{bmatrix} u \\ v \end{bmatrix} = ik\rho\, e^{ik\rho}\begin{bmatrix} c \\ d \end{bmatrix} . \qquad (4.17)$$

Assuming the additional constraints

$$\lim_{\rho \to 0} \rho \frac{\partial}{\partial\rho}\begin{bmatrix} u \\ v \end{bmatrix} = \lim_{\rho \to \infty}\begin{bmatrix} u \\ v \end{bmatrix} = \lim_{\rho \to \infty} \frac{\partial}{\partial\rho}\begin{bmatrix} u \\ v \end{bmatrix} = 0 , \qquad (4.18)$$

which are consistent with the edge and radiation conditions for ρ and ϕ-directed components of the scattered fields, then application of the K-L transform to (4.17) leads to a differential equation for the transforms $(\tilde{u},\tilde{v})$ [21];

$$\left[\frac{d^2}{d\phi^2} + \nu^2\right]\begin{bmatrix} \tilde{u}(\nu,\phi) \\ \tilde{v}(\nu,\phi) \end{bmatrix} = \frac{2i\nu\, e^{-i(\pi/2)\nu}}{\sin \nu\pi}\begin{bmatrix} c \\ d \end{bmatrix} . \qquad (4.19)$$

The general solution to this equation can be written in the form

$$\begin{bmatrix} \tilde{u}(\nu,\phi) \\ \tilde{v}(\nu,\phi) \end{bmatrix} = \begin{bmatrix} A(\nu) \\ C(\nu) \end{bmatrix} \cos\nu\phi + \begin{bmatrix} B(\nu) \\ D(\nu) \end{bmatrix} \sin \nu\phi + \frac{2i\, e^{-i(\pi/2)\nu}}{\nu \sin \nu\pi}\begin{bmatrix} c \\ d \end{bmatrix} \qquad (4.20)$$

where the arbitrary functions A,B,C, and D are independent of ϕ. The task is to find specific values of A,B,C, and D such that $(\tilde{u},\tilde{v})$ satisfy the appropriate transformed boundary conditions. In terms of (u,v), the conditions are:

<u>E-Polarization</u>:

$$\left.\frac{\partial u_1}{\partial \phi}\right|_{\phi=\psi} - \left.\frac{\partial u_2}{\partial \phi}\right|_{\phi=\psi} + \frac{2ik\rho}{\eta} \left. u_1\right|_{\phi=\psi} = -\frac{2ik\rho}{\eta}\left[e^{ik\rho\cos(\psi-\phi_0)} - ce^{ik\rho}\right] \tag{4.21a}$$

$$\left. u_1\right|_{\phi=\psi} - \left. u_2\right|_{\phi=\psi} = 0 \tag{4.21b}$$

$$\left.\frac{\partial u_1}{\partial \phi}\right|_{\phi=2\pi-\psi} - \left.\frac{\partial u_2}{\partial \phi}\right|_{\phi=-\psi} - \frac{2ik\rho}{\eta} \left. u_1\right|_{\phi=2\pi-\psi} = \frac{2ik\rho}{\eta}\left[e^{-ik\rho\cos(\psi+\phi_0)} - ce^{ik\rho}\right] \tag{4.21c}$$

$$\left. u_1\right|_{\phi=2\pi-\psi} - \left. u_2\right|_{\phi=-\psi} = 0 \ . \tag{4.21d}$$

H-Polarization:

$$\left.\frac{\partial v_1}{\partial \phi}\right|_{\phi=\psi} + \frac{ik\eta\rho}{2}\left[\left. v_1\right|_{\phi=\psi} - \left. v_2\right|_{\phi=\psi}\right] = -ik\rho\left[\sin(\psi - \phi_0)e^{-ik\rho\cos(\psi-\phi_0)} - \frac{\eta}{2}(d_1 - d_2)e^{ik\rho}\right] \tag{4.22a}$$

$$\left.\frac{\partial v_1}{\partial \phi}\right|_{\phi=\psi} - \left.\frac{\partial v_2}{\partial \phi}\right|_{\phi=\psi} = 0 \tag{4.22b}$$

$$\left.\frac{\partial v_1}{\partial \phi}\right|_{\phi=2\pi-\psi} - \frac{ik\eta\rho}{2}\left. v_1\right|_{\psi=2\pi-\psi} - \left. v_2\right|_{\phi=-\psi}$$

$$= ik\rho\left[\sin(\psi + \phi_0)e^{ik\rho\cos(\psi+\phi_0)} - \frac{\eta}{2}(d_1 - d_2)e^{ik\rho}\right] \tag{4.22c}$$

$$\left.\frac{\partial v_1}{\partial \phi}\right|_{\phi=2\pi-\psi} - \left.\frac{\partial v_2}{\partial \phi}\right|_{\phi=-\psi} = 0 \tag{4.22d}$$

The K-L transform of equations (4.21) and (4.22) are obtained directly by multiplying each equation by $(1/\rho)H_\nu^{(1)}(k\rho)$ and integrating along $(0,\infty)$. The results are:

E-Polarization:

$$\left.\frac{d\tilde{u}_1}{d\phi}\right|_{\phi=\psi} - \left.\frac{d\tilde{u}_2}{d\phi}\right|_{\phi=\psi} + \frac{2ik}{\eta}\int_0^\infty u_1\Big|_{\phi=\psi} H_\nu^{(1)}(k\rho)d\rho$$

$$= \frac{4i}{\eta}\;\frac{e^{-i(\pi/2)\nu}}{\sin\nu\pi}\left[c\nu - \frac{\sin\nu(\pi-\phi_0+\psi)}{\sin(\phi_0-\psi)}\right] \tag{4.23a}$$

$$\tilde{u}_1\Big|_{\phi=\psi} - \tilde{u}_2\Big|_{\phi=\psi} = 0 \tag{4.23b}$$

$$\left.\frac{d\tilde{u}_1}{d\phi}\right|_{\phi=2\pi-\psi} - \left.\frac{d\tilde{u}_2}{d\phi}\right|_{\phi=-\psi} - \frac{2ik}{\eta}\int_0^\infty u_1\Big|_{2\pi-\psi} H_\nu^{(1)}(k\rho)d\rho$$

$$= -\frac{4i}{\eta}\;\frac{e^{-i(\pi/2)\nu}}{\sin\nu\pi}\left[c\nu - \frac{\sin\nu(\pi-\phi_0-\psi)}{\sin(\phi_0+\psi)}\right] \tag{4.23c}$$

$$\tilde{u}_1\Big|_{\phi=2\pi-\psi} - \tilde{u}_2\Big|_{\phi=-\psi} = 0 \tag{4.23d}$$

H-Polarization:

$$\left.\frac{d\tilde{v}_1}{d\phi}\right|_{\phi=\psi} + \frac{ik\eta}{2}\int_0^\infty \left(v_1\Big|_{\phi=\psi} - v_2\Big|_{\phi=\psi}\right)H_\nu^{(1)}(k\rho)d\rho$$

$$= 2i\,\frac{e^{-i(\pi/2)\nu}}{\sin\nu\pi}\left[\frac{\eta}{2}(d_1-d_2)\nu + \sin\nu(\pi-\phi_0+\psi)\right] \tag{4.24a}$$

$$\left.\frac{d\tilde{v}_1}{d\phi}\right|_{\phi=\psi} - \left.\frac{d\tilde{v}_2}{d\phi}\right|_{\phi=\psi} = 0 \tag{4.24b}$$

$$\left.\frac{d\tilde{v}_1}{d\phi}\right|_{\phi=2\pi-\psi} - \frac{ikn}{2}\int_0^\infty \left(v_1\Big|_{\phi=2\pi-\psi} - v_2\Big|_{\phi=-\psi}\right) H_\nu^{(1)}(k\rho)d\rho$$

$$= -2i\,\frac{e^{-i(\pi/2)\nu}}{\sin \nu\pi}\left[\frac{n}{2}(d_1 - d_2)\nu - \sin \nu(\pi - \phi_0 - \psi)\right] \quad (4.24c)$$

$$\left.\frac{d\tilde{v}_1}{d\phi}\right|_{\phi=2\pi-\psi} - \left.\frac{d\tilde{v}_2}{d\phi}\right|_{\phi=-\psi} = 0 \quad (4.24d)$$

Some note regarding the regions in the complex ν plane for which equation (4.20), and boundary conditions (4.23) and (4.24) are defined is necessary at this point.

For $(u,v) = O(\rho^\delta)$, $\delta > 0$, it is shown in Appendix A that $(\tilde{u},\tilde{v})$, and hence the left-hand side of (4.20), are defined for $|\mathrm{Re}\ \nu| < \delta$. Inasmuch as the unknowns A,B,C and D are determined, the RHS of (4.20) may or may not provide an analytic continuation of $(\tilde{u},\tilde{v})$ into any additional regions where it is properly defined.

From similar considerations, the LHS of (4.23) and (4.24) are also defined for $|\mathrm{Re}\ \nu| < \delta$, while the RHS are analytic in $|\mathrm{Re}\ \nu| < 1$, having poles at ν equal to a nonzero integer.

It is important to note the various regions of analyticity of the transformed quantities, since the inversion contour of (4.4b,c) must lie within the intersection of these regions.

With these caveats in mind, equation (4.20) may be substituted into the transformed boundary conditions in an attempt to determine the unknown coefficients A,B,C, and D (each defined in Regions 1 and 2).

From (4.23b,d),

$$A_1(\nu)\cos\nu\psi + B_1(\nu)\sin\nu\psi - A_2(\nu)\cos\nu\psi - B_2(\nu)\sin\nu\psi = 0$$

$$A_1(\nu)\cos\nu(2\pi-\psi) + B_1(\nu)\sin\nu(2\pi-\psi) - A_2(\nu)\cos\nu\psi + B_2(\nu)\sin\nu\psi = 0 ,$$

implying

$$A_2(\nu) = \frac{\cos\nu(\pi-\psi)}{\cos\nu\psi}\,[A_1(\nu)\cos\nu\pi + B_1(\nu)\sin\nu\pi] \quad (4.25a)$$

$$B_2(\nu) = \frac{\sin\nu(\pi-\psi)}{\sin\nu\psi}\,[A_1(\nu)\sin\nu\pi - B_1(\nu)\cos\nu\pi] \quad (4.25b)$$

Similar results can be derived from (4.24b,d), i.e.,

$$C_2(\nu) = -\frac{\sin\nu(\pi-\psi)}{\sin\nu\psi}\,[C_1(\nu)\cos\nu\pi + D_1(\nu)\sin\nu\pi] \quad (4.26a)$$

$$D_2(\nu) = -\frac{\cos\nu(\pi-\psi)}{\cos\nu\psi}\,[C_1(\nu)\sin\nu\pi - D_1(\nu)\cos\nu\pi] \quad (4.26b)$$

Substitution of (4.25, 26) into the remaining boundary conditions (4.23a,c) and (4.24a,c) leads to the four rather complicated equations below:

<u>E-Polarization</u>:

$$\frac{\nu\sin\nu\pi}{\cos\nu\psi\,\sin\nu\psi}\,[-\sin\nu(\pi-\psi)A_1(\nu) + \cos\nu(\pi-\psi)\,B_1(\nu)]$$

$$+ \frac{2ik}{\eta}\int_0^\infty u_1\Big|_{\phi=\psi} H_\nu^{(1)}(k\rho)d\rho = K(\nu,\psi) \quad (4.27a)$$

$$\frac{\nu \sin \nu\pi}{\cos \nu\psi \sin \nu\psi} [-\sin \nu(\pi + \psi) A_1(\nu) + \cos \nu(\pi + \psi) B_1(\nu)]$$

$$- \frac{2ik}{\eta} \int_0^{\infty} u_1 \Big|_{\phi=2\pi-\psi} H_\nu^{(1)}(k\rho) d\rho = -K(\nu,-\psi) \qquad (4.27b)$$

<u>H-Polarization:</u>

$$-\nu[C_1(\nu) \sin \nu\psi - D_1(\nu) \cos \nu\psi] + \frac{ik\eta}{2} \int_0^{\infty} \left(v_1 \Big|_{\phi=\psi} - v_2 \Big|_{\phi=\psi} \right)$$

$$\cdot H_\nu^{(1)}(k\rho) d\rho = L(\nu,\psi) \qquad (4.28a)$$

$$-\nu[C_1(\nu) \sin \nu(2\pi - \psi) - D_1(\nu) \cos \nu(2\pi - \psi)]$$

$$- \frac{ik\eta}{2} \int_0^{\infty} \left(v_1 \Big|_{\phi=2\pi-\psi} - v_2 \Big|_{\phi=-\psi} \right) H_\nu^{(1)}(k\rho) d\rho = - L(\nu,-\psi)$$

$$(4.28b)$$

The functions $K(\nu,\psi)$ and $L(\nu,\psi)$ are given by

$$K(\nu,\psi) = \frac{4i}{\eta} \frac{e^{-i(\pi/2)\nu}}{\sin \nu\pi} \left[c\nu - \frac{\sin \nu(\pi - \phi_0 + \psi)}{\sin(\phi_0 - \psi)} \right] , \qquad (4.29a)$$

$$L(\nu,\psi) = 2i \frac{e^{-i(\pi/2)\nu}}{\sin \nu\pi} \left[\frac{\eta}{2} (d_1 - d_2)\nu + \operatorname{sgn} \psi \sin \nu(\pi - \phi_0 + \psi) \right]$$

$$(4.29b)$$

Equations (4.27) and (4.28), defined in the strip $|\mathrm{Re}\ \nu| <$ min $(\delta,1)$, are a pair of coupled equations for the remaining unknowns A_1, B_1, C_1, and D_1, and their solution provides all that is necessary for determining the unknown transformed fields $(\tilde{u},\tilde{v})$. However, these equations cannot be solved explicitly in their present form, due to the presence of the "untransformed" fields (u,v) under the integral signs.

