RAYLEIGH SCATTERING*

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*This work was supported by the U.S. Air Force Office of Scientific Research under Grants 81-0156 and 82-0112.

1. INTRODUCTION

The scattering of acoustic and electromagnetic waves from bounded objects whose dimensions are small compared with the length of the incident wave has been the subject of considerable study for more than a century. Lord Rayleigh found this problem of continuing interest and his contributions in this area (e.g., Rayleigh, 1881, 1897) provide the foundation on which almost all subsequent work is based. It is only fitting that his name grace the subject. Some idea of the history of this area of scientific inquiry and its use in light scattering may be gained from Twersky (1964) and the books of van de Hulst (1957) and Kerker (1969); and much relevant material, especially concerning spherical scatterers, is contained in the fascinating study of Logan (1965).

Despite this long history, it was not until relatively recently that a rigorous mathematical definition of Rayleigh scattering was attempted (Kleinman, 1965, 1978). Although there is not necessarily universal agreement, it is generally accepted that in three-dimensional problems, Rayleigh scattering concerns the determination of the first nonvanishing term in a series expansion in powers of wave number of a relevant quantity of interest such as the scattered field or far field coefficient. This is the definition used here and made precise in Section 2.

The present study is concerned with the determination of the first or Rayleigh term in the far field coefficient for a variety of scattering problems. The unifying thread is the use of so-called polarizability tensors to characterize the Rayleigh term in the far field coefficient. These tensors, which are symmetric for isotropic scatterers, enable one to express the scattered field in terms of quantities which depend only on the geometry and constitutive parameters of the scatterer. Methods for finding these tensor elements as solutions of integral equations are presented and numerical results are given which show the dependence of these quantities on the geometry of the scatterer. In Section 3 the case of perfectly conducting scatterers with nonzero volume is considered whereas Section 4 treats the corresponding case of zero volume scatterers. Section 5 deals with homogeneous dielectrics and Section 6 discusses the effect produced when the scatterer is dissipative or dispersive in the particular cases of lossy dielectrics and plasmas. Section 7 treats Rayleigh scattering of acoustic waves using the mechanism of polarizability tensors.

In Section 7 reference is made to the first two terms in the low frequency expansion for an acoustical hard body and to the first three terms for a soft body, and it is noted that these higher order terms are all specified by the potential functions needed for the calculation of the leading terms alone, a fact discovered by Van Bladel (1968). The determination of higher order terms in the electromagnetic case has also been studied. For dielectric bodies, Jones (1980) has shown that the second term is specified by the functions needed for the first term, a result similar to that for an acoustically hard body, but surprisingly the perfectly conducting body offers greater difficulty. It appears (Senior, 1982) that to obtain the first two far field terms requires slightly more than one near field term. An extra interior potential problem must be solved, though the full second near field term is not necessary.

A method which reduces the task of finding terms of any order in the low frequency expansion to problems in potential theory was proposed by Stevenson (1953a) and refined by Kleinman (1967a, 1978). The procedure is quite involved and serious questions can be raised about the wisdom of such higher order calculations. If the higher order terms can be obtained analytically, as Stevenson (1953b) did for the second non-trivial term for ellipsoids (in electromagnetics) and Asvestas and Kleinman (1970) did with all terms for spheroids (in acoustics), the results can be used to good advantage in calculating scattering properties. But if, as is more often the case for irregularly-shaped bodies, one must resort to a numerical solution of the required potential problems at each stage, then it is not clear whether it is advantageous to do so rather than solving the appropriate frequency dependent integral equation at the desired frequencies. Of course, if one were able to associate individual terms in a low frequency expansion with particular geometric or constitutive properties of the scatterer, or if the calculation of higher order terms could be reduced to the solution of certain canonical problems, independent of the incident field direction, as was possible for the first term using polarizability tensors, then the incentive for obtaining higher order terms would be much greater. Unfortunately, such results are not presently available, and in their absence the compromise between the desirable and the realistic leads one to concentrate on the first term in the low frequency expansion. Though admittedly limited to low frequencies, such results have proven to be quite useful in this range and they may be systematically presented for a wide variety of problems. It is such a presentation that is attempted here.

2. DEFINITIONS, FORMULATION AND REPRESENTATIONS

Although scattering problems are most frequently formulated in the form of boundary value or transition problems for the Helmholtz or Maxwell equations, it is also customary and useful to employ integral representations of the fields, both to reformulate the scattering problem as an integral equation over the scattering surface and to represent the far field in terms of the surface field. These integral representations are a central tool in low frequency analysis and in this section we state the relevant differential equation formulations, present some integral representations and relations, and define the quantities of physical interest.

The scattering problems considered herein involve a time harmonic wave incident upon a bounded surface B with finite (possibly zero) volume.

2.1 Notation

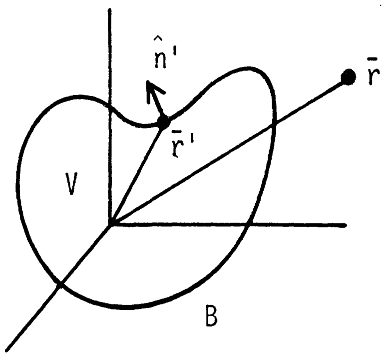

Fig. 1

Let B denote a bounded simply connected piecewise smooth surface which divides R^3 into two regions, the interior, int B, of finite volume V and an unbounded exterior, ext B. The surface B is assumed to have a continuous turning normal almost everywhere.

A precise mathematical chracterization of the scattering surface is surprisingly involved (e.g., Günter, 1967; Müller, 1969). For smooth surfaces one can require that each point on the surface be the center of a local coordinate system in terms of which the surface is given locally as $z = f(x,y)$ where f is either three times differentiable (Werner, 1962), twice differentiable (Müller & Niemeyer, 1961), or differentiable with Hölder continuous derivatives (Günter, 1967). Non-smooth surfaces are similarly defined except that piecewise differentiability rather than differentiability is required. We denote by $\bar{r}$ (usually in ext B) and $\bar{r}'$ (usually on B) vectors of magnitudes r and r', respectively, and rectangular coordinates (x,y,z) and (x',y',z') with respect to a Cartesian coordinate system centered in int B. A carat ^ will denote a vector of unit magnitude, so $\hat{n}'$ denotes a unit normal at $\bar{r}'$, directed from B to ext B.

2.2 Scalar Scattering

We assume a time harmonic incident field $u^{inc}(\bar{r})$, where a factor $e^{-i\omega t}$ is suppressed, due to sources in ext B located either at infinity (plane waves) or at finite points in ext B (point sources). In either case u^{inc} will satisfy the Helmholtz equation

$$(\nabla^2 + k^2)u^{inc} = 0 \qquad (1)$$

at all points in int B and almost all points in ext B. If ω is the angular frequency of the incident field, λ is the wavelength and c is the velocity of propagation, then the wave number k satisfies the relations

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{c} \tag{2}$$

and it is this parameter, properly normalized to make it dimensionless, which is usually taken to be small in the low frequency regime.

The presence of the scatterer B disturbs the incident field and gives rise to a scattered field which is merely the difference between the total field and that which would have been present were the body absent. Thus

$$u^t = u^{inc} + u^s \quad , \tag{3}$$

where the superscripts denote total, incident and scattered, respectively. It is the task of finding u^s for prescribed u^{inc} which constitutes the scattering problem. In all the scalar problems considered, the scattered field satisfies the Helmholtz equation and the radiation condition:

$$(\nabla^2 + k^2)u^s = 0 \quad , \quad \bar{r} \in \text{ext } B \tag{4}$$

$$\lim_{r \to \infty} r \left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \text{ uniformly in all directions } \hat{r} \quad . \tag{5}$$

Depending on the problem, u^s may satisfy either of the following conditions on B:

Dirichlet or Acoustically Soft:

$$u^s = -u^{inc} , \qquad \bar{r} \in B \ . \tag{6}$$

Neumann or Acoustically Hard:

$$\frac{\partial u^s}{\partial n} = - \frac{\partial u^{inc}}{\partial n} , \qquad \bar{r} \in B \ . \tag{7}$$

These boundary conditions can be written slightly more simply in terms of u^t using (3). If B is smooth then (4), (5) and either of (6) or (7) comprise a well-posed classical boundary value problem, i.e., u^s is twice differentiable in ext B and the first derivatives are continuous and bounded up to the surface B. If B has corners or edges, an additional condition (finite energy) must be imposed, which may be written as

$$\int_{\Omega} (|u^s|^2 + |\nabla u^s|^2) \, d\tau < \infty \tag{8}$$

for every bounded subdomain $\Omega \subset$ ext B. This replaces the condition of differentiability up to the boundary and the Neumann boundary condition is imposed only where the normal exists.

The boundary value problems enumerated have a common feature— no energy penetrates inside B. When waves do exist in int B we have a so-called transmission problem. When the interior medium is homogeneous, the only case considered here, the total field satisfies (4) and (5) and, in addition, the following:

$$(\nabla^2 + k_1^2)u^t = 0 \quad , \qquad \bar{r} \in \text{int } B \tag{9}$$

and transmission (also called transition) conditions

$$u_+^t = u_-^t \quad , \qquad \bar{r} \in B \tag{10}$$

and

$$\frac{\partial u^t}{\partial n_+} = \rho \frac{\partial u^t}{\partial n_-} \quad , \qquad \bar{r} \in B \quad , \tag{11}$$

where + and - denote the limits from ext B and int B respectively, ρ denotes the ratio of densities of the two media and k_1 is the wave number in the interior:

$$k_1 = \frac{\omega}{c_1} \quad , \quad \rho = \frac{\rho_+}{\rho_-} \quad . \tag{12}$$

While it is customary to write u^{inc} and u^s separately (rather than u^t) in ext B, it is also usual to treat u^t as one quantity in int B since u^{inc} does not satisfy (10) unless $k_1 = k$.

By a trivial scaling of the fields, the conditions on B can be rewritten so that the field has a jump and the normal derivatives are continuous.

2.3 Electromagnetic Scattering

Again we assume a time harmonic incident field ($\bar{E}^{inc}$, $\bar{H}^{inc}$) which satisfies the homogeneous Maxwell equations everywhere in int B and almost everywhere in ext B:

$$\nabla \times \bar{E}^{inc} = ikZ_0\bar{H}^{inc} \quad , \quad \nabla \times \bar{H}^{inc} = -ikY_0\bar{E}^{inc} \tag{13}$$

where Z_0 and Y_0 are the intrinsic impedance and admittance respectively of free space. In terms of the constitutive parameters ε_0 (permittivity) and μ_0 (permeability),

$$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \quad , \quad Y_0 = \frac{1}{Z_0} \quad , \quad k = \omega\sqrt{\varepsilon_0\mu_0} \tag{14}$$

where, as before, ω is the frequency of the incident wave and the factor $e^{-i\omega t}$ is suppressed from all field quantities. Here $\bar{E}$ and $\bar{H}$ (with appropriate superscripts) denote electric and magnetic fields respectively, vector-valued functions of position $\bar{r}$.

The presence of the surface B induces the scattered field, and we write

$$\bar{E}^t = \bar{E}^{inc} + \bar{E}^s \quad , \quad \bar{H}^t = \bar{H}^{inc} + \bar{H}^s \quad , \qquad \bar{r} \in \text{ext B} \quad . \tag{15}$$

The scattered fields satisfy the homogeneous Maxwell equations

$$\nabla \times \bar{E}^s = ikZ_0\bar{H}^s \quad , \quad \nabla \times \bar{H}^s = -ikY_0\bar{E}^s \quad , \qquad \bar{r} \in \text{ext B} \tag{16}$$

and the radiation condition

$$\lim_{r \to \infty} \bar{r} \times (\nabla \times \bar{E}^s) + ikr\bar{E}^s = \lim_{r \to \infty} \bar{r} \times (\nabla \times \bar{H}^s) + ikr\bar{H}^s = 0 \qquad (17)$$

uniformly in all directions $\hat{r}$.

The scattering problem is that of finding $\bar{E}^s$ and $\bar{H}^s$ when the incident field is specified. If the boundary B does not permit energy to penetrate, we have the perfectly conducting boundary conditions:

$$\begin{aligned} \hat{n} \times \bar{E}^s &= -\hat{n} \times \bar{E}^{inc} , \\ \hat{n} \cdot \bar{H}^s &= -\hat{n} \cdot \bar{H}^{inc} , \end{aligned} \qquad \bar{r} \in B . \qquad (18)$$

The classical scattering problem then consists of finding $(\bar{E}^s, \bar{H}^s)$ such that (15) through (18) are satisfied. If B is smooth, the field will be differentiable up to the boundary; however, if there are corners, the additional requirement of finite energy must be imposed, namely

$$\int_{\Omega} (|E^s|^2 + |H^s|^2) \, d\tau < \infty \qquad (19)$$

for every bounded $\Omega \subset$ ext B.

When B is the surface of a scatterer which allows energy to penetrate, we have a transmission problem. Assume that the interior of B is homogeneous and characterized by constitutive parameters

$\varepsilon_1, \mu_1, \sigma_1$, where σ_1 is the conductivity, which vanishes for pure dielectrics. Then in addition to (15), (16) and (17), the following equations must be satisfied:

$$\nabla \times \bar{E}^t = i\omega\mu_1 \bar{H}^t \quad , \quad \nabla \times \bar{H}^t = (-i\omega\varepsilon_1 + \sigma_1)\, \bar{E}^t$$

$$\bar{r} \in \text{int } B \quad , \qquad (20)$$

and the transmission conditions:

$$\begin{aligned} \hat{n} \times \bar{E}^t_+ &= \hat{n} \times \bar{E}^t_- \quad , \quad \hat{n}\cdot\varepsilon_0 \bar{E}^t_+ = \hat{n}\cdot\left(\varepsilon_1 + i\,\frac{\sigma_1}{\omega}\right) \bar{E}^t_- \\ & \qquad\qquad\qquad\qquad\qquad\qquad \bar{r} \in B \quad , \qquad (21) \\ \hat{n} \times \bar{H}^t_+ &= \hat{n} \times \bar{H}^t_- \quad , \quad \hat{n}\cdot\mu_0 \bar{H}^t_+ = \hat{n}\cdot\mu_1 \bar{H}^t_- \end{aligned}$$

where + and - again denote limiting values from ext B and int B respectively.

Equations (20) and (21) show that the frequency ω is perhaps a more appropriate expansion parameter than the wave number. However, for a variety of reasons it has become quite common to introduce complex frequency-dependent parameters

$$\begin{aligned} \tilde{\varepsilon} &= \varepsilon_1 + i\,\frac{\sigma_1}{\omega} \\ k_1 &= \omega\sqrt{\tilde{\varepsilon}\mu_1} \qquad\qquad (22) \\ Z_1 &= \sqrt{\mu_1/\tilde{\varepsilon}} = 1/Y_1 \end{aligned}$$

and to write

$$\nabla \times \bar{E}^t = ik_1 Z_1 \bar{H}^t \quad , \quad \nabla \times \bar{H}^t = -ik_1 Y_1 \bar{E} \qquad \bar{r} \in \text{int } B \tag{23}$$

$$\hat{n} \cdot \varepsilon_0 \bar{E}^t_+ = \hat{n} \cdot \tilde{\varepsilon}\, \bar{E}^t_- \; , \qquad \bar{r} \in B \tag{24}$$

while the remaining equations in (21) remain intact. The quantity

$$m = \left(\frac{k_1}{k}\right)^{1/2} \tag{25}$$

is called the complex index of refraction.

2.4 Integral Representations

Integral representations derived from Green's theoreom or some other variation of the divergence theorem play a central role in scattering at all frequencies including those of present concern.

If u^s satisfies (4) and (5) then

$$\frac{1}{4\pi} \int_B \left\{ u^s(\bar{r}') \frac{\partial}{\partial n'} \left(\frac{e^{ikR}}{R}\right) - \frac{e^{ikR}}{R} \frac{\partial u^s}{\partial n'} \right\} dS' = \alpha(\bar{r}) u^s(\bar{r}) \tag{26}$$

whereas if u^{inc} satisfies (4) in int B, then

$$\frac{1}{4\pi} \int_B \left\{ u^{inc}(\bar{r}') \frac{\partial}{\partial n'} \left(\frac{e^{ikR}}{R}\right) - \frac{e^{ikR}}{R} \frac{\partial u^{inc}}{\partial n'} \right\} dS' = (\alpha(\bar{r}) - 1) u^{inc}(\bar{r}) \tag{27}$$

where $R = |\bar{r} - \bar{r}'|$ and

$$\alpha(\bar{r}) = - \lim_{\varepsilon \to 0} \frac{1}{4\pi} \int_{B_\varepsilon} \frac{\partial}{\partial n'} \left(\frac{1}{R}\right) dS' \tag{28}$$

and B_ε is that part of the surface of a sphere of radius ε and center $\bar{r}$ lying in ext B. If B is smooth, then

$$\alpha(\bar{r}) = \begin{cases} 1 , & \bar{r} \in \text{ext } B \\ 1/2 , & \bar{r} \in B \\ 0 , & \bar{r} \in \text{int } B \end{cases} , \qquad (29)$$

whereas if B has corners or edges, (29) still holds at other points (int B, ext B and smooth points of B), but (28) must be used when $\bar{r}$ is a corner or edge point. The function $\alpha(\bar{r})$ is a measure of the solid angle subtended at P by B (e.g., Mikhlin, 1970). Combining (26) and (27), we obtain

$$\frac{1}{4\pi} \int_B \left\{ u^t(\bar{r}') \frac{\partial}{\partial n'} \left(\frac{e^{ikR}}{R} \right) - \frac{e^{ikR}}{R} \frac{\partial}{\partial n'} u^t(\bar{r}') \right\} dS' = \alpha(\bar{r}) u^t(\bar{r}) - u^{inc}(\bar{r}) . \qquad (30)$$

The corresponding equations for electromagnetic fields are well known (e.g., Stratton, 1941; Jones, 1964; Müller, 1969). If $(\bar{E}^s, \bar{H}^s)$ satisfy Maxwell's equations (16) and the radiation condition (17) in ext B, then

$$\frac{1}{4\pi} \int_B \left\{ ikZ_0 \frac{e^{ikR}}{R} \hat{n}' \times \bar{H}^s(r') + \nabla' \left(\frac{e^{ikR}}{R} \right) \hat{n}' \cdot \bar{E}^s(\bar{r}') - \nabla' \left(\frac{e^{ikR}}{R} \right) \times \left(\hat{n}' \times \bar{E}^s(\bar{r}') \right) \right\} dS' = \alpha(\bar{r}) \bar{E}^s(\bar{r}) \qquad (31a)$$

and

$$\frac{1}{4\pi} \int_B \left\{ -ikY_o \frac{e^{ikR}}{R} \hat{n}' \times \bar{E}^s(\bar{r}') + \nabla' \left(\frac{e^{ikR}}{R} \right) \hat{n}' \cdot \bar{H}^s(\bar{r}') - \nabla' \left(\frac{e^{ikR}}{R} \right) \right.$$

$$\left. \times \left(\hat{n}' \times \bar{H}^s(\bar{r}') \right) \right\} dS' = \alpha(\bar{r}) \bar{H}^s(\bar{r}) \quad , \qquad (31b)$$

whereas if $(\bar{E}^{inc}, \bar{H}^{inc})$ satisfy Maxwell's equations (16) in int B, then

$$\frac{1}{4\pi} \int_B \left\{ ikZ_o \frac{e^{ikR}}{R} \left(\hat{n}' \times \bar{H}^{inc}(\bar{r}') \right) + \nabla' \left(\frac{e^{ikR}}{R} \right) \hat{n}' \cdot \bar{E}^{inc}(\bar{r}') \right.$$

$$\left. - \nabla' \left(\frac{e^{ikR}}{R} \right) \times (\hat{n}' \times \bar{E}^{inc}(\bar{r}') \right\} dS' = (\alpha(\bar{r}) - 1) \bar{E}^{inc}(\bar{r}) \quad , \qquad (32a)$$

$$\frac{1}{4\pi} \int_B \left\{ -ikY_o \frac{e^{ikR}}{R} \hat{n}' \times \bar{E}^{inc}(\bar{r}') + \nabla' \left(\frac{e^{ikR}}{R} \right) \hat{n}' \cdot \bar{H}^{inc}(\bar{r}') \right.$$

$$\left. - \nabla' \left(\frac{e^{ikR}}{R} \right) \times \left(\hat{n}' \times \bar{H}^{inc}(\bar{r}') \right) \right\} dS' = (\alpha(\bar{r}) - 1) \bar{H}^{inc}(\bar{r}) \quad . \qquad (32b)$$

Combining (31) and (32) we have

$$\frac{1}{4\pi}\int_B \left\{ ikZ_0 \frac{e^{ikR}}{R} \hat{n}' \times \bar{H}^t(\bar{r}') + \nabla' \left(\frac{e^{ikR}}{R}\right) \hat{n}' \cdot \bar{E}^t(\bar{r}') \right.$$

$$\left. - \nabla' \left(\frac{e^{ikR}}{R}\right) \times \left(\hat{n}' \times \bar{E}^t(\bar{r}')\right) \right\} dS' = \alpha(\bar{r})\, \bar{E}^t(\bar{r}) - \bar{E}^{inc}(\bar{r}) \quad , \tag{33a}$$

$$\frac{1}{4\pi}\int_B \left\{ -ikY_0 \frac{e^{ikR}}{R} \hat{n}' \times \bar{E}^t(\bar{r}') + \nabla' \left(\frac{e^{ikR}}{R}\right) \hat{n}' \cdot \bar{H}^t(\bar{r}') \right.$$

$$\left. - \nabla' \left(\frac{e^{ikR}}{R}\right) \times \left(\hat{n}' \times \bar{H}^t(\bar{r}')\right) \right\} dS' = \alpha(\bar{r})\bar{H}^t(\bar{r}) - \bar{H}^{inc}(r) \quad . \tag{33b}$$

It should be noted that the boundary values of the field quantities occurring in these representations are all limits from ext B, e.g., u^t_+, $\partial u^t/\partial n^+$, $\bar{E}^t_+$, etc.

One consequence of the representations just presented is that if $\bar{r} \in$ ext B then

$$u^s(\bar{r}) = \frac{1}{4\pi}\int_B \left\{ u(\bar{r}') \frac{\partial}{\partial n'} \left(\frac{e^{ikR}}{R}\right) - \frac{e^{ikR}}{R} \frac{\partial u}{\partial n'} \right\} dS' \tag{34}$$

where u in the integrand may be taken to be u^s or u^t provided a consistent choice is made for u and $\partial u/\partial n'$. Similarly, for $\bar{r} \in$ ext B

$$\bar{E}^s(\bar{r}) = \frac{1}{4\pi} \int_B \left\{ ikZ_0 \frac{e^{ikR}}{R} \hat{n}' \times \bar{H}(\bar{r}') + \nabla' \left(\frac{e^{ikR}}{R}\right) \hat{n}' \cdot \bar{E}(\bar{r}') \right.$$

$$\left. - \nabla' \left(\frac{e^{ikR}}{R}\right) \times \left(\hat{n}' \times \bar{E}(\bar{r}')\right) \right\} dS' , \quad (35a)$$

$$\bar{H}^s(\bar{r}) = \frac{1}{4\pi} \int_B \left\{ -ikY_0 \frac{e^{ikR}}{R} \hat{n}' \times \bar{E}(\bar{r}') + \nabla' \left(\frac{e^{ikR}}{R}\right) \hat{n}' \cdot \bar{H}(\bar{r}') \right.$$

$$\left. - \nabla' \left(\frac{e^{ikR}}{R}\right) \times \left(\hat{n}' \times \bar{H}(\bar{r}')\right) \right\} dS' , \quad (35b)$$

where $(\bar{E}, \bar{H})$ may be chosen either as $(\bar{E}^s, \bar{H}^s)$ or $(\bar{E}^t, \bar{H}^t)$ in the integrands. An alternate form may be obtained by taking the curl of (35a) and (35b) and using Maxwell's equations, and is

$$\bar{E}^s(\bar{r}) = \frac{1}{4\pi} \nabla \times \int_B \frac{e^{ikR}}{R} \hat{n}' \times \bar{E}(\bar{r}') dS'$$

$$+ \frac{iZ_0}{4\pi k} \nabla \times \nabla \times \int_B \hat{n}' \times \bar{H}(\bar{r}') \frac{e^{ikR}}{R} dS' , \quad (36a)$$

$$\bar{H}^s(\bar{r}) = \frac{1}{4\pi} \nabla \times \int_B \frac{e^{ikR}}{R} \hat{n}' \times \bar{H}(\bar{r}') \, dS' - \frac{iY_o}{4\pi k} \nabla \times \nabla \times \int_B \hat{n}' \times \bar{E}(\bar{r}') \frac{e^{ikR}}{R} \, dS' \qquad (36b)$$

for $\bar{r} \in$ ext B, where the quantities in the integrands can be either total or scattered fields provided the choice is made consistently.

2.5 The Far Field

Expressions for the far field may be readily derived from the integral representations by employing the asymptotic form of e^{ikR}/R for large r, i.e.,

$$\frac{e^{ikR}}{R} = \frac{e^{ikr-ik\hat{r}\cdot\bar{r}'}}{r} + O(r^{-2}) \quad . \qquad (37)$$

From (34) we have

$$u^s(\bar{r}) = -\frac{e^{ikr}}{4\pi r} \int_B e^{-ik\hat{r}\cdot\bar{r}'} \left(ik\hat{n}'\cdot\hat{r}u(\bar{r}') + \frac{\partial u}{\partial n'} \right) dS' + O(r^{-2}) \quad , \qquad (38)$$

and from (36a,b) we have (Kleinman, 1967b)

$$\bar{E}^s(\bar{r}) = \frac{e^{ikr}}{4\pi r} \, ik \; \hat{r} \times \int_B \hat{n}' \times \bar{E}(\bar{r}') \, e^{-ik\hat{r}\cdot\bar{r}'} \, dS' - Z_o \, \hat{r} \times \hat{r} \times \int_B \hat{n}' \times \bar{H}(\bar{r}') \, e^{-ik\hat{r}\cdot\bar{r}'} \, dS' + O(r^{-2}) \qquad (39)$$

$$\bar{H}^s(\bar{r}) = Y_o \hat{r} \times \bar{E}^s(\bar{r}) + O(r^{-2}) \quad . \tag{40}$$

This can be written slightly more conveniently for low frequency purposes as

$$\bar{E}^s(\bar{r}) = \frac{e^{ikr}}{4\pi r} k^2 \left\{ \hat{r} \times \hat{r} \times \int_B e^{-ik\hat{r}\cdot\bar{r}'} \bar{r}'[Z_o \hat{r}\cdot\hat{n}' \times \bar{H}(\bar{r}') - \hat{n}' \cdot \bar{E}(\bar{r}')]dS' \right.$$

$$\left. - \hat{r} \times \int_B e^{-ik\hat{r}\cdot\bar{r}'} \bar{r}'[\hat{r}\cdot\hat{n}' \times \bar{E}(\bar{r}') + Z_o \hat{n}' \cdot \bar{H}(\bar{r}')]dS' \right\} + O(r^{-2}) \tag{41}$$

with (40) still holding. As before, the quantities u, $\partial u/\partial n'$ and $(\bar{E},\bar{H})$ appearing in the integrands of these expressions may be taken to be either total or scattered fields.

It is often convenient to introduce so-called scattering or far field coefficients in both the scalar and vector cases as the coefficients of $e^{ikr}/4\pi r$ in the far field. Thus

$$S(\hat{r}) = - \int_B e^{-ik\hat{r}\cdot\bar{r}'} \left(ik\hat{n}'\cdot\hat{r}\, u(\bar{r}') + \frac{\partial u}{\partial n'} \right) dS' \,, \tag{42}$$

$$\bar{S}(\hat{r}) = ik \left\{ \hat{r} \times \int_B \hat{n}' \times \bar{E}(\bar{r}') e^{-ik\hat{r}\cdot\bar{r}'} dS' \right.$$

$$\left. - Z_o \hat{r} \times \hat{r} \times \int_B \hat{n}' \times \bar{H}(\bar{r}') e^{-ik\hat{r}\cdot\bar{r}'} \right\} dS' \quad , \tag{43}$$

in terms of which

$$u^s(\bar{r}) = \frac{e^{ikr}}{4\pi r} S(\hat{r}) \tag{44}$$

and

$$\bar{E}^s(\bar{r}) = \frac{e^{ikr}}{4\pi r} \bar{S}(\hat{r}) \quad . \tag{45}$$

Note that $\bar{S}(\hat{r})$ has no radial component and an alternate definition of $\bar{S}(\hat{r})$ is available from (41).

While the expressions thus far are valid for any incident field we now restrict attention to plane wave incidence where the wave propagates in a direction $\hat{k}$. Thus, in the scalar case,

$$u^{inc}(\bar{r}) = e^{ik\hat{k}\cdot\bar{r}} \quad , \tag{46}$$

and in the electromagnetic case,

$$\bar{E}^{inc}(\bar{r}) = \hat{a}\, e^{ik\hat{k}\cdot\bar{r}} \quad , \tag{47}$$

$$\bar{H}^{inc}(\bar{r}) = Y_o \hat{b}\, e^{ik\hat{k}\cdot\bar{r}} \quad , \tag{48}$$

where the vectors $\hat{a}$, $\hat{b}$, $\hat{k}$ are mutually orthogonal and

$$\hat{b} = \hat{k} \times \hat{a} \quad , \quad \hat{a} \cdot \hat{k} = 0 \quad . \tag{49}$$

We consider only linearly polarized waves with $\hat{a}$ the electric field direction (polarization) and $\hat{b}$ the magnetic field direction. In this case the scattering coefficients are written so as to exhibit the incident field dependence: $S(\hat{r},\hat{k})$ in the scalar case and $\bar{S}(\hat{r},\hat{k},\hat{a})$ in the electromagnetic case. Observe that the incident fields are also chosen to have unit amplitude.

In addition to the fields themselves, of interest are measures of the power, scattered and absorbed. We therefore define the following quantities for plane wave incidence:

Differential Scattering (or Radar) Cross Section

$$\sigma(\hat{r}) = \lim_{r\to\infty} 4\pi r^2 \frac{|u^s|^2}{|u^{inc}|^2} = \frac{1}{4\pi} |S(r,k)|^2 \text{ (scalar)} \tag{50}$$

$$\sigma(\hat{r}) = \lim_{r\to\infty} 4\pi r^2 \frac{|\bar{E}^s|^2}{|\bar{E}^{inc}|^2} = \frac{1}{4\pi} |\bar{S}(\hat{r},\hat{k},\hat{a})|^2 \text{ (electromagnetic);} \tag{51}$$

Total Scattering Cross Section

$$\sigma_T = \frac{1}{4\pi}\int_\Omega \sigma(\hat{r})\, d\Omega = \int_{B_\infty} |u^s|^2\, dS$$

$$= \frac{1}{k}\int_{B_\infty} u^{s*} \frac{\partial u^s}{\partial n}\, dS \quad \text{(scalar)} \tag{52}$$

$$\sigma_T = \frac{1}{4\pi}\int_\Omega \sigma(\hat{r})\, d\Omega = \int_{B_\infty} |\bar{E}^s|^2\, dS$$

$$= Z_0 \operatorname{Re}\int_{B_\infty} \hat{n}\cdot\bar{E}^s \times \bar{H}^{s*}\, dS \quad \text{(electromagnetic)} , \tag{53}$$

where Ω is a unit sphere, and B_∞ is a sphere whose radius goes to infinity with $\hat{n} = \hat{r}$ on B_∞, and * denotes the complex conjugate.

Absorption Cross Section

$$\sigma_A = \frac{1}{k} \text{Im} \int_B u_+^t \frac{\partial u^{t*}}{\partial n_+} dS \quad \text{(scalar)} , \tag{54}$$

$$\sigma_A = Z_0 \text{ Re} \int_B \hat{n} \cdot \bar{H}^t \times \bar{E}^{t*} dS \quad \text{(electromagnetic)} . \tag{55}$$

Since $\hat{n} \times \bar{E}^t$ and $\hat{n} \times \bar{H}^t$ are continuous through B (23), $\hat{n} \cdot \bar{H}^t \times \bar{E}^{t*}$ may be taken as the limiting values of $\bar{E}^{t*}$ and $\bar{H}^t$ from either int B or ext B.

Extinction Cross Section

$$\sigma_{ext} = \sigma_T + \sigma_A . \tag{56}$$

There is a remarkable relation between σ_{ext} and S known as the forward scattering theorem (Van de Hulst, 1957) which states

$$\sigma_{ext} = \frac{1}{k} \text{Im } S(\hat{k},\hat{k}) \quad \text{(scalar)} \tag{57}$$

$$\sigma_{ext} = \frac{1}{k} \text{Im } \hat{a} \cdot \bar{S}(\hat{k},\hat{k},\hat{a}) \quad \text{(electromagnetic)} \tag{58}$$

and we observe that in the scalar case for Dirichlet (6) or Neumann (7) boundary conditions, or for the transmission problem (10) through (12) with k_1 and ρ real, $\sigma_A = 0$. Similarly, in the electromagnetic case for the perfectly conducting boundary conditions (18) or for the transmission problem with $\sigma_1 = 0$ (i.e., k_1 real), we again have $\sigma_A = 0$, so that σ_{ext} may be replaced by σ_T in these cases.

2.6 Rayleigh Scattering--Plane Wave Incidence

The central idea in low frequency scattering is to approximate the field quantities by a finite number of terms of a series expansion in powers of k, or more conveniently, ik. The determination of the coefficients, which are of course functions of position but independent of k, is still formidable. Nevertheless, as Rayleigh observed, much useful information can be obtained from a knowledge of only the first nonvanishing term in this expansion. In the present context we shall call this the Rayleigh term and by Rayleigh scattering we mean this one term approximation to the scattered fields.

In this approximation, the near and far fields display markedly different behavior. The regions are distinguished in the following sense. Although, as noted above, we consider expansions in powers of ik, in actual fact the expansion parameter is dimensionless, and there are two critical dimensionless parameters, ka and kr, where a denotes some characteristic length of the scattering surface B, e.g., diameter, and r, the distance to the field point. While low frequency scattering is concerned with small values of ka, the near field is characterized by small kr as well, while in the far field kr is large.

Near Field

For plane wave incidence we have the expansions

$$u^{inc}(\bar{r}) = \sum_{n=o}^{\infty} (ik)^n u_n^{inc}(\bar{r}) \quad , \quad u_n^{inc} = \frac{(\hat{k}\cdot\bar{r})^n}{n!}$$

$$\bar{E}^{inc}(\bar{r}) = \sum_{n=o}^{\infty} (ik)^n \bar{E}_n^{inc}(\bar{r}) \quad , \quad \bar{E}_n^{inc}(\bar{r}) = \hat{a}\,\frac{(\hat{k}\cdot\bar{r})^n}{n!} \tag{59}$$

$$\bar{H}^{inc}(\bar{r}) = \sum_{n=o}^{\infty} (ik)^n \bar{H}_n^{inc}(\bar{r}) \quad , \quad \bar{H}_n^{inc}(\bar{r}) = Y_o\hat{b}\,\frac{(\hat{k}\cdot\bar{r})^n}{n!}$$

where we note that these series converge for all kr. For small ka and kr we have expansions of the scattered fields

$$u^s(\bar{r}) = \sum_{n=o}^{\infty} (ik)^n u_n^s(\bar{r}) \quad , \tag{60}$$

$$\bar{E}^s(\bar{r}) = \sum_{n=o}^{\infty} (ik)^n \bar{E}_n^s(\bar{r}) \quad , \tag{61}$$

$$\bar{H}^s(\bar{r}) = \sum_{n=o}^{\infty} (ik)\bar{H}_n^s(\bar{r}) \quad , \tag{62}$$

for $\bar{r} \in$ ext B, where the coefficients u_n^s, $\bar{E}_n^s$, and $\bar{H}_n^s$ are independent of k. The expansion coefficients will be real valued in the Dirichlet, Neumann and perfectly conducting cases as well as transmission problems for real k_1. In the electromagnetic transmission problem with nonzero conductivity, a low frequency expansion in powers of $i\omega$ again leads to real coefficients when the introduction of a complex index of refraction is avoided.

Considering only the lowest order terms, the Helmholtz equation and Maxwell equations imply that

$$\nabla^2 u_0^s = \nabla^2 u_1^s = 0 \tag{63}$$

$$r \in \text{ext } B \quad .$$

$$\nabla \times \bar{E}_0^s = \nabla \times \bar{H}_0^s = 0 \tag{64}$$

Moreover, since $\bar{E}^s$ and $\bar{H}^s$ are divergence free, there are scalar potential functions Φ_0^s and Ψ_0^s such that

$$\bar{E}_0^s(\bar{r}) = -\nabla\Phi_0^s(\bar{r}) \quad , \quad \nabla^2\Phi_0^s = 0 \tag{65}$$

$$\bar{r} \in \text{ext } B \quad ,$$

$$\bar{H}_0^s(\bar{r}) = -Y_0\nabla\Psi_0^s \quad , \quad \nabla^2\Psi_0^s = 0 \tag{66}$$

so that not only are the first two terms in the expansion of the scalar scattered field solutions of Laplace's equation, but the lowest order electromagnetic terms are expressible in terms of scalar potential functions. We also remark that through the use of (34) and (35) it may be ascertained that u_0^s, Φ_0^s and Ψ_0^s all decay at least as fast as $1/r$ as $r \to \infty$, i.e., they are "regular" at ∞. Furthermore, since

$$u_0^{inc} = 1 \quad , \quad u_1^{inc} = \hat{k}\cdot\bar{r} \quad , \tag{67}$$

$$\bar{E}_0^{inc} = \hat{a} = \nabla(\hat{a}\cdot\bar{r} + c) \quad , \tag{68}$$

$$\bar{H}_0^{inc} = Y_0\hat{b} = Y_0\nabla\,(\hat{b}\cdot\bar{r} + c_1) \quad , \tag{69}$$

the incident fields are also expressible in terms of scalar potential functions, i.e.,

$$\nabla^2 u_0^{inc} = \nabla^2 u_1^{inc} = 0 \tag{70}$$

and

$$\nabla^2(\hat{a}\cdot\bar{r} + c) = \nabla^2(\hat{b}\cdot\bar{r} + c_1) = 0 \ . \tag{71}$$

Thus, defining coefficients of the expansion of the total fields

$$u_n^t = u_n^s + u_n^{inc} \ , \quad \bar{E}_n^t = \bar{E}_n^s + \bar{E}_n^{inc} \text{ and } \bar{H}_n^t = \bar{H}_n^s + \bar{H}_n^{inc} \tag{72}$$

we have

$$\nabla^2 u_0^t = \nabla^2 u_1^t = 0 \ , \quad \bar{r} \in \text{ext } B \tag{73}$$

$$\bar{E}_0^t = -\nabla\Phi_0^t \tag{74}$$

$$\bar{H}_0^t = -Y_0\nabla\Psi_0^t \tag{75}$$

where

$$\Phi_0^t = \Phi_0^s - \hat{a}\cdot\bar{r} - c \tag{76}$$

$$\Psi_0^t = \Psi_0^s - \hat{b}\cdot\bar{r} - c_1 \tag{77}$$

and

$$\nabla^2\Phi_0^t = \nabla^2\Psi_0^t = 0 \ , \quad \bar{r} \in \text{ext } B \ . \tag{78}$$

Far Field

When the low frequency expansions of the fields are used in the expressions for the scattering coefficients (42) and (43) there result

$$S(\hat{r},\hat{k}) = -\sum_{n=0}^{\infty}\sum_{m=0}^{n} (ik)^n \int_B \frac{(-\hat{r}\cdot\bar{r}')^{n-m}}{(n-m)!}\left(ik\hat{n}'\cdot\hat{r}u_m + \frac{\partial u_m}{\partial n'}\right) dS' \qquad (79)$$

in the scalar case and

$$\bar{S}(\hat{r},\hat{k},\hat{a}) = \sum_{n=1}^{\infty}\sum_{m=0}^{n} (ik)^{n+1} \int_B \frac{(-\hat{r}\cdot\bar{r}')^{n-m}}{(n-m)!} [\hat{r} \times (\hat{n}' \times \bar{E}_m(\bar{r}')$$

$$- Z_0\, \hat{r} \times \hat{r} \times (\hat{n}' \times \bar{H}_m(\bar{r}'))]\, dS' \qquad (80)$$

or, with (43),

$$\bar{S}(\hat{r},\hat{k},\hat{a}) = k^2 \sum_{n=0}^{\infty}\sum_{m=0}^{n} (ik)^n \int_B \frac{(-\hat{r}\cdot\bar{r}')^{n-m}}{(n-m)!} \quad \hat{r} \times (\hat{r} \times \bar{r}')$$

$$\left\{Z_0\hat{r}\cdot\hat{n}' \times \bar{H}_m(\bar{r}') - \hat{n}'\cdot\bar{E}_m(\bar{r}') - (\hat{r} \times \bar{r}')[\hat{r}\cdot\hat{n}' \times \bar{E}_m(\bar{r}') + Z_0\hat{n}'\cdot\bar{H}_m(\bar{r}')]\right\} dS'$$

(81)

in the electromagnetic case. Again we call attention to the fact that either the total or scattered fields may be used in the integrands.

These expressions for the scattering coefficient may be rewritten as

$$S(\hat{r},\hat{k}) = \sum_{n=0}^{\infty} (ik)^n S_n(\hat{r},\hat{k}) \qquad (82)$$

and

$$\bar{S}(\hat{r},\hat{k},\hat{a}) = k^2 \sum_{n=o}^{\infty} (ik)^n \, \bar{S}_n(\hat{r},\hat{k},\hat{a})$$

with

$$S_n(\hat{r},\hat{k}) = - \sum_{m=o}^{n-1} \int_B \frac{(-\hat{r}\cdot\bar{r}')^{n-1-m}}{(n-1-m)!} \hat{n}'\cdot\hat{r} \, u_m \, dS' - \sum_{m=o}^{n} \int_B \frac{(-\hat{r}\cdot\bar{r}')^{n-m}}{(n-m)!} \frac{\partial u_m}{\partial n} dS' \, ,$$

$$n \geq 1 \qquad (83)$$

$$= - \int_B \frac{\partial u_o}{\partial n'} dS' \, , \qquad n = 0$$

and

$$\bar{S}_n(\hat{r},\hat{k},\hat{a}) = \sum_{m=o}^{n} \int_B \frac{(-\hat{r}\cdot\bar{r}')^{n-m}}{(n-m)!} \Big\{ \hat{r} \times (\hat{r} \times \bar{r}') \, [Z_o \hat{r}\cdot\hat{n}' \times \bar{H}_m(\bar{r}') - \hat{n}'\cdot \bar{E}_m(\bar{r}')]$$

$$- (\hat{r} \times \bar{r}')[\hat{r}\cdot\hat{n} \times \bar{E}_m(\bar{r}') + Z_o \hat{n}'\cdot \bar{H}_m(\bar{r}')] \Big\} \, dS' \qquad (84)$$

where $S_n(\hat{r},\hat{k})$ and $\bar{S}_n(\hat{r},\hat{k},\hat{a})$ are <u>real</u> scalar and vector-valued functions respectively in all cases where u_m or $\bar{E}_m$ and $\bar{H}_m$ are real. This fact enables us to expand the differential scattering cross sections (50) and (51) in the form

$$\sigma(\hat{r}) = \frac{1}{4\pi} \sum_{n=o}^{\infty} (-1)^n \, k^{2n} \sum_{m=o}^{2n} S_m(\hat{r},\hat{k}) S_{2n-m}(\hat{r},\hat{k}) \qquad (85)$$

in the scalar case and

$$\sigma(\hat{r}) = \frac{k^4}{4\pi} \sum_{n=o}^{\infty} (-1)^n k^{2n} \sum_{m=o}^{2n} \bar{S}_m(\hat{r},\hat{k},\hat{a}) \cdot \bar{S}_{2n-m}(\hat{r},\hat{k},\hat{a}) \tag{86}$$

in the electromagnetic case.

When only the lowest order terms in k are retained, we have

$$S(\hat{r},\hat{k}) = -\int_B \frac{\partial u}{\partial n'} \, dS' + \mathcal{O}(k) \tag{87}$$

and

$$\bar{S}(\hat{r},\hat{k},\hat{a}) = k^2 \int_B \Big\{ \hat{r} \times (\hat{r} \times \bar{r}') \, [Z_o \hat{r} \cdot n' \times \bar{H}_o(\bar{r}') - \hat{n}' \cdot \bar{E}_o(\bar{r}')]$$

$$- (\hat{r} \times \bar{r}')[\hat{r} \cdot \hat{n}' \times \bar{E}_o(\bar{r}') + Z_o \hat{n}' \cdot \bar{H}_o(\bar{r}')] \Big\} \, dS' + \mathcal{O}(k^3) \quad . \tag{88}$$

Expressed in terms of potentials and dipole moments this becomes (Kleinman, 1973)

$$\bar{S}(\hat{r},\hat{k},\hat{a}) = -k^2 \left(\frac{\hat{r}}{\varepsilon_o} \times \hat{r} \times \bar{p} + Z_o \hat{r} \times \bar{m} \right) + \mathcal{O}(k^3) \tag{89}$$

where the electric dipole moment is

$$\bar{p} = \varepsilon_o \int_B \left\{ \hat{n}' \Phi_o(\bar{r}') - \bar{r}' \frac{\partial}{\partial n'} \Phi_o(\bar{r}') \right\} dS' \quad , \tag{90}$$

the magnetic dipole moment is

$$\bar{m} = Y_o \int_B \left\{ \hat{n}' \Psi_o(\bar{r}') - \bar{r}' \frac{\partial}{\partial n'} \Psi_o(\bar{r}') \right\} dS' \quad , \tag{91}$$

and either total or scattered potentials may be consistently employed. We remark that these expressions for the scattering coefficients remain valid for dipole as well as plane wave sources.

When these lowest order approximations to the far field coefficient are employed in the expressions for the scattering cross section (50) to (56), (85) and (86) we find, in the scalar case,

$$\sigma_T = \sigma(\hat{r}) = \frac{S_0^2}{4\pi} + O(k) \tag{92}$$

which expresses the fact that if the constant $S_0 \neq 0$ the scattering is isotropic to this order. In addition, it can be shown that

$$\sigma_A = O(k^2) \quad , \tag{93}$$

and hence

$$\sigma_{ext} = \sigma_T = \sigma(\hat{r}) \quad . \tag{94}$$

In the electromagnetic case the scattering is a superposition of fields due to electric and magnetic dipoles and is not isotropic. From (89) we find

$$\sigma(\hat{r}) = \frac{k^4}{4\pi\varepsilon_0^2}\left(|\hat{r} \times \bar{p}|^2 + \frac{2}{c}\,\hat{r}\cdot\bar{p} \times \bar{m} + \frac{1}{c^2}\,|\hat{r} \times \bar{m}|^2\right) + O(k^6) \quad , \tag{95}$$

$$\sigma_T = \frac{k^4}{6\pi\varepsilon_0^2}\left(|\bar{p}|^2 + \frac{1}{c^2}\,|\bar{m}|^2\right) + O(k^6) \quad , \tag{96}$$

where c is the velocity of light.

When $\sigma_A = 0$ the forward scattering theorem implies that

$$\hat{a}\cdot\bar{S}_1(\hat{k},\hat{k},\hat{a}) = 0 \tag{97}$$

and

$$\hat{a}\cdot\bar{S}_3(\hat{k}\cdot\hat{k}\cdot\hat{a}) = \frac{1}{6\pi\varepsilon_0^2}\left(|\bar{p}|^2 + \frac{1}{c^2}|\bar{m}|^2\right) \quad . \tag{98}$$

To compute the electric and magnetic dipole moments it is necessary to specify the geometry and composition of the body. There are a variety of cases to be considered and we start with the simple problem of a perfectly conducting body.

3. PERFECTLY CONDUCTING BODY

3.1 Formulation

Let B be the surface of a closed perfectly conducting body of non-zero volume V. The boundary conditions on the zeroth order scattered fields at the surface are then

$$\begin{aligned} \hat{n} \times \bar{E}_o^s &= -\hat{n} \times \bar{E}_o^{inc} \\ \hat{n} \cdot \bar{H}_o^s &= -\hat{n} \cdot \bar{H}_o^{inc} \end{aligned} \qquad (99)$$

and these are independent. In terms of the potentials Φ^s, Ψ^s (see (65) and (66)) and the corresponding incident field potentials Φ^{inc}, Ψ^{inc} (throughout this section the subscript o will be omitted)

$$\left. \begin{aligned} \Phi^s &= -\Phi^{inc} + c \\ \frac{\partial \Psi^s}{\partial n} &= -\frac{\partial \Psi^{inc}}{\partial n} \end{aligned} \right\} \quad \bar{r} \in B , \qquad (100)$$

where the constant c must be chosen to satisfy the zero induced charge condition

$$\int_B \frac{\partial \Phi^s}{\partial n'}\, dS' = 0 \tag{101}$$

for an isolated conductor. In the case of two or more separate (disjoint) bodies, (101) must be satisfied on each. The unknown potentials Φ^s and Ψ^s are solutions of Laplace's equation in ext B and are $O(r^{-2})$ as $r \to \infty$.

It is a trivial matter to construct integral equations for the potentials. Since Φ^s is an exterior potential, Green's theorem gives (set k = 0 in (26))

$$\frac{1}{4\pi} \int_B \left\{ \Phi^s \frac{\partial}{\partial n'} \left(\frac{1}{R}\right) - \frac{1}{R} \frac{\partial \Phi^s}{\partial n'} \right\} dS' = \alpha(\bar{r}) \Phi^s(\bar{r}) \tag{102}$$

where $\alpha(\bar{r})$ is defined in (28). Similarly, if χ is an interior potential inside B (set k = 0 in (27))

$$\frac{1}{4\pi} \int_B \left\{ \chi \frac{\partial}{\partial n'} \left(\frac{1}{R}\right) - \frac{1}{R} \frac{\partial \chi}{\partial n'} \right\} dS' = (\alpha(\bar{r}) - 1)\chi(\bar{r}) \quad , \tag{103}$$

and if (103) with $\chi = \Phi^{inc} - c$ is added to (102), we obtain

$$\frac{1}{4\pi} \int_B \frac{1}{R} \frac{\partial}{\partial n'} (\Phi^s + \Phi^{inc} - c) dS' = \Phi^{inc} - c - \alpha(\bar{r})(\Phi^s + \Phi^{inc} - c) \quad . \tag{104}$$

In particular, for $\bar{r} \subseteq B$,

$$\frac{1}{4\pi} \int_B \frac{1}{R} \frac{\partial \Phi^t}{\partial n'}\, dS' = \Phi^{inc}(\bar{r}) - c \tag{105}$$

which is an integral equation with which to determine the total field potential $\Phi^t = \Phi^s + \Phi^{inc} - c$ on B. An alternative integral equation may be obtained by taking the normal derivative of (104) from ext B, giving

$$\frac{\partial \Phi^t}{\partial n} + \frac{1}{2\pi} \int_B \frac{\partial \Phi^t}{\partial n'} \frac{\partial}{\partial n} \left(\frac{1}{R} \right) dS' = 2 \frac{\partial \Phi^{inc}}{\partial n} . \qquad (106)$$

For the magnetostatic potential we likewise have

$$\frac{1}{2\pi} \int_B \Psi^t \frac{\partial}{\partial n'} \left(\frac{1}{R} \right) dS' = \Psi^t(\bar{r}) - 2\Psi^{inc}(\bar{r}) , \qquad (107)$$

$\bar{r} \in B$, and this is an integral equation of the second kind for $\Psi^t = \Psi^s + \Psi^{inc}$.

From the solutions of (105) and (107) and the boundary conditions (100), the electric and magnetic dipole moments can be computed using (90) and (91), respectively.

3.2 Plane Wave Incidence

The above results can be simplified if the incident field is a plane linearly polarized wave. From (68) and (69), the zeroth order approximations are

$$\bar{E}_0^{inc} = \hat{a} , \quad \bar{H}_0^{inc} = Y_0 \hat{b}$$

implying

$$\Phi^{inc} = -\hat{a} \cdot \bar{r} , \quad \Psi^{inc} = -\hat{b} \cdot \bar{r} , \qquad (108)$$

and the dependence of the electric and magnetic dipole moments on $\hat{a}$ and $\hat{b}$ can be made explicit by writing

$$\Phi^s = \sum_{i=1}^{3} (\hat{a}\cdot\hat{x}_i)\Phi_i^s \quad , \quad \Psi^s = \sum_{i=1}^{3} (\hat{b}\cdot\hat{x}_i)\Psi_i^s \tag{109}$$

where x_i, $i = 1,2,3$, are Cartesian coordinates. If the incident and total potentials are expanded in a similar manner, the boundary conditions (100) become

$$\begin{aligned} \Phi_i^s &= x_i + c_i \\ \frac{\partial \Psi_i^s}{\partial n} &= \hat{n}\cdot\hat{x}_i \end{aligned} \tag{110}$$

on B, with c_i chosen to satisfy

$$\int_B \frac{\partial \Phi_i^s}{\partial n'}\, dS' = 0 \quad . \tag{111}$$

The boundary condition on Φ_i^s is now independent of $\hat{a}$ and we can therefore write

$$\bar{p} = \varepsilon_0 \bar{\bar{P}}\cdot\hat{a} \tag{112}$$

where $\bar{\bar{P}}$ is the electric polarizability tensor whose elements P_{ij}:

$$P_{ij} = \int_B \left(\hat{n}'\cdot\hat{x}_i\Phi_j^s - x_i' \frac{\partial \Phi_j^s}{\partial n'}\right) dS' \tag{113}$$

are functions only of the geometry of the body. Since $\Phi_j^s = \Phi_j^t + x_j + c_j$, P_{ij} can also be written as

$$P_{ij} = - \int_B x_i' \frac{\partial \Phi_j^t}{\partial n'} dS' \qquad (114)$$

and yet another form is

$$P_{ij} = V\delta_{ij} + \int_{\text{ext } B} \nabla'\Phi_i^s \cdot \nabla'\Phi_j^s \, dV' \quad . \qquad (115)$$

This makes explicit the symmetry of $\bar{\bar{P}}$, which therefore has at most six independent elements. A related tensor is $\bar{\bar{Q}}$ with elements

$$Q_{ij} = \int_{\text{ext } B} \nabla'\Phi_i^s \cdot \nabla'\Phi_j^s \, dV' = - \int_B x_i' \frac{\partial \Phi_j^s}{\partial n'} dS' \qquad (116)$$

in terms of which

$$\bar{\bar{P}} = V\bar{\bar{I}} + \bar{\bar{Q}} \qquad (117)$$

where $\bar{\bar{I}}$ is the identity tensor, and $\bar{\bar{Q}}$ has been called (Schiffer and Szegö, 1949; Payne, 1967) the polarization tensor for an isolated conductor.

Similarly, the magnetic dipole moment can be written as

$$\bar{m} = -Y_0 \bar{\bar{M}} \cdot \hat{b} \qquad (118)$$

where $\bar{\bar{M}}$ is the magnetic polarizability tensor with elements

$$M_{ij} = -\int_B \left(\hat{n}'\cdot\hat{x}_i \Psi_j^s - x_i' \frac{\partial \Psi_j^s}{\partial n'} \right) dS' \quad . \tag{119}$$

These are also functions only of the body geometry, and since $\Psi_j^s = \Psi_j^t + x_j$,

$$\begin{aligned} M_{ij} &= -\int_B \hat{n}'\cdot\hat{x}_i \Psi_j^t \, dS' \\ &= V\delta_{ij} + \int_{\text{ext } B} \nabla'\Psi_i^s \cdot \nabla'\Psi_j^s \, dV' \end{aligned} \tag{120}$$

from which the symmetry of $\bar{\bar{M}}$ is apparent. The tensor therefore has at most six independent elements and if $\bar{\bar{W}}$ is such that

$$W_{ij} = \int_{\text{ext } B} \nabla'\Psi_i^s \cdot \nabla'\Psi_j^s \, dV' = -\int_B \hat{n}'\cdot\hat{x}_i \Psi_j^s \, dS' \; , \tag{121}$$

we have

$$\bar{\bar{M}} = V\bar{\bar{I}} + \bar{\bar{W}} \quad . \tag{122}$$

The tensor $\bar{\bar{W}}$ is identical to that which arises in the study of the irrotational flow of an incompressible inviscid fluid past the rigid surface B, where it is termed the added or virtual mass tensor (Schiffer and Szegö, 1949; Payne, 1967). Although (122) is due to Taylor (1928), the fact that $\bar{\bar{M}}$ is the magnetic polarizability tensor was not noted.

The tensors $\bar{\bar{P}}$ and $\bar{\bar{M}}$ are of most concern to us. The only non-axially symmetric body for which explicit results are available is the ellipsoid. If the coordinate axes are chosen to coincide with the principal axes of the ellipsoid whose equation then becomes

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1 \quad , \tag{123}$$

the tensors diagonalize and

$$P_{ii} = \frac{1}{L_i} \quad , \qquad M_{ii} = \frac{1}{L_i - 1} \quad , \tag{124}$$

$i = 1,2,3$, where

$$L_1 = \frac{1}{2} a_1 a_2 a_3 \int_0^\infty (s^2 + a_1^2)^{-3/2} (s^2 + a_2^2)^{-1/2} (s^2 + a_3^2)^{-1/2} \, ds \tag{125}$$

(Van de Hulst, 1957). L_2 and L_3 are given by the same formula with cyclical changes, and

$$L_1 + L_2 + L_3 = 1 \quad . \tag{126}$$

In the special case $a_1 = a_2$, the ellipsoid becomes a spheroid having the x_3 axis as its axis of symmetry (see §3.4).

More generally, the elements of $\bar{\bar{P}}$ can be computed if $\partial\Phi_i^t/\partial n$ can be determined, and from (105), (108), and (109) an integral equation for this is

$$\frac{1}{4\pi}\int_B \frac{1}{R}\,\frac{\partial\Phi_i^t}{\partial n'}\,dS' = -x_i - c_i \quad , \tag{127}$$

while a second equation is

$$\frac{\partial\Phi_i^t}{\partial n} + \frac{1}{2\pi}\int_B \frac{\partial\Phi_i^t}{\partial n'}\,\frac{\partial}{\partial n}\left(\frac{1}{R}\right) dS' = -2\hat{n}\cdot\hat{x}_i \quad . \tag{128}$$

The corresponding equation for Ψ_i^t is (see (107))

$$\frac{1}{2\pi}\int_B \Psi_i^t\,\frac{\partial}{\partial n'}\left(\frac{1}{R}\right) dS' = \Psi_i^t(\bar{r}) + 2x_i \quad . \tag{129}$$

To solve (127) it is helpful to write

$$\Phi_i^t = c_i\Phi^{t(o)} + \Phi_i^{t(1)} \tag{130}$$

where

$$\frac{1}{4\pi}\int_B \frac{1}{R}\,\frac{\partial\Phi^{t(o)}}{\partial n'}\,dS' = -1 \tag{131}$$

$$\frac{1}{4\pi}\int_B \frac{1}{R}\,\frac{\partial\Phi_i^{t(1)}}{\partial n'}\,dS' = -x_i \quad . \tag{132}$$

From (130) and (111)

$$c_i\int_B \frac{\partial\Phi^{t(o)}}{\partial n'}\,dS' + \int_B \frac{\partial\Phi_i^{t(1)}}{\partial n'}\,dS' = 0$$

and hence

$$\Phi_i^t = \Phi_i^{t(1)} + \frac{\varepsilon_o}{C}\int_B \frac{\partial\Phi_i^{t(1)}}{\partial n'}\,dS'\,\Phi^{t(o)} \tag{133}$$

where

$$C = -\varepsilon_o \int_B \frac{\partial \Phi^{t(o)}}{\partial n'} dS' \tag{134}$$

is the electrostatic capacity of the isolated body. If the body has two (or more) electrically distinct parts, the constant c_i may differ on each, and (101) must be enforced separately on each part.

The numerical solution of these equations for a particular class of bodies is discussed in §3.4, but some information about the tensor elements can be obtained analytically. As shown by Keller et al (1972)

$$\begin{aligned} P_{ij} &= 3\left\{V\delta_{ij} + \int_{\delta V} \hat{x}_i \cdot \nabla' \Phi_j^s \, dV'\right\} \\ M_{ij} &= \frac{3}{2}\left\{V\delta_{ij} + \int_{\delta V} \hat{x}_i \cdot \nabla' \Psi_j^s \, dV'\right\} \end{aligned} \tag{135}$$

where δV is the volume exterior to B but interior to the smallest sphere containing B. Thus, for a sphere,

$$\bar{\bar{P}} = 3V\bar{\bar{I}} \quad , \qquad \bar{\bar{M}} = \frac{3}{2} V\bar{\bar{I}} \quad , \tag{136}$$

and if $\delta V \neq 0$, the volume integrals in (135) play the role of "shape corrections". They can be used to estimate (or bound) the change in the dipole moments produced by a small departure from sphericity.

The tensor elements also satisfy certain inequalities. By application of Schwarz's inequality to (115) and (120) it follows that for fixed i and j

$$(P_{ii} - V)(P_{jj} - V) \geq (P_{ij} - V\delta_{ij})^2$$

$$(M_{ii} - V)(M_{jj} - V) \geq (M_{ij} - V\delta_{ij})^2 \qquad (137)$$

$$(P_{ii} - V)(M_{ii} - V) \geq V^2$$

and

$$P_{ii}, M_{ii} \geq V \qquad (138)$$

(Schiffer and Szegö, 1949), where i,j = 1,2,3 and repeated suffices do not imply summation. In addition

$$\sum_{i=1}^{3} (P_{ii} - V) \sum_{j=1}^{3} (M_{jj} - V) \geq 9V^2 \qquad (139)$$

and still other inequalities are quoted by Payne (1967), but more striking are the results for an axially symmetric body.