Two methods for simplifying (4.27) and (4.28) by eliminating the "untransformed" fields are presented in the following subsections. In 4.2.1, an approach leading to second order difference equations is developed. The technique parallels that used by Lebedev and Skal'skaya [23] in solving the impedance wedge problem. A second, more general method leading to Fredholm integral equations of the second kind for the unknowns is described in 4.2.2. The method, developed by the author, allows the well-established theory of linear integral equations to be applied toward solving the equations.

4.2.1 Difference Equation Method

The method to be presented requires that one of the following two conditions is satisfied by the unknown functions $(\tilde{u},\tilde{v})$;

(1) $\delta > 1$, or

(2) $(\tilde{u},\tilde{v})$ may be analytically continued into the region $|\mathrm{Re}\ \nu| < \delta'$, $\delta' > 1$.

The net result of either of these requirements is that $(\tilde{u},\tilde{v})$ or their continuations are analytic in the strip $|\mathrm{Re}\ \nu| < 1 + \varepsilon$, $\varepsilon > 0$. For the moment it will be assumed that such is the case. Whether it is indeed true will be discussed in another part of this work.

Equations (4.27) and (4.28) are now multiplied by $e^{i(\pi/2)\nu}$, and then ν is replaced first by $(1 + \nu)$ and then by $(1 - \nu)$. The resulting set of equations in $(1 + \nu)$ have a common region of analyticity with the corresponding set of equations in $(1 - \nu)$, which from the above requirements is given by $|\mathrm{Re}\,\nu| < \varepsilon$. By subtracting one set from the other, and making use of the identity

$$e^{i(\pi/2)(1+\nu)}\,H^{(1)}_{1+\nu}(k\rho) - e^{i(\pi/2)(1-\nu)}\,H^{(1)}_{1-\nu}(k\rho) = \frac{2i\nu}{k\rho}\,e^{i(\pi/2)\nu}\,H^{(1)}_{\nu}(k\rho) \tag{4.30}$$

there results a new system of equations for the unknowns A_1, B_1, C_1, D_1:

E-Polarization:

$$p_+(1+\nu)F(1+\nu) - p_+(1-\nu)F(1-\nu) + q_+(1+\nu)G(1+\nu)$$

$$- q_+(1-\nu)G(1-\nu) - \frac{4}{\eta}\cos\psi\;F(\nu) + \sin\nu\psi G(\nu) = \frac{8i}{\eta}\,\frac{\cos\nu(\pi-\phi_0+\psi)}{\sin\nu\pi} \tag{4.31a}$$

$$p_-(1+\nu)F(1+\nu) - p_-(1-\nu)F(1-\nu) + q_-(1+\nu)G(1+\nu)$$

$$- q_-(1-\nu)G(1-\nu) + \frac{4}{\eta}[\cos\nu(2\pi-\psi)F(\nu) + \sin\nu(2\pi-\psi)G(\nu)]$$

$$= -\frac{8i}{\eta}\,\frac{\cos\nu(\pi-\phi_0-\psi)}{\sin\nu\pi} \tag{4.31b}$$

where

$$p_{\pm}(\nu) = -\frac{\sin \nu\pi \sin \nu(\pi \mp \psi)}{\sin \nu\psi \cos \nu\psi}$$

$$q_{\pm}(\nu) = \frac{\sin \nu\pi \cos \nu(\pi \mp \psi)}{\sin \nu\psi \cos \nu\psi}$$

and

$$F(\nu) = \nu\, e^{i(\pi/2)\nu} A_1(\nu)$$

$$G(\nu) = \nu\, e^{i(\pi/2)\nu} B_1(\nu) \quad .$$

<u>H-Polarization</u>:

$$\sin(1+\nu)\psi F'(1+\nu) - \sin(1-\nu)\psi F'(1-\nu) - \cos(1+\nu)\psi G'(1+\nu)$$

$$+ \cos(1-\nu)\psi G'(1-\nu) + \eta[q_+(\nu)F'(\nu) - p_+(\nu)G'(\nu)]$$

$$= 4i\, \frac{\sin(\phi_0 - \psi)\cos \nu(\pi - \phi_0 + \psi)}{\sin \nu\pi} \qquad (4.32a)$$

$$\sin[(1+\nu)(2\pi - \psi)]F'(1+\nu) - \sin[(1-\nu)(2\pi - \psi)]F'(1-\nu)$$

$$- \cos[(1+\nu)(2\pi - \psi)]G'(1+\nu) + \cos[(1-\nu)(2\pi - \psi)]G'(1-\nu)$$

$$- \eta[q_-(\nu)F'(\nu) - p_-(\nu)G'(\nu)] = 4i\, \frac{\sin(\phi_0 + \psi)\cos \nu(\pi - \phi_0 - \psi)}{\sin \nu\pi} \quad (4.32b)\ .$$

where

$$F'(\nu) = e^{i(\pi/2)\nu}\, C_1(\nu)$$

$$G'(\nu) = e^{i(\pi/2)\nu}\, D_1(\nu) \quad .$$

Equations (4.31) and (4.32) each constitute a pair of coupled second order linear functional equations for $F(\nu)$,$G(\nu)$ and $F'(\nu)$, $G'(\nu)$, respectively. Although one may observe several symmetries associated with the equations, there is no straightforward means for finding their solutions known to the author.

Certainly the task of developing expressions for F and G (and likewise F' and G') would be easier if the functional equations could be uncoupled.

To this end, the decomposition derived in Section 2.2 is applied to the transformed functions $(\tilde{u},\tilde{v})$. From the equivalent of equation (2.18), one can define

$$\begin{bmatrix} \tilde{u}_1(\nu,\phi) \\ \tilde{v}_1(\nu,\phi) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \tilde{u}^e_1(\nu,\phi) + \tilde{u}^o_1(\nu,\phi) \\ \tilde{v}^e_1(\nu,\phi) + \tilde{v}^o_1(\nu,\phi) \end{bmatrix}, \tag{4.33}$$

where

$$\begin{bmatrix} \tilde{u}^{e,o}_1 \\ \tilde{v}^{e,o}_1 \end{bmatrix} = \begin{bmatrix} \tilde{u}_1(\nu,\phi) \pm \tilde{u}_1(\nu,2\pi - \phi) \\ \tilde{v}_1(\nu,\phi) \pm \tilde{v}_1(\nu,2\pi - \phi) \end{bmatrix}. \tag{4.34}$$

The plus (minus) signs correspond to the symmetric (antisymmetric) components. It is convenient to write, analogous to (4.20),

$$\begin{bmatrix} \tilde{u}_1^{e,o} \\ \tilde{v}_1^{e,o} \end{bmatrix} = \begin{bmatrix} A_1^{e,o}(\nu) \\ C_1^{e,o}(\nu) \end{bmatrix} \cos\nu\phi + \begin{bmatrix} B_1^{e,o}(\nu) \\ D_1^{e,o}(\nu) \end{bmatrix} \sin\nu\phi + \frac{2ie^{-i(\pi/2)\nu}}{\nu \sin\nu\pi} \begin{bmatrix} c_1^{e,o} \\ d_1^{e,o} \end{bmatrix} \quad . \tag{4.35}$$

Substituting (4.35) into (4.33) leads to the following results:

$$A_1 = \frac{1}{2}[A_1^e + A_1^o] \quad ; \tag{4.36a}$$

$$B_1 = \frac{1}{2}[B_1^e + B_1^o] \quad , \tag{4.36b}$$

and similarly for C_1 and D_1. In addition, from (4.34)

$$c_1^e = 2c$$

$$d_1^e = 2d_1$$

$$c_1^o = d_1^o = 0 \quad .$$

From either (4.34) or the transforms of (2.24c,c') and (2.25c,c'), it is easily shown that

$$\left.\frac{d\tilde{u}_1^e}{d\phi}\right|_{\phi=\pi} = \left.\frac{d\tilde{v}_1^e}{d\phi}\right|_{\phi=\pi} = 0 \quad ,$$

$$\left.\tilde{u}_I^o\right|_{\phi=\pi} = \left.\tilde{v}_1^o\right|_{\phi=\pi} = 0 \quad ,$$

which is equivalent to

$$\begin{bmatrix} B_1^e \\ D_1^e \end{bmatrix} = \tan \nu\pi \begin{bmatrix} A_1^e \\ C_1^e \end{bmatrix} , \qquad (4.37a)$$

$$\begin{bmatrix} B_1^o \\ D_1^o \end{bmatrix} = - \cot \nu\pi \begin{bmatrix} A_1^o \\ C_1^o \end{bmatrix} . \qquad (4.37b)$$

By substituting (4.36) and (4.37) into the appropriate functional equations (4.31) and (4.32), and defining

$$[F_e(\nu), F_e'(\nu)] = \nu\, e^{i(\pi/2)\nu} \tan \nu\pi [A_1^e(\nu), C_1^e(\nu)] \qquad (4.38a)$$

$$[F_o(\nu), F_o'(\nu)] = \nu\, e^{i(\pi/2)\nu} [A_1^o(\nu), C_1^o(\nu)] \qquad (4.38b)$$

one can obtain, after considerable algebraic manipulation, a set of four uncoupled, linear, second order functional equations:

<u>E-Polarization</u>:

$$\frac{F_e(1+\nu)}{\cos(1+\nu)\psi} - \frac{F_e(1-\nu)}{\cos(1-\nu)\psi} - \frac{4}{\eta} \frac{\cos \nu(\pi-\psi)}{\sin \nu\pi} F_e(\nu)$$

$$= \frac{16i}{\eta} \frac{\cos \nu(\pi - \phi_o) \cos \nu\psi}{\sin \nu\pi} \qquad (4.39a)$$

$$\frac{F_o(1+\nu)}{\sin(1+\nu)\psi} - \frac{F_o(1-\nu)}{\sin(1-\nu)\psi} + \frac{4}{\eta}\,\frac{\sin\nu(\pi-\psi)}{\sin\nu\pi}\,F_o(\nu)$$

$$= \frac{16i}{\eta}\,\frac{\sin\nu(\pi-\phi_o)\sin\nu\psi}{\sin\nu\pi} \qquad (4.39b)$$

H-Polarization:

$$\frac{\sin[(1+\nu)(\pi-\psi)]}{\sin(1+\nu)\pi}F_e'(1+\nu) - \frac{\sin[(1-\nu)(\pi-\psi)]}{\sin(1-\nu)\pi}F_e'(1-\nu)$$

$$-\eta\frac{F_e'(\nu)}{\sin\nu\psi} = -\frac{4i}{\sin\nu\pi}[\sin(\phi_o-\psi)\cos\nu(\pi-\phi_o+\psi)$$

$$-\sin(\phi_o+\psi)\cos\nu(\pi-\phi_o-\psi)] \qquad (4.40a)$$

$$\frac{\cos[(1+\nu)(\pi-\psi)]}{\sin(1+\nu)\pi}F_o'(1+\nu) - \frac{\cos[(1-\nu)(\pi-\psi)]}{\sin(1-\nu)\pi}F_o'(1-\nu)$$

$$+\eta\frac{F_o'(\nu)}{\cos\nu\psi} = \frac{4i}{\sin\nu\pi}[\sin(\phi_o-\psi)\cos\nu(\pi-\phi_o+\psi)$$

$$+\sin(\phi_o+\psi)\cos\nu(\pi-\phi_o-\psi)] \qquad (4.40b)$$

Equations (4.39) and (4.40) are the fundamental functional equations for the resistive wedge scattering problem. While an equation of similar type has been solved by Lebedev and Skal'skaya in [23], the author has not been able to generate a solution using their method.

A general solution to any of equations (4.39,40) can be written in the form

$$F(\nu) = P_1(\nu)F^{(1)}(\nu) + P_2(\nu)F^{(2)}(\nu) + F_p(\nu) \quad , \tag{4.41}$$

where $F^{(1)}, F^{(2)}$ are the two independent solutions of the corresponding homogeneous (RHS = 0) equation, and $F_p(\nu)$ is a particular solution of the inhomogeneous equation. P_1 and P_2 are arbitrary even periodic functions with period unity.

Determination of a unique solution to (4.41) requires restricting the solution to a particular class of functions satisfying a specified analyticity condition, along with a prescribed behavior as $|\mathrm{Im}\ \nu| \to \infty$.

More specifically, it is possible to write

$$F_e(\nu) = -4ic + \nu e^{-i(\pi/2)\nu} \sin \nu\pi \int_0^\infty u_1^e\Big|_{\phi=\pi} H_\nu^{(1)}(k\rho)\,\frac{d\rho}{\rho} \tag{4.42a}$$

$$F_o(\nu) = \nu e^{-i(\pi/2)\nu} \sin \nu\pi \int_0^\infty u_1^o\Big|_{\phi=\pi} H_\nu^{(1)}(k\rho)\,\frac{d\rho}{\rho} \tag{4.42b}$$

with identical expressions for F_e' in terms of v_1^e, d_1, and for $F_o'(\nu)$ in terms of v_1^o, where

$$\begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{2}\begin{bmatrix} u^e + u^o \\ v^e + v^o \end{bmatrix} \quad . \tag{4.43}$$

From (4.39-4.43) and behavior of $(u_1|_{\phi=\pi}, v_1|_{\phi=\pi})$ at $\rho = 0$ and for $\rho \to \infty$, the following conditions are satisifed by F_e, F_o, F'_e, F'_o:

1. F_e, F'_o are even functions of ν, while

   F'_e, F_o are odd functions of ν.

2. The function $F_e(\nu)$ is analytic in the strip $|\text{Re}\ \nu| < \delta$, where $u_1^e = O(\rho^\delta)$, $\delta > 0$, for $\rho \to 0$. Similar results hold for F_o, F'_e, F'_o in terms of u_1^o, v_1^e, v_1^o, respectively.

3. The function $F_e(\nu)$ approaches zero when $|\text{Re}\ \nu| < \delta$ and $|\text{Im}\ \nu| \to \infty$, provided $u_1^e|_{\phi=\pi}$ contains no geometrical optics scattered fields (recall the decomposition (2.8)), which is always true for $\phi_o - 2\psi > 0$. Similar results hold for F_o, F'_e, and F'_o. In [23], Lebedev and Skal'skaya write $F^e(\nu) = F^{eg}(\nu) + F^{ed}(\nu)$, equivalent to (2.8), and then state that $F^{ed}(\nu) \to 0$ for $|\text{Re}\ \nu| < \delta$, $|\text{Im}\ \nu| \to \infty$ for all ϕ, ψ. This is unnecessary, since by solving the problem for $\phi_o - 2\psi > 0$, (whereby $F^{eg} = 0$), the solution for $\phi_o - 2\psi \leq 0$ can be obtained via an appropriate analytic continuation. Further discussions regarding the behavior as $|\text{Im}\ \nu| \to \infty$ can be found in Appendix A.

Conditions (1-3) above are sufficient to uniquely determine the solution (4.41).

When $\eta = 0$, equations (4.39) can be solved directly, yielding the known solutions for a perfectly conducting wedge (E-polarization)

$$F_e(\nu)\Big|_{\eta=o} = -4i\,\frac{\cos\nu(\pi - \phi_o)\cos\nu\psi}{\cos\nu(\pi - \psi)} \tag{4.44a}$$

$$F_o(\nu)\Big|_{\eta=o} = 4i\,\frac{\sin\nu(\pi - \phi_o)\sin\nu\psi}{\sin\nu(\pi - \psi)} \tag{4.44b}$$

Although the solution to (4.40) (H-polarization) with $\eta = 0$ cannot be as easily deduced, the resulting equations can be solved via Fourier transform techniques, yielding the correct expressions.

Furthermore, it is relatively straightforward to solve (4.39-40) for the special cases $\psi = 0, \pi/2$, corresponding to a resistive half-plane and full plane, respectively. Both solutions are known, the latter being the simple geometrical optics reflected and transmitted plane waves, while the former having been given in [27,28].

The lack of available methods for solving functional equations of the type (4.39-40) has prevented the author from obtaining an exact solution for arbitrary values of ψ. However, since (4.39-40) were derived from (4.27-28) based on the assumption that $(\tilde{u},\tilde{v})$ were analytic in a strip $|\mathrm{Re}\ \nu| < \delta$, with $\delta \geq 1$, it is reasonable to ask if a different set of equations, replacing (4.39-40), can be derived in the more general situation where $\delta > 0$. Such a derivation is presented in the next section.

4.2.2 Integral Equation Method

The starting point for this section will be equations (4.27-28). It is beneficial to immediately express them in terms of symmetric and antisymmetric components. This is achieved by substitution of (4.36-38), then adding and subtracting (4.27a) and (4.27b), and similarly, (4.28a) and (4.28b). The net results are the following:

E-Polarization:

$$\frac{F_e(\nu)}{\cos \nu\psi} + \frac{2ik}{\eta} e^{i(\pi/2)\nu} \int_0^\infty u_1^e\Big|_{\phi=\psi} H_\nu^{(1)}(k\rho)d\rho$$

$$= \csc \nu\pi \left[2c\nu - \frac{\sin \nu(\pi - \phi_0 + \psi)}{\sin(\pi - \phi_0 + \psi)} - \frac{\sin \nu(\pi - \phi_0 - \psi)}{\sin(\pi - \phi_0 - \psi)} \right] \qquad (4.45a)$$

$$\frac{F_0(\nu)}{\sin \nu\psi} - \frac{2ik}{\eta} e^{i(\pi/2)\nu} \int_0^\infty u_1^0\Big|_{\phi=\psi} H_\nu^{(1)}(k\rho)d\rho$$

$$= \frac{4i}{\eta} \csc \nu\pi \left[\frac{\sin \nu(\pi - \phi_0 + \psi)}{\sin(\pi - \phi_0 + \psi)} - \frac{\sin \nu(\pi - \phi_0 - \psi)}{\sin(\pi - \phi_0 - \psi)} \right] \qquad (4.45b)$$

H-Polarization:

$$\frac{\sin \nu(\pi - \psi)}{\sin \nu\pi} F_e'(\nu) + \frac{ik\eta}{2} e^{i(\pi/2)\nu} \int_0^\infty [v_1^e - v_2^e]\Big|_{\phi=\psi} H_\nu^{(1)}(k\rho)d\rho$$

$$= 2i \csc \nu\pi[\eta(d_1 - d_2) + \sin \nu(\pi - \phi_0 + \psi) - \sin \nu(\pi - \phi_0 - \psi)] \qquad (4.46a)$$

$$\frac{\cos \nu(\pi - \psi)}{\sin \nu\pi} F_0'(\nu) - \frac{ik\eta}{2} e^{i(\pi/2)\nu} \int_0^\infty [v_1^0 - v_2^0]\Big|_{\phi=\psi} H_\nu^{(1)}(k\rho)d\rho$$

$$= -2i \csc \nu\pi[\sin \nu(\pi - \phi_0 + \psi) + \sin \nu(\pi - \phi_0 - \psi)] \qquad (4.46b)$$

From this point on it will be assumed that $\phi_0 - 2\psi > 0$ (implying $\psi < \pi/2$), which for reasons stated earlier allows $F_e(\nu) \to 0$ for $|\mathrm{Im}\ \nu| \to \infty$, $|\mathrm{Re}\ \nu| < \delta$. Similar behavior is exhibited by F_0, F_e', and F_0'.

In [23] it was shown that by decomposing u_1^e in geometrical optics and diffracted components, i.e.,

$$u_1^e = u_1^{eg} + u_1^{ed} \quad , \tag{4.47}$$

the transformed quantity $\tilde{u}_1^e - \tilde{u}_1^{eg}$ satisfies all the conditions of Theorem 2, and hence the following representation is valid

$$u_1^e - u_1^{eg} = -\frac{i}{4} \int_{-i\infty}^{i\infty} \mu \sin \mu\pi \, [\tilde{u}_1^e(\mu) - \tilde{u}_1^{eg}(\mu)] e^{i\mu\pi} H_\mu^{(1)}(k\rho) d\mu \, . \tag{4.48}$$

Therefore

$$\int_0^\infty [u_1^e - u_1^{eg}] H_\nu^{(1)}(k\rho) d\rho = -\frac{i}{4} \int_0^\infty H_\nu^{(1)}(k\rho) \cdot \int_{-i\infty}^{i\infty} \mu \sin \mu\pi [\tilde{u}_1^e - \tilde{u}_1^{eg}] e^{i\pi\mu} \, H_\mu^{(1)}(k\rho) \, d\mu \, d\rho \tag{4.49}$$

From the conditions of Theorem 2 along with results from [21], it is allowable to exchange the order of integration in (4.49). With the aid of the identity

$$e^{i(\pi/2)x} \, H_x^{(1)}(k\rho) = \frac{2}{\pi i} K_x(-ik\rho) \quad ,$$

the integration on ρ can be expressed in terms of the integral [52]

$$e^{-i(\pi/2)\nu}\int_0^\infty K_\nu(-ik\rho)K_\mu(-ik\rho)d\rho \quad = \quad \frac{i\pi^2}{2k}\, e^{-i(\pi/2)\nu}[\cos\mu\pi + \cos\nu\pi]^{-1}$$

$$|\mathrm{Re}\,\mu| + |\mathrm{Re}\,\nu| < 1 \quad ; \qquad \mathrm{Im}\, k > 0 \quad .$$

The net result is

$$\int_0^\infty [u_1^e - u_1^{eg}]\, H_\nu^{(1)}(k\rho)d\rho \quad = \quad -\frac{1}{2k}\, e^{-i(\pi/2)\nu} \quad \cdot$$

$$\int_{-i\infty}^{i\infty} \frac{\mu \sin\mu\pi[\tilde{u}_1^e - \tilde{u}_1^{eg}]e^{i(\pi/2)\nu}}{\cos\mu\pi + \cos\nu\pi}\, d\mu \qquad (4.50)$$

Jones has shown [22] that the integral

$$\tilde{u}_1^{ge}(\mu) \quad = \int_0^\infty u_1^{ge} H_\mu^{(1)}(k\rho)\, \frac{d\rho}{\rho} \qquad (4.51)$$

is uniformly convergent for Re $\mu = 0$, and hence (4.51) may be substituted into (4.50) and the order of integration exchanged; therefore

$$\int_0^\infty [u_1^e - u_1^{ge}]H_\nu^{(1)}(k\rho)d\rho \quad = \quad -\frac{1}{2k}\, e^{-i(\pi/2)\nu}\int_{-i\infty}^{i\infty} \frac{\mu\sin\mu\pi\, \tilde{u}_1^e e^{i(\pi/2)\mu}}{\cos\mu\pi + \cos\nu\pi}\, d\mu$$

$$-\frac{1}{2k}\, e^{-i(\pi/2)\nu}\int_0^\infty u_1^{ge}\,\frac{d\rho}{\rho}\int_{-i\infty}^{i\infty} \frac{\sin\mu\pi}{\cos\mu\pi + \cos\nu\pi}\, \mu H_\mu^{(1)}(k\rho)e^{i(\pi/2)\mu}\, d\mu$$

$$(4.52)$$

By means of the identity

$$\mu\, e^{i(\pi/2)\mu} H_\mu^{(1)}(k\rho) = \frac{k\rho}{\pi}\,[K_{\mu-1}(-ik\rho) - K_{\mu+1}(-ik\rho)] \quad ,$$

it follows that

$$\int_{-i\infty}^{i\infty} \frac{\sin \mu\pi\, e^{i(\pi/2)\mu}}{\cos \mu\pi + \cos \nu\pi}\, \mu H_\mu^{(1)}(k\rho)\, d\mu$$

$$= \frac{k\rho}{\pi} \int_{-i\infty}^{i\infty} \frac{\sin \mu\pi}{\cos \mu\pi + \cos \nu\pi} \cdot [K_{\mu-1}(-ik\rho) - K_{\mu+1}(-ik\rho)]\, d\mu$$

$$= \frac{k\rho}{\pi}\left(\int_{-i\infty-1}^{i\infty-1} + \int_{i\infty+1}^{-i\infty+1} \right) \frac{\sin \mu\pi}{\cos \mu\pi - \cos \nu\pi}\, K_\mu(-ik\rho)\, d\mu \qquad (4.53)$$

The function $K_\mu(-ik\rho)$ is analytic within the strip $|\text{Im}\ \mu| < 1$, for $\text{Im}\ k > 0$, and $|K_\mu(-ik\rho)| \to 0$ as $|\text{Im}\ \mu| \to \infty$ within the strip. Therefore, (4.53) may be evaluated using residue theory, whereby the terms on the LHS and RHS of (4.52) containing u_1^{ge} cancel, leaving

$$\int_0^\infty u_1^e H_\nu^{(1)}(k\rho)\, d\rho = -\frac{1}{2k}\, e^{-i(\pi/2)\nu} \int_{-i\infty}^{i\infty} \frac{\mu \sin \mu\pi\, e^{i(\pi/2)\mu}}{\cos \mu\pi + \cos \nu\pi}\, \tilde{u}_1^e(\mu)\, d\mu \qquad (4.54)$$

$$|\text{Re}\ \nu| + |\text{Re}\ \mu| < 1 \quad ; \quad \text{Im}\ k > 0 \quad .$$

It is easily shown that

$$\tilde{u}_1^e(\mu) = \frac{e^{-i(\pi/2)\mu}}{\mu \sin \mu\pi}\, [F_e(\mu) \cos \mu(\pi - \phi) + 4ic] \quad , \qquad (4.55)$$

and in addition,

$$\int_{-i\infty}^{i\infty} \frac{4ic}{\cos \mu\pi + \cos \nu\pi}\, d\mu = - \frac{8c\nu}{\sin \nu\pi} \quad . \tag{4.56}$$

Substitution of (4.54-56) into (4.45a) results in an integral equation for $F_e(\nu)$, viz

$$F_e(\nu) - \frac{i}{\eta}\int_{-i\infty}^{i\infty} \frac{\cos \mu(\pi - \psi)\, \cos \nu\psi}{\cos \mu\pi + \cos \nu\pi} F_e(\mu) d\mu$$

$$= - \frac{4i}{\eta}\, \frac{\cos \nu\psi}{\sin \nu\pi} \left[\frac{\sin \nu(\pi - \phi_0 + \psi)}{\sin(\phi_0 - \psi)} + \frac{\sin \nu(\pi - \phi_0 - \psi)}{\sin(\phi_0 + \psi)} \right] \tag{4.57}$$

$$|\mathrm{Re}\ \nu| + |\mathrm{Re}\ \mu| < 1 \quad .$$

Equations analogous to (4.54-56) exist for the functions $u_1^o, v_1^o, v_2^e, v_1^o$, and v_1^o; hence it is possible to generate similar integral equations from (4.45b) and (4.46) for F_o, F_e', and F_o'.