3.3 Axial Symmetry

The polarizability tensors simplify if the body has one or more planes of symmetry which coincide with the x_i planes. Consider, for example, the tensor $\bar{\bar{P}}$ (the properties of $\bar{\bar{M}}$ are identical). If the body is symmetrical about (say) the $x_1 = 0$ plane, the symmetry of the total potentials Φ_i^t, i = 1,2,3, shows that $P_{21} = P_{31} = 0$, implying $P_{12} = P_{13} = 0$. The tensor then has at most the five non-zero elements P_{11} and P_{ij} with i,j = 2,3, of which only four are independent. If the body is also symmetrical about the $x_2 = 0$ or $x_3 = 0$ planes, $P_{23} = 0$ implying $P_{32} = 0$, producing a diagonal tensor.

We are now left with only three non-zero and, in general, independent elements, and we remark that symmetry about a third perpendicular plane produces no further simplification.

A special case of symmetry about two perpendicular planes is axial symmetry. If B has an axis of symmetry which is taken to be the z (= x_3) axis, $\bar{\bar{P}}$ and $\bar{\bar{M}}$ are diagonal ($P_{ij} = 0 = M_{ij}$, $i \neq j$) and

$$P_{11} = P_{22} \quad , \quad M_{11} = M_{22} \tag{140}$$

from the invariance to a $\pi/2$ rotation about the z-axis. It would now appear that just four scalar quantities, functions only of the geometry of B, are sufficient to specify the tensors and, hence, the Rayleigh scattering for any incident plane wave. In actual fact, there are only three such independent quantities. This was shown by Payne (1956) using a proof based on the relation between flow potentials and stream functions, and also by Karp (1956). They obtained the identity

$$M_{33} = \frac{1}{2} P_{11} \tag{141}$$

provided B is simply connected. In practical terms this means that the axial component of the electric dipole moment (P_{33}) and a transverse component of each of the electric and magnetic dipole moments (P_{11} and M_{11}) are sufficient to specify the scattering in its entirety.

What is more, even the three independent tensor elements that remain are constrained by certain inequalities. From (138)

$$P_{33} \ , \ M_{11} \ , \ M_{33} \geq V \tag{142}$$

and hence, from (141),

$$P_{11} \geq 2V \ . \tag{143}$$

Also, from the last of (137),

$$\begin{aligned} (P_{11} - V)(M_{11} - V) &\geq V^2 \\ \left(\frac{1}{2} P_{11} - V\right) (P_{33} - V) &\geq V^2 \end{aligned} \tag{144}$$

which serve to establish lower bounds on M_{11} and P_{33} once P_{11} is determined, and other inequalities which can be deduced are (Payne, 1956):

$$\begin{aligned} 2P_{11} + P_{33} &\geq 9V \\ P_{11} + 4M_{11} &\geq 9V \\ (P_{11} - V)(P_{11} + 2P_{33} - 3V) &\geq 12V^2 \\ (M_{11} - V)(P_{11} + M_{11} - 3V) &\geq \frac{3}{4} V^2 \end{aligned} \tag{145}$$

All of these are optimum in the sense that equality holds for at least one body (a sphere).

Two other inequalities that have been derived are (Kleinman and Senior, 1972)

$$P_{11} + M_{11} \geq 4V \tag{146}$$

$$P_{11} + 2P_{33} \geq 8V \tag{147}$$

and though the bounds are optimum, the optimum shape is no longer a sphere. In (147) equality obtains for an oblate spheroid having $\ell/w = 0.5$ where ℓ is the body length in the direction of the symmetry axis and w is the maximum dimension in a perpendicular direction, whereas in (146) equality is approached by a prolate spheroid as $\ell/w \to \infty$. There are, in addition, a variety of inequalities that can be established by variational and other techniques (see, for example, Schiffer and Szegö, 1949), and which can likewise serve to estimate (or bound) the tensor elements.

These relations can be used to provide lower bounds on the radar cross section of the body. From (89), (112), (118) and (141), the scattering coefficient is

$$\bar{S}(\hat{r},\hat{k},\hat{a}) = -k^2\left\{P_{11}\hat{r} \times (\hat{r} \times \hat{a}) + (P_{33} - P_{11})(\hat{a}\cdot\hat{z})\hat{r} \times (\hat{r} \times \hat{z}) - M_{11}\hat{r} \times \hat{b} - \left(\frac{1}{2}P_{11} - M_{11}\right)(\hat{b}\cdot\hat{z})\hat{r} \times \hat{z}\right\} + O(k^3) \tag{148}$$

from which the radar cross section follows using (51). A number of special cases have been considered by Kleinman and Senior (1972) and two examples will suffice. For axial incidence ($\hat{k} = -\hat{z}$, say), the back scattering ($\hat{r} = \hat{z}$) cross section is

$$\sigma = \frac{k^4}{4\pi} (P_{11} + M_{11})^2 \qquad (149)$$

and (146) now shows that

$$\sigma \geq \frac{4}{\pi} k^4 V^2 , \qquad (150)$$

whereas the forward scattering ($\hat{r} = -\hat{z}$) cross section is

$$\sigma = \frac{k^4}{4\pi} (P_{11} - M_{11})^2 \qquad (151)$$

whose lower bound is simply zero. Both bounds are optimum and are approached, for example, by a prolate spheroid as $\ell/w \to \infty$. For the total (integrated) cross section σ_T, we have

$$\sigma_T = \frac{k^4}{6\pi} \left\{ P_{11}^2 + M_{11}^2 + (P_{33}^2 - P_{11}^2)(\hat{a}\cdot\hat{z})^2 + \left(\frac{1}{4} P_{11}^2 - M_{11}^2\right) (\hat{b}\cdot\hat{z}) \right\} \qquad (152)$$

(Kleinman and Senior, 1972), valid for all angles of incidence.

3.4 Analytical and Numerical Results

There are a few simple axially symmetric geometries for which analytical expressions for the tensor elements have been obtained either by the method of images or by expansion of the potentials in terms of the characteristic functions for an appropriate system of curvilinear coordinates.

One of these shapes is a spheroid, prolate or oblate. The non-zero tensor elements can be written as ratios of Legendre functions of the first and second kinds, and by inserting the expressions for these, the results for a prolate spheroid having interfocal distance 2d and radial spheroidal variable $\xi (1 \leq \xi \leq \infty)$ are found to be

$$
\begin{aligned}
P_{11} &= -2\left\{\frac{1}{2}\,\xi(\xi^2 - 1)\ln\frac{\xi + 1}{\xi - 1} - \xi^2\right\}^{-1} V \\
P_{33} &= \left\{\frac{1}{2}\,\xi(\xi^2 - 1)\ln\frac{\xi + 1}{\xi - 1} - \xi^2 + 1\right\}^{-1} V \qquad (153) \\
M_{11} &= 2\left\{\frac{1}{2}\,\xi(\xi^2 - 1)\ln\frac{\xi + 1}{\xi - 1} - \xi^2 + 2\right\}^{-1} V
\end{aligned}
$$

(Stevenson, 1953b; Ruck et al, 1970) where

$$V = \frac{4\pi d^3}{3}\,\xi(\xi^2 - 1)$$

is the volume. In terms of ξ the length-to-width ratio of the body is

$$\ell/w = \xi(\xi^2 - 1)^{-1/2} \quad ,$$

ranging from infinity for a long thin spheroid ($\xi = 1$) down to unity for a sphere ($\xi = \infty$). The analogous results for an oblate spheroid can be obtained by replacing d by -id and ξ by $i\xi$, where now $0 \leq \xi \leq \infty$. The length-to-width ratio is then

$$\ell/w = \xi(\xi^2 + 1)^{-1/2}$$

and varies from unity for a sphere ($\xi = \infty$) down to zero for a disk ($\xi = 0$). In Figs. 2 through 4 P_{11}/V, P_{33}/V and M_{11}/V are plotted as functions of ℓ/w for $0.1 \leq \ell/w \leq 10$.

From an examination of data for the spheroid, Siegel (1958) derived an approximate expression for the backscattering cross section of an arbitrary body of revolution at axial incidence in the form

$$\sigma = \frac{4}{\pi} k^4 V^2 F^2 \tag{154}$$

where F is a shape factor given by

$$F = 1 + \frac{1}{\pi\gamma} e^{-\gamma} \tag{155}$$

and γ is a "measure" of the elongation or length-to-width ratio. We observe that (154) is in agreement with the rigorous lower bound (150), and that F is actually an approximation to $(P_{11} + M_{11})/(4V)$.

Analytical expressions for some or all of the tensor elements have also been derived (Schiffer and Szegö, 1949) for an anchor ring (or torus), spindle (or ogive), two spheres, and a generalized "lens" which reduces to a hemisphere or a hemispherical thin shell (or bowl) in special cases, but only in a few instances have numerical results been obtained. Thus, for two equal spheres in contact with their centers on the z axis,

$$\frac{P_{11}}{V} = 3 \sum_{n-1}^{\infty} \frac{(-1)^{n-1}}{n^3} = 2.705$$

$$\frac{P_{33}}{V} = \frac{8}{3}\,\frac{P_{11}}{V} = 7.212 \quad .$$

For a (solid) hemisphere

$$\frac{P_{11}}{V} = 6\left(2 - \frac{59}{27\sqrt{3}}\right) = 4.430$$

$$\frac{P_{33}}{V} = \frac{2}{9\sqrt{3}}\left\{\frac{64}{3} - \frac{25}{16}(\sqrt{3}+1)\right\} = 2.189 \quad ,$$

and for a hemispherical bowl of infinitesimal thickness

$$\frac{P_{11}}{V'} = 3 + \frac{4}{\pi} = 4.273$$

$$\frac{P_{33}}{V'} = 3 - \frac{8}{\pi} + \frac{6}{2+\pi} = 1.621$$

where V' is the volume of the solid hemisphere. From a comparison of these, it is clear that the concavity has relatively little effect on P_{11}.

For an arbitrary axially symmetric body, a program has been written (Senior and Ahlgren, 1972, 1973) to solve the integral equations (129), (131) and (132) by the moment method and then evaluate the expressions (114) and (120) for the tensor elements P_{11}, P_{33} and M_{11}. In terms of the cylindrical polar coordinates ρ,ϕ,z the surface is defined as $\rho = \rho(z)$, and to simplify the specification it is assumed that the profile $\rho(z)$ is made up of straight line and circular arc segments. Re-entrant shapes are permitted, as well as configurations consisting of two bodies either electrically separated or joined by a wire of infinitesimal thickness along the z axis.

To illustrate the results obtained, Figs. 2 through 4 show the normalized tensor elements P_{11}/V, P_{33}/V and M_{11}/V as functions of the length-to-width ratio ℓ/w for rounded cones (circular sector, subtending an angle θ, rotated about a side), ogives (circular segment rotated about a chord), and lenses (equal abutting circular segments rotated about the bisector). The cone data are in good agreement with those obtained by mode-matching techniques (Senior, 1971), and other data for satellite and missile-like geometries have been given by Kleinman and Senior (1975). The analogous results for spheroids computed from (153) are also presented in Figs. 2 through 4, and it is clear that for many practical purposes the spheroid provides an adequate approximation to the tensor elements of other bodies.

However, this is not always true. For bodies which are re-entrant or are highly asymmetric left to right, the actual volume and length-to-width ratio are not necessarily the most effective parameters to use when selecting the corresponding spheroid, and Senior (1973) has examined the use of other parameters, such as the weighted surface integral $1/2\int \rho \, dS$ in place of the volume. As an example of a non-convex body, Table 1 gives data for a sequence of 'scooped-out' hemispheres (see Fig. 5) all of which have $\ell/w = 0.5$.

Table 1: Tensor elements for a 'scooped-out' hemisphere (see Fig. 5).

V' is the hemispherical volume

Δ	V	$\frac{P_{11}}{V}$	$\frac{P_{33}}{V}$	$\frac{M_{11}}{V}$	$\frac{P_{11}}{V'}$	$\frac{P_{33}}{V'}$	$\frac{M_{11}}{V'}$
1	2.094	4.424	2.185	1.371	1.424	2.185	1.371
0.9	1.937	4.728	2.222	1.935	4.400	2.054	1.789
0.8	1.776	5.117	2.299	2.120	4.339	1.950	1.797
0.6	1.433	6.273	2.628	2.597	4.291	1.798	1.776
0.4	1.039	8.584	3.422	3.483	4.258	1.697	1.727
0.2	0.570	15.56	5.974	6.044	4.234	1.625	1.645
0.1	0.299	29.60	11.16	10.84	4.221	1.592	1.546
0.05	0.154	57.22	21.34	19.24	4.210	1.570	1.416

When the tensor elements are normalized to the actual volume, all increase without limit as $\Delta \to 0$, but when normalized to the volume V' of the smallest enclosing convex body, i.e., the hemisphere, the results are much closer to those for an oblate spheroid having $\ell/w = 0.5$, viz.

$$\frac{P_{11}}{V} = 4.230 \quad , \quad \frac{P_{33}}{V} = 1.897 \quad , \quad \frac{M_{11}}{V} = 1.310 \quad .$$

There are also instances where no obvious normalization is effective. Table 2 shows data for two spheres each of radius 0.5 and distance Δ apart joined by an infinitesimally thin wire along the z axis. Only a minimum number of sampling points (10 on each spherical arc) were used in the moment method, and the accuracy of the data can be judged by comparing the values of P_{11}/V and P_{33}/V for $\Delta = 0$ with the exact results for two spheres in contact. As $\Delta(= \ell/w - 2)$ increases, P_{33}/V increases and does so more rapidly than expected for a long thin body, while P_{11}/V and M_{11}/V approach the values for a single sphere. If the spheres are electrically disconnected by removing the wire, only P_{33} is affected, and the new values are P'_{33}. In neither case does a spheroid provide an accurate approximation.

Table 2: Tensor elements for two spheres Δ apart ($\ell/w = 2 + \Delta$)

Δ	$\frac{P_{11}}{V}$	$\frac{P_{33}}{V}$	$\frac{M_{11}}{V}$	$\frac{M_{33}}{V}$
0	2.702	7.237	1.605	--
0.02	2.715	7.403	1.604	4.557
0.05	2.732	7.655	1.592	4.210
0.1	2.759	8.086	1.579	3.922
0.5	2.891	12.02	1.528	3.299
1.0	2.950	18.19	1.511	3.142
2.0	2.988	34.86	1.505	3.038
3.5	2.996	70.99	1.502	3.017
5.0	2.998	120.6	1.500	3.015
8.0	3.002	260.1	1.501	3.008
10.0	3.002	383.1	1.501	3.002

4. PERFECTLY CONDUCTING FLAT PLATE

A particular case of the body discussed above is a perfectly conducting flat plate of infinitesimal thickness and, hence, volume. This is a geometry which is of interest in its own right, but also, via Babinet's principle, in connection with the transmission of an electromagnetic wave through the complementary aperture in a perfectly conducting screen. It is therefore appropriate to consider it separately.

4.1 Tensor Elements

An infinitesimally thin perfectly conducting flat plate B of finite dimensions lies in the plane z = 0 and is illuminated by the plane linearly polarized electromagnetic wave (47) and (48). Since the tangential components of the total electric field and the normal components of the total magnetic field are zero on the plate, the scattering coefficient is (see (88))

$$\bar{S}(\hat{r},\hat{k},\hat{a}) = k^2 \int_B \hat{r}\times(\hat{r}\times\hat{r}') \left\{ Z_0 \hat{r}\cdot\hat{z}\, \bar{H}_0(\bar{r}') \Big|_-^+ - \hat{z}\cdot\bar{E}_0(\bar{r}') \Big|_-^+ \right\} dS' + O(k^3) \tag{156}$$

where the vertical line denotes the discontinuity across the plate. Hence, from (90) and (91), the electric and magnetic dipole moments are

$$\bar{p} = -\varepsilon_0 \int_B \bar{r}' \frac{\partial}{\partial z'} \Phi_0(\bar{r}') \Big|_-^+ dS' \tag{157}$$

$$\bar{m} = Y_0 \hat{z} \int_B \psi_0(\bar{r}') \Big|_-^+ dS' \tag{158}$$

respectively, where either scattered or total field potentials can be used. Clearly, $\bar{p}$ has components only parallel to the plate and $\bar{m}$ has just the single component in the normal direction.

Integral equations with which to determine the required potentials can be obtained in a manner similar to that described in Chapter 3. We again introduce the partial potentials Φ_i and Ψ_i (see (109)) with $x_3 = z$, and (127) then gives

$$\frac{1}{4\pi} \int_B \frac{1}{R} \frac{\partial \Phi_i}{\partial z'} \Big|_-^+ dS' = -x_i - c_i \tag{159}$$

where c_i is a constant chosen to satisfy the zero induced charge condition

$$\int_B \frac{\partial \Phi_i}{\partial z'} \Big|_-^+ dS' = 0 \quad . \tag{160}$$

In terms of the solution of (159) the elements P_{ij} of the electric polarizability tensor $\bar{\bar{P}}$ (see (112) are

$$P_{ij} = - \int_B x_i' \frac{\partial \Phi_j}{\partial z'} \Big|_-^+ dS' \tag{161}$$

from which it follows that $P_{3j} = 0$. Since the tensor is symmetric, i.e., $P_{ij} = P_{ji}$, we have $P_{ij} = 0$, $i,j = 3$, and the tensor has at most three independent elements.

For the magnetostatic potential, $\Psi_i^S(\bar{r}) = 0$ unless $i = 3$, and then

$$\Psi\Big|_-^+ = (\hat{b}\cdot\hat{z})\psi_3\Big|_-^+ \quad . \tag{162}$$

The representation

$$\psi_3(\bar{r}) = \frac{1}{4\pi}\int_B \frac{\partial}{\partial z'}\left(\frac{1}{R}\right)\psi_3\Big|_-^+ dS' \tag{163}$$

then leads to the integro-differential equation

$$\frac{1}{4\pi}\frac{\partial^2}{\partial z^2}\int_B \frac{1}{R}\psi_3\Big|_-^+ dS' = -1 \quad , \tag{164}$$

from which $\psi_3\big|_-^+$ can be found, e.g., by converting the second normal derivative to a surface Laplacian. The resulting magnetic polarizability tensor $\overline{\overline{M}}$ has just the single non-zero element

$$M_{33} = -\int_B \psi_3\Big|_-^+ dS' \quad . \tag{165}$$

4.2 Analytical and Numerical Results

One of the few geometries for which analytic expressions for the tensor elements are available is the elliptical disk (see, for example, Collin, 1960). If the equation of the perimeter is

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$$

and

$$e = (1 - b^2/a^2)^{1/2}$$

is the ellipticity,

$$P_{11} = \frac{4\pi}{3} a^3 e^2 \left\{ K(e^2) - E(e^2) \right\}^{-1} \quad (166)$$

$$P_{22} = \frac{4\pi}{3} a^3 e^2 \left\{ (1 - e^2)^{-1} E(e^2) - K(e^2) \right\}^{-1} \quad (167)$$

$$M_{33} = \frac{4\pi}{3} a^3 (1 - e^2) \left\{ E(e^2) \right\}^{-1} \quad (168)$$

with $P_{ij} = 0$, $i \neq j$, where $K(e^2)$ and $E(e^2)$ are the complete elliptic integrals of the first and second kinds respectively as defined by Abramowitz and Stegun (1964). We note that

$$\frac{1}{M_{33}} = \frac{1}{P_{11}} + \frac{1}{P_{22}} \quad . \quad (169)$$

Normalized versions of these elements are plotted in Figs. 6 and 7 as functions of the length-to-width ratio ℓ/w (= a/b) for $1 \leq \ell/w \leq 10$ and will be discussed later. In the particular case of zero ellipticity when the disk is circular, (166-168) give

$$P_{11} = P_{22} = \frac{16}{3} a^3 \quad , \qquad M_{33} = \frac{8}{3} a^3 \quad , \tag{170}$$

where a is the radius of the disk.

There have been numerous attempts to develop isoperimetric and other bounds on the tensor elements, but the only ones that have been rigorously established are based on the fact that for a solid body

$$P = \sum_{i=1}^{3} \sum_{j=1}^{3} P_{ij}(\hat{a}\cdot\hat{x}_i)(\hat{a}\cdot\hat{x}_j) \quad \text{and} \quad M = \sum_{i=1}^{3} \sum_{j=1}^{3} M_{ij}(\hat{b}\cdot\hat{z}_i)(\hat{b}\cdot\hat{x}_j)$$

are increasing set functions (Schiffer and Szegö, 1949). By regarding the plate as the limit of a right prism, we then have

$$\begin{aligned} P'_{ii} \le P_{ii} \le P''_{ii} & \\ & \qquad (i = 1,2) \\ M'_{33} \le M_{33} \le M''_{33} & \end{aligned} \tag{171}$$

where the single and double primed quantities refer to the largest inscribed and smallest circumscribed circular disks respectively.

If the plate differs significantly from a circular disk, the bounds given in (171) are too loose to be helpful, but other bounds have been postulated. A common assumption is that of all plates of given area A, the circular disk has the largest average virtual mass (see §3.2) and smallest average polarization. Since, for a circular disk,

$$P_{11} = P_{22} = \frac{16}{3}\left(\frac{A}{\pi}\right)^{3/2} \ , \quad M_{33} = \frac{8}{3}\left(\frac{A}{\pi}\right)^{3/2}$$

(see (170)), it follows that

$$P_{av} = \frac{1}{2}(P_{11} + P_{22}) \geq \frac{16}{3} r^3 \ , \quad M_{33} \leq \frac{8}{3} r^3$$

where $r = (A/\pi)^{1/2}$ is the "area" radius. From a study of various types of symmetrization, Jaggard and Papas (1978) have also proposed that

$$P_{av} \leq \frac{16}{3} r'^3 \ , \qquad M_{33} \geq \frac{8}{3} \frac{r^4}{r'}$$

where $r' = (\text{perimeter})/(2\pi)$ is the "perimeter" radius, and thus

$$\frac{16}{3} r^3 \leq P_{av} \leq \frac{16}{3} r^3 \left(\frac{r'}{r}\right)^3 \tag{172}$$

$$\frac{8}{3} r^3 \left(\frac{r}{r'}\right) \leq M_{33} \leq \frac{8}{3} r^3 \ .$$

In each case, the upper and lower bounds are identical for a circular disk and since, for an elliptical disk,

$$A = \pi a^2 (1 - e^2)^{1/2} \ , \qquad \text{perimeter} = 4aE(e^2) \ ,$$

M_{33} is then equal to the lower bound for all ellipticities. In a subsequent paper, Jaggard (1979) suggested that the numerical factor 8/3 in the lower bound for M_{33} should be replaced by the slightly smaller quantity $\pi^2/4$ to encompass the known data for narrow ($\ell/w << 1$) plates.

In Fig. 6 $P_{11}/(2N)$, $P_{22}/(2N)$ and $P_{av}/(2N)$ for elliptical disks are plotted as functions of the length-to-width ratio, where $N = 8r^3/3$. The upper bound $(r'/r)^3$ is also shown, and by appropriate identification of ℓ and w, these data can be used to estimate the tensor elements for other plates. The analogous plot of M_{33}/N is shown in Fig. 7.

During the last few years several programs have been written to solve the integral equations (159) and (164) by the moment method and, hence, compute the tensor elements. In most cases the results presented are for the normalized dipole moments of the corresponding aperture, and the quantities ν_{mx}, ν_{my} and τ_{av} computed by De Meulenaere and Van Bladel (1977) and others are related to the tensor elements for a plate as follows:

$$\nu_{mx} = \frac{P_{11}}{2A^{3/2}} \quad , \quad \nu_{my} = \frac{P_{22}}{2A^{3/2}} \quad , \quad \tau_{av} = \frac{M_{33}}{2A^{3/2}} \quad .$$

Data for rectangles, "rounded" rectangles (i.e., rectangles with semi-circular ends), diamonds and crosses have been computed by De Meulenaere and Van Bladel (1977) and De Smedt and Van Bladel (1980), and presented graphically as functions of ℓ/w. For the convex shapes at least, the results are similar to those for the corresponding ellipse, with the values for an elliptical disk tending to provide a lower bound for ν_{mx} and ν_{my}.

Additional data for these and other shapes have been provided by Okon and Harrington (1980). Thus, for a square

$$P_{11} = P_{22} = 1.032\ A^{3/2} \quad , \quad M_{33} = 0.452\ A^{3/2} \quad \text{(O\&H)}$$

$$= 1.038\ A^{3/2} \quad \text{(DS\&VB)} \quad ;$$

for a 5:1 rectangle

$$P_{11} = 4.022\ A^{3/2} \quad , \quad P_{22} = 0.373\ A^{3/2} \quad , \quad M_{33} = 0.315\ A^{3/2} \quad \text{(O\&H)}$$

$$= 0.320\ A^{3/2} \quad \text{(DM\&VB)}$$

and for a 2:1 diamond

$$P_{11} = 1.975\ A^{3/2} \quad , \quad P_{22} = 0.606\ A^{3/2} \quad , \quad M_{33} = 0.417\ A^{3/2} \quad \text{(O\&H)}$$

$$= 0.406\ A^{3/2} \quad \text{(DM\&VB)}$$

The letters in parentheses are the initials of the authors cited above, and the accuracy which is claimed for the data is about 1 percent. The results are in accordance with the bounds given in (172).

The simplest example of a transition problem is a homogeneous dielectric body immersed in free space. One application of the results is in the study of scattering by atmospheric particles, and for analytical purposes it is assumed here that the permittivity and permeability are independent of frequency, though possibly complex.

5. HOMOGENEOUS DIELECTRIC BODY

Let B be the surface of a homogeneous isotropic dielectric body of permittivity ϵ_1 and permeability μ_1, illuminated by the linearly polarized plane electromagnetic wave (47), (48). The electric and magnetic dipole moments are given in (90), (91) where the potentials can be taken to be the exterior scattered ones, and if these are now written in the form (109), the electric polarizability tensor $\bar{\bar{P}}$ can be introduced via (112). Its elements P_{ij} are shown in (113) and are functions only of the geometry and permittivity of the body. There is similarly a magnetic polarizibility tensor $\bar{\bar{M}}$ (see (118)) whose elements M_{ij} are given in (119), and these are functions only of the geometry and permeability of the body.

5.1 General Polarizability Tensor

When the conditions on the electrostatic and magnetostatic potentials are examined, it is seen that $\bar{\bar{P}}$ and $\bar{\bar{M}}$ are particular cases of a general polarizability tensor $\bar{\bar{X}}(\tau)$ such that

$$\bar{\bar{P}} = \bar{\bar{X}}(\epsilon_r) \quad , \qquad \bar{\bar{M}} = -\bar{\bar{X}}(\mu_r) \tag{173}$$

(Senior, 1976) where ε_r and μ_r are the relative permittivity and permeability respectively of the dielectric. $\bar{\bar{X}}(\tau)$ depends only on the geometry and the material parameter τ, and its elements X_{ij} are

$$X_{ij} = \int_B \left(\hat{n}'\cdot\hat{x}_i \Phi_j^s - x_i' \frac{\partial \Phi_j^s}{\partial n'} \right) dS' \qquad (174)$$

where Φ_j^s is an exterior scattered potential (the total potential could be used instead) satisfying the boundary conditions

$$\Phi_j^s = \Phi_j + x_j \qquad (175)$$

$$\bar{r} \in B \quad ,$$

$$\frac{\partial \Phi_j^s}{\partial n} = \tau \frac{\partial \Phi_j}{\partial n} + \hat{n}\cdot\hat{x}_j \qquad (176)$$

where Φ_j is an interior potential. Clearly (176) guarantees the fulfillment of the zero induced charge condition

$$\int_B \frac{\partial \Phi_j^s}{\partial n'} \, dS' = 0 \ . \qquad (177)$$

In the limiting case of a perfectly conducting body, $\bar{\bar{P}} = \bar{\bar{X}}(\infty)$, $\bar{\bar{M}} = -\bar{\bar{X}}(0)$, and thus the data presented in Section 3 also specify the tensor $\bar{\bar{X}}$ for these two extreme values of τ.

From the definition (174) it might appear that we require a knowledge of both Φ_j^s and $\partial\Phi_j^s/\partial n$ on B to calculate X_{ij}, but this is not in fact so. From (176) we have

$$\int_B x_i' \frac{\partial \Phi_j^s}{\partial n'} dS' = \tau \int_{\text{int } B} \nabla' \cdot (x_i' \nabla' \Phi_j) dV' + \int_{\text{int } B} \nabla' \cdot (x_i' \hat{x}_j) dV'$$

by applying the divergence theorem. But

$$\nabla' \cdot (x_i' \hat{x}_j) = \delta_{ij} ,$$

where δ_{ij} is the Kronecker delta function and

$$\nabla' \cdot (x_i' \nabla' \Phi_j) = \hat{x}_i \cdot \nabla' \Phi_j$$

since $\nabla^2 \Phi_j = 0$ in V. Hence, by a further application of the divergence theorem,

$$\int_B x_i' \frac{\partial \Phi_j^s}{\partial n'} dS' = \tau \int_B \hat{n}' \cdot \hat{x}_i \Phi_j \, dS' + V\delta_{ij} ,$$

and on using (175) to eliminate Φ_j in favor of Φ_j^s,

$$X_{ij} = (1 - \tau) \int_B \hat{n}' \cdot \hat{x}_i \Phi_j^t \, dS' \qquad (178)$$

where

$$\Phi_j^t = \Phi_j^s - x_j \qquad (179)$$

is the total potential outside B. Equation (178) is identical to the (corrected) expression (3.76) of Van Bladel (1964), and we note that

$$X_{ij}(1) = 0 \quad . \tag{180}$$

This shows that if the material is non-magnetic, i.e., $\mu_r = 1$, there is no magnetic dipole moment, and similarly if $\varepsilon_r = 1$, the electric dipole moment is zero.

In a similar manner to the above, the following alternative expression can be derived:

$$X_{ij} = \frac{1 - \tau}{\tau} \int_B x_i' \frac{\partial \Phi_j^t}{\partial n'} \, dS' \tag{181}$$

and from (178) and (181) it is a simple matter to deduce two other forms involving the interior potential Φ_j.