Prior to presenting the integral equations, it is convenient for later analyses to put $\nu = i\tau$, $\mu = i\tau'$, and define

$$[f_e(\tau), f_o(\tau)] = [F_e(i\tau), F_o(i\tau)] \quad ,$$

$$[f_e'(\tau), f_o'(\tau)] = [F_e'(i\tau), F_o'(i\tau)] \quad . \tag{4.58}$$

In terms of the new unknowns, the integral equations can be written as*

E-Polarization:

$$f_e(\tau) + \frac{2}{\eta}\int_0^\infty \frac{ch(\pi - \psi)\tau' ch\,\psi\tau}{ch\,\pi\tau' + ch\,\pi\tau}\, f_e(\tau')d\tau'$$

$$= -\frac{4i}{\eta}\,\frac{ch\,\psi\tau}{sh\,\pi\tau}\left[\frac{sh(\pi - \phi_0 + \psi)\tau}{\sin(\phi_0 - \psi)} + \frac{sh(\pi - \phi_0 - \psi)\tau}{\sin(\phi_0 + \psi)}\right] \tag{4.59a}$$

$$f_o(\tau) + \frac{2}{\eta}\int_0^\infty \frac{sh(\pi - \psi)\tau'\, sh\,\psi\tau}{ch\,\pi\tau' + ch\,\pi\tau}\, f_o(\tau')d\tau'$$

$$= -\frac{4}{\eta}\,\frac{sh\,\psi\tau}{sh\,\pi\tau}\left[\frac{sh(\pi - \phi_0 + \psi)\tau}{\sin(\phi_0 - \psi)} - \frac{sh(\pi - \phi_0 - \psi)\tau}{\sin(\phi_0 + \psi)}\right] \tag{4.59b}$$

H-Polarization:

$$f_e'(\tau) + \frac{\eta}{2}\int_0^\infty \frac{sh\,\pi\tau'}{sh\,\psi\tau'}\,\frac{sh\,\pi\tau}{sh(\pi - \psi)\tau}\,\frac{f_e'(\tau')}{ch\,\pi\tau' + ch\,\pi\tau}\,d\tau'$$

$$= 4i\,\frac{ch(\pi - \phi_0)\tau\, sh\,\psi\tau}{sh(\pi - \psi)\tau} \tag{4.60a}$$

$$f_o'(\tau) + \frac{\eta}{2}\int_0^\infty \frac{sh\,\pi\tau'}{ch\,\psi\tau'}\,\frac{sh\,\pi\tau}{ch(\pi - \psi)\tau}\,\frac{f_o'(\tau')}{ch\,\pi\tau' + ch\,\pi\tau}\,d\tau'$$

$$= 4\,\frac{sh(\pi - \phi_0)\tau\, ch\,\psi\tau}{ch(\pi - \psi)\tau}\,. \tag{4.60b}$$

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*The functions sh( ), ch( ), represent the hyperbolic sine and cosine functions, respectively.

The even or odd property of each unknown has been used to reduce the integration interval to the semi-infinite line $[0,\infty)$.

Equations (4.59-60) are the fundamental equations for the resistive wedge scattering problem formulated via the K-L transform technique. The author is unaware of any derivation of this type in the literature on the transformation. The equations are Fredholm integral equations of the second kind (when $\eta \neq 0$).

All four equations can be written in the form

$$f(\tau) - \lambda \int_0^\infty K(\tau,\tau')f(\tau')d\tau' = g(\tau) \quad ; \qquad 0 \leq \tau \leq \infty \ . \qquad (4.61)$$

The functions $f(\tau)$, $g(\tau)$ are simultaneously either even or odd functions, while $K(\tau,\tau')$ is simultaneously either even or odd in both τ and τ'. The parameter λ is proportional to either η (H-polarization) or η^{-1} (E-polarization). The kernels of the Fredholm integrals are all bounded in the quarter-plane $0 \leq \tau < \infty$, $0 \leq \tau' < \infty$. In addition, the following order relations hold

$$|K(\tau,\tau')| = O(e^{-\psi|\text{Re}\ \tau|}) \quad ; \qquad |\text{Re}\ \tau| \to \infty \qquad (4.62a)$$

for τ' = constant. Likewise

$$|K(\tau,\tau')| = O(e^{-(\pi-\psi)|\text{Re}\ \tau'|}) \quad ; \qquad |\text{Re}\ \tau'| \to \infty \qquad (4.62b)$$

for τ = constant. As a result, it can be shown that

$$\int_0^\infty |K(\tau,\tau')|\ d\tau\ <\ \infty\ ;\qquad \text{for real } \tau'$$

$$\int_0^\infty |K(\tau,\tau')|\ d\tau'\ <\ \infty\ ;\qquad \text{for real } \tau \tag{4.63}$$

i.e., the kernel $K(\tau,\tau')$ is absolutely integrable in both τ and τ'. However, it is not integrable in the quarter-plane, that is to say

$$\int_0^\infty \int_0^\infty |K(\tau,\tau')|^2\ d\tau\ d\tau'$$

is unbounded. Furthermore, the inhomogeneous terms $g(\tau)$ of the integral equations (4.59-60) satisfy

$$|g(\tau)|\ =\ 0(e^{-(\phi_0-2\psi)|\mathrm{Re}\ \tau|})\ ;\quad |\mathrm{Re}\ \tau| \to \infty\ . \tag{4.64}$$

Since it has been assumed that $\phi_0 - 2\psi > 0$ for this analysis, it is also true that

$$\int_0^\infty |g(\tau)| d\tau < \infty\ . \tag{4.65}$$

The analyticity properties of the kernels and inhomogeneous terms, represented by $K(\tau,\tau')$ and $g(\tau)$ respectively, are also easily determined, and can be summed up as follows:

Lemma 1:

There exists a $\delta > 0$ such that $g(\tau)$ is analytic in the strip $|\mathrm{Im}\ \tau| < \delta$, and $g(\tau)$ decays exponentially to zero as $|\mathrm{Re}\ \tau| \to \infty$ uniformly within the strip.

Similarly, for the same δ, the kernel $K(\tau,\tau')$ is analytic, as a function of both τ and τ', in the region

$$\mathcal{D} = \left\{ |\mathrm{Im}\ \tau| < \delta \ \cup \ |\mathrm{Im}\ \tau'| < \delta \right\} .$$

In addition, $K(\tau,\tau')$ decays exponentially to zero as $|\mathrm{Re}\ \tau| \to \infty$, uniformly in $|\mathrm{Im}\ \tau| < \delta$, provided τ' is held to a constant within the strip $|\mathrm{Im}\ \tau'| < \delta$. Similar behavior holds as a function τ' when τ is constant.

The properties described in Lemma 1 will be used in Chapter V in order to develop an iterative solution to equations (4.59-60).

Once again the author has not been able to derive a closed form solution for the unknowns represented by $f(\tau)$, except in the special cases $\eta = 0$ or $\psi = 0, \pi/2$. However, the formulation of the problem in terms of Fredholm integral equations of the second kind allows that vast wealth of knowledge [37,38,53,54] regarding these equations to come into play, particularly for iteratively generating convergent series solutions.

In the next chapter, an iterative series solution is described based upon the method of successive approximations (Neumann series expansion) from linear operator theory, and the convergence of the series is discussed.

CHAPTER V. THE METHOD OF SUCCESSIVE APPROXIMATIONS

5.1 Review of Linear Operator Theory

Before deriving a series solution of the integral equations (4.59-60), it is appropriate to review some aspects of linear operator theory which ensure the convergence of the series. Most of the results are taken directly from References 55 through 57.

A complex linear vector space is a set X together with the set of complex numbers Z, such that for all $x, y \in X$ and $a, b \in Z$:

(1) $x + y = y + x \in X$

(2) $ax \in X$

(3) $a(x + y) = ax + ay$ and $(a + b)x = ax + bx$

(4) $a(bx) = (ab)x$

(5) $1 \cdot x = x$.

A normed linear vector space is a linear vector space X together with a function $||x||$ on X, such that for all $x, y \in X$ and $a \in Z$:

(1) $||x|| \geq 0$ and $||x|| = 0$ if and only if $x \equiv 0$

(2) $||ax|| = |a| \cdot ||x||$

(3) $||x + y|| \leq ||x|| + ||y||$.

By defining the function $g(x,y) = ||x - y||$, for $x, y \in X$, the normed linear vector space X is made a metric space with metric g.

If X is a metric space with metric g defined as above, then a Cauchy sequence is a sequence x_n in X such that for each real $\beta > 0$, there exists a positive integer N for which

$$g(x_n, x_m) = ||x_n - x_m|| < \beta \quad \text{whenever } n,m \geq N \ .$$

A metric space X is said to be complete if for every Cauchy sequence x_n in X there exists a $y \in X$ having

$$y = \lim_{n \to \infty} x_n \ .$$

A complete normed linear vector space X is called a Banach space.

If X is a Banach space, then a linear operator T is a function from X onto X which is linear, i.e.,

$$T(ax + by) = aTx + bTy$$

for all $x,y \in X$, $a,b \in Z$. The linear operator T on the Banach space X is said to be bounded if these exists some $M \in [0,\infty)$ such that

$$||Tx|| \leq M||x|| \quad \text{for all } x \in X \ .$$

The norm of the bounded linear operator T is defined as

$$||T|| = \sup_{x \in A} \frac{||Tx||}{||x||} ,$$

where $A = \{x \in X |\ x \neq 0\}$.

If T is a bounded linear operator on X, it is possible to define the nth iterated operator T^n, $n \geq 1$, by the inductive equations

$$T^1 x = Tx$$

$$T^2 x = T(Tx)$$

$$T^{n+1} x = T(T^n x)$$

for all $x \in X$. It is easily shown that

$$||T^n|| \leq ||T||^n .$$

With this background it is possible to present the following theorem.

<u>Theorem 3</u>:

If T is a bounded linear operator on the Banach space X, with $||T|| < 1$, then the series

$$\sum_{n=1}^{\infty} T^n$$

converges in operator norm to a unique linear operator T_0. Furthermore, the operator

$$L = I + T_0 = I + \sum_{n=1}^{\infty} T^n$$

is defined on X, where I is the identity operator $Ix = x$, and from $||T^n|| \leq ||T||^n$ along with

$$(I - T)(I + \sum_{p=1}^{n} T^p) = I - T^{n+1} = (I + \sum_{p=1}^{n} T^p)(I - T) ,$$

it follows that

$$(I - T)L = I = L(I - T)$$

and hence I - T is one-to-one onto X with bounded inverse

$$(I - T)^{-1} = L = I + \sum_{n=1}^{\infty} T^n .$$

Theorem 3 allows a solution to the Fredholm integral equations (4.59-60) to be formulated as a convergent series.

5.2 Series Solution to the Fredholm Integral Equations

The general form for the Fredholm integral equations can be written, as before,

$$f(\tau) - \lambda \int_0^{\infty} K(\tau,\tau')f(\tau')d\tau' = g(\tau) , \qquad (5.1)$$

where $f(\tau)$ represents the desired unknown. Using operator notation, (5.1) can be written as

$$(I - T)f = g , \qquad (5.2)$$

where the linear operator T is given by

$$Tf = \lambda \int_0^\infty K(\tau,\tau')f(\tau')d\tau' \quad . \tag{5.3}$$

It is desirable to select a Banach space X such that T is a bounded operator on this space. The results of Theorem 3 can then be applied to equation (5.2) to generate a series solution for $f(\tau)$.