To compute the tensor elements it is sufficient to determine either Φ_j^t or $\partial\Phi_j^t/\partial n$ on B. By adding τ times (103) with $\chi = \Phi_j$ to (103) with $\chi = x_j$ and then subtracting the result from (102) with Φ^S replaced by Φ_j^s, we obtain

$$(1 - \tau) \frac{1}{4\pi} \int_B (\Phi_j^s - x_j') \frac{\partial}{\partial n'} \left(\frac{1}{R}\right) dS' = \tau\Phi_j^s + (1 - \tau)[x_j + \alpha(\bar{r})(\Phi_j^s - x_j)] \; .$$

In particular, for $\bar{r} \in B$,

$$\Phi_j^t(\bar{r}) = -\frac{2}{1 + \tau} x_j + \frac{1 - \tau}{1 + \tau} \frac{1}{2\pi} \int_B \Phi_j^t(\bar{r}') \frac{\partial}{\partial n'} \left(\frac{1}{R}\right) dS' \tag{182}$$

which is an integral equation for Φ_j^t. Knowing Φ_j^t, X_{ij} can then be computed from (178), and even for a body of arbitrary shape, solution of the three integral equations (182) with j = 1,2,3 is sufficient to specify the tensor elements.

We can also derive an integral equation for the normal derivative of the total potential on B. If (103) with $\chi = \Phi_j + x_j$ is subtracted from (102) with Φ^s replaced by Φ_j^s,

$$\frac{1-\tau}{\tau}\,\frac{1}{4\pi}\int_B \frac{1}{R}\,\frac{\partial\Phi_j^t}{\partial n'}\,dS' = \Phi_j^t + x_j \quad , \qquad (183)$$

and on taking the normal derivative of this, we have

$$\frac{1-\tau}{\tau}\,\frac{1}{4\pi}\int_B \frac{\partial\Phi_j^t}{\partial n'}\,\frac{\partial}{\partial n}\left(\frac{1}{R}\right) dS' = \frac{1}{2}\,\frac{1+\tau}{\tau}\,\frac{\partial\Phi_j^t}{\partial n} + \hat{n}\cdot\hat{x}_j$$

for $\bar{r} \in B$. An integral equation for $\partial\Phi_j^t/\partial n$ is therefore

$$\frac{\partial\Phi_j^t}{\partial n} = -\frac{2\tau}{1+\tau}\,\hat{n}\cdot\hat{x}_j + \frac{1-\tau}{1+\tau}\,\frac{1}{2\pi}\int_B \frac{\partial\Phi_j^t}{\partial n'}\,\frac{\partial}{\partial n}\left(\frac{1}{R}\right) dS' \quad , \qquad (184)$$

and once $\partial\Phi_j^t/\partial n$ has been found, X_{ij} can be computed from (181). By integrating (184) over the entire surface B, it can be verified that the zero induced charge condition (177) is satisfied. Since the eigenvalues of (182) and (184) are all real with moduli not less than unity, the equations have unique solutions if τ is complex or τ is real and positive.

5.2 Properties of $\overline{\overline{X}}$

The potentials and, hence, the elements X_{ij} are real if τ is. If we use the boundary conditions (175) and (176) to eliminate $\hat{x}_i$ and $\hat{n}\cdot\hat{x}_i$ from (174), the expression for X_{ij} becomes

$$X_{ij} = \int_B \left\{ \Phi_j \frac{\partial \Phi_i^s}{\partial n'} + \Phi_i \frac{\partial \Phi_j^s}{\partial n'} \right\} dS' - \int_B \left\{ \tau\Phi_j \frac{\partial \Phi_i}{\partial n'} + \Phi_i^s \frac{\partial \Phi_j^s}{\partial n'} \right\} dS' + V\delta_{ij}$$

and by applying the divergence theorem to the second integral we find

$$X_{ij} = \int_B \left\{ \Phi_j \frac{\partial \Phi_i^s}{\partial n'} + \Phi_i \frac{\partial \Phi_j^s}{\partial n'} \right\} dS' + \int_{\text{ext } B} \nabla'\Phi_i^s\cdot\nabla'\Phi_j^s \, dV'$$

$$- \tau \int_{\text{int } B} \nabla'\Phi_i\cdot\nabla'\Phi_j \, dV' + V\delta_{ij} \qquad (185)$$

which is symmetric in i and j. $\overline{\overline{X}}$ is therefore a symmetric tensor, real if τ is, having at most six independent elements.

In certain special cases the tensor simplifies considerably. If B is symmetric about the $x_1 = 0$ plane, it is evident from (182) that

$$\Phi_1^t(-x_1,x_2,x_3) = -\Phi_1^t(x_1,x_2,x_3)$$

and

$$\Phi_j^t(-x_1,x_2,x_3) = \Phi_j^t(x_1,x_2,x_3) \quad , \qquad j = 2,3 \quad . \qquad (186)$$

To every point on B there now corresponds another where $\hat{n}\cdot\hat{x}_2$ is the same but Φ_1^t is reversed in sign, and if $B_+(B_-)$ is that portion of the surface for which $x_1 \geq (\leq) 0$,

$$X_{21} = (1 - \tau) \left\{ \int_{B+} \hat{n} \cdot \hat{x}_2 \Phi_1^t \, dS + \int_{B-} \hat{n} \cdot \hat{x}_2 \Phi_1^t \, dS \right\} = 0 \quad .$$

Similarly $X_{31} = 0$ and therefore $X_{12} = X_{13} = 0$. It follows that $\overline{\overline{X}}$ has at most the nonzero elements

$$\begin{aligned} X_{11} &= 2(1 - \tau) \int_{B+} \hat{n} \cdot \hat{x}_1 \phi_1^t \, dS \\ X_{ij} &= 2(1 - \tau) \int_{B+} \hat{n} \cdot \hat{x}_i \Phi_j^t \, dS \quad , \quad i,j = 2,3 \end{aligned} \tag{187}$$

and of these no more than four are independent.

If, in addition, the body is symmetrical about a second perpendicular plane which we take to be the plane $x_2 = 0$, then

$$\Phi_2^t(x_1, -x_2, x_3) = -\Phi_2^t(x_1, x_2, x_3) \tag{188}$$

and

$$\Phi_j^t(x_1, -x_2, x_3) = \Phi_j^t(x_1, x_2, x_3) \quad , \quad j = 1,3$$

in which case $X_{12} = X_{32} = 0$, implying $X_{21} = X_{23} = 0$. The tensor is now diagonal and has at most the three nonzero elements

$$X_{ii} = 4(1 - \tau) \int_{B++} \hat{n} \cdot \hat{x}_i \Phi_i^t \, dS \quad , \qquad i = 1,2,3 \tag{189}$$

where B_{++} is that portion of the surface having $x_1, x_2 \geq 0$. In general all three elements are distinct, but in the special case of symmetry about the x_3 axis, $X_{22} = X_{11}$. Symmetry about a third perpendicular plane $x_3 = 0$ produces no further simplification to the tensor's form.

Bounds on the tensor elements can be developed even for a body of arbitrary shape. From (178) and (181)

$$(1 + \tau)X_{ij} = (1 - \tau)\left\{\int_B \left(\hat{n}'\cdot\hat{x}_i\Phi_j + x_i' \frac{\partial\Phi_j^s}{\partial n'}\right) dS' - V\delta_{ij}\right\}$$

where we have used the boundary conditions (175) and (176) to replace the total potential, and by using these again to eliminate $\hat{n}'\cdot\hat{x}_i$ and x_i', we obtain

$$\frac{\tau + 1}{\tau - 1} X_{ij} - V\delta_{ij} = \tau \int_{\text{int } B} \nabla'\Phi_i\cdot\nabla'\Phi_j \, dV' + \int_{\text{ext } B} \nabla'\Phi_i^s\cdot\nabla'\Phi_j^s \, dV' \quad . \tag{190}$$

From Schwarz's inequality

$$\left\{\int_{\text{int } B} \nabla'\Phi_i\cdot\nabla'\Phi_j \, dV'\right\}^2 \le \int_{\text{int } B} |\nabla'\Phi_i|^2 \, dV' \int_{\text{int } B} |\nabla'\Phi_j|^2 \, dV' \quad , \tag{191}$$

and similarly for the second integral in (190). By squaring both sides of (190) and then applying the inequality to the new right-hand side, we finally obtain

$$\left(X_{ij} - \frac{\tau - 1}{\tau + 1} V\delta_{ij}\right)^2 \le \left(X_{ii} - \frac{\tau - 1}{\tau + 1} V\right)\left(X_{jj} - \frac{\tau - 1}{\tau + 1} V\right) \tag{192}$$

for τ real and non-negative. This fundamental inequality is the analogue of those for the electric and magnetic polarizability tensors in the case of perfect conductivity (see (137)), and degenerates to them on putting $\tau = 0$ or ∞.

We can also develop upper and lower bounds on the diagonal elements. Since

$$\int_{\text{int } B} |\hat{x}_i + \gamma\nabla'\Phi_i|^2 \, dV' = V + \frac{2\gamma}{1-\tau} X_{ii} + \gamma^2 \int_{\text{int } B} |\nabla'\Phi_i|^2 \, dV'$$

for any γ, combination with (190) gives

$$\frac{\tau+1 - 2\tau/\gamma}{\tau - 1} X_{ii} = \frac{\gamma^2 - \tau}{\gamma^2} V + \frac{\tau}{\gamma^2} \int_{\text{int } B} |\hat{x}_i + \gamma\nabla'\Phi_i|^2 \, dV' + \int_{\text{ext } B} |\nabla'\Phi^S|^2 \, dV' ,$$

implying

$$\frac{\tau+1 - 2\tau/\gamma}{\tau - 1} X_{ij} \geq \frac{\gamma^2 - \tau}{\gamma^2} V$$

for real γ and real $\tau > 0$. Upper and lower bounds on X_{ii} are now obtained by taking $\gamma = 1$ and τ respectively, viz

$$\tau - 1 \geq \frac{X_{ii}}{V} \geq \frac{\tau - 1}{\tau} . \qquad (193)$$

It is obvious that these bound X_{ii} extremely closely if τ is near to unity, and whatever geometry dependence then exists must be minimal. The bounds are, moreover, optimum, in the sense that the upper one is achieved by the element X_{33} for a vanishingly thin prolate spheroid, and the lower one by the same elements for a vanishingly thin oblate spheroid. Other bounds have been derived by Jones (1979, 1980) by considering the "content matrix" whose elements C_{ij} are related to the X_{ij} by

$$X_{ij} = (1 - \tau)\left\{(1 - \tau)C_{ij} + V\delta_{ij}\right\} . \tag{194}$$

5.3 Analytical and Numerical Results

A body for which analytical expressions for the tensor elements are available is the ellipsoid. If the body is defined by (119) so that the principal axes coincide with the coordinate axes x_i, the tensor is diagonal, and

$$X_{ii}(\tau) = V\frac{\tau - 1}{L_i\tau + 1 - L_i} \tag{195}$$

where the L_i are given by (121) with cyclical changes of the suffices. Some computed values of the L_i are quoted by Van de Hulst (1957). From (126) and (190)

$$\left(\frac{1}{X_{11}} + \frac{1}{X_{22}} + \frac{1}{X_{33}}\right)V = \frac{\tau + 2}{\tau - 1} . \tag{196}$$

A special case of an ellipsoid is a spheroid, prolate or oblate. If the x_3 axis is chosen as the axis of symmetry, $L_2 = L_1$ so that $X_{22} = X_{11}$, and the tensor elements can be written as ratios of Legendre functions of the first and second kinds. On inserting the expressions for these functions, the results for a prolate spheroid are found to be (see §3.4)

$$X_{11}(\tau) = -2\left\{\frac{1}{2}\xi(\xi^2 - 1)\ln\frac{\xi + 1}{\xi - 1} - \xi^2 - \frac{2}{\xi - 1}\right\}^{-1} V \qquad (197)$$

$$X_{33}(\tau) = \left\{\frac{1}{2}\xi(\xi 2 - 1)\ln\frac{\xi + 1}{\xi - 1} - \xi^2 + \frac{\tau}{\tau - 1}\right\}^{-1} V \qquad (198)$$

The analogous results for an oblate spheroid can be obtained as indicated in §3.4.

From (197) and (198) it is evident that

$$X_{33}(\tau) = -\frac{1}{2}X_{11}(2/\tau - 1) \qquad (199)$$

for all spheroids, prolate and oblate, and this enables us to deduce the values of X_{11} and X_{33} for $\tau < 1$ from those of X_{33} and X_{11} for the corresponding $\tau > 1$. Equation (199) is a generalization of (141) for a perfectly conducting body of revolution about the x_3 axis, but is not valid for bodies other than a spheroid.

Figures 8 and 9 show $X_{11}(\tau)$ and $X_{33}(\tau)$ computed using (197) and (198) respectively as functions of ℓ/w, $0.1 \leq \ell/w \leq 10$, for a variety of (real) $\tau \geq 0$. For any given τ, X_{11}/V is a decreasing function of ℓ/w, decreasing from $\tau - 1$ for a disk, through $3(\tau - 1)/(\tau + 2)$ for a sphere, to $2(\tau - 1)/(\tau + 1)$ for a "rod", whereas X_{33}/V is an increasing function from $(\tau - 1)/\tau$ through $3(\tau - 1)/(\tau + 2)$ to $\tau - 1$.

For a number of other bodies computer programs have been written to solve the integral equations (182) or (184) by the moment method and, hence, obtain the tensor elements. In general it is found that (182) is superior to (184) from a numerical standpoint and most programs have employed the former. One such program, designated Dielcom (Senior and Willis, 1982), is applicable to any body of revolution about the x_3 axis whose profile can be constructed from straight line and circular arc segments. Concave (re-entrant) shapes can be treated, as well as multiply connected ones, e.g., a torus. Bodies having two or more disjoint portions can also be considered provided their material prameters are the same and they have a common axis of symmetry. A similar program has been written for rectangular parallelepipeds (Herrick and Senior, 1977).

Using these programs data have been obtained for a variety of geometry and material combinations. If $\tau \geq 0$ it is found that the tensor elements are relatively insensitive to the details of the body's shape, and in many instances they can be adequately approximated by the elements for a spheroid having the same τ, volume and length to width ratio ℓ/w. This is illustrated in Figs. 10 and 11 showing the ratios of the tensor elements for a parallelepiped of square cross section to those of a spheroid. The maximum difference is no more than 16 percent. Similar plots of the parallelepiped values normalized to those for a right circular cylinder are given by Herrick and Senior (1977), and in this case the maximum difference is 10 percent. From these it is concluded that

$$\begin{aligned} &\frac{X_{11}}{V}\begin{pmatrix}\text{square}\\ \text{cylinder}\end{pmatrix} \lessgtr \frac{X_{11}}{V}\begin{pmatrix}\text{circular}\\ \text{cylinder}\end{pmatrix} , \qquad (\tau \lessgtr 1) \\ &\frac{X_{33}}{V}\begin{pmatrix}\text{square}\\ \text{cylinder}\end{pmatrix} < \frac{X_{33}}{V}\begin{pmatrix}\text{circular}\\ \text{cylinder}\end{pmatrix} , \qquad (\tau \geq 0) \end{aligned} \quad . \qquad (200)$$

Since a circle can be viewed as the limit of an n-sided regular polygon as $n \to \infty$, it seems reasonable to conjecture that the tensor elements of a cylinder whose cross section is an n-sided regular polygon with $n > 4$ lie between those of the corresponding square and circular cylinders.

Equations (195) through (199) and the above-mentioned computer programs are also applicable if τ is complex, in which case the tensor elements are complex. If Re $\tau \geq 0$ a spheroid may then suffice to approximate the real and imaginary parts of X_{11} and X_{33}, leading to an estimate of the absorption cross section σ_A (see (55)). The manner in which σ_A depends on body shape has been discussed by Senior (1980), but we caution that if Re $\tau < 0$ both the scattering and the absorption can be extremely dependent on the body's shape. Many crystalline materials have permittivities whose real parts are negative at frequencies in the infrared and optical ranges.

6. HOMOGENEOUS DISPERSIVE BODIES

Slightly more complicated physical situations involve dispersive targets such as lossy dielectrics and overdense plasmas. The resulting transition problem involves a frequency dependent permittivity. This does not make the low frequency analysis more difficult, in fact, in some sense it is simpler than for pure dielectrics, but the nature of the low frequency expansion must be reexamined in order to understand its meaning.

6.1 Lossy Dielectrics

As in Section 5, assume that the linearly polarized plane electromagnetic wave (47) and (48) illuminates a homogeneous isotropic body B which, however, is now characterized by the permittivity ε_1, permeability μ_1, and nonzero conductivity σ_1. Using the complex permittivity defined in (22), the analytic results of Section 5 apply immediately in the present case. That is, the electric and magnetic polarizability tensors may be expressed in terms of one general polarizability tensor $\bar{\bar{\chi}}(\tau)$ given by (173) except now

$$\varepsilon_r = \frac{\tilde{\varepsilon}}{\varepsilon_o} = \left(\varepsilon_1 + i\,\frac{\sigma_1}{\omega}\right)\frac{1}{\varepsilon_o} \quad . \tag{201}$$

The elements of this general tensor may be determined as in Section 5, however the parameter τ will be complex when $\tau = \varepsilon_r$. Thus Section 5.1 is applicable in entirety as are the equations in Sections 5.2 and 5.3, but the inequalities and upper and lower bounds on the tensor elements apply only for real τ. Nevertheless the integral equations for the tensor elements (182) and (184) are valid in the present case even when τ is complex as are the explicit results for the ellipsoid (195)-(199).

The question we consider here then is not the construction of the tensor elements in the first term of the expansion but the nature of the expansion itself. Combining (89), (112), (118) and (173), the scattering coefficient is given by

$$\bar{S}(\hat{r},\hat{k},\hat{a}) = -k^2\{\hat{r}\times[\hat{r}\times(\bar{\bar{X}}(\varepsilon_r)\cdot\hat{a})] + \hat{r}\times(\bar{\bar{X}}(\mu_r)\cdot\hat{b})\} + \mathcal{O}(k^3) \tag{202}$$

and, with (95),

$$\sigma(\hat{r}) = \frac{k^4}{4\pi}\{|\hat{r}\times(\bar{\bar{X}}(\varepsilon_r)\cdot\hat{a})|^2 + |\hat{r}\times(\bar{\bar{X}}(\mu_r)\cdot\hat{b})|^2$$

$$+ 2\,\mathrm{Re}\,\hat{r}\cdot(\bar{\bar{X}}(\varepsilon_r)\cdot\hat{a})\times(\bar{\bar{X}}(\mu_r)\cdot\hat{b})\} \; . \tag{203}$$

However, ε_r is now a function ω, and since $k = \omega\sqrt{\varepsilon_o\mu_o}$, ε_r can be considered to be a function of k as well. With ε_r as defined in (201) we see that this may be written as

$$\varepsilon_r = \frac{\varepsilon_1}{\varepsilon_o} + i\,\frac{\sigma_1}{\omega\varepsilon_o} = \frac{\varepsilon_1}{\varepsilon_o} + i\,\frac{\sigma_1}{kY_o} \; , \tag{204}$$

from which it is straightforward to establish that

$$\frac{1 - \varepsilon_r}{\varepsilon_r} = -1 + O\left(\frac{k}{\sigma_1}\right) \quad . \tag{205}$$

$$\frac{1 - \varepsilon_r}{1 + \varepsilon_r} = -1 + O\left(\frac{k}{\sigma_1}\right) \tag{206}$$

and

$$\frac{\varepsilon_r}{1 + \varepsilon_r} = 1 + O\left(\frac{k}{\sigma_1}\right) \quad . \tag{207}$$

Now with (181) and (205) we find that

$$\chi_{ij}(\varepsilon_r) = -\int_B x_i' \frac{\partial \Phi_j^t}{\partial n'} \, dS' + O\left(\frac{k}{\sigma_1}\right) \tag{208}$$

where $\partial \Phi_j^t / \partial n'$ is the solution of (184), and with (206) and (207), the integral equation (184) may be written as

$$\frac{\partial \Phi_j^t}{\partial n} = -2\hat{n} \cdot \hat{x}_j - \frac{1}{2\pi} \int_B \frac{\partial \Phi_j^t}{\partial n'} \frac{\partial}{\partial n} \left(\frac{1}{R}\right) dS' + O\left(\frac{k}{\sigma_1}\right) \quad . \tag{209}$$

Comparison with (128) reveals that to order $O(k/\sigma_1)$, $\partial \Phi_j^t / \partial_n$ satisfies the same integral equation as in the perfectly conducting case. Similarly, comparison of (114) with (208) shows that the tensor elements in the lossy case are the same as those in the perfectly conducting case to this order in k/σ_1. Thus, if σ_1 is fixed and nonzero, it follows that

$$\chi_{ij}(\varepsilon_r) = \chi_{ij}(\infty) + O(k) \quad . \tag{210}$$

and the far field coefficient (202) becomes

$$\bar{S}(\hat{r},\hat{k},\hat{a}) = -k^2\{\hat{r} \times [\hat{r} \times (\bar{\bar{X}}(\infty) \cdot \hat{a})] + \hat{r} \times (\bar{\bar{X}}(\mu_r) \cdot \hat{b})\} + O(k^3) \quad . \tag{211}$$

Although they differ in appearance, (202) and (211) are both correct to the same asymptotic order in k for fixed nonzero σ_1. Equation (211) has the advantage of simplicity, especially when the tensor elements are known in the perfectly conducting case, although if they have to be computed, the numerical solution of the integral equations in either case is of the same order of difficulty regardless of the value of τ or ε_r. Equation (202) has the advantage of being uniformly valid in σ_1 for all values of σ_1 in $[0,\infty]$ including the endpoints and should definitely be employed when kY_0/σ_1 is large. When kY_0/σ_1 is small either formula may be employed. The precise definitions of "large" and "small" will depend on the accuracy desired and will also depend on $\varepsilon_1/\varepsilon_0$.

For example, if B is a sphere of volume V and $\mu_r = 1$, (202) yields

$$\bar{S}_1 = -3k^2V \, \hat{r} \times (\hat{r} \times \hat{a}) \, \frac{1 + \frac{kV_0}{i\sigma_1}\left(\frac{\varepsilon_1}{\varepsilon_0} - 1\right)}{1 + \frac{kV_0}{i\sigma_1}\left(2 + \frac{\varepsilon_1}{\varepsilon_0}\right)} \tag{212}$$

whereas (211) becomes

$$\bar{S}_2 = -3k^2V \, \hat{r} \times (\hat{r} \times \hat{a}) \quad . \tag{213}$$

A measure of the relative difference in these results is provided by

$$\left|\frac{\bar{S}_1 - \bar{S}_2}{\bar{S}_1}\right| = 3\,\frac{kY_0}{\sigma_1}\left\{1 + \left(\frac{kY_0}{\sigma_1}\right)^2\left(\frac{\varepsilon_1}{\varepsilon_0} - 1\right)^2\right\}^{-1/2} \tag{214}$$

This quantity will be less than 0.1 for all values of kV_0/σ_1 if $\varepsilon_1/\varepsilon_o > 29$, but if $\varepsilon_1/\varepsilon_o = 1$, it is less than 0.1 only if $kY_0/\sigma_1 <$ 1/30.

Note that in the case of nonmagnetic lossy materials ($\mu_r = 1$ and $\sigma_1 > 0$), (211) becomes simply

$$\bar{S}(\hat{r},\hat{k},\hat{a}) = -k^2\hat{r} \times [\hat{r} \times (\bar{\bar{X}}(\infty) \cdot \hat{a})] + O(k^3) \quad . \tag{215}$$

Thus for these materials the electric dipole moment is the same as if the material were perfectly conducting whereas the magnetic dipole moment is zero, the same as if the material were free space. For these materials, (202) also has only an electric dipole contribution.

6.2 Overdense Plasmas

Assume now that a plane wave illuminates a homogeneous isotropic body which is characterized by a frequency dependent relative permittivity

$$\varepsilon_r = 1 - \frac{\omega_p^2}{\omega(\omega + i\nu_c)} \tag{216}$$

where ω is the frequency of the incident plane wave, ω_p is the plasma frequency and ν_c is the collision frequency. This is the standard model of a so-called overdense plasma as an equivalent lossy dielectric (e.g., Kerker, 1969; p. 185). If for convenience we define parameters

$$k_p = \frac{\omega_p}{c} \tag{217}$$

and

$$k_\nu = \frac{\nu_c}{c} \tag{218}$$

then the equivalent relative permittivity may be written as

$$\varepsilon_r = 1 - \frac{k_p^2}{k(k + ik_c)} = \frac{\frac{k}{k_p}\left(\frac{k}{k_p} + i\,\frac{k_c}{k_p}\right) - 1}{\frac{k}{k_p}\left(\frac{k}{k_p} + i\,\frac{k_c}{k_p}\right)} \quad . \tag{219}$$

Corresponding to (205) - (207), it may now easily be shown that regardless of whether or not $k_c = 0$

$$\frac{1 - \varepsilon_r}{\varepsilon_r} = -1 + O\left(\frac{k}{k_p}\right) \tag{220}$$

$$\frac{1 - \varepsilon_r}{1 + \varepsilon_r} = -1 + O\left(\frac{k}{k_p}\right) \tag{221}$$

and

$$\frac{\varepsilon_r}{1 + \varepsilon_r} = 1 + O\left(\frac{k}{k_p}\right) \quad . \tag{222}$$

Thus the analysis in Section 6.1 may be repeated and it is seen that (202) and (203) remain valid as does (211) for fixed nonzero k_p. Additional constraints must now be placed on ε_r or k_p and k_c to ensure that (182) or (184) is uniquely solvable; that is, we require that $(1 - \varepsilon_r)/(1 + \varepsilon_r)$ is not an eigenvalue of the integral equation.

As in Section 6.1 there is a choice between (202), with ε_r given by (216), and (211), where both are accurate to order $O(k^3)$ for fixed nonzero k_p. In this case it appears that (202) is preferable unless k is considerably smaller than k_p regardless of k_c.

For example, when B is a sphere of volume V and $\mu_r = 1$ then (202) yields

$$\bar{S}_1 = -3k^2V\, \hat{r} \times (\hat{r} \times \hat{a}) \left\{1 - 3\,\frac{k}{k_p}\left(\frac{k}{k_p} + i\,\frac{k_c}{k_p}\right)\right\}^{-1} \tag{223}$$

whereas (211) becomes

$$\bar{S}_2 = -3k^2V\, \hat{r} \times (\hat{r} \times \hat{a}) \tag{224}$$

and a measure of the difference in these results is

$$\left|\frac{\bar{S}_1 - \bar{S}_2}{\bar{S}_1}\right| = 3\left|\frac{k}{k_p}\right| \left\{\left(\frac{k}{k_p}\right)^2 + \left(\frac{k_c}{k_p}\right)^2\right\}^{1/2} . \tag{225}$$

This quantity is less than 0.1 for $k/k_p < 0.32$ when $k_c/k_p = 0$, but when $k_c/k_p = 1$ we require $k/k_p < 0.1$, and when $k_c/k_p = 10$ we require $k/k_p < 0.01$ to guarantee that (225) is still less than 0.1.

Again, (202) is valid uniformly in k_p including $k_p = 0$, whereas (211) is not.

One interesting application of (211) is to the case of a perfectly conducting surface B coated with either a lossy nonmagnetic dielectric, $\sigma_1 \neq 0$ and $\mu_r = 1$, or an overdense plasma with $k_c \neq 0$ and $\mu_r = 1$. If the surface of the coating is denoted by C, it has been shown (Kleinman and Senior, 1975) that the far field coefficient is given by

$$\bar{S}(\hat{r},\hat{k},\hat{a}) = -k^2\{\hat{r} \times [\hat{r} \times (\bar{\bar{X}}_C(\infty) \cdot \hat{a})] + \hat{r} \times (\bar{\bar{X}}_B(0) \cdot \hat{b}) + (k^3) \tag{226}$$

where $\bar{\bar{X}}_C(\infty)$ is the electric polarizability tensor associated with the perfectly conducting surface C while $\bar{\bar{X}}_B(0)$ is the magnetic polarizability tensor associated with the perfectly conducting surface B.

Finally we note that Kerker (1975) considered the case of a dielectric ellipsoid with a dielectric or plasma coating and showed that it was possible to find a condition on the two relative permittivities which caused the k^2 term in the far field to vanish.

7. ACOUSTICALLY SOFT, HARD AND PENETRABLE BODIES

Scalar (acoustic) scattering problems are usually easier to solve than vector (electromagnetic) ones, and this is true also at low frequencies. The analysis is more straightforward than in the electromagnetic case, and the potential functions that occur are identical to those previously discussed.

For simplicity we shall confine attention to a plane wave incident on a soft or hard body at the surface of which the boundary condition is (6) or (7) respectively, or a penetrable body for which the transmission conditions (10) and (11) are applicable. Since, for a soft body, the leading term in the low frequency expansion is both frequency and aspect independent, the next two terms are derived as well.

7.1 Soft Body

A finite closed acoustically soft body B is illuminated by the plane acoustic wave $u^{inc}(\bar{r})$ given in (46). For $\bar{r} \in$ ext B the resulting scattered field is (34), and when the boundary condition (6) is imposed,

$$u^s(\bar{r}) = -\frac{1}{4\pi} \int_B \frac{e^{ikR}}{R} \frac{\partial u^t}{\partial n'} dS' \quad . \tag{227}$$

In the particular case of $\bar{r} \in B$, we have

$$\frac{1}{4\pi} \int_B \frac{e^{ikR}}{R} \frac{\partial u^t}{\partial n'} dS' = u^{inc}(\bar{r}) \tag{228}$$

which is an integral equation for $\partial u^t/\partial n$.

For small k the exponential in (228) can be expanded in powers of ik, and when the incident and scattered fields are similarly expanded (see (59) and (60)), the first three terms yield

$$\frac{1}{4\pi}\int_B \frac{1}{R}\frac{\partial u_0^t}{\partial n'}\,dS' = 1 \tag{229}$$

$$\frac{1}{4\pi}\int_B \frac{1}{R}\frac{\partial u_1^t}{\partial n'}\,dS' = \hat{k}\cdot\bar{r} - \frac{1}{4\pi}\int_B \frac{\partial u_0^t}{\partial n'}\,dS' \tag{230}$$

$$\frac{1}{4\pi}\int_B \frac{1}{R}\frac{\partial u_2^t}{\partial n'}\,dS' = \frac{1}{2}(\hat{k}\cdot\bar{r})^2 - \frac{1}{8\pi}\int_B R\,\frac{\partial u_0^t}{\partial n'}\,dS' - \frac{1}{4\pi}\int_B \frac{\partial u_1^t}{\partial n'}\,dS'. \tag{231}$$

Comparison of (229) with (131) shows that

$$\frac{\partial u_0^t}{\partial n} = -\frac{\partial \Phi^{t(o)}}{\partial n} \tag{232}$$

and hence

$$\int_B \frac{\partial u_0^t}{\partial n'}\,dS' = \frac{C}{\varepsilon_o} \tag{233}$$

where C is the electrostatic capacity of an isolated perfectly conducting body B immersed in free space.