In order to specify the Banach space X it is beneficial to recall the properties of $f(\tau)$ and $g(\tau)$, as they must belong to the space. In particular, from Chapter IV and Appendix A,

1. $f(\tau)$, $g(\tau)$ are analytic in a strip $|\mathrm{Im}\ \tau| < \delta$, $\delta > 0$ and
2. $f(\tau)$, $g(\tau) \to 0$ as $|\mathrm{Re}\ \tau| \to \infty$ uniformly within the strip

for $\phi_0 - 2\psi > 0$, implying $\psi < \pi/2$.

If the region $-\infty < \tau < \infty$ is denoted by a, then from (1) and (2) above it follows that

$$\sup_{\tau \in a} |f(\tau)| < \infty ,$$

and similarly for $g(\tau)$, i.e., f and g are bounded in a.

Therefore, consider the linear vector space X defined as the set of bounded functions of τ analytic in the strip $|\mathrm{Im}\ \tau| < \delta$ and hence continuous on the region a. The norm of $f(\tau) \in X$ is defined as

$$||f|| = \sup_{\tau \in a} |f(\tau)| \quad . \tag{5.4}$$

From a theorem of topology (see for example, pp. 84, 108, 216 of [57]), it follows that X is complete, and hence X is a Banach space.

In addition, from (5.3) and the definition of the norm it follows that $Tx \in X$ for all $x \in X$, and thus T is indeed a linear operation from X into itself.

With the norm as defined in (5.4), it is possible to show that the operator T is bounded for the kernels $K(\tau,\tau')$ occurring in (4.59-60). In particular,

$$\begin{aligned} ||Tf|| &= \sup_{\tau \in a} \left|\lambda \int_0^\infty K(\tau,\tau')f(\tau')d\tau'\right| \\ &\leq \sup_{\tau \in a} |\lambda| \int_0^\infty |K(\tau,\tau')|\,d\tau' \;\; \sup_{\tau \in a}|f(\tau)| \\ &\leq |\lambda| \; ||f|| \;\; \sup_{\tau \in a} \int_0^\infty |K(\tau,\tau')|\,d\tau' \; . \end{aligned} \tag{5.5}$$

Consider the various kernels of (4.59-60).

E-Polarization:

$$\int_0^\infty |K_e(\tau,\tau')|\,d\tau' = \text{ch}\,\psi\tau \int_0^\infty \frac{\text{ch}(\pi - \psi)\tau'}{\text{ch}\,\pi\tau' + \text{ch}\,\pi\tau}\,d\tau'$$

$$= \frac{\text{ch}\,\psi\tau\;\text{sh}(\pi - \psi)\tau}{\text{sh}\,\pi\tau\;\sin\psi} = \frac{1}{2}\csc\psi\left[1 - \frac{\text{sh}(\pi - 2\psi)\tau}{\text{sh}\,\pi\tau}\right]$$

$$\int_0^\infty |K_o(\tau,\tau')|\,d\tau' = \text{sh}\,\psi\tau \int_0^\infty \frac{\text{sh}(\pi - \psi)\tau'}{\text{ch}\,\pi\tau' + \text{ch}\,\pi\tau}\,d\tau'$$

$$\leq \text{ch}\,\psi\tau \int_0^\infty \frac{\text{ch}(\pi - \psi)\tau'}{\text{ch}\,\pi\tau' + \text{ch}\,\pi\tau}\,d\tau'$$

$$= \frac{1}{2}\csc\psi\left[1 - \frac{\text{sh}(\pi - 2\psi)\tau}{\text{sh}\,\pi\tau}\right]$$

implying

$$\sup_{\tau \in a} \int_0^\infty |K_e(\tau,\tau'),K_o(\tau,\tau')|d\tau' \leq \frac{1}{2}\csc\psi \quad . \tag{5.6}$$

Likewise,

<u>H-Polarization</u>:

$$\int_0^\infty |K_e'(\tau,\tau')|d\tau' = \frac{\text{sh }\pi\tau}{\text{sh}(\pi - \psi)\tau} \int_0^\infty \frac{\text{sh }\pi\tau'}{\text{sh }\psi\tau'} [\text{ch }\pi\tau' + \text{ch }\pi\tau]^{-1}\, d\tau'$$

$$\leq \frac{\pi}{\psi} \frac{\text{sh }\pi\tau}{\text{sh}(\pi - \psi)\tau} \int_0^\infty \frac{\text{ch}(\pi - \psi)\tau'}{\text{ch }\pi\tau' + \text{ch }\pi\tau}\, d\tau'$$

$$= \frac{\pi}{\psi} \frac{\text{sh }\pi\tau}{\text{sh}(\pi - \psi)\tau} \frac{\text{sh}(\pi - \psi)\tau}{\text{sh }\pi\tau \sin\psi} = \frac{\pi}{\psi}\csc\psi$$

$$\int_0^\infty |K_o'(\tau,\tau')|d\tau' = \frac{\text{sh }\pi\tau}{\text{ch}(\pi - \psi)\tau} \int_0^\infty \frac{\text{sh }\pi\tau'}{\text{ch }\psi\tau'} [\text{ch }\pi\tau' + \text{ch }\pi\tau]\, d\tau'$$

$$\leq 2\frac{\text{sh }\pi\tau}{\text{ch}(\pi - \psi)\tau} \int_0^\infty \frac{\text{sh}(\pi - \psi)\tau'}{\text{ch }\pi\tau' + \text{ch }\pi\tau}\, d\tau'$$

$$\leq 2\frac{\text{sh }\pi\tau}{\text{ch}(\pi - \psi)\tau} \frac{\text{sh}(\pi - \psi)\tau}{\text{sh }\pi\tau \sin\psi} = 2\csc\psi\ \text{th}(\pi - \psi)\tau$$

implying

$$\sup_{\tau \in a} \int_0^\infty |K_e'(\tau,\tau'),K_o'(\tau,\tau')|d\tau' \leq \frac{\pi}{\psi}\csc\psi \quad . \tag{5.7}$$

The results above have made use of the assumption $\phi_o - 2\psi > 0$ which requires $\psi < \pi/2$.

It is now possible to derive a set of bounds for the norms of the various operators. Let the operator T associated with the kernel $K_e(\tau,\tau')$ be denoted by T_e, and so on. From the definition of the norm of a linear operator, it follows that

$$||T_e,T_o|| \leq \frac{1}{2}|\lambda| \csc\psi \tag{5.8}$$

$$||T_e',T_o'|| \leq \frac{\pi}{\psi}|\lambda'| \csc\psi \tag{5.9}$$

where

$$\lambda = -\frac{2}{\eta}$$

$$\lambda' = -\frac{\eta}{2} .$$

In order to make use of Theorem 3, it is required that $||T|| < 1$, and hence from (5.8-9)

$$|\eta| > \csc\psi \; ; \qquad \text{for } T_e,T_o \; , \tag{5.10a}$$

$$|\eta| < \frac{2\psi}{\pi}\sin\psi \; ; \qquad \text{for } T_e',T_o' \; . \tag{5.10b}$$

Equation (5.10) provides sufficient conditions on η for the convergence of the series in Theorem 3.

Specifically, the solutions to (4.59-60) in the Banach space X can be written as follows:

$$f_e(\tau) = g_e(\tau) + \sum_{n=1}^{\infty} T_e^n g_e(\tau) \tag{5.11a}$$

$$f_o(\tau) = g_o(\tau) + \sum_{n=1}^{\infty} T_o^n g_o(\tau) \tag{5.11b}$$

where

$$T_{e,o}^n g_{e,o} = \left(-\frac{2}{\eta}\right)^n \int_0^\infty K_{e,o}^n(\tau,\tau') g_{e,o}(\tau')\, d\tau'$$

and

$$|\eta| > \csc\psi \ .$$

<u>H-Polarization</u>:

$$f_e'(\tau) = g_e'(\tau) + \sum_{n=1}^{\infty} T_e'^n g_e'(\tau) \tag{5.12a}$$

$$f_o'(\tau) = g_o'(\tau) + \sum_{n=1}^{\infty} T_o'^n g_o'(\tau) \tag{5.12b}$$

where

$$T_{e,o}'^n g_{e,o}' = \left(-\frac{\eta}{2}\right)^n \int_0^\infty K_{e,o}'^n(\tau,\tau') g_{e,o}'(\tau')\, d\tau'$$

and

$$|\eta| < \frac{2\psi}{\pi} \sin\psi \ .$$

The iterated kernel $K^n(\tau,\tau')$ is defined via

$$K^1(\tau,\tau') = K(\tau,\tau')$$

$$K^{n+1}(\tau,\tau') = \int_0^\infty K^n(\tau,\tau'')K(\tau'',\tau')\,d\tau'' \quad , \quad n \geq 2 \quad .$$

Several observations regarding the solutions (5.11-12) can be made:

1. For E-polarization the solution is in the form of a power series in η^{-1}, and is convergent outside the disc $|\eta| \leq \csc\psi$. For H-polarization, the series is in powers of η, convergent within the disc $|\eta| < (2\psi/\pi)\sin\psi$.

2. It may be possible to extend the regions of convergence of (5.11-12) beyond the limits expressed in (1) above, since the limits are simply sufficient conditions for the convergence of the series.

3. The two regions of convergence for E- and H-polarization do not overlap, and hence cannot be used simultaneously to solve the problem with an arbitrarily polarized incident field.

4. The series fail to converge for $\psi = 0$, at least in the sufficient sense. This is consistent with Senior's results [10] which indicated that for the half-plane an expansion in terms of η also contained contributions of order $\eta \ln \eta$, and thus could not be expressed as a simple power series.

5. For E-polarization, in the limit $\eta \to \infty$, the unknowns f_e, $f_o \to 0$. This is consistent with the fact that for $\eta \to \infty$, the wedge ceases to exist, and hence the scattered field is zero.

Likewise, for H-polarization and $\eta = 0$, the result is $f_e', f_o' = g_e', g_o'$, the known solutions for the perfectly conducting wedge.

6. The series (5.11-12) are uniformly convergent for $\tau \in (-\infty, \infty)$ within the bounds on η by virtue of the norm (5.4). When applying the inverse transform to the unknowns f_e, f_o, etc., it is therefore possible to exchange the order of integration and summation. This allows an approximate solution to be generated, with each successive term increasing the accuracy of the approximation.

The final forms for the unknown transformed fields $(\tilde{u}, \tilde{v})$ are given by:

E-Polarization:

$$\tilde{u}_1(\nu) = \frac{e^{-i(\pi/2)\nu}}{2\nu \sin \nu\pi} [f_e(-i\nu)\cos \nu(\pi - \phi) + f_o(-i\nu)\sin \nu(\pi - \phi) + 2ic] \tag{5.13a}$$

$$\tilde{u}_2(\nu) = \frac{e^{-i(\pi/2)\nu}}{2\nu \sin \nu\pi}\left[f_e(-i\nu) \frac{\cos \nu(\pi - \psi)}{\cos \nu\psi} \cos \nu\phi + f_o(-i\nu) \frac{\sin \nu(\pi - \psi)}{\sin \nu\psi} \sin \nu\phi + 2ic \right] \tag{5.13b}$$

H-Polarization:

$$\tilde{v}_1(\nu) = \frac{e^{-i(\pi/2)\nu}}{2\nu \sin \nu\pi}\left[f_e'(-i\nu)\cos \nu(\pi - \phi) + f_o'(-i\nu)\sin \nu(\pi - \phi) + 2id_1\right] \tag{5.14a}$$

$$\tilde{v}_2(\nu) = -\frac{e^{-i(\pi/2)\nu}}{2\nu \sin \nu\pi}\left[f_e'(-i\nu) \frac{\sin \nu(\pi - \psi)}{\sin \nu\psi} \cos \nu\phi + f_o'(-i\nu) \frac{\cos \nu(\pi - \psi)}{\cos \nu\psi} \sin \nu\phi + 2id_2 \right], \tag{5.14b}$$

where the substitution $\tau = -i\nu$ has been used.

Recall that to invert (5.13-14) directly requires restraining ϕ to regions where the geometrical optics fields are zero. Otherwise, the transforms of these fields must be subtracted prior to inversion (see equation (4.48) and Reference 23). Having already assumed the condition $\phi_0 - 2\psi = \zeta > 0$, the appropriate interval on ϕ for which the geometrical optics field is zero is given by

$$|\pi - \phi| < \zeta \tag{5.15}$$

in Region 1, i.e., for $\tilde{u}_1, \tilde{v}_1$. In Region 2, no such interval exists. Since it is a difficult task to determine the geometrical optics field in the interior of the wedge, especially for small values of ψ, the half-angle of the wedge, one must first determine (u_1, v_1) from $(\tilde{u}_1, \tilde{v}_1)$ in the interval (5.15), and then analytically continue the result to the surface of the wedge. From the boundary conditions it is then possible to determine (u_2, v_2) on the surface of the wedge, and hence everywhere inside Region 2. A more straightforward procedure is to use Jones' inversion formula (4.4c), and to deform the contour of integration in such a manner which allows ε to go to zero. This procedure is outlined in [22]. In fact, Jones' formula can be used to directly invert all of (5.13-14) without need for a restriction such as (5.15).