The second integral equation (230) then becomes

$$\frac{1}{4\pi}\int_B \frac{1}{R}\frac{\partial u_1^t}{\partial n'}\,dS' = \sum_{i=1}^{3}(\hat{k}\cdot\hat{x}_i)x_i - \frac{C}{4\pi\varepsilon_o}\ , \tag{234}$$

and comparison with (131) and (132) shows

$$\frac{\partial u_1^t}{\partial n} = - \sum_{i=1}^{3} (\hat{k}\cdot\hat{x}_i) \frac{\partial \Phi_i^{t(1)}}{\partial n} + \frac{C}{4\pi\varepsilon_o} \frac{\partial \Phi^{t(o)}}{\partial n} , \tag{235}$$

implying

$$\int_B \frac{\partial u_1^t}{\partial n'} dS' = - \sum_{i=1}^{3} (\hat{k}\cdot\hat{x}_i) \int_B \frac{\partial \Phi_i^{t(1)}}{\partial n'} dS' - \frac{1}{4\pi}\left(\frac{C}{\varepsilon_o}\right)^2 . \tag{236}$$

However, as first proved by Van Bladel (1968), we can also evaluate the left-hand side of (236) without knowing u_1^t. By expanding both sides of (227) in powers of ik, it is seen that $u_1^s + C/(4\pi\varepsilon_o)$ is an exterior potential function. Hence, from the boundary value of $\Phi^{s(o)}$(= 1 on B) and the reciprocity theorem for exterior potentials,

$$\begin{aligned}\int_B \frac{\partial u_1^s}{\partial n'} dS' &= \int_B \Phi^{s(o)} \frac{\partial}{\partial n'}\left(u_1^s + \frac{C}{4\pi\varepsilon_o}\right) dS' \\ &= \int_B \left(u_1^s + \frac{C}{4\pi\varepsilon_o}\right) \frac{\partial \Phi^{s(o)}}{\partial n'} dS' \\ &= \int_B \left(- \hat{k}\cdot\bar{r}' + \frac{C}{4\pi\varepsilon_o}\right) \frac{\partial \Phi^{s(o)}}{\partial n'} dS' .\end{aligned}$$

But

$$\frac{\partial u_o^{inc}}{\partial n'} = 0 , \quad \int_B \frac{\partial u_1^{inc}}{\partial n'} dS' = 0$$

and therefore

$$\int_B \frac{\partial u_1^t}{\partial n'}\, dS' = - \int_B \hat{k}\cdot\bar{r}'\, \frac{\partial \Phi^{t(o)}}{\partial n'}\, dS' - \frac{1}{4\pi}\left(\frac{C}{\varepsilon_o}\right)^2 , \qquad (237)$$

a result which can be demonstrated alternatively by applying the reciprocity theorem to $\Phi_i^{s(1)}$ and $\Phi^{s(o)}$.

Using (237) the integral equation (231) becomes

$$\frac{1}{4\pi}\int_B \frac{1}{R}\, \frac{\partial u_2^t}{\partial n'}\, dS' = \frac{1}{2}(\hat{k}\cdot\bar{r})^2 + f(\bar{r})$$

with

$$f(\bar{r}) = \frac{1}{4\pi}\int_B \left(\frac{1}{2}R + \hat{k}\cdot\bar{r}'\right) \frac{\partial \Phi^{t(o)}}{\partial n'}\, dS' + \left(\frac{C}{4\pi\varepsilon_o}\right)^2 , \qquad (238)$$

but we can also find $\int_B (\partial u_2^t/\partial n')dS'$ knowing only $\partial\Phi^{t(o)}/\partial n$. Since $u_2^s - f(\bar{r})$ is an exterior potential,

$$\int_B \frac{\partial u_2^s}{\partial n'}\, dS' = \int_B \frac{\partial f}{\partial n'}\, dS' + \int_B \Phi^{s(o)} \frac{\partial}{\partial n'}\left\{u_2^s - f(\bar{r}')\right\} dS'$$

$$= \int_B \frac{\partial f}{\partial n'}\, dS' + \int_B \left\{u_2^s - f(\bar{r}')\right\} \frac{\partial \Phi^{s(o)}}{\partial n'}\, dS' .$$

In addition

$$\int_B \frac{\partial u_2^{inc}}{\partial n'}\, dS' = \int_B (\hat{k}\cdot\bar{r}')(\hat{k}\cdot n')dS' = V$$

and hence

$$\int_B \frac{\partial u_2^t}{\partial n'}\, dS' \;=\; V + \int_B \frac{\partial f}{\partial n'}\, dS' - \int_B \left\{ f(\bar{r}') + \frac{1}{2}(\hat{k}\cdot\bar{r}')^2 \right\} \frac{\partial \Phi^{t(o)}}{\partial n'}\, dS' \quad . \tag{239}$$

This is as far as we can go using the zeroth order potentials alone.

From (79) and the boundary condition (6), the scattering coefficient is

$$S(\hat{r},\hat{k}) \;=\; -\int_B \frac{\partial u_0^t}{\partial n'}\, dS' + ik \int_B \left(\hat{r}\cdot\bar{r}' \frac{\partial u_0^t}{\partial n'} - \frac{\partial u_1^t}{\partial n'} \right) dS' + (ik)^2 S_2(\hat{r},\hat{k}) + O(k^3) \tag{240}$$

with

$$S_2(\hat{r},\hat{k}) \;=\; -\int_B \left\{ \frac{1}{2}(\hat{r}\cdot\bar{r}')^2 \frac{\partial u_0^t}{\partial n'} - \hat{r}\cdot\bar{r}' \frac{\partial u_1^t}{\partial n'} + \frac{\partial u_2^t}{\partial n'} \right\} dS'. \tag{241}$$

From (232), (233) and (237) it follows immediately that

$$S(\hat{r},\hat{k}) \;=\; -\frac{C}{\varepsilon_o} + ik \left\{ \frac{1}{4\pi}\left(\frac{C}{\varepsilon_o}\right)^2 + (\hat{k} - \hat{r})\cdot \int_B \bar{r}' \frac{\partial \Phi^{t(o)}}{\partial n'}\, dS' \right\} + O(k^2) \quad , \tag{242}$$

as originally derived by Van Bladel (1968). We observe that (242) involves only the zeroth order electrostatic potential $\Phi^{t(o)}$. The integral term is absent in the forward scattering direction $\hat{r} = \hat{k}$ and, in addition, the integral itself vanishes if the origin of coordinates is chosen at the center of gravity of the charge distribution—a location which is obvious for a symmetric body, but which in general can be found only when $\partial\Phi^{t(o)}/\partial n$ has been determined. In either case

$$S(\hat{r},\hat{k}) = -\frac{C}{\varepsilon_0}\left(1 - ik\frac{C}{4\pi\varepsilon_0}\right) + O(k^2) \tag{243}$$

and the electrostatic capacity alone now specifies two terms in the expansion.

The term $O(k^2)$ in (240) intrinsically involves u_1^t (or $\Phi_i^{t(1)}$) and from (232), (235) and (239)

$$S_2(\hat{r},\hat{k}) = \int_B \left\{ f(\bar{r}') + \frac{1}{2}(\hat{k}\cdot\bar{r}')^2 + \frac{1}{2}(\hat{r}\cdot\bar{r}')^2 + \frac{C}{4\pi\varepsilon_0}\hat{r}\cdot\bar{r}' \right\} \frac{\partial\Phi^{t(o)}}{\partial n'} dS'$$

$$- V - \int_B \frac{\partial f}{\partial n'} dS' - \sum_{i=1}^{3} (\hat{k}\cdot\hat{x}_i)\hat{r}\cdot\int_B \bar{r}' \frac{\partial\Phi_i^{t(1)}}{\partial n'} dS' .$$

This can be simplified as follows. From (133)

$$\frac{\partial\Phi_i^{t(1)}}{\partial n} = \frac{\partial\Phi_i^t}{\partial n} - \frac{\varepsilon_o}{C}\int \frac{\partial\Phi_i^{t(1)}}{\partial n'} dS' \frac{\partial\Phi^{t(o)}}{\partial n} ,$$

and by applying the reciprocity theorem to $\Phi_i^{t(1)}$ and $\Phi^{t(o)}$, we have

$$\int_B \frac{\partial\Phi_i^{t(1)}}{\partial n'} dS' = \int_B x_i' \frac{\partial\Phi^{t(o)}}{\partial n'} dS' .$$

Hence

$$\sum_{i=1}^{3} (\hat{k}\cdot\hat{x}_i)\hat{r}\cdot\int_B \bar{r}' \frac{\partial\Phi_i^{t(1)}}{\partial n'} dS' = -\hat{k}\cdot\bar{\bar{P}}\cdot\hat{r}$$

$$- \frac{\varepsilon_o}{C}\left[\int_B \hat{k}\cdot\bar{r}' \frac{\partial\Phi^{t(o)}}{\partial n'} dS'\right]\left[\int_B \hat{r}\cdot\bar{r}' \frac{\partial\Phi^{t(o)}}{\partial n'} dS'\right]$$

where $\bar{\bar{P}}$ is the electric polarizability tensor (see §3.2), and the final expression for $S_2(\hat{r},\hat{k})$ is

$$S_2(\hat{r},\hat{k}) = \hat{k}\cdot\bar{\bar{P}}\cdot\hat{r} - V - \frac{C}{4\pi\varepsilon_0}\left[\frac{1}{4\pi}\left(\frac{C}{\varepsilon_0}\right)^2 + (\hat{k} - \hat{r})\cdot\int_B \bar{r}'\, \frac{\partial\Phi^{t(o)}}{\partial n'}\, dS'\right]$$
$$+ \frac{1}{2}\int_B \left\{(\hat{k}\cdot\bar{r}')^2 + (\hat{r}\cdot\bar{r}')^2\right\} \frac{\partial\Phi^{t(o)}}{\partial n'}\, dS'$$
$$+ \frac{\varepsilon_0}{C}\left[\int_B \hat{k}\cdot\bar{r}'\, \frac{\partial\Phi^{t(o)}}{\partial n'}\, dS'\right]\left[\int_B \hat{r}\cdot\bar{r}'\, \frac{\partial\Phi^{t(o)}}{\partial n'}\, dS'\right]$$
$$- \frac{1}{8\pi}\iint_B \left(\frac{\partial R}{\partial n'} - R\,\frac{\partial\Phi^{t(o)}}{\partial n'}\right)\frac{\partial\Phi^{t(o)}}{\partial n}\, dS\, dS' \quad . \qquad (244)$$

This can be evaluated knowing only the quantities involved in determining the electric dipole moment for the perfectly conducting body.

7.2 Hard Body

If the plane acoustic wave (46) illuminates an acoustically hard body, application of the boundary condition (7) to the representation (30) gives

$$\frac{1}{2\pi}\int_B u^t(\bar{r}')\, \frac{\partial}{\partial n'}\left(\frac{e^{ikR}}{R}\right) dS' = u^t(\bar{r}) - 2u^{inc}(\bar{r}) \qquad (245)$$

for $\bar{r} \in B$, which is an integral equation for u^t. For small k we can again expand the total and incident fields in powers of ik, and when the exponential is similarly expanded, the terms $O(k^0)$ are found to be

$$\frac{1}{2\pi}\int_B u_0^t(\bar{r}')\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)dS' = u_0^t(\bar{r}) - 2 \quad . \tag{246}$$

Since $\partial u_0^{inc}/\partial n = 0$, it follows that $u_0^s(\bar{r}) = 0$, and the solution of (246) is therefore $u_0^t(\bar{r}) = u_0^{inc}(\bar{r}) = 1$. The terms $O(k)$ in (245) then yield the integral equation

$$\frac{1}{2\pi}\int_B u_1^t(\bar{r}')\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)dS' = u^t(\bar{r}) - 2\hat{k}\cdot\bar{r} \tag{247}$$

and by comparison with (129), the solution of this is

$$u_1^t(\bar{r}) = -\sum_{i=1}^{3}(\hat{k}\cdot\hat{x}_i)\Psi_i^t(\bar{r}) \tag{248}$$

where Ψ_i^t is a magnetostatic potential such that $\partial\Psi_i^s/\partial n = \hat{n}\cdot\hat{x}_i$ on B.

From (42) and the boundary condition (7), the scattering coefficient is

$$S(\hat{r},\hat{k}) = -ik\int_B \hat{n}'\cdot\hat{r}\ u^t(\bar{r}')\ e^{-ik\hat{r}\cdot\bar{r}'}\ dS' \tag{249}$$

and since $u_0^t(\bar{r}) = 1$, expansion of the integral in powers of ik gives

$$S(\hat{r},\hat{k}) = k^2\int_B [u^t(\bar{r}') - \hat{r}\cdot\bar{r}'](\hat{n}'\cdot\hat{r})dS' + O(k^3) \quad . \tag{250}$$

The integral is easily evaluated. By applying the divergence theorem

$$\int_B (\hat{r}\cdot\bar{r}')(\hat{n}'\cdot\hat{r})\ dS' = V$$

and, from (248),

$$\int_B u^t(\bar{r}')\hat{n}'\cdot\hat{r}\, dS' = -\sum_{i=1}^{3} (\hat{k}\cdot\hat{x}_i) \int_B \Psi_i^t(\bar{r}')\hat{n}'\cdot\hat{r}\, dS'$$

$$= \hat{k}\cdot\bar{\bar{M}}\cdot\hat{r} \tag{251}$$

where $\bar{\bar{M}}$ is the magnetic polarizability tensor defined in §3.2. Hence

$$S(\hat{r},\hat{k}) = k^2(\hat{k}\cdot\bar{\bar{M}}\cdot\hat{r} - V) + O(k^3) , \tag{252}$$

and we remark that the magnetostatic potentials Ψ_i^t necessary to compute $\bar{\bar{M}}$ are also sufficient (Van Bladel, 1968) to determine the next term (proportional to k^3) in the expansion of $S(\hat{r},\hat{k})$.

7.3 Penetrable Body

If the plane acoustic wave (46) illuminates a penetrable body, the low frequency expansions are

$$u = \sum_{n=o}^{\infty} (ik)^n u_n \qquad \text{for } \bar{r} \in \text{ext } B$$

and (253)

$$u^t = \sum_{n=o}^{\infty} (ik)^n \nu^n u_n^t \qquad \text{for } \bar{r} \in \text{int } B ,$$

where the factor

$$\nu = k_1/k \tag{254}$$

appears because the field in int B is analytic in k_1 rather than k. In ext B, u and u_n may be taken to be either u^t and u_n^t, u^s and u_n^s or u^{inc} and u_n^{inc}. The transmission conditions (10) and (11) give

$$u^t_{n_+} = \nu^n u^t_{n_-} \quad \text{and} \quad \frac{\partial u^t_n}{\partial n_+} = \rho\nu^n \frac{\partial u^t_n}{\partial n_-} \tag{255}$$

and the Helmholtz equations imply

$$\nabla^2 u_n = u_{n-2} \quad , \tag{256}$$

where u_n may be u^t_n, u^s_n or u^{inc}_n, and $u_{-2} = u_{-1} = 0$.

When these results are used in conjunction with the integral representation (30) it is easy to verify that u^s_0 and u^s_1 are both solutions of Laplace's equation which are regular at infinity, whereas u^t_0 and u^t_1 are regular solutions of Laplace's equation in int B. In fact, the zeroth-order term is trivially found to be

$$u^t_0 = 1 \quad , \tag{257}$$

and for u_1 the transmission conditions (255) yield

$$\begin{aligned} u^t_{1_+} &= \hat{k}\cdot\bar{r} + u^s_1 = \nu u^t_{1_-} \\ \frac{\partial u^t_1}{\partial n_+} &= \hat{n}\cdot\hat{k} + \frac{\partial u^s_1}{\partial n_+} = \rho\nu \frac{\partial u^t_1}{\partial n_-} \quad . \end{aligned} \tag{258}$$

With the notation of Section 5 (c.f. (175) and (176)) it follows that

$$u^s_1 = -\sum_{j=1}^{3} \hat{k}\cdot\hat{x}_j \Phi^s_j \quad , \qquad \bar{r} \in \text{ext B} \tag{259}$$

and

$$\nu u_1^t = -\sum_{j=1}^{3} \hat{k}\cdot\hat{x}_j \Phi_j \quad , \qquad \bar{r} \in \text{int } B \tag{260}$$

and these may be employed in (83) to obtain the far field coefficient. Since

$$S_0(\hat{r},\hat{k}) = -\int_B \frac{\partial u_0^t}{\partial n_+'} \, dS' \quad ,$$

(257) implies

$$S_0(\hat{r},\hat{k}) = 0 \quad . \tag{261}$$

The next term is

$$\begin{aligned} S_1(\hat{r},\hat{k}) &= -\int_B \left\{ \hat{n}'\cdot\hat{r}\; u_0^t - \hat{r}\cdot\bar{r}' \frac{\partial u_0^t}{\partial n_+'} + \frac{\partial u_1^t}{\partial n_+'} \right\} dS' \\ &= -\int_B \left\{ \hat{n}'\cdot\hat{r} + \rho\nu \frac{\partial u_1^t}{\partial n_-'} \right\} dS' \end{aligned}$$

which also vanishes on using the divergence theorem and (256). Thus

$$S_1(\hat{r},\hat{k}) = 0 \quad . \tag{262}$$

The first nonzero term is

$$\begin{aligned} S_2(\hat{r},\hat{k}) = \int_B \Bigg\{ & (\hat{n}'\cdot\hat{r})(\hat{r}\cdot\bar{r}')u_0^t - \hat{n}'\cdot\hat{r}\; u_{1_+}^t - \frac{1}{2}(\hat{r}\cdot\bar{r}')^2 \frac{\partial u_0^t}{\partial n_+'} \\ & + \hat{r}\cdot\bar{r}' \frac{\partial u_1^t}{\partial n_+'} - \frac{\partial u_2^t}{\partial n_+'} \Bigg\} dS' \end{aligned}$$

$$= \int_B \left\{ (\hat{n}'\cdot\hat{r})(\hat{r}\cdot\bar{r}') - \rho\nu^2 \frac{\partial u_2^t}{\partial n'_-} - \hat{n}'\cdot\hat{r}\left(\hat{k}\cdot\bar{r}' - \sum_{j=1}^{3} \hat{k}\cdot\hat{x}_j \Phi_j^s\right) \right.$$

$$\left. + \hat{r}\cdot\bar{r}'\left(\hat{n}'\cdot\hat{k} - \sum_{j=1}^{3} \hat{k}\cdot\hat{x}_j \frac{\partial \Phi_j^s}{\partial n'_+}\right)\right\} dS' \quad . \tag{263}$$

Using the divergence theorem it can be shown that

$$\int_B \left\{ (\hat{n}'\cdot\hat{r})(\hat{r}\cdot\bar{r}') - (\hat{n}'\cdot\hat{r})(\hat{k}\cdot\bar{r}') + (\hat{n}'\cdot\hat{k})(\hat{r}\cdot\bar{r}') \right\} dS' = V$$

and, with (256),

$$\int_B \frac{\partial u_2^t}{\partial n'_-} dS' = \int_{\text{int } B} \nabla^2 u_2^t \, dV = \int_{\text{int } B} u_0^t \, dV = V \quad .$$

Thus

$$S_2(\hat{r},\hat{k}) = (1 - \rho\nu^2)V + \sum_{i,j=1}^{3} (\hat{r}\cdot\hat{x}_i)(\hat{k}\cdot\hat{x}_j) \int_B \left\{ \hat{n}'\cdot\hat{x}_i \Phi_j^s - x_i' \frac{\partial \Phi_j^s}{\partial n'} \right\} dS' \tag{264}$$

and by using the polarizability tensor (174) with τ replaced by ρ,

$$S_2(\hat{r},\hat{k}) = (1 - \rho\nu^2)V + \sum_{i,j=1}^{3} (\hat{r}\cdot\hat{x}_i)(\hat{k}\cdot\hat{x}_j)X_{ij} \quad . \tag{265}$$

The far field coefficient is therefore

$$S(\hat{r},\hat{k}) = -k^2 \left\{ (1 - \rho\nu^2)V + \hat{r}\cdot\bar{\bar{X}}(\rho)\cdot\hat{k} \right\} + O(k^3) \quad . \tag{266}$$

The polarizability tensor $\bar{\bar{X}}$ is symmetric and all of the results of Section 5 are directly applicable. When the interior density becomes infinite, $\rho \to 0$ (see (12)), and since $\bar{\bar{X}}(0) = -\bar{\bar{M}}$, the above expression reduces to the hard body result (252).

The formulas of this section as well as those of §§7.1 and 7.2 have been derived by Jones (1979b), who also obtained expressions for the k^3 term. It should be noted that the tensor elements C_{ij} defined by Jones are related to those presented here by

$$C_{ij} = -\frac{1}{(1-\rho)^2} X_{ij} - \frac{1}{1-\rho} V\delta_{ij} \quad . \tag{267}$$

7.4 Analytical and Numerical Results for a Soft Body

The preceding results make evident the mathematical similarity of the electromagnetic and acoustic scattering problems at low frequencies, and the results (242) and (252) are also valid for a plate or open shell of infinitesimal thickness and volume. In the far field of a soft body, the first three terms in the low frequency expansion can all be expressed in terms of the potential functions involved in computing the electric dipole moment for the corresponding perfectly conducting body, and for a hard body the first two terms are similarly related to the magnetic dipole moment.

The higher order terms provide more detail about the scattering and, in the case of a soft body, are necessary to reveal any dependence on the directions of incidence and observation. However, little information about them is available. Numerical studies have focussed almost exclusively on the leading terms alone, and for those simple bodies which are amenable to analytical solution, the natural choice of the origin of coordinates eliminates the integral term in (242) and the term proportional to k^3 in (252).

For a soft ellipsoid, Williams (1971) has determined the cross section in terms of Lamé functions, and from his analysis it is possible to extract expressions for the terms of the first three orders in (242). A special case is that of a spheroid, and for axial incidence ($\hat{k} = -\hat{z}$) Senior (1960) developed the low frequency expansion through terms $O(k^5)$ for soft and hard bodies. In particular, for a soft sphere of radius a,

$$S(\hat{r},\hat{k}) = -4\pi a \left\{ 1 - ika + (ika)^2 \left(\frac{2}{3} + \hat{r}\cdot\hat{z} \right) + O(k^3) \right\} \qquad (268)$$

(Bowman et al, 1969). Another special case of an ellipsoid is an elliptical disk, and the solution for normal incidence on a soft disk has been obtained through terms $O(k^2)$ by Williams (1970).

To the leading term in the far field, the scattering of a hard body is determined by the magnetic polarizability tensor $\bar{\bar{M}}$ and the volume, and data for the tensor elements M_{ij} have been discussed in §3.4 for a solid body of non-zero volume and in §4.2 for a flat plate. It is therefore sufficient to confine attention to a soft body for which the determining factor (see (243)) is the electrostatic capacity.

Of all the electromagnetic parameters for a body, it is probable that the capacity has been the most widely studied. Analytical expressions are available for a number of solid and planar configurations. Many other geometries have been treated numerically, and a variety of bounds have been established or proposed. It is therefore impossible to do more than cite a few of the results.

For a prolate spheroid with interfocal distance 2d and radial spheroidal variable ξ

$$\frac{C}{\varepsilon_o} = 4\pi d \left\{\frac{1}{2} \log \frac{\xi + 1}{\xi - 1}\right\}^{-1} \tag{269}$$

(Senior, 1973), leading to the well-known result

$$\frac{C}{\varepsilon_o} = 4\pi a \tag{270}$$

for a sphere of radius a. The corresponding formula for an oblate spheroid is

$$\frac{C}{\varepsilon_o} = 4\pi d \left\{\tan^{-1} \frac{1}{\xi}\right\}^{-1} , \tag{271}$$

reducing to

$$\frac{C}{\varepsilon_o} = 8a \tag{272}$$

for a circular disk of radius a. For two spheres of radius a in contact

$$\frac{C}{\varepsilon_o} = 4\pi a \sum_{n=0}^{\infty} \left(\frac{1}{n + 1/2} - \frac{1}{n + 1} \right) = 17.421 \text{ a} \quad . \qquad (273)$$

For a solid hemisphere of radius a

$$\frac{C}{\varepsilon_o} = 8\pi a \left(1 - \frac{1}{\sqrt{3}} \right) = 10.622 \text{ a} \qquad (274)$$

and for the corresponding hemispherical bowl of infinitesimal thickness (and volume)

$$\frac{C}{\varepsilon_o} = 4\pi a \left(1 + \frac{\pi}{2} \right) = 32.306 \text{ a} \qquad (275)$$

(Schiffer and Szegö, 1949).

Since C/ε_o has the dimensions of length, a possible normalized factor is $4\pi(3V/4\pi)^{1/3}$. As conjectured by Poincaré (1903) and subsequently proved by Szegö (1930), a sphere has the minimum electrostatic capacity of all solids of equal volume. The capacity normalized in the above manner therefore has a lower bound of unity and is, in fact, just the (normalized) equivalent radius $\tilde{a}$, i.e., the ratio of the radius of a sphere having the same capacity as the body to the radius of the sphere having the same volume.

The equivalent radius for prolate and oblate spheroids is plotted as a function of the length-to-width ratio ℓ/w (see §3.4) in Fig. 12. For two spheres in contact, $\tilde{a}$ = 1.100 with ℓ/w = 2, and for a hemisphere, $\tilde{a}$ = 1.065 with ℓ/w = 0.5, and these data are included in Fig. 12. As $\ell/w \to 0$, $\tilde{a} \to \infty$, but this is due to the vanishing of the normalization factor rather than to any intrinsic property of C/ε_0. As $\ell/w \to \infty$, however, C/ε_0 becomes infinite logarithmically, and because of this there is no simple geometric quantity with which to normalize the capacity to produce a variation within finite nonzero bounds.

A normalization factor that is sometimes convenient has been proposed by Senior (1973) and is $L = \pi(\ell + w)$, i.e., 2π times the average of the body's length and width. For a spheroid $C/(\varepsilon_0 L)$ is a monotonically decreasing function of ℓ/w, decreasing from $4/\pi$ for a thin disk ($\ell/w = 0$), through 1 for a sphere, to 0 for a long thin spheroid ($\ell/w = \infty$), and this quantity is plotted in Fig. 13. As proved by Szegö (1931),

$$\frac{C}{\varepsilon_0} \leq 2\pi D \tag{276}$$

where D is the maximum separation of any two points on the surface, and since $D < \ell + w$, it follows that

$$\frac{C}{\varepsilon_0 L} < 2 \quad . \tag{277}$$

The bound is not an optimum one.

For an arbitrary axially-symmetric body, the program (Senior and Ahlgren, 1972, 1973) written to compute the elements of the electric and magnetic polarizability tensors also furnishes the capacity C/ε_0. To illustrate the results obtained, the normalized capacity $C/(\varepsilon_0 L)$ for various rounded cones, ogives and lenses (see §3.4) and right circular cylinders have been included in Fig. 13. It is evident that the spheroid provides a reasonable approximation to the data.

For the "scooped-out" hemisphere shown in Fig. 5, $\ell/w = 0.5$ and $L = 3\pi$ regardless of Δ, and the computed values of $C/(\varepsilon_0 L)$ decrease monotonically from 1.127 for $\Delta = 1$ (complete hemisphere) to 1.084 for $\Delta = 0.05$. The corresponding (oblate) spheroid has $C/(\varepsilon_0 L) = 1.053$. Additional data for two identical spheres joined by a wire along the axis, and for two identical hemispheres, plane sides facing and similarly joined, are given in Fig. 14, and in these cases the spheroid simulation is increasingly in error as ℓ/w get larger.

The capacity of a flat plate has also been widely studied and is of interest in connection with transmission through the complementary aperture in an acoustically hard screen. Since the thickness and, hence, the volume are both zero, the preceding results are irrelevant to a plate, but as conjectured by Lord Rayleigh (1896) and proved by Polya and Szegö (1951), of all plates of a given area A the circular disk has the minimum capacity. Thus, from (272),

$$\frac{C}{\varepsilon_0} \geq 8\left(\frac{A}{\pi}\right)^{1/2} \tag{278}$$

with equality (of course) for the circle.

The behavior of C under various types of symmetrization has been studied by many authors and, in particular, it has been shown (Polya and Szegö, 1951) that symmetrization with respect to a line never increases the capacity. Since the process does not increase the perimeter of the plate, it supports the often-proposed but still unproved conjecture that of all plates with a given perimeter P the circular disk has the largest capacity. If this is true, then (Jaggard and Papas, 1968)

$$8r \leq \frac{C}{\varepsilon_0} \leq 8r' \tag{279}$$

where

$$r = \left(\frac{A}{\pi}\right)^{1/2} \tag{280}$$

is the inner (area) radius and

$$r' = \frac{P}{2\pi} \tag{281}$$

is the outer (perimeter) radius. A convenient normalization factor for the capacity is therefore 8r. The capacity so normalized is never less than unity and, if the above-mentioned conjecture is valid, is bounded above by r'/r.

One of the few geometries for which an exact expression for the capacity is available is the elliptical disk for which

$$\frac{C}{\varepsilon_o} = \frac{4\pi a}{K(e^2)} \tag{282}$$

where a is the semi major axis, e is the ellipticity and $K(e^2)$ is the complete elliptic integral of the first kind. Since

$$A = \pi a^2(1 - e^2)^{1/2} \text{ and } P = 4aE(e^2)$$

where $E(e^2)$ is the complete elliptic integral of the second kind,

$$r = a(1 - e^2)^{1/4} \text{ and } r' = \frac{2a}{\pi} E(e^2) \quad . \tag{283}$$

Figure 15 shows the normalized capacity $C/(8r\varepsilon_o)$ as a function of the length-to-width ratio $\ell/w = (1 - e^2)^{-1/2}$ for $1 \leq \ell/w \leq 10$. According to (279) the upper bound is

$$\frac{r'}{r} = \frac{2}{\pi}(1 - e^2)^{-1/4} E(e^2) \quad ,$$

and is clearly satisfied. For an arbitrary plate of overall maximum dimension ℓ and largest perpendiuclar dimension w, these data can be used to estimate the capacity.

During the last few years several programs have been written to solve the integral equation (229) for a flat plate and, hence, compute the capacity. One of the more accurate is that of Okon and Harrington (1979) who have obtained data for a variety of shapes including the elliptical and circular disks, a square, an isosceles triangle and a regular hexagon. To judge from the disk data, the accuracy of the "asymptotic" values is about one percent. For a square it is found that

$$\frac{C}{8r\varepsilon_o} = 1.019$$

(cf the value 1.113 obtained by Maxwell, 1879), and this is consistent with the upper bound r'/r = 1.128. For the triangle

$$\frac{C}{8r\varepsilon_o} = 1.082$$

compared with the upper bound r'/r = 1.362, but for the hexagon

$$\frac{C}{8r\varepsilon_o} = 0.981$$

which violates the rigorous lower bound. It would therefore appear that even for convex shapes the accuracy of existing data is not sufficient for a detailed study of how the shape affects the capacity.

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LEGENDS FOR FIGURES

Fig. 1: Geometry.

Fig. 2: Normalized tensor element P_{11}/V as a function of the length-to-width ratio of the body for prolate and oblate spheroids (——), rounded cones of half-angle $\theta < \pi/2$ (----) and $\theta > \pi/2$ (— —), and ogives and lenses (⊙ ⊙).

Fig. 3: Normalized tensor element P_{33}/V as a function of the length-to-width ratio of the body for prolate and oblate spheroids (——), rounded cones of half-angle $\theta < \pi/2$ (----) and $\theta > \pi/2$ (— —), and ogives and lenses (⊙ ⊙).

Fig. 4: Normalized tensor element M_{11}/V as a function of the length-to-width ratio of the body for prolate and oblate spheroids (——), rounded cones of half-angle $\theta < \pi/2$ (----) and $\theta > \pi/2$ (— —), and ogives and lenses (⊙ ⊙).

Fig. 5: 'Scooped-out' hemisphere ($\Delta = 1 - \tan \theta/2$).

Fig. 6: Normalized tensor elements for elliptical disks, where $N = 8r^3/3$.

Fig. 7: Normalized tensor element for elliptical disks, where $N = 8r^3/3$. The element is equal to its postulated lower bound.

Fig. 8: $X_{11}(\tau)/V$ for a spheroid as a function of the length-to-width ratio ℓ/w for different τ.

Fig. 9: $X_{33}(\tau)/V$ for a spheroid as a function of the length-to-width ratio ℓ/w for different τ.

Fig. 10: The tensor element $X_{11}(\tau)$ normalized to that of a spheroid of same volume and material parameter τ. $A_{11} = X_{11}$ (rectangular parallelepiped)/X_{11} (spheroid).

Fig. 11: The tensor element $X_{33}(\tau)$ normalized to that of a spheroid of same volume and material parameter τ. $A_{33} = X_{33}$ (rectangular parallelpiped)/X_{33} (spheroid).

Fig. 12: Equivalent radius $a = (1/4\pi)(4\pi/3V)^{1/3}\ C/\varepsilon_0$ for a spheroid (——), for a hemisphere (⊙) and for two spheres in contact (×).

Fig. 13: Normalized capacity $C/\varepsilon_0 L$ for spheroids (——), rounded cones (— —,----), ogives and lenses (⊙ ⊙) and right circular cylinders (× ×) (see §3.4).

Fig. 14: Normalized capacity $C/\varepsilon_0 L$ for prolate spheroids (——), two identical joined spheres (⊙ ⊙), and two joined hemispheres back-to-back (× ×).

Fig. 15: Normalized capacity for an elliptical plate (———) compared with the upper bound r'/r (— —).

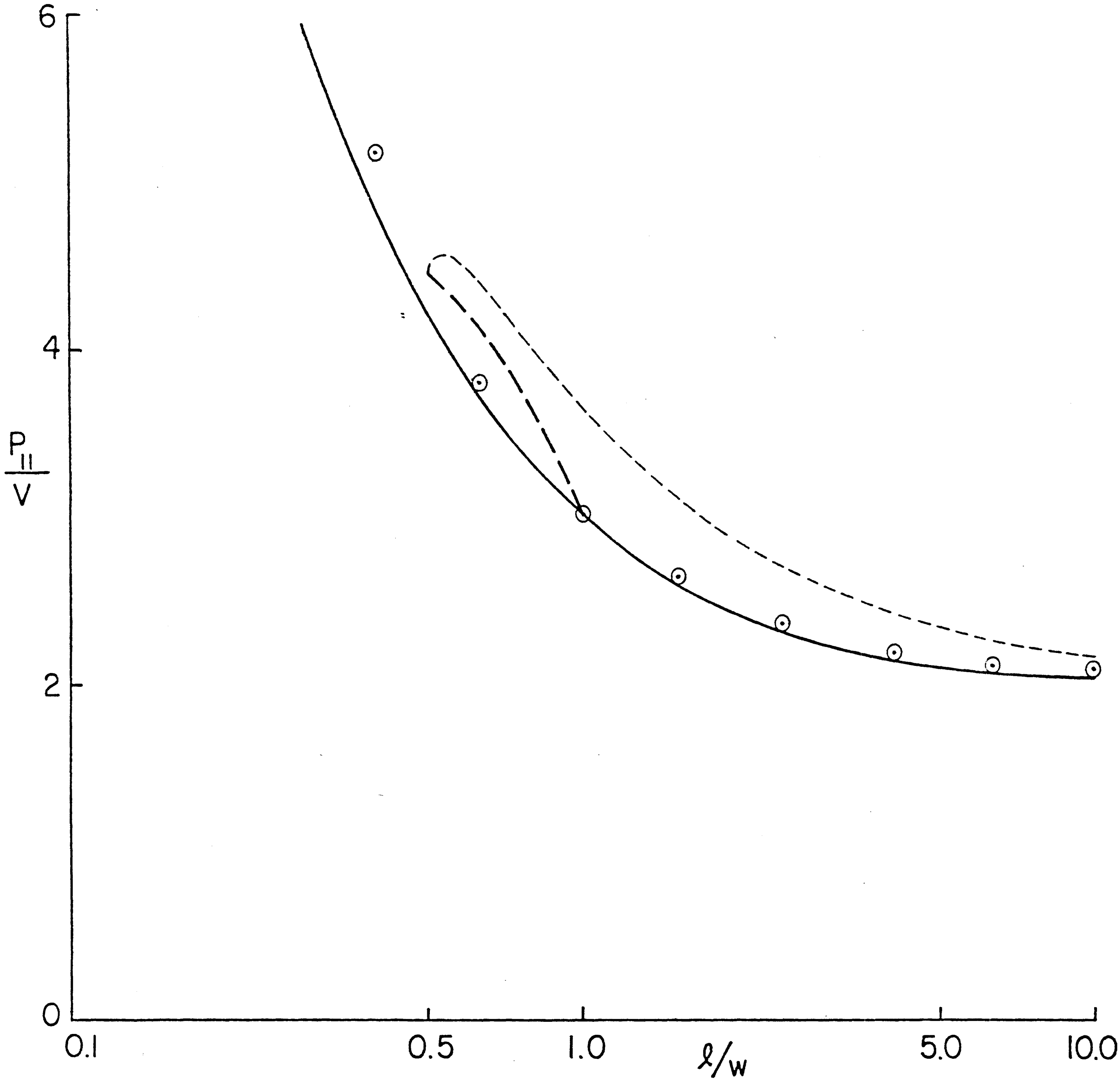


Fig. 2: Normalized tensor element P_{11}/V as a function of the length-to-width ratio of the body for prolate and oblate spheroids (——), rounded cones of half-angle $\theta < \pi/2$ (----) and $\theta > \pi/2$ (— —), and ogives and lenses (⊙ ⊙).

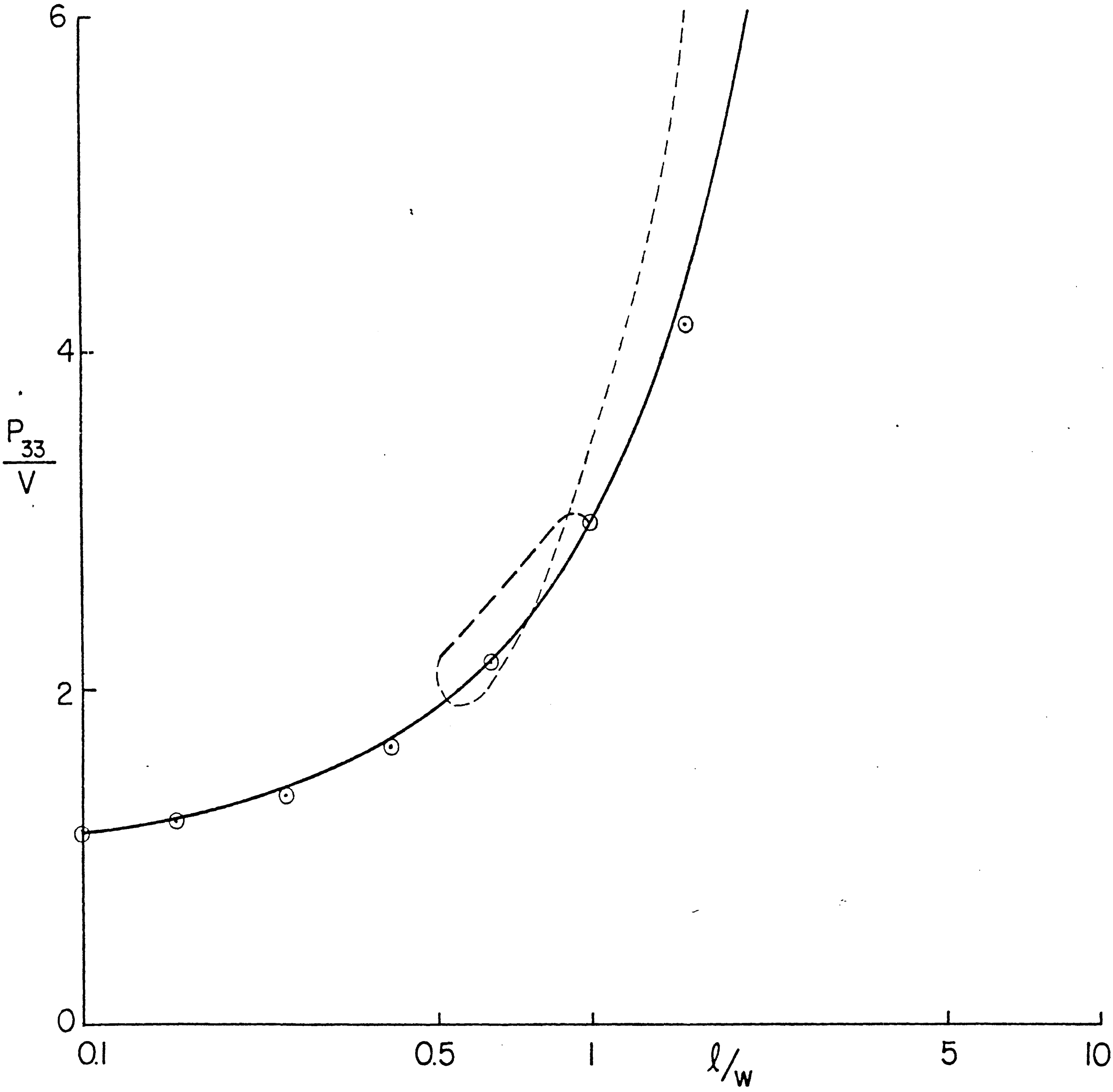


Fig. 3: Normalized tensor element P_{33}/V as a function of the length-to-width ratio of the body for prolate and oblate spheroids (——), rounded cones of half-angle $\theta < \pi/2$ (----) and $\theta > \pi/2$ (— —), and ogives and lenses (⊙ ⊙).

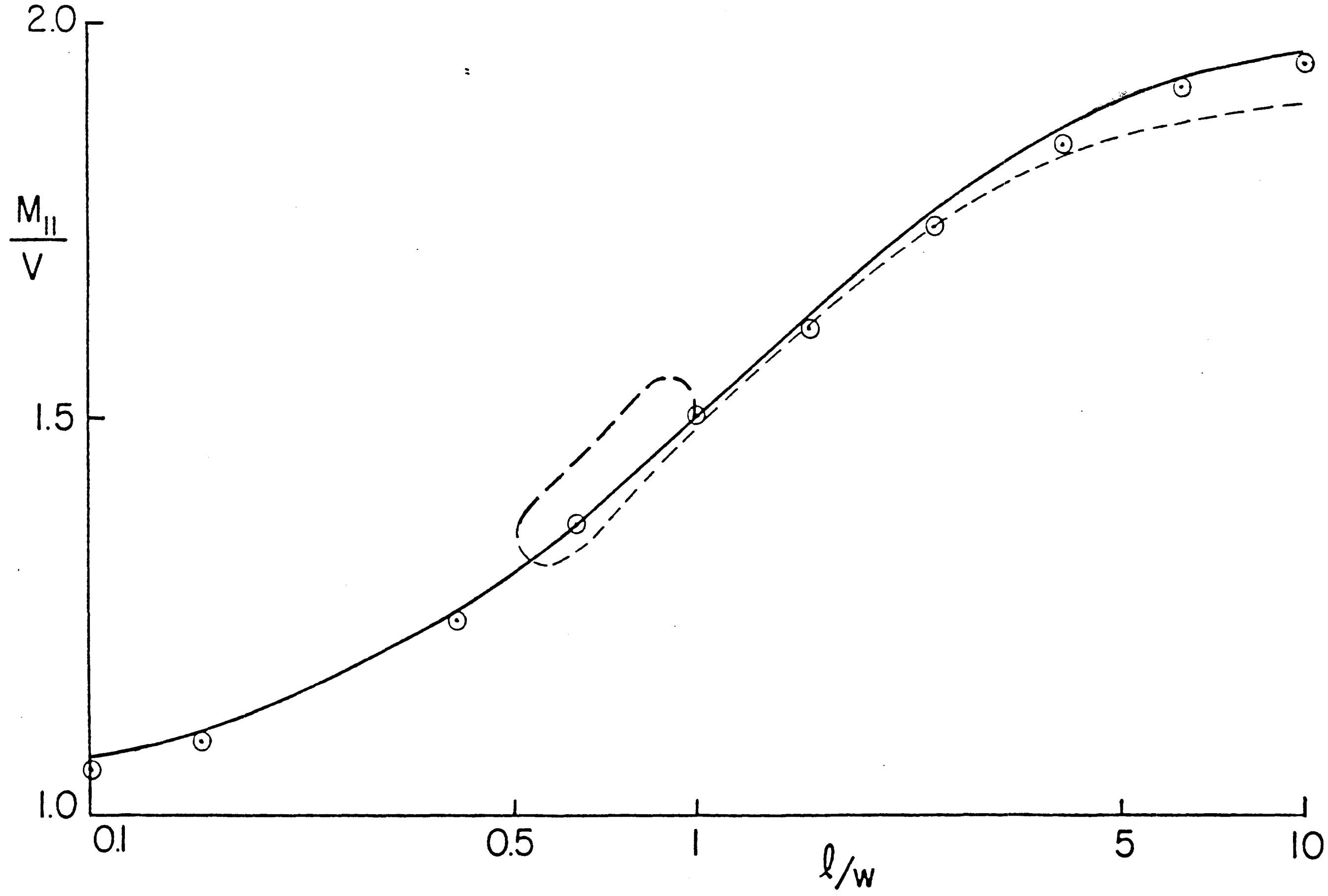


Fig. 4: Normalized tensor element M_{11}/V as a function of the length-to-width ratio of the body for prolate and oblate spheroids (——), rounded cones of half-angle $\theta < \pi/2$ (- - - -) and $\theta > \pi/2$ (— —), and ogives and lenses (⊙ ⊙).

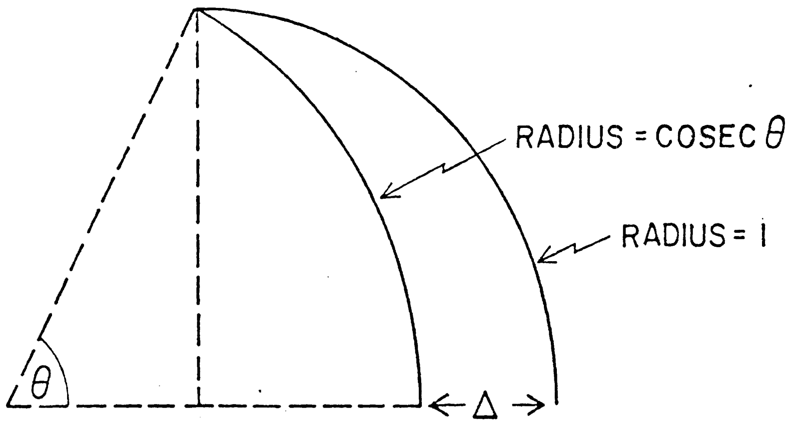


Fig. 5: 'Scooped-out' hemisphere ($\Delta = 1 - \tan\theta/2$).

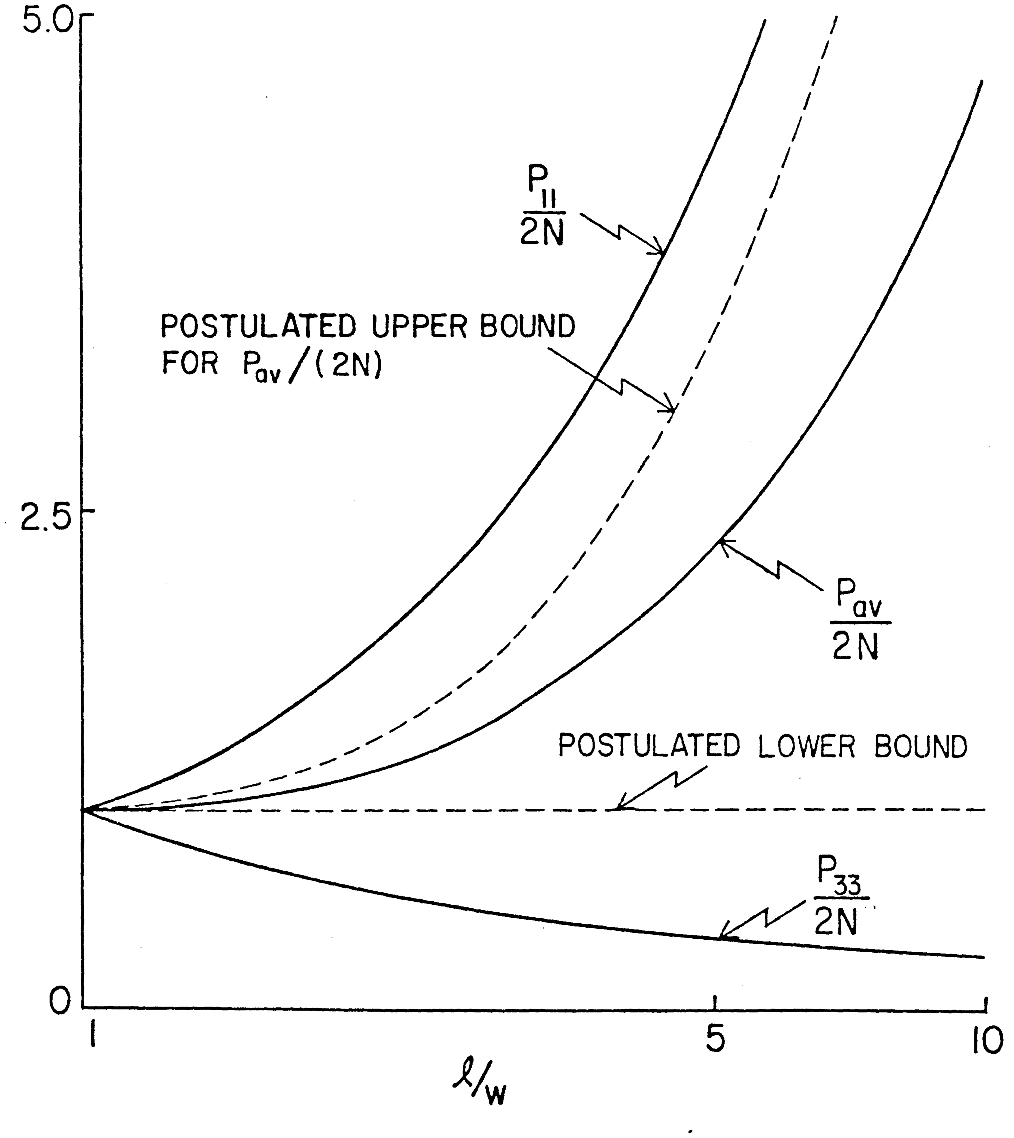


Fig. 6: Normalized tensor elements for elliptical disks, where $N = 8r^3/3$.

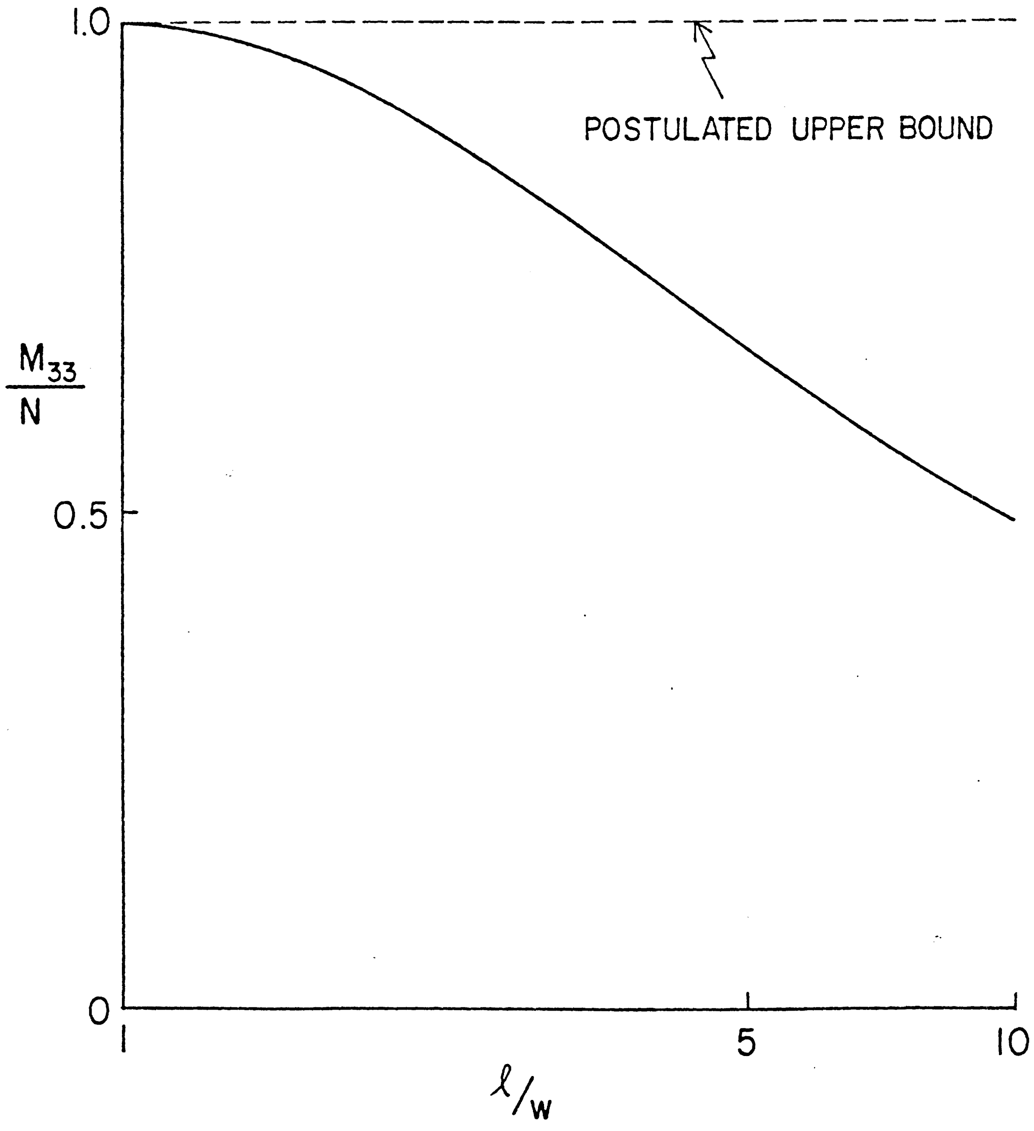


Fig. 7: Normalized tensor element for elliptical disks, where $N = 8r^3/3$. The element is equal to its postulated lower bound.

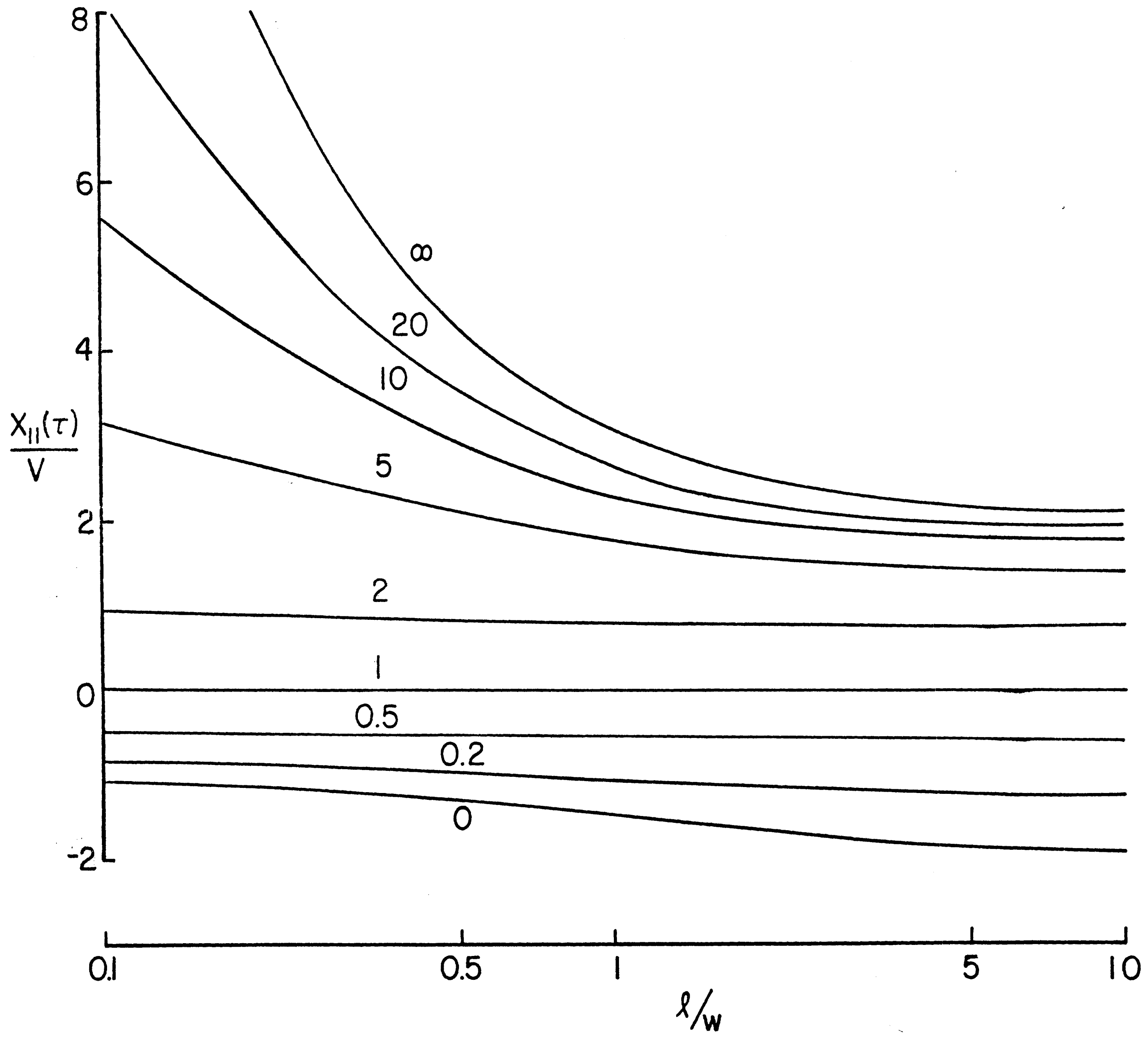


Fig. 8: $X_{11}(\tau)/V$ for a spheroid as a function of the length-to-width ratio ℓ/w for different τ.