It would next be desirable to complement the series solutions (5.11-12) with similar results for small η (E-polarization) or large η (H-polarization). Such an approach is considered in the next section.

5.3 Operator Theory for the Difference Equations

For the purposes of this discussion it is sufficient to consider one of the integral equations (4.59-60), say (4.59a).

It is easily shown that the corresponding difference equation (4.39a), which was derived under the assumption that $F_e(\nu)$ was analytic in a strip $|\mathrm{Re}\ \nu| < \delta$, where $\delta > 1$, can be obtained from (4.59a) under the same assumption. By replacing τ by alternately $\tau + i$ and $\tau - i$ in (4.59a), it follows that:

$$f_e(\tau + i) + \frac{2}{\eta} \int_0^\infty \frac{\mathrm{ch}(\pi - \psi)\tau' \ \mathrm{ch}\ \psi(\tau + i)}{\mathrm{ch}\ \pi\tau' - \mathrm{ch}\ \pi\tau} f_e(\tau')d\tau' = g_e(\tau + i) \tag{5.16a}$$

$$f_e(\tau - i) + \frac{2}{\eta} \int_0^\infty \frac{\mathrm{ch}(\pi - \psi)\tau' \ \mathrm{ch}\ \psi(\tau - i)}{\mathrm{ch}\ \pi\tau' - \mathrm{ch}\ \pi\tau} f_e(\tau')d\tau' = g_e(\tau - i) \quad . \tag{5.16b}$$

Equation (4.59a) is valid for $|\mathrm{Im}\ \tau| + |\mathrm{Im}\ \tau'| < 1$, which becomes $|\mathrm{Im}\ \tau| < 1$ since τ' is real. It then follows that (5.16a) is valid for $-2 < |\mathrm{Im}\ \tau| < 0$, while (5.16b) is valid for $0 < |\mathrm{Im}\ \tau| < 2$.

From the relation (4.58) between $f_e(\tau)$ and $F_e(\nu)$, where $\nu = i\tau$, the requirement that $F_e(\nu)$ is analytic in $|\mathrm{Re}\ \nu| < \delta$, $\delta > 1$, implies $f_e(\tau)$ is analytic in $|\mathrm{Im}\ \tau| < \delta$. This behavior allows one to consider the limits as $|\mathrm{Im}\ \tau| \to 0^+, 0^-$ in (5.16a,b), respectively. From the formulas of Plemelj (see for example, p. 232 of [58]),

$$\lim_{\operatorname{Im}\tau \to 0^+,0^-} \int_0^\infty \frac{\operatorname{ch}(\pi - \psi)\tau' \operatorname{ch}\psi(\tau \pm i)}{\operatorname{ch}\pi\tau' - \operatorname{ch}\pi\tau} f_e(\tau')\, d\tau'$$

$$= \mp i \frac{\operatorname{ch}\psi(\tau \pm i)\operatorname{ch}(\pi - \psi)\tau}{\operatorname{sh}\pi\tau} f_e(\tau) + \int_0^\infty \frac{\operatorname{ch}(\pi - \psi)\tau' \operatorname{ch}\psi(\tau \pm i)}{\operatorname{ch}\pi\tau' - \operatorname{ch}\pi\tau} \cdot f_e(\tau')\, d\tau' \tag{5.17}$$

Division of (5.16a,b) by $\operatorname{ch}\psi(\tau \pm i)$, taking the limit as $|\operatorname{Im}\tau| \to 0^+,0^-$, respectively, and making use of (5.17) leads to

$$\frac{f_e(\tau + i)}{\operatorname{ch}\psi(\tau + i)} - \frac{2i}{\eta} \frac{\operatorname{ch}(\pi - \psi)\tau}{\operatorname{sh}\pi\tau} f_e(\tau) + \int_0^\infty \frac{\operatorname{ch}(\pi - \psi)\tau'}{\operatorname{ch}\pi\tau' - \operatorname{ch}\pi\tau} f_e(\tau')d\tau' = \frac{g_e(\tau + i)}{\operatorname{ch}\psi(\tau + i)}$$

$$\frac{f_e(\tau - i)}{\operatorname{ch}\psi(\tau - i)} + \frac{2i}{\eta} \frac{\operatorname{ch}(\pi - \psi)\tau}{\operatorname{sh}\pi\tau} f_e(\tau) + \int_0^\infty \frac{\operatorname{ch}(\pi - \psi)\tau'}{\operatorname{ch}\pi\tau' - \operatorname{ch}\pi\tau} f_e(\tau')d\tau' = \frac{g_e(\tau - i)}{\operatorname{ch}\psi(\tau - i)}$$

Subtracting the former from the latter and inserting the expression for $g_e(\tau)$ gives

$$\frac{f_e(\tau - i)}{\operatorname{ch}\psi(\tau - i)} - \frac{f_e(\tau + i)}{\operatorname{ch}\psi(\tau + i)} + \frac{4i}{\eta} \frac{\operatorname{ch}(\pi - \psi)}{\operatorname{sh}\pi\tau} f_e(\tau) = \frac{16}{\eta} \frac{\operatorname{ch}(\pi - \phi_0)\tau \operatorname{ch}\psi\tau}{\operatorname{sh}\pi\tau} \tag{5.18}$$

which easily reduces to (4.39a) by letting $\tau = -i\nu$, $f_e(-i\nu) = F_e(\nu)$, and recalling that $F_e(\nu)$ is an even function.

The above derivation suggests an interesting result from operator theory. In (5.2) the integral equation was written as

$$(I - T_e)f_e = g_e \quad , \tag{5.19}$$

with T_e being an integral operator. Equation (5.18) is then equivalent to

$$T_e^{-1}\,[(I - T_e)f_e] = (T_e^{-1} - I)f_e = T_e^{-1}g_e \tag{5.20}$$

where the operator T_e^{-1} is the left inverse of T_e, provided the space of functions is restricted to those which are analytic in a strip $|\mathrm{Im}\,\tau| < \delta$, $\delta > 1$. It is also possible to show that T_e^{-1} is a right inverse under the same restrictions. Explicitly, T_e^{-1} is defined by

$$T_e^{-1}f_e = \frac{\eta}{4i}\,\frac{\mathrm{sh}\,\pi\tau}{\mathrm{ch}(\pi - \psi)\tau}\left[\frac{f_e(\tau - i)}{\mathrm{ch}\,\psi(\tau - i)} - \frac{f_e(\tau + i)}{\mathrm{ch}\,\psi(\tau + i)}\right] \tag{5.21}$$

where f_e is even and analytic in $|\mathrm{Im}\,\tau| < \delta$, $\delta > 1$. Obviously, T_e^{-1} is a difference operator.

Equation (5.20) can be written as

$$(I - T_e^{-1})f_e = -T_e^{-1}g_e \tag{5.22}$$

which leads one to ask if the results of Theorem 3 can be used to generate a series solution for f_e in terms of the iterated operator $(T_e^{-1})^n$, which is essentially a series in powers of η. Such a series would indeed complement the earlier result, equation (5.11a), which expresses f_e as a power series in η^{-1}.

However, the author has not been able to develop a suitable bound for the norm of T_e^{-1}, and hence is not able to take advantage of the results of the theorem. Nonetheless, with a suitable choice of a Banach space of functions and operator norm, it may be possible to prove the convergence of a series in T_e^{-1}. Indeed, the author has shown that the first two terms of such a series produce the correct geometrical optics fields. Furthermore, when $\eta = 0$, the series reduces to a single term which is the correct solution for the perfectly conducting wedge. Thus it appears that the series, even if it is not convergent, may provide an asymptotic representation as $\eta \to 0$. Similar conclusions regarding the other operators T_o, T_e', T_o' can be made.

CHAPTER VI. COMMENTS AND CONCLUSIONS

The electromagnetic scattering of an arbitrarily polarized plane wave normally incident upon a resistive wedge has been formulated via a pair of related function-theoretic techniques, the method of Maliuzhinets [12-14] and the Kontorovich-Lebedev transform method [20-22], both of which have been successfully used to solve scattering problems in a single wedge-shaped region.

The goal of the author was twofold: obviously to find an exact solution to the resistive wedge problem, and secondly, to point out the similarities between the two function-theoretic techniques as well as the complexity that arises in applying them to a two-region problem.

With regard to the latter of these goals, the author has shown that, as with past applications, both methods lead to a set of difference (or functional) equations for the various unknowns. However, if these methods are simply applied directly, as in the single region problem, the presence of nonzero fields in two regions produces a coupling of the unknowns in the functional equations (see equations (3.9-10) and (4.31-32), for example). By means of a decomposition into symmetric and antisymmetric components, the equations were successfully uncoupled, although they remained sufficiently complex to prevent the determination of closed form exact solutions.

One reason for this shortcoming is the lack of a systematic technique for solving difference equations of order greater than one.

To this end, the author has developed a novel procedure which, under less restrictive conditions, replaces the difference equations of the K-L method with Fredholm integral equations of the second kind. This allows a large number of techniques to come into play for the determination of exact and approximate solutions.

One such technique, the method of successive approximations, is shown to lead to uniformly convergent power series solutions for certain values of the normalized resistivity η. In particular, for E-polarization, the series converge when η is large, while for H-polarization, they converge when η is small. This behavior prevents the use of both polarizations simultaneously for generating the solution to an arbitrarily polarized incident field.

Although an attempt to alleviate this problem was made by demonstrating that the integral operator and difference operator were inverses of each other under restricted conditions, the author was not able to bound the difference operator over the space of functions being considered, and hence could not prove a series generated with this operator would converge. Nonetheless, the series does exhibit proper behavior under certain circumstances, leaving open the question as to whether a bound for the operator does exist. Certainly, this is an area open to future work. A parallel effort investigating more accurate bounds for the integral operators, thereby extending the radius of convergence of the series, is also worth pursuing.

In addition, the author is hoping to communicate the results of this work, particularly the functional equations (3.32-33) and

(4.39-40), to several authorities on functional equations for their suggestions regarding possible solutions.

Finally, the reader may find this work conspicuously lacking in numerical or graphical results. This is not an accident, as the purpose of the author was to explore the possibility of extending proven function-theoretic techniques toward finding an exact solution for the resistive wedge, and in the process, to illustrate the nature of the complications that arise when these techniques are applied to a two-region problem. The author believes that the presentation of numerical data based on an approximate solution is not consistent with these goals, and hence has reserved this area for pursuit in the future.

APPENDIX . THEOREMS FOR THE METHODS OF MALIUZHINETS AND THE KONTOROVICH-LEBEDEV TRANSFORM

A.1 The Method of Maliuzhinets

The material in this section is derived entirely from the works of Maliuzhinets [13-15,51]. The basis for his method is the representation of a function $S(\rho,\phi)$ of the polar coordinates (ρ,ϕ) in the form of a Sommerfeld integral, viz.

$$S(\rho,\phi) = \frac{1}{2\pi i} \int_{\gamma} e^{ik\rho\cos\alpha}\, s(\alpha - \phi)d\alpha \qquad \text{(A.1)}$$

where γ is a contour in the complex α plane consisting of two loops, γ_1 and γ_2. The contour approaches infinity within the shaded regions where the real part of $ik\rho\cos\alpha$ is negative for positive real k, as shown in Fig. 3.1, and reproduced for convenience in Fig. A.1.

In [14], Maliuzhinets establishes conditions for the existence of a unique solution $s(\alpha)$ to the integral equation (A.1) for $S(\rho,\phi)$ satisfying certain boundedness conditions.

Theorem A.1:

Let M,a,b,c,d be positive numbers, and let ε,m be numbers satisfying

$$0 < \varepsilon < \pi \quad .$$

$$|\arg m| \leq \frac{\pi}{2}$$

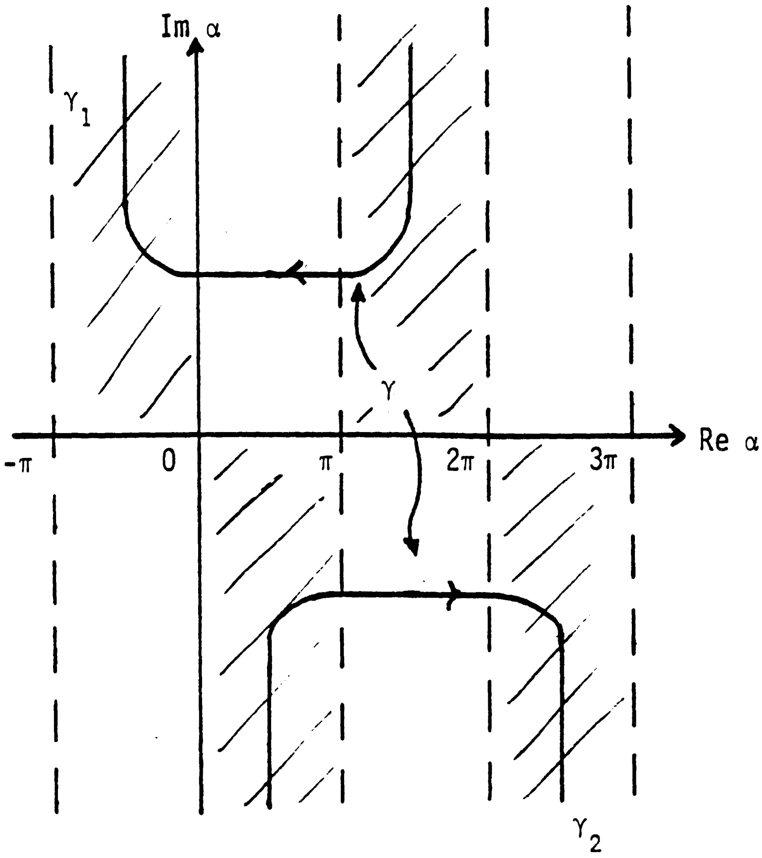

Fig. A.1: Contour of Integration for the Representation of a Function by a Sommerfeld Integral.

Let $F(\rho)$ be a function satisfying the inequality

$$|F(\rho)| < M|\rho|^{-1+a} e^{b|\rho|}$$

for positive values of ρ, and also in the entire region

$$0 < |\rho| < \infty$$
$$|\arg \rho| < \varepsilon_1 \quad , \quad 0 < \varepsilon_1 < \pi \quad ,$$

where this function is analytic. Consider the integral equation

$$F(\rho) = \frac{1}{2\pi i} \int_{\gamma'} e^{m\rho\cos\alpha} f(\alpha)\, d\alpha \quad , \qquad \text{(A.2)}$$

where the contour γ' is made up of two loops, γ_1' and γ_2'. The loop γ_1 consists of the two half lines

$$\operatorname{Re} \alpha = \arg m \pm \left(\varepsilon + \frac{\pi}{2}\right)$$
$$\operatorname{Im} \alpha \geq d$$

and the line segment $\operatorname{Im} \alpha = d$. The loop γ_2' is symmetric with respect to γ_1' about $\alpha = 0$ (see Fig. A.2).

Then there exists one and only one solution $f(\alpha)$ to (A.2) which is analytic on and within the contour γ' except at infinitely distant points, and which satisfies the additional constraints

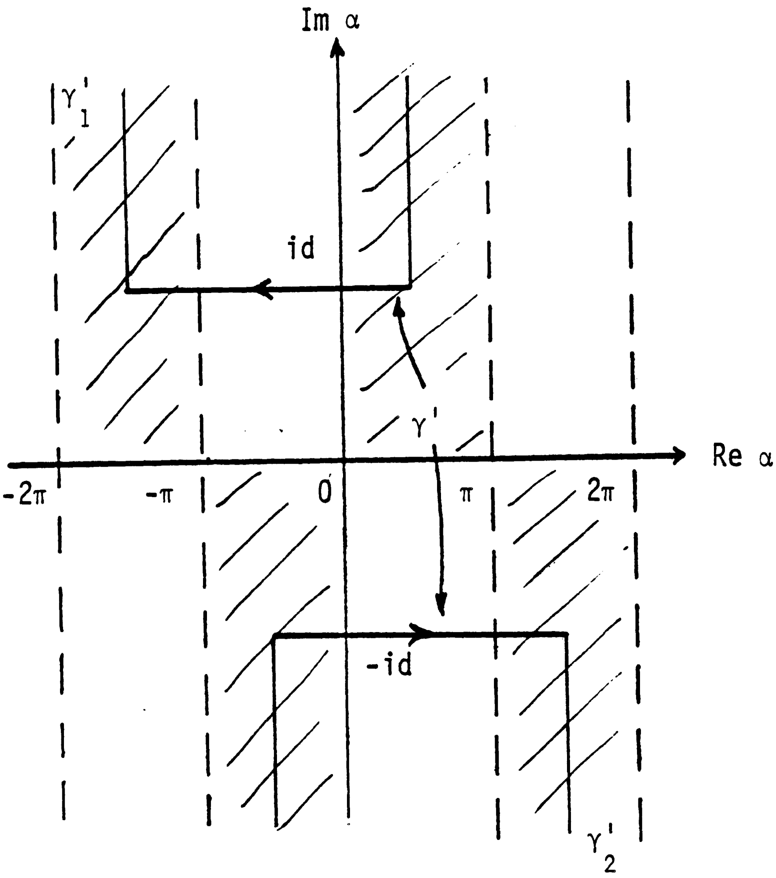

Fig. A.2: The Contour of Integration γ' in the Complex α Plane.

$$|f(\alpha)| < M_1 e^{(1-a_1)|\text{Im}\,\alpha|} \quad ; \quad M_1, a_1 > 0 \tag{A.3}$$
$$f(\alpha) = -f(-\alpha) \ .$$

This function is represented by the integral

$$f(\alpha) = -\frac{m \sin\alpha}{2} \int_0^\infty F(\rho) e^{-m\rho\cos\alpha} \, d\alpha \tag{A.4}$$

for $\text{Re}(m\cos\alpha) > 0$. For this function, $a_1 = a$.

Proof:

In view of the fact that $f(\alpha)$ is odd, (A.2) can be written as

$$F(\rho) = \frac{1}{\pi i} \int_{\gamma_1'} e^{m\rho\cos\alpha} f(\alpha) \, d\alpha \tag{A.5}$$

By making a change of variables, $W = e^{i(\arg m)}\cos\alpha$, and defining

$$g(W) = -2 \frac{f(\alpha)}{\sin\alpha} e^{-i(\arg m)} \tag{A.6}$$

equation (A.5) becomes

$$F(\rho) = \frac{1}{2\pi i} \int_\Gamma e^{|m|\rho W} g(W) \, dW \ , \tag{A.7}$$

where Γ is the image of γ_1' under the transformation of variables. The contour Γ intersects the real axis between zero and ch d, and coincides at infinity with the rays $\arg W = \pm(\varepsilon + \frac{\pi}{2})$, as shown in Fig. (A.3).

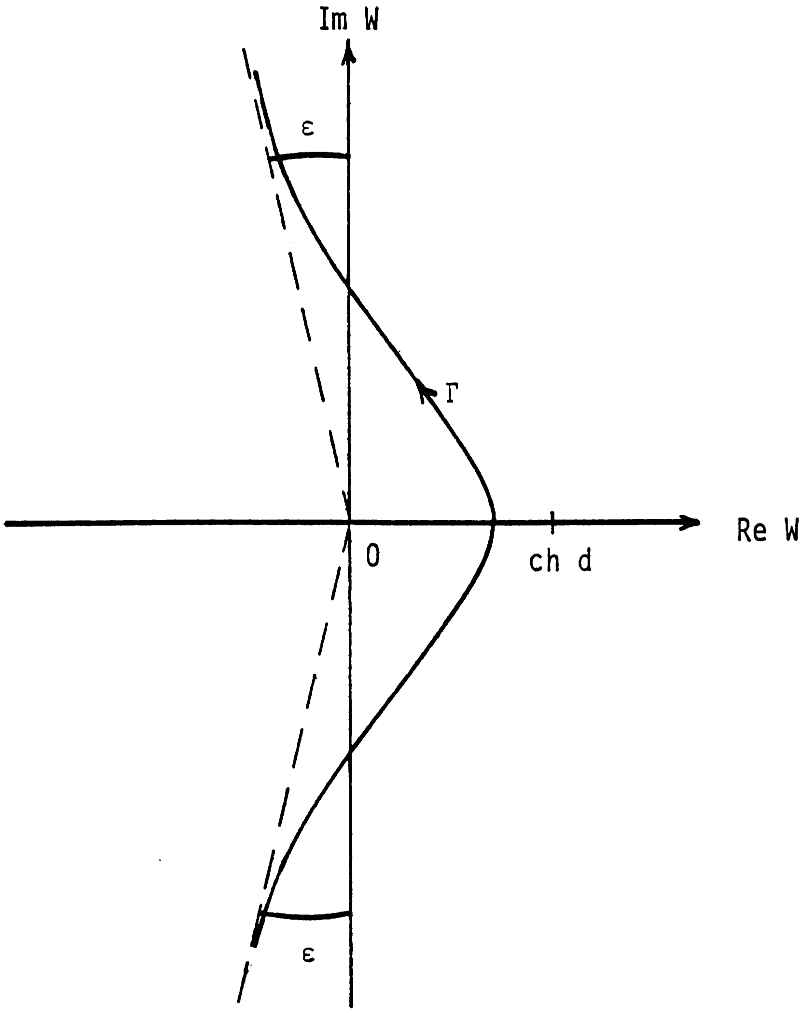

Fig. A.3: Contour of Integration Γ in the Complex W Plane.

The function $g(W)$ is analytic to the right of Γ, which coincides with the interior of the loop γ_1', where $f(\alpha)/\sin\alpha$ is analytic. From (A.3) it follows that

$$|g(W)| < 4M_1 |W|^{-a_1} \quad , \quad |W| \to \infty \quad , \quad |\arg W| < \varepsilon + \frac{\pi}{2} \; . \qquad \text{(A.8)}$$

Multiplying (A.7) by $e^{-|m|\rho W}$, where Re $W > ch\ d$, and integrating over $(0,\infty)$ on ρ gives

$$\int_0^\infty F(\rho)e^{-|m|\rho W}\, d\rho = \frac{1}{2\pi i}\int_0^\infty d\rho \int_\Gamma e^{|m|\rho(W_1-W)}\, g(W_1)\, dW_1 \quad , \qquad \text{(A.9)}$$

where the integration is taken as a limit at the endpoints. By virtue of (A.8) and the fact that $\mathrm{Re}(W_1-W) < 0$, the order of integration may be exchanged, and the limit taken, which leads to

$$\int_0^\infty F(\rho)e^{-|m|\rho W}\, d\rho = -\frac{1}{2\pi i|m|}\int_\Gamma \frac{g(W_1)}{W - W_1}\, dW_1 = \frac{g(W)}{|m|} \; , \qquad \text{(A.10)}$$

where, thanks to (A.8), the integral in (A.10) has been evaluated from the theory of residues. Transforming W back to α,

$$\int_0^\infty F(\rho)\, e^{-m\rho\cos\alpha}\, d\rho = -\frac{2f(\alpha)}{m\sin\alpha}$$

which is the desired result given in (A.4).

In order to show that the assumptions regarding the analyticity of $f(\alpha)$, along with the conditions (A.3), are indeed true, consider equation (A.10) as a definition of the function $g(W)$.

Because of the conditions that $F(\rho)$ be analytic for $|\arg \rho| < \varepsilon_1$, it follows that the integration for $\rho > c$ in (A.10) may be displaced to an arbitrary half line $|\arg \rho| = \varepsilon_1 > \varepsilon$ (by choice), from which it follows that $g(W)$ is analytic for large $|W|$, provided $|\arg W| < \varepsilon + \pi/2$. Defining $f(\alpha)$ via (A.6), one then concludes that for sufficiently large d, $f(\alpha)$ is analytic in $\operatorname{Im} \alpha \geq d$, $|\operatorname{Re} \alpha - \arg m| < \varepsilon + \pi/2$, which corresponds to the interior of the loop γ_1'.

In addition, because it has been assumed that

$$|F(\rho)| < M|\rho|^{-1+a} e^{b|r|} ,$$

it can be shown via (A.10) that

$$\frac{|g(W)|}{|m|} < \int_0^\infty |F(\rho)| \, e^{-\rho|mW|\cos(\arg W)} \, d\rho$$

$$< M|mW|^{-a} \int_0^\infty x^{-1+a} \, e^{-x[\cos(\arg W) - b/|mW|]} \, dx .$$

Since the integral is bounded for sufficiently large $|W|$,

$$|g(W)| < M_2 |W|^{-a} , \quad M_2 > 0 ,$$

confirming (A.8), and since

$$\frac{f(\alpha)}{\sin \alpha} = -2Wg(W)$$

it follows that

$$|f(\alpha)| < M_1 \, e^{(1-a_1)|\mathrm{Im}\alpha|}$$

with $a_1 = a$, as assumed. Thus the proof is complete.

By means of the substitutions

$$F(\rho) = S(\rho,\phi)$$

$$f(\alpha') = \frac{1}{2}\,[s(\alpha' + \pi - \phi) - s(-\alpha' + \pi - \phi)]$$

$$M = ik \text{ (implying Im } k > 0\text{)}$$

and replacing $\alpha' = \alpha - \pi$, equation (A.5) becomes

$$S(\rho,\phi) = \frac{1}{2\pi i} \int_{\gamma_1} e^{ik\rho\cos\alpha}\,[s(\alpha - \phi) - s(-\alpha + 2\pi - \phi)]\,d\alpha \quad .$$

From Fig. A.1 and the relation between the contours γ_1 and γ_2, the expression

$$S(\rho,\phi) = \frac{1}{2\pi i} \int_{\gamma} e^{ik\rho\cos\alpha}\, s(\alpha - \phi)\,d\alpha$$

is obtained, which agrees with (A.1).