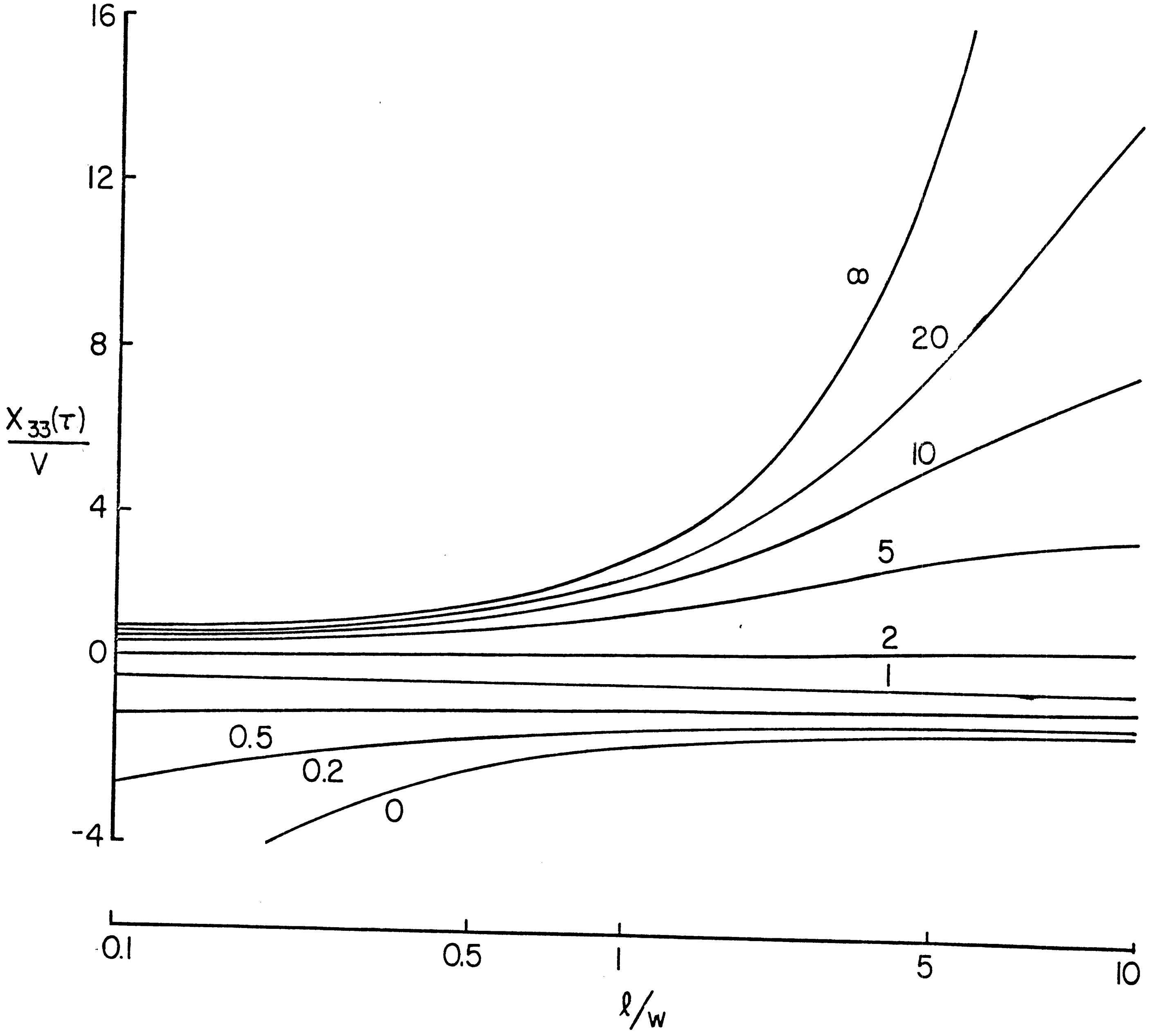
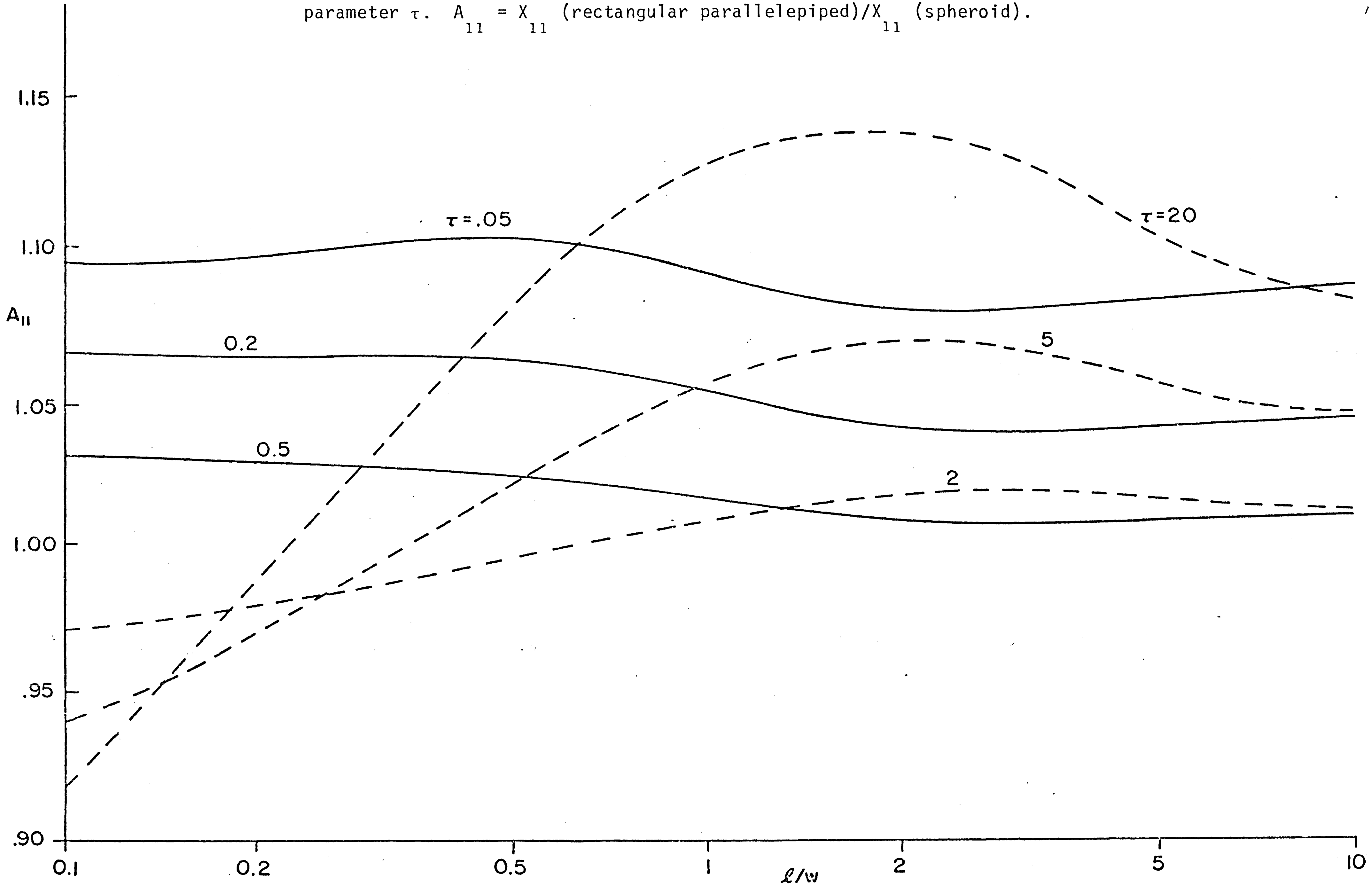
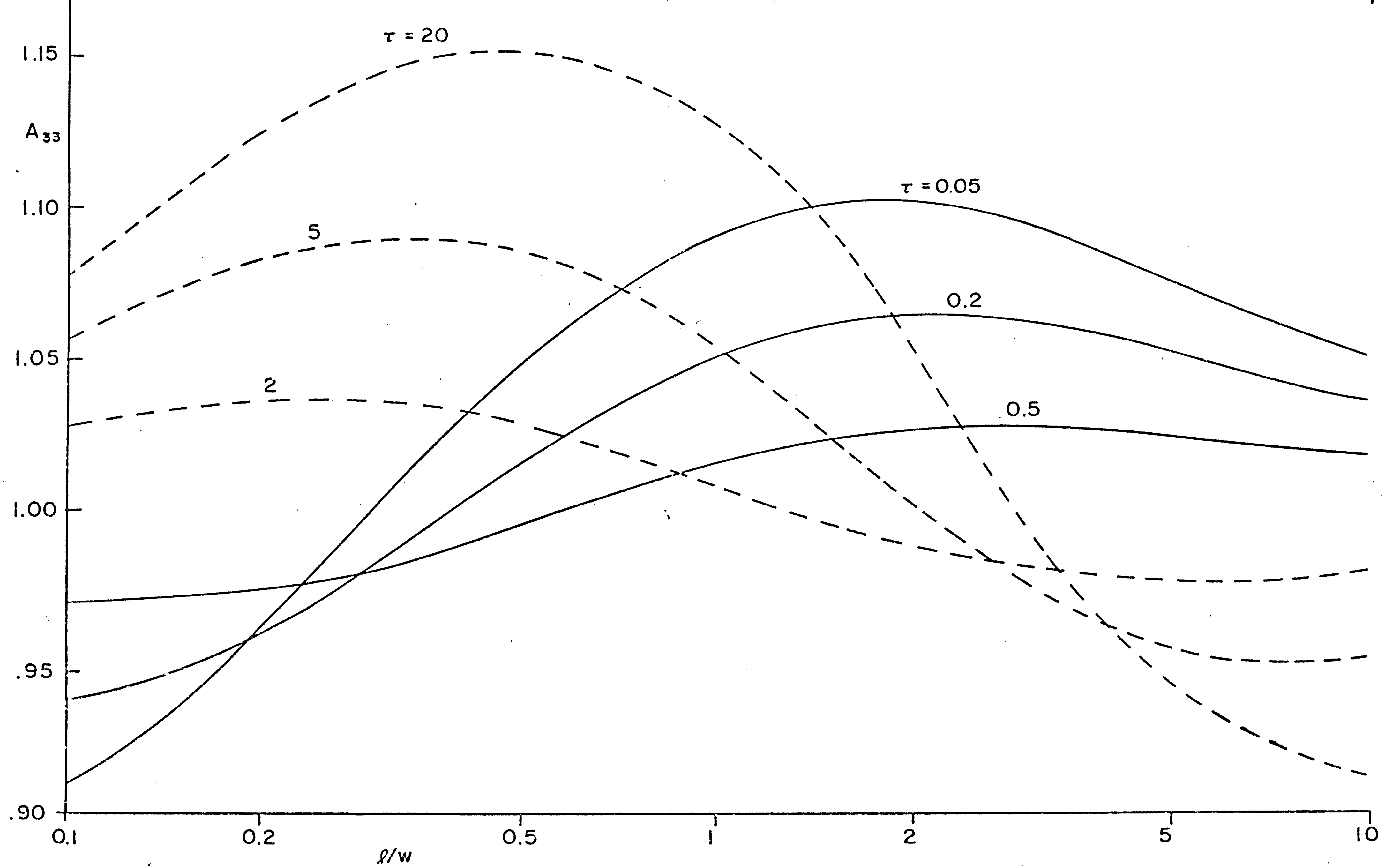


Fig. 9: $X_{33}(\tau)/V$ for a spheroid as a function of the length-to-width ratio ℓ/w for different τ.

Fig. 10: The tensor element $X_{11}(\tau)$ normalized to that of a spheroid of same volume and material parameter τ. $A_{11} = X_{11}$ (rectangular parallelepiped)/X_{11} (spheroid).



parameter τ. $A_{33} = X_{33}$ (rectangular parallelepiped)/X_{33} (spheroid).



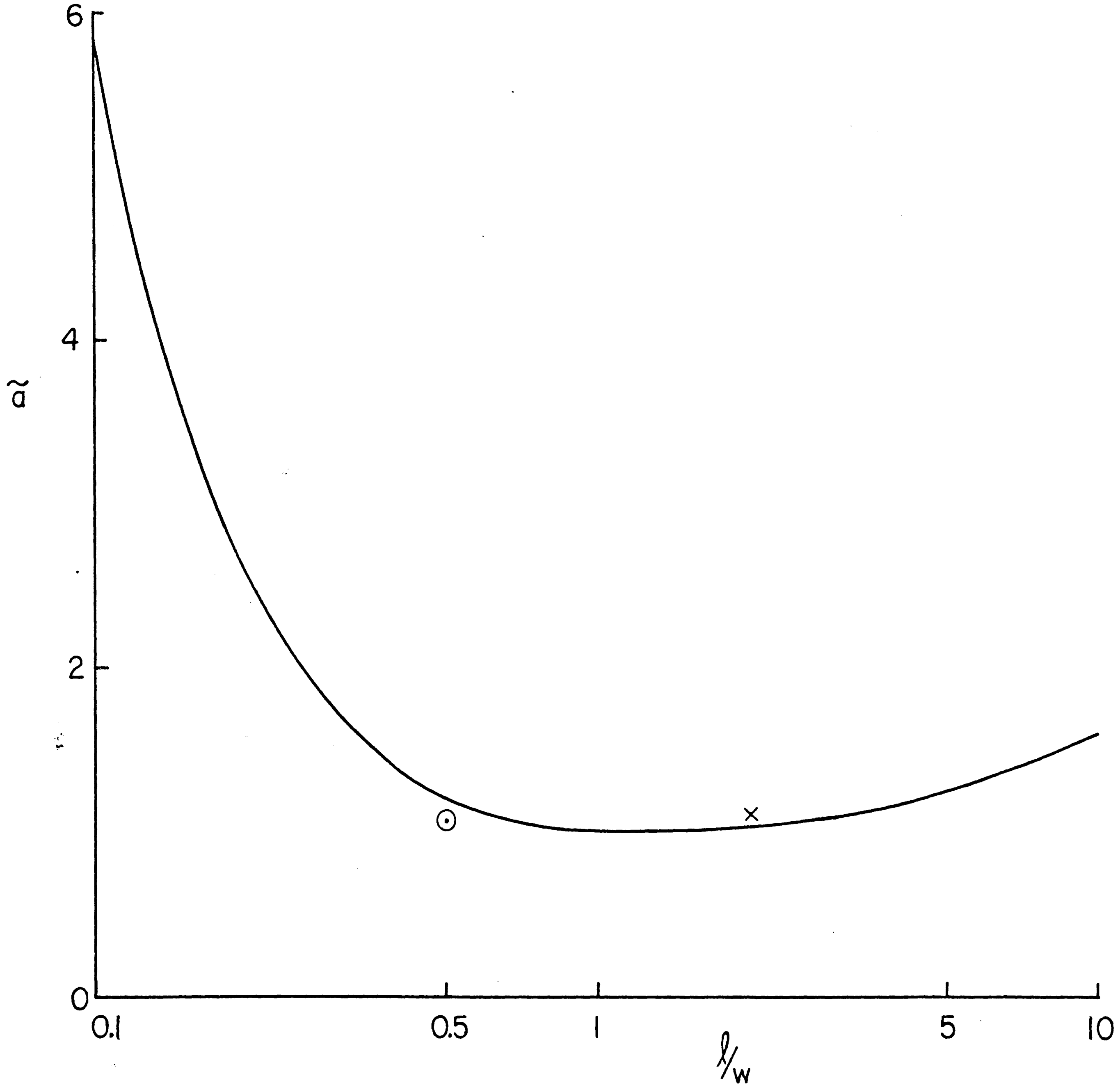


Fig. 12: Equivalent radius $\tilde{a} = (1/4\pi)(4\pi/3V)^{1/3}\ C/\varepsilon_0$ for a spheroid (——), for a hemisphere (⊙) and for two spheres in contact (×).

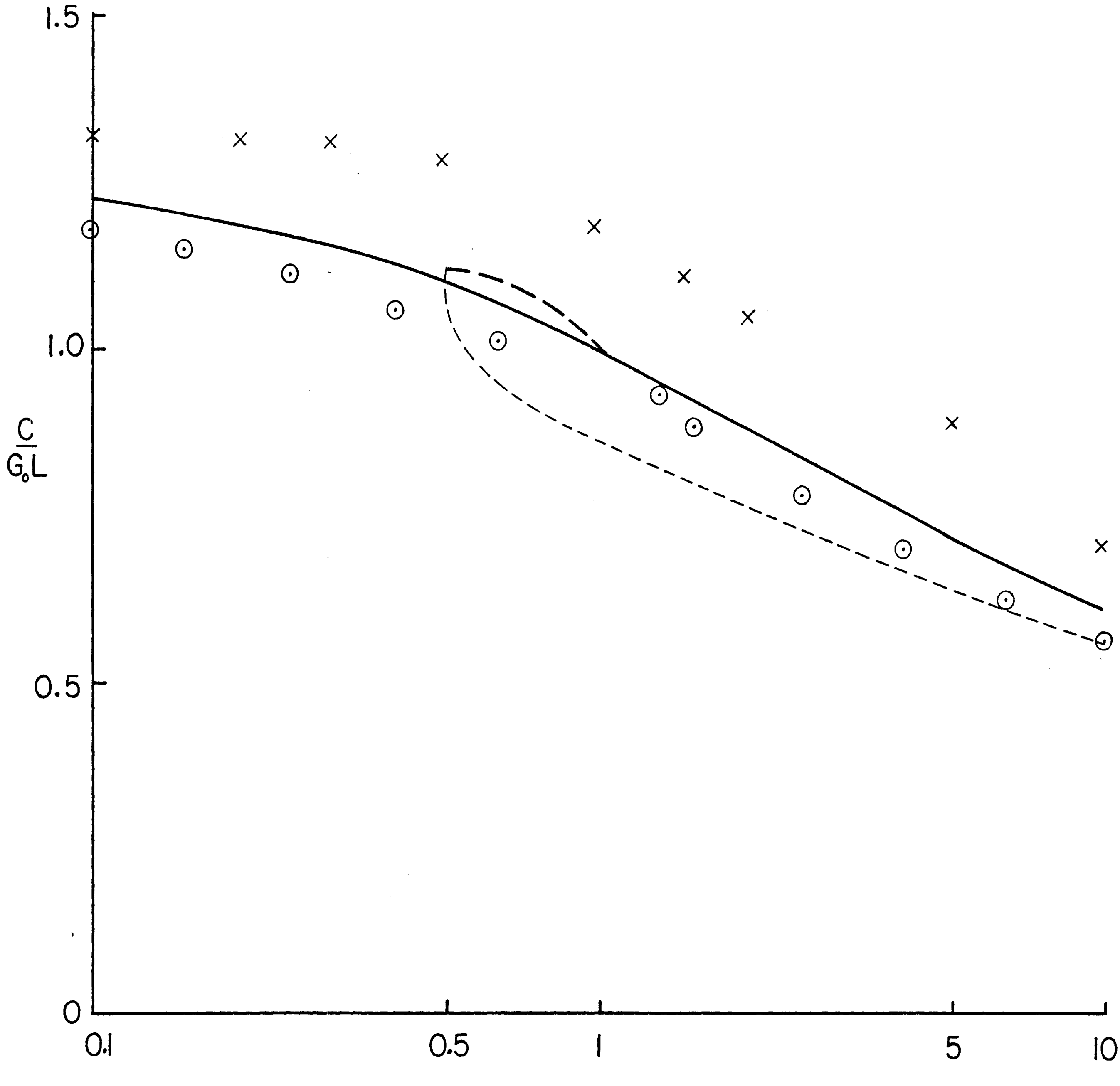


Fig. 13: Normalized capacity $C/\varepsilon_0 L$ for spheroids (——), rounded cones (— —,- - -), ogives and lenses (⊙ ⊙) and right circular cylinders (× ×) (see §3.4).

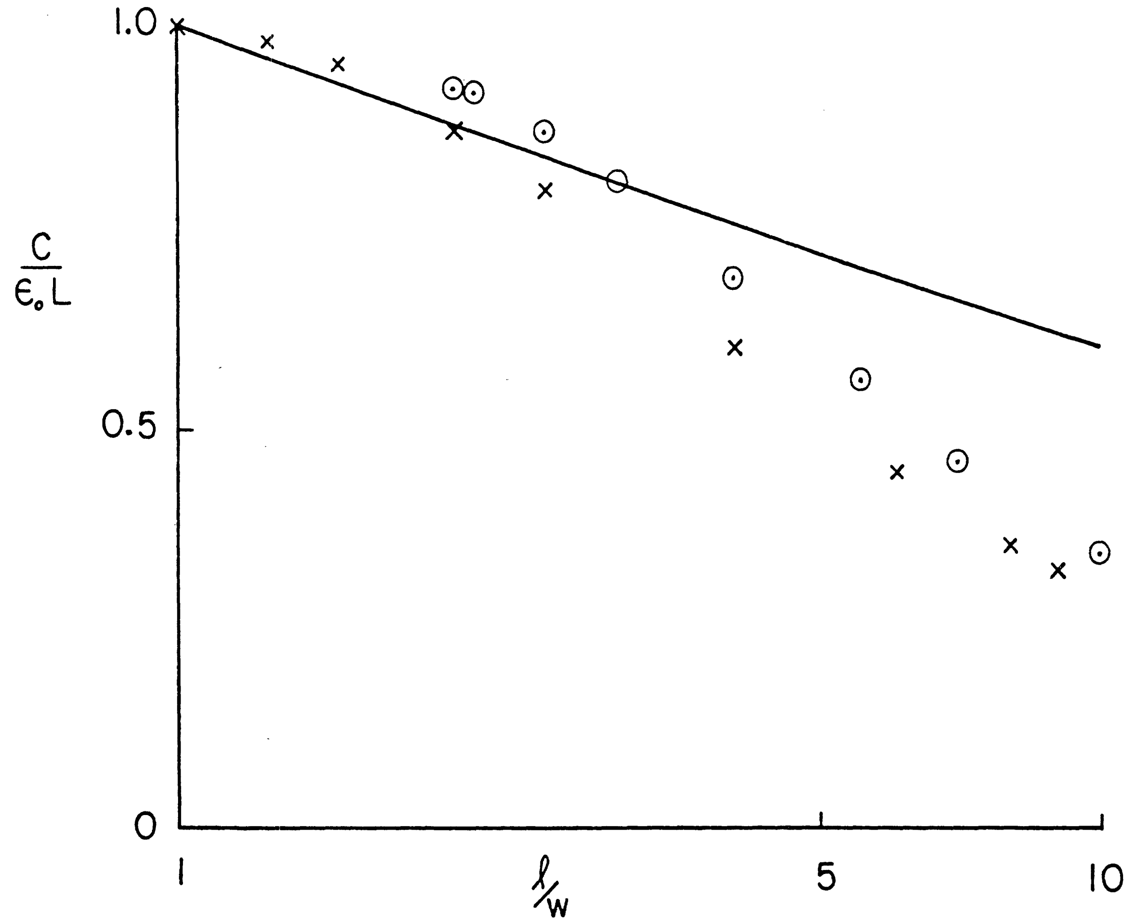


Fig. 14: Normalized capacity $C/\varepsilon_0 L$ for prolate spheroids (———), two identical joined spheres (⊙ ⊙), and two joined hemispheres back-to-back (× ×).

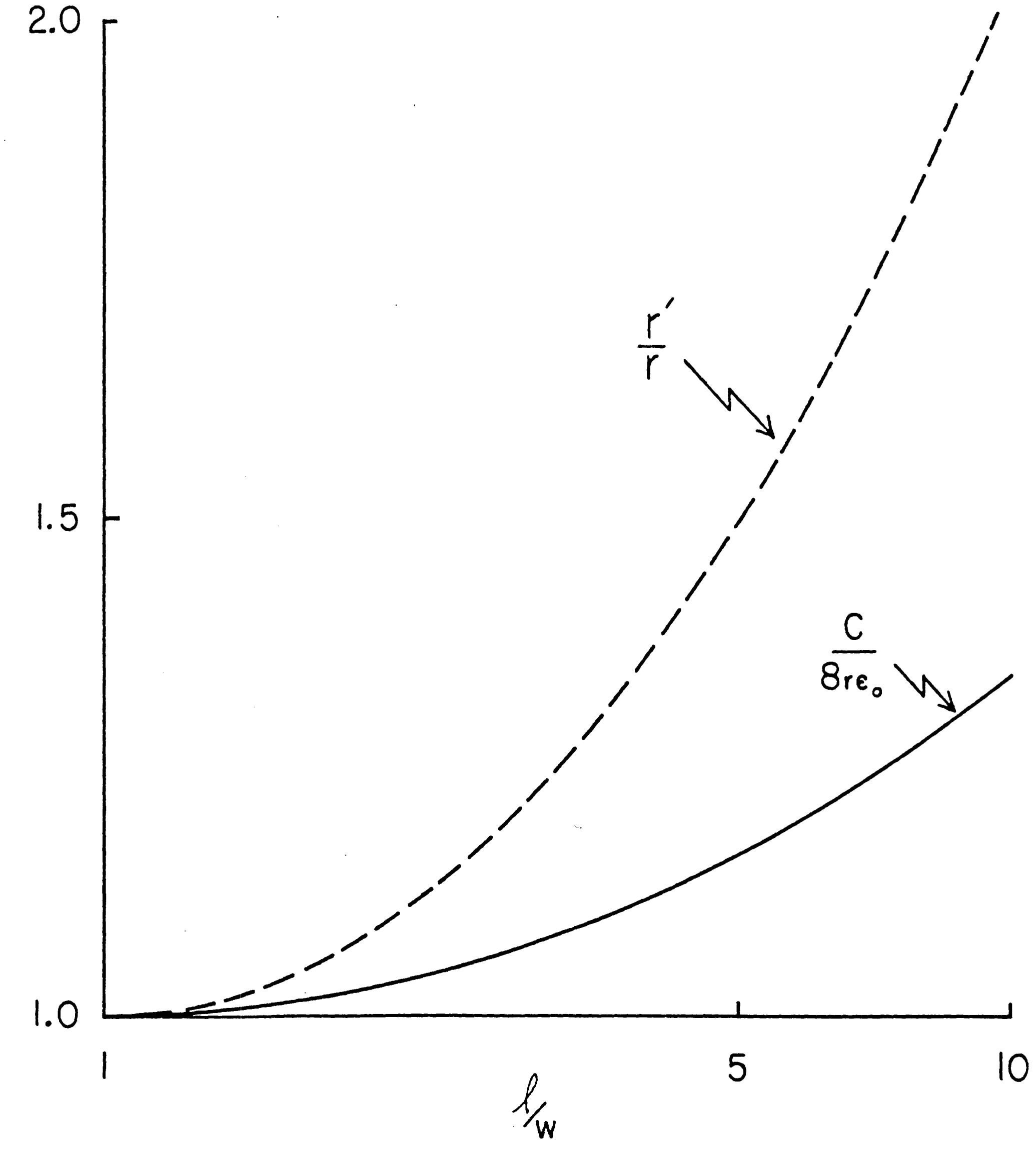


Fig. 15: Normalized capacity for an elliptical plate (———) compared with the upper bound r'/r (– – –).

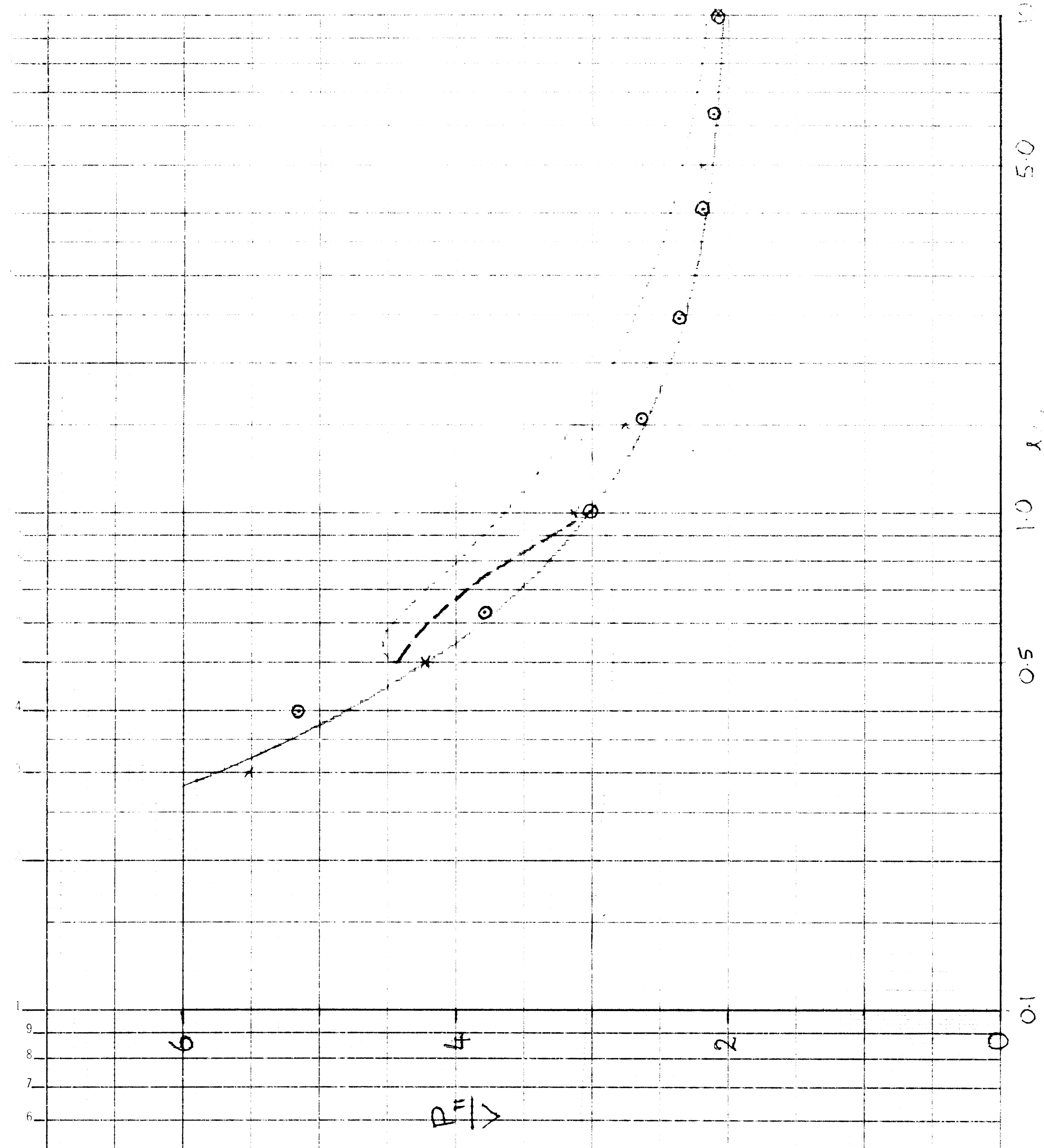


Fig. 3-1: Normalized tensor element P_{11}/V as a function of the length-to-width ratio of the body for prolate and oblate spheroids (—), rounded cones of half-angle $\theta < \pi/2$ (----) and $\theta > \pi/2$ (– –), ogive and lens (⊙ ⊙ ⊙), and right circular cylinders (x x x).