An important result of Theorem A.1 is that an odd solution to the equation

$$\frac{1}{2\pi i} \int_{\gamma'} f(\alpha')e^{m\rho\cos\alpha'} d\alpha' = 0$$

must vanish identically; consequently, a solution to the equation

$$\int_{\gamma} e^{ik\rho\cos\alpha} s(\alpha - \phi)d\alpha = 0 \tag{A.12}$$

must satisfy the condition

$$s(\alpha - \phi) = s(2\pi - \alpha - \phi) \quad , \tag{A.13}$$

i.e., the coefficient of the exponential in the integral must be even about $\alpha = \pi$. This result is used extensively in formulating the functional equations for the unknowns in Chapter III.

The justification of a representation such as (A.1) for the unknown fields satisfying the Helmholtz equation

$$(\nabla^2 + k^2)S(\rho,\phi) = 0$$

is discussed in considerable detail in [51], and will not be reproduced here for the sake of brevity. However, it is worthwhile noting that the bounds placed upon the function $F(\rho)$ in Theorem A.1 include the functions $S(\rho,\phi)$ satisfying the edge and radiation conditions discussed in Chapter II. Furthermore, from [51], the value of $S(\rho,\phi)$ at $\rho = 0$, provided $S(\rho,\phi)$ is bounded, is given by

$$S(0,\phi) \;=\; 2is(i\infty) \;=\; -2is(-i\infty) \quad , \tag{A.14}$$

and does not depend on ϕ, as expected. The boundedness at $\rho = 0$ implies $a \geq 1$, which also implies, from (A.3), that $s(i\infty) < \infty$, consistent with (A.14).

The representation (A.1) is simply a superposition of elementary plane wave solutions to the Helmholtz equation. This is more readily seen by replacing α by $\alpha - \pi + \phi$, whereby

$$S(\rho,\phi) \;=\; \frac{1}{2\pi i} \int_{\gamma_\phi} e^{-ik\rho\cos(\alpha+\phi)}\, s(\alpha - \pi)\, d\alpha \tag{A.15}$$

where γ_ϕ is simply the contour γ displaced an amount $\pi-\phi$ to the right. As discussed in [51], (A.15) represents a set of plane waves incident from a direction $2\pi - \alpha$, where α is complex, allowing for evanescent (decaying) waves.

Recalling the geometry for the scattering of a plane wave by a resistive wedge, the incident field is given by

$$(U^i,V^i) \;=\; e^{-ik\rho\cos(\phi_0-\phi)} \quad . \tag{A.16}$$

Maliuzhinets shows in [51] that discrete plane waves, given by the geometrical optics fields, correspond to poles of $s(\alpha-\pi)$ lying within the strip $\pi - \phi < \text{Re}\,\alpha < 3\pi - \phi$. From (A.16), it follows that $s(\alpha-\pi)$ has a pole at $\alpha = 2\pi - \phi_0$ with residue unity. Furthermore, the field (A.16) is the only geometrical optics field incident from within

Region 1 ($\psi \leq \phi \leq 2\pi - \psi$), as all other fields appear to emanate from within the wedge (i.e., Region 2).

Therefore, it may be stated that $s_1(\alpha)$ (representing the field in Region 1) satisfies an additional constraint for the wedge problem, that is

$$s_1\,(\alpha - \pi) - (\alpha - 2\pi + \phi_0)^{-1} \tag{A.17}$$

is analytic for $\psi \leq \text{Re}\,\alpha \leq 2\pi - \psi$. This condition is stated in Chapter III as a means of uniquely determining a solution to the functional equations derived for the unknowns.

A.2 The Kontorovich-Lebedev Transform

The requirements for the existence of the K-L transform and its inverse are rigorously described in [21,22,59], each of which prescribes a set of conditions for a function $f(\rho)$ or $\tilde{f}(\nu)$ in order for the integral formulas to converge. The results of [21] have been reproduced without proof in Theorem 2, Chapter IV of this work.

Rather than repeat any of these derivations here, the author will simply state the necessary results, and then apply them to the particular functions being considered in Chapter IV in order to justify the assumptions made therein.

Specifically, a set of total fields (U,V), representing the solution to the scattering by a resistive wedge of an E- or H-polarized incident plane wave, were shown in Chapter II to have certain boundedness properties as ρ approaches zero and infinity.

Near the vertex of the wedge

$$(U,V) = O(\rho^{\delta}) \quad , \quad \delta \geq 0 \quad , \quad \rho \to 0 \quad , \qquad \text{(A.18)}$$

which allows (U,V) to be at most a nonzero constant at $\rho = 0$. Furthermore, by writing,

$$(U,V) = (U^g + U^d, V^g + V^d) \qquad \text{(A.19)}$$

i.e., as a sum of geometrical optics and diffracted fields, the radiation condition requires

$$\lim_{\rho \to \infty} \rho^{1/2}\left[\frac{\partial}{\partial\rho} - ik\right]\begin{bmatrix} U^d \\ V^d \end{bmatrix} = 0 \qquad \text{(A.20)}$$

which implies that (U^d,V^d) decay at least as rapidly as $\rho^{-1/2}$ as $\rho \to \infty$. In addition, several of the inversion theorems assume $\text{Im } k > 0$, in which case, writing

$$(U,V) = (U^i + U^s, V^i + V^s) \quad ,$$

the scattered fields (U^s,V^s) behave as $e^{-b\rho}$ as $\rho \to \infty$, where $b > 0$.

It is convenient to summarize these results into a set of assumptions regarding (U,V):

1. There exist A, $\rho_0 > 0$ such that

$$|U,V| \leq A\rho^{\delta} \quad , \qquad \delta \geq 0 \tag{A.21}$$

for $\rho \leq \rho_0$.

2. There exist B,C > 0 such that

$$|U^d,V^d| \leq B\rho^{-1/2} e^{ik\rho} \tag{A.22a}$$

$$|U^g,V^g| \leq C \tag{A.22b}$$

for $\rho > \rho_0$.

Since the value of $|U^i,V^i|$ is unity for all ρ, it follows that (U^s,V^s) satisfy (A.21) above. Inasmuch as the scattered fields contain geometrical optics terms, (U^s,V^s) also satisfy (A.22b).

In Chapter IV, the modified scattered fields

$$\begin{bmatrix} u \\ \\ v \end{bmatrix} = \begin{bmatrix} U^s + c\, e^{ik\rho} \\ \\ V^s + d\, e^{ik\rho} \end{bmatrix}$$

where $c = -U^s\ (\rho = 0)$, $d = -V^s\ (\rho = 0)$, were defined. This implies that (u,v) satisfy (A.21) with δ strictly greater than zero. In a manner analogous to (A.19), it is possible to separate the geometrical optics terms from (u,v), (as was done in [23]), viz

$$u = u^g + u^d = (U^{sg} + c^g e^{ik\rho}) + (U^{sd} + c^d e^{ik\rho}) \tag{A.24}$$

and similarly for v. Note that u^g, v^g both satisfy (A.21) with $\delta > 0$, in addition to (A.22b).

Consider the K-L transform of (u,v), given by

$$(\tilde{u},\tilde{v}) = \int_0^\infty (u,v)H_\nu^{(1)}(k\rho)\,\frac{d\rho}{\rho} \quad . \tag{A.25}$$

Because $H_\nu^{(1)}(k\rho) = O(\rho^{-|Re\nu|})$, it follows from (A.21) and (A.22b) that the integral (A.25) is uniformly convergent for $|Re\ \nu| < \delta$, and thus $(\tilde{u},\tilde{v})$ are analytic functions in the strip $|Re\ \nu| < \delta$. In addition, from [23] it can be shown that $(\tilde{u}^d,\tilde{v}^d)$ approach zero as $|Im\ \nu| \to \infty$, $|Re\ \nu| < \delta$. Unfortunately, the geometrical optics transformed fields do not exhibit such behavior.

In particular, a general form for (u^g,v^g) is

$$(u^g,v^g) = \sum_{n=1}^{N} a_n\left[e^{-ik\rho\cos\beta_n} - e^{ik\rho}\right] \tag{4.26}$$

where $\beta_n = \phi - \phi_n$, ϕ_n is the negative direction of propagation of the plane wave, and a_n,N are constants. The transform is then given by

$$(\tilde{u}^g,\tilde{v}^g) = \sum_{n=1}^{N} a_n \int_0^\infty (e^{-ik\rho\cos\beta} - e^{ik\rho})H_\nu^{(1)}(k\rho)\ \frac{d\rho}{\rho}$$

$$= \sum_{n=1}^{N} 2ia_n\ e^{-i(\pi/2)}\ \frac{1 - \cos\nu(\pi \mp \beta_n)}{\nu\sin\nu\pi} \quad . \tag{A.27}$$

Note that the second order pole at $\nu = 0$ is cancelled by the second order zero of $[1 - \cos\nu(\pi - B_n)]$. Therefore (u^g,v^g) are analytic in $|Re\ \nu| < 1$. The minus sign corresponds to $0 \leq \beta_n \leq 2\pi$, the plus sign to $-2\pi \leq \beta_n \leq 0$. It is easy to show that $(\tilde{u}^g,\tilde{v}^g)$ become unbounded for $\pi/2 < |\beta_n| < 3\pi/2$ when $\nu \to -i\infty$. It is for this reason that one cannot perform the inversion

$$(u,v) = -\frac{1}{2}\int_{-i\infty}^{i\infty} (\tilde{u},\tilde{v})\nu\, J_\nu(k\rho)\, d\rho$$

directly. Instead it is the integral

$$(u - u^g,\ v - v^g) = -\frac{1}{2}\int_{-i\infty}^{i\infty} (\tilde{u} - \tilde{u}^g,\ \tilde{v} - \tilde{v}^g)\nu J_\nu(k\rho)\, d\rho \tag{A.28}$$

which must be considered. Jones avoided this problem in [22] by defining the inversion via

$$(u,v) = -\lim_{\varepsilon\to 0^+} \frac{1}{2}\int_{-i\infty}^{i\infty} e^{\varepsilon\nu^2} (\tilde{u},\tilde{v})\nu J_\nu(k\rho)\, d\rho \quad , \tag{A.29}$$

allowing the inclusion of the geometrical optics terms.

It should be noted that should there exist intervals in ϕ for which (u^g,v^g) are zero, then the inversion may be carried out for these values of ϕ, and the result analytically continued for other values. This method neatly generates the geometrical optics terms.

In Section 4.2.2, the derivation of the integral equations for the various unknowns made use of a formula of the form

$$\int_0^\infty u^g H_\nu^{(1)}(k\rho)\, d\rho = -\frac{1}{2k} e^{-i(\pi/2)\nu}\int_{-i\infty}^{i\infty} \frac{\mu \sin \mu\pi\ \tilde{u}^g(\mu) e^{i(\pi/2)\mu}}{\cos \mu\pi + \cos \nu\pi}\, d\mu\ . \tag{A.30}$$

It is desirable to explicitly show this result.

By inserting (A.26) into the LHS of (A.30) one obtains

$$\int_0^\infty u^g H_\nu^{(1)}(k\rho)\, d\rho = \sum_{n=1}^{N} \frac{2a_n}{k}\,\frac{e^{-i(\pi/2)\nu}}{\sin \nu\pi}\left[\nu + \frac{\sin \nu(\pi - |\beta_n|)}{\sin|\beta_n|}\right]. \tag{A.31}$$

Similarly, substitution of (A.22) into the RHS of (A.30) gives

$$-\frac{1}{2k}\, e^{-i(\pi/2)\nu} \int_{-i\infty}^{i\infty} \frac{\mu \sin \mu\pi \; e^{-i(\pi/2)\mu}}{\cos \mu\pi + \cos \nu\pi}\, \tilde{u}^{g}(\mu)\, d\mu$$

$$= \sum_{n=1}^{N} \frac{2a_n}{k}\, e^{-i(\pi/2)\nu} \int_{0}^{\infty} \frac{1 - \text{ch}(\pi - |\beta_n|)\tau'}{\text{ch}\, \pi\tau' + \cos \nu\pi}\, d\tau'$$

$$= \sum_{n=1}^{N} \frac{2a_n}{k}\, \frac{e^{-i(\pi/2)\nu}}{\sin \nu\pi} \left[\nu + \frac{\sin \nu(\pi - |\beta_n|)}{\sin |\beta_n|} \right] \qquad (A.32)$$

and hence (A.30) is indeed valid.*

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*This result was derived in Chapter IV in a slightly different manner, based on the uniform convergence of the transform representation of $\tilde{u}^{g}$. The results are equivalent.

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