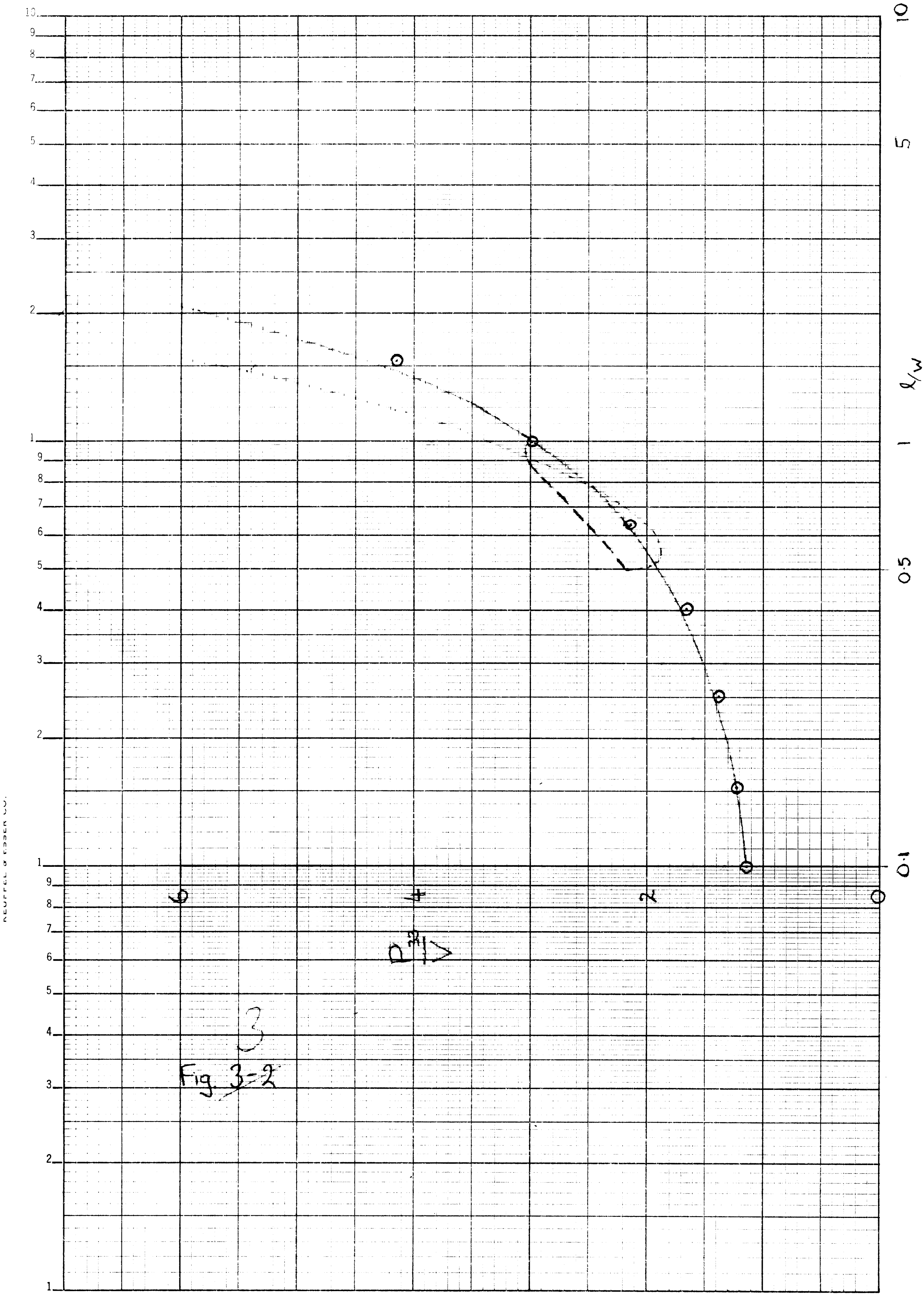


Fig. 3

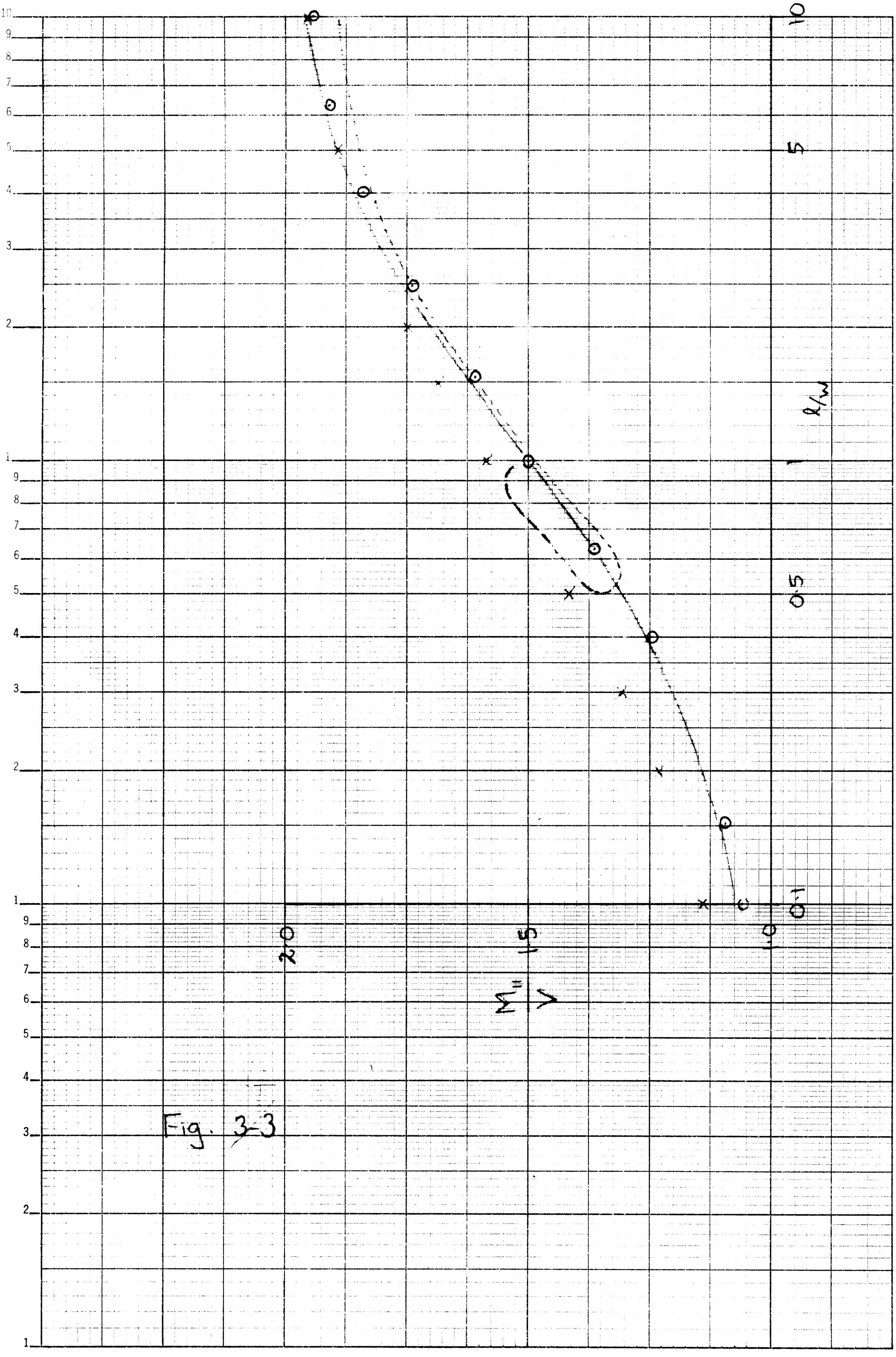


Fig. 3-3

Δ	V	$\frac{P_{11}}{V}$	$\frac{P_{33}}{V}$	$\frac{M_{11}}{V}$	$\frac{P_{11}}{V'}$	$\frac{P_{33}}{V'}$	$\frac{M_{11}}{V'}$
1	2.094	4.424	2.185	1.371	1.424	2.185	1.371
0.9	1.937	4.728	2.222	1.935	4.400	2.054	1.787
0.8	1.776	5.117	2.299	2.120	4.339	1.950	1.797
0.6	1.433	[illegible]	2.628	2.597	4.291	1.798	1.776
0.4	1.039	8.684	3.422	3.483	4.258	1.677	1.727
0.2	0.570	15.56	[illegible]	6.044	4.234	1.625	1.645
0.1	0.299	29.60	11.16	10.84	4.221	1.592	1.546
0.05	0.154	57.22	21.34	19.24	4.210	1.590	1.416

Table ~~3-1~~ 1: Tensor elements for a 'scooped-out' hemisphere (see Fig. 3-4). V' is the hemispherical volume.

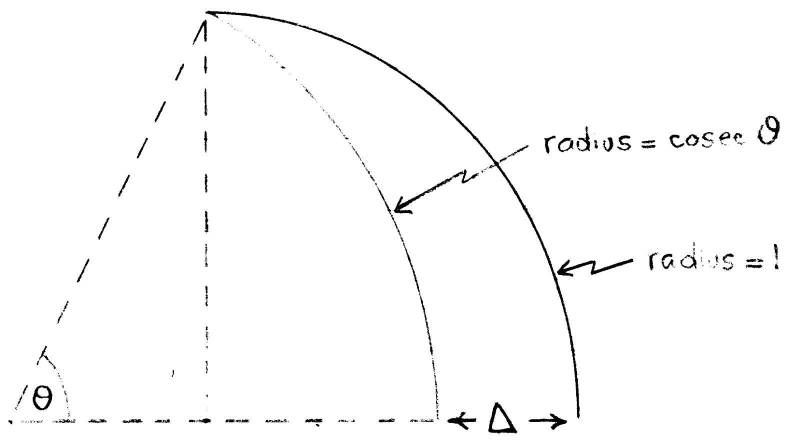


Fig. ~~3-4~~ 5: 'Scooped-out' hemisphere ($\Delta = 1 - \tan \theta/2$)

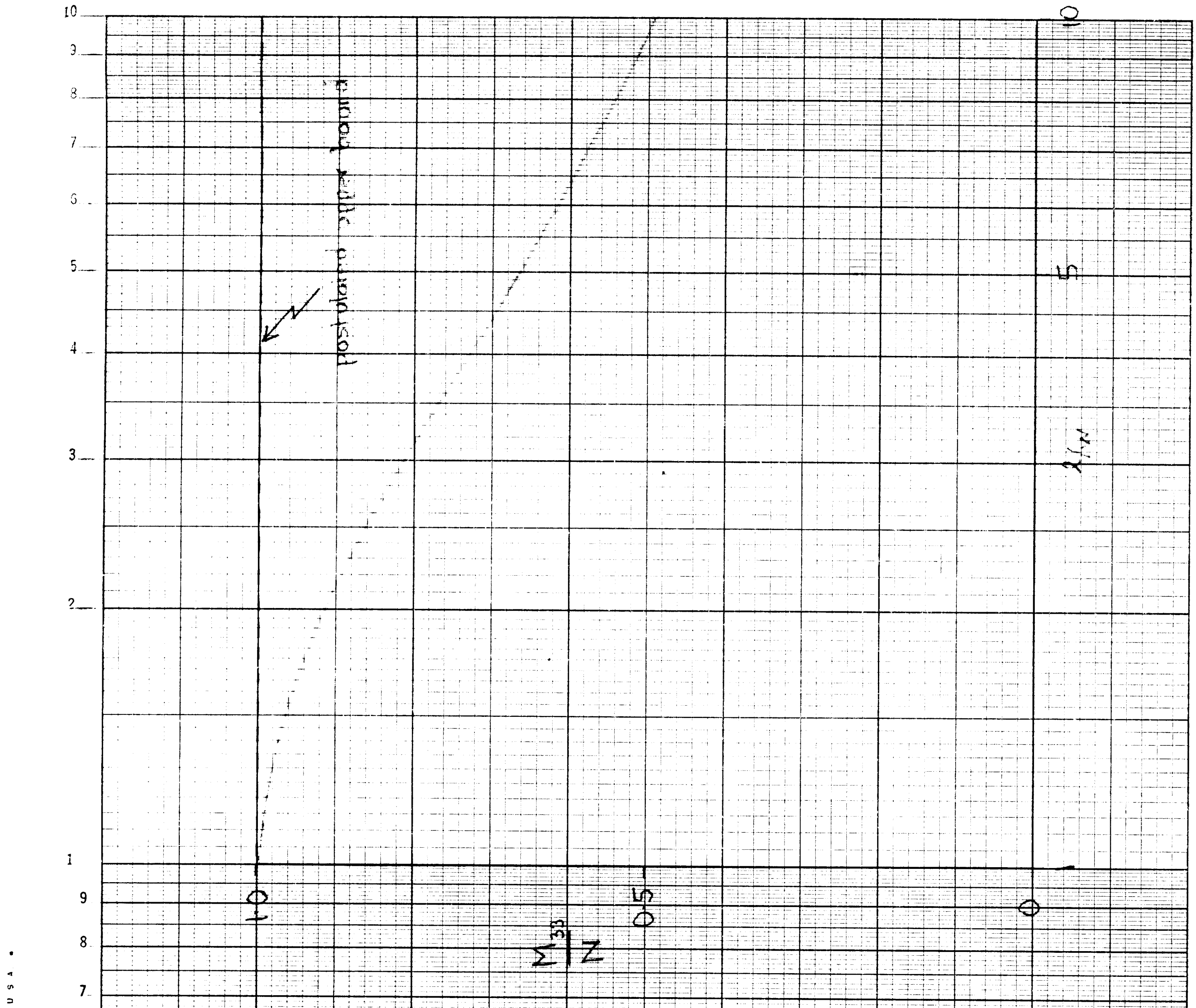
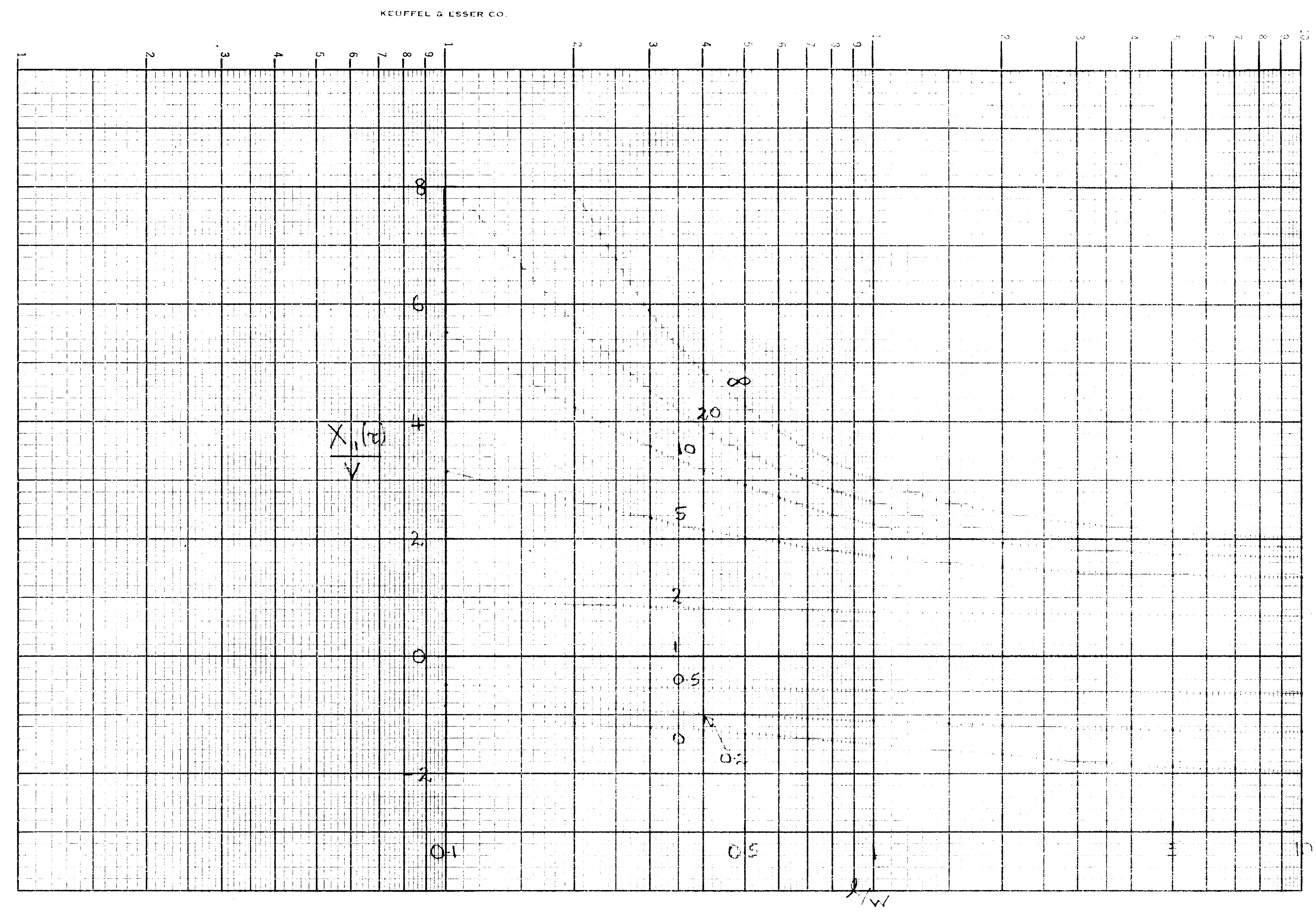
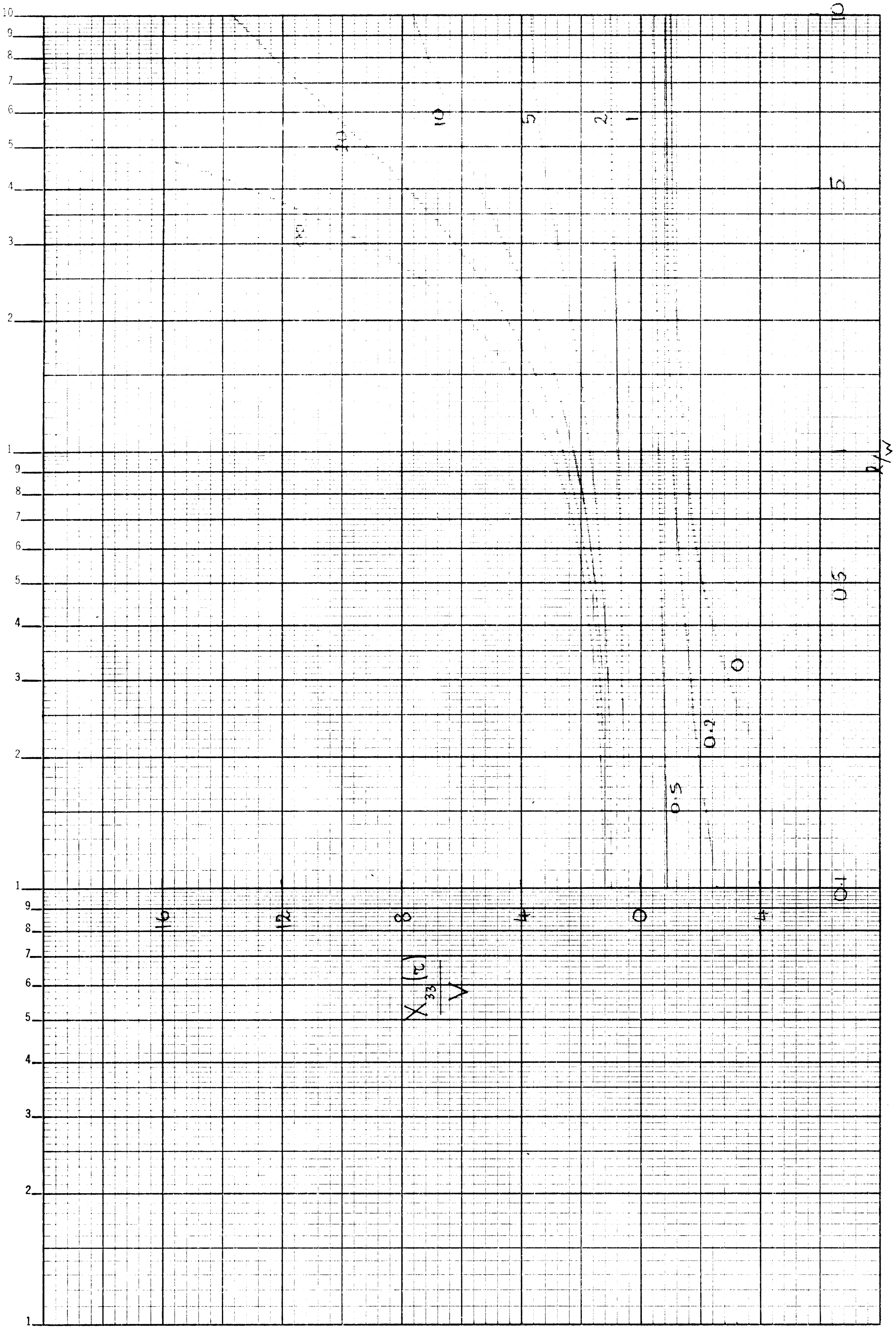


Fig. 7: Normalized tensor element for elliptical disks, where $N = 8r^3/3$. The element is equal to its postulated lower bound.


KEUFFEL & ESSER CO.
X11(τ)/V
ℓ/W
∞
20
10
5
2
1
0.5
0
0.2
0.1
0.5
1
10


X33(τ)/V
R/w
16
12
8
4
0
4
10
5
2
1
0.5
0.2
0
0.1
0.5

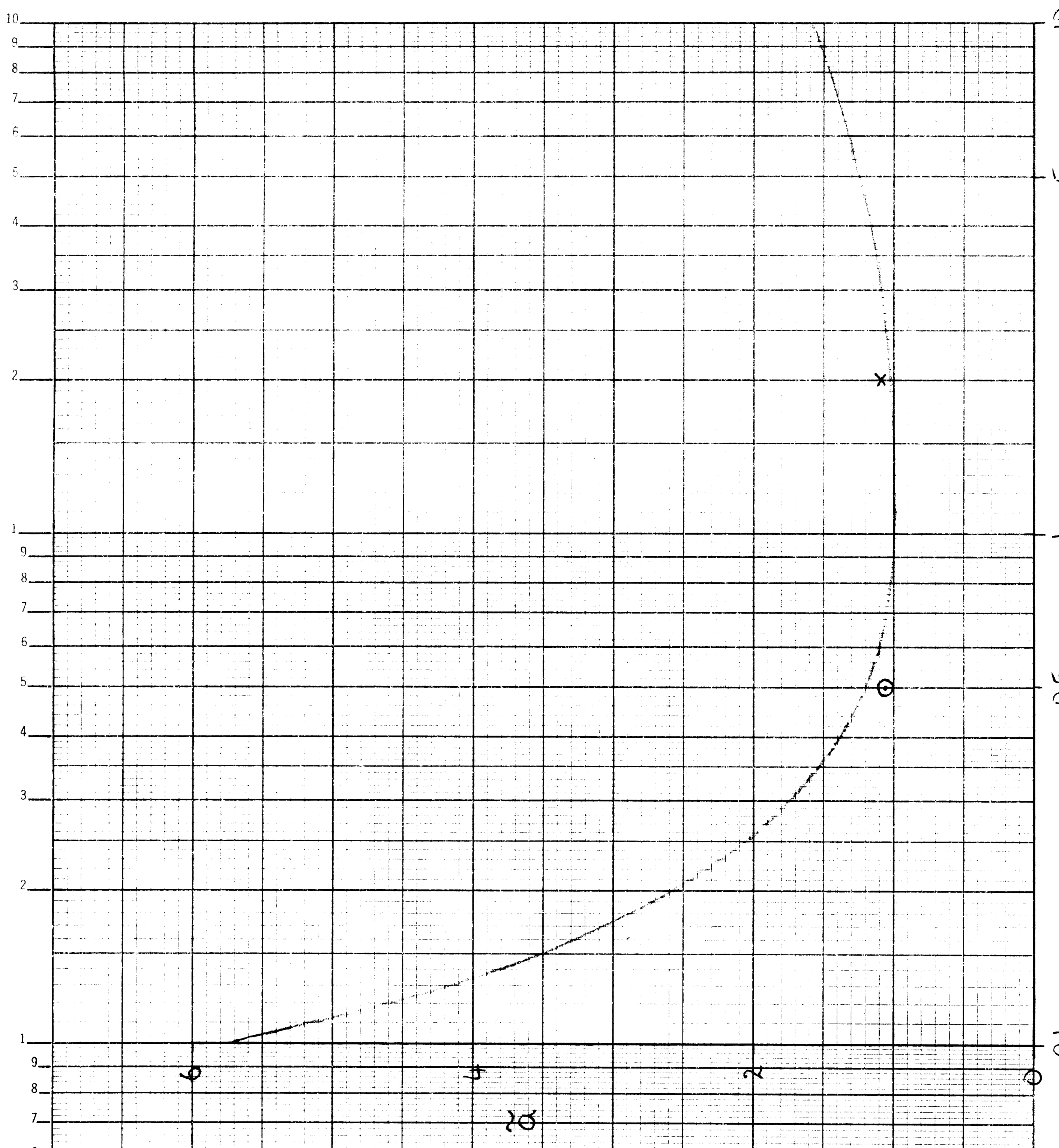


Fig. ~~7-1~~ 12 : Equivalent radius $\tilde{a} = \frac{1}{4\pi}\left(\frac{4\pi}{3V}\right)^{1/3}\frac{C}{\epsilon_0}$ for a spheroid (—), for a hemisphere (⊙) and for two spheres in contact (×).

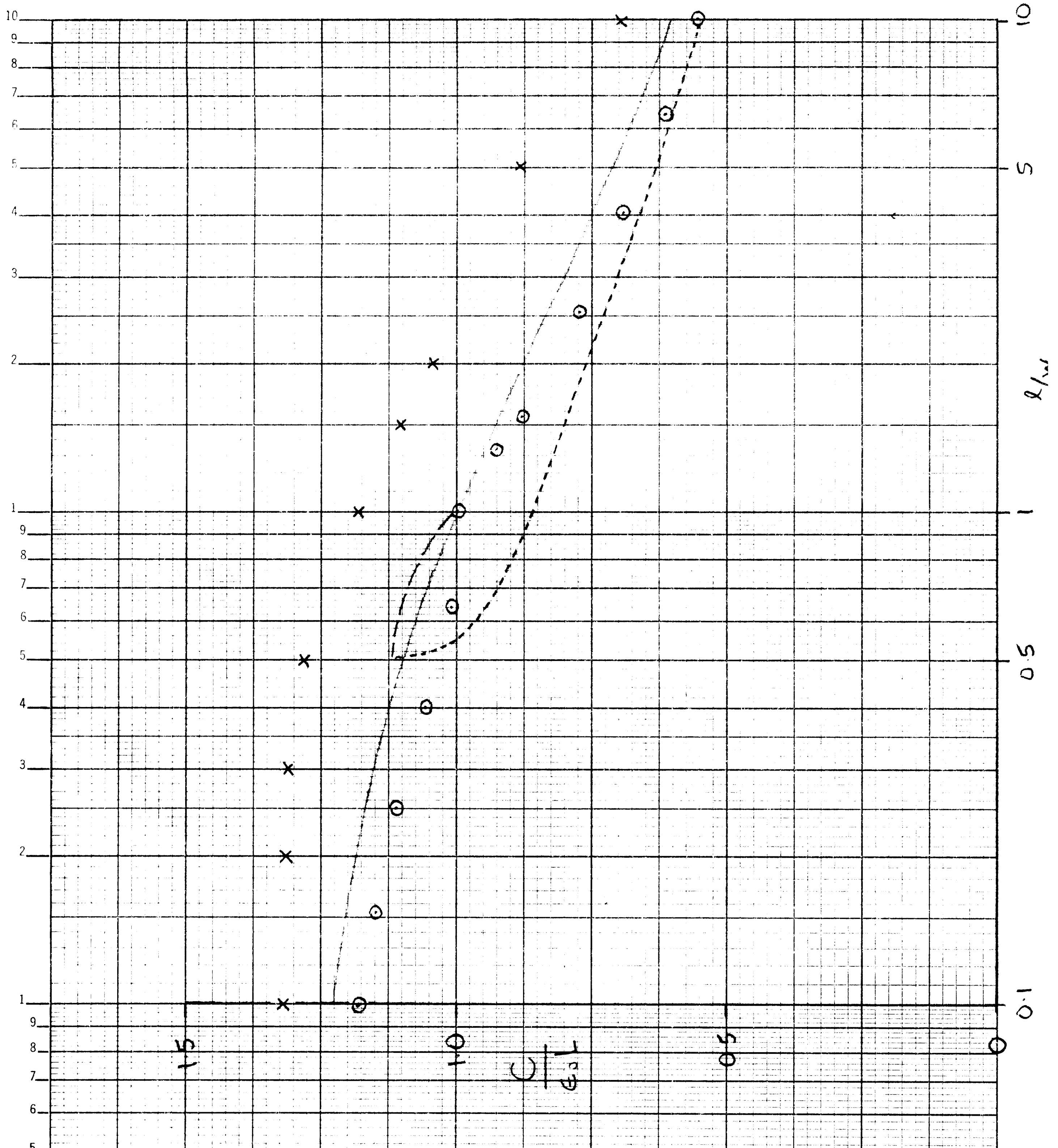


Fig. 13-2 : Normalized capacity $C/\epsilon_0 L$ for spheroids (——), rounded cones (– –, - - -), ogives and lenses (ooo) and right circular cylinders (x x x) (see § 3·4).

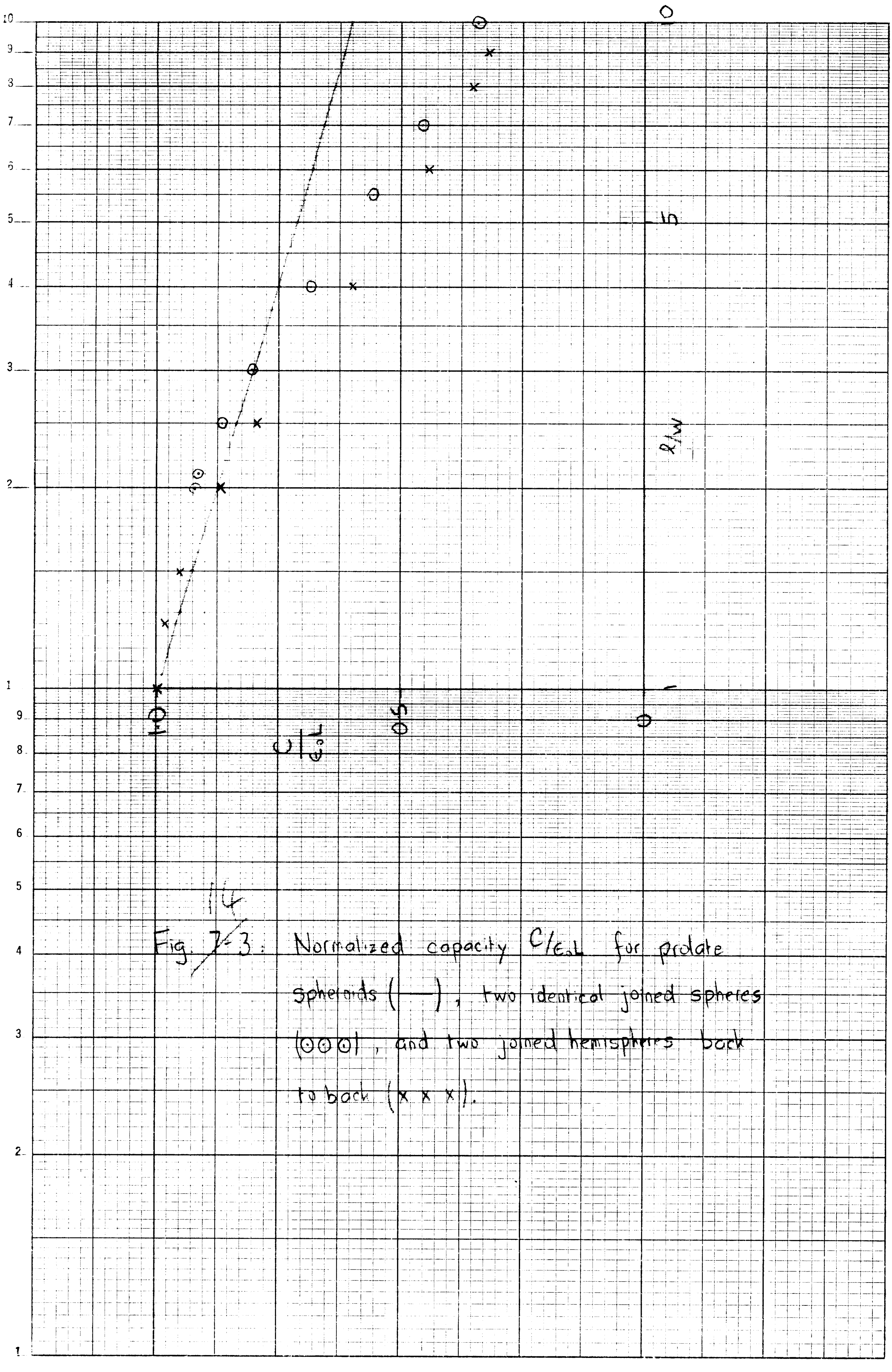


Fig. ~~7~~ 14-3: Normalized capacity $C/\varepsilon_0 L$ for prolate spheroids (——), two identical joined spheres (⊙⊙⊙), and two joined hemispheres back to back (x x x).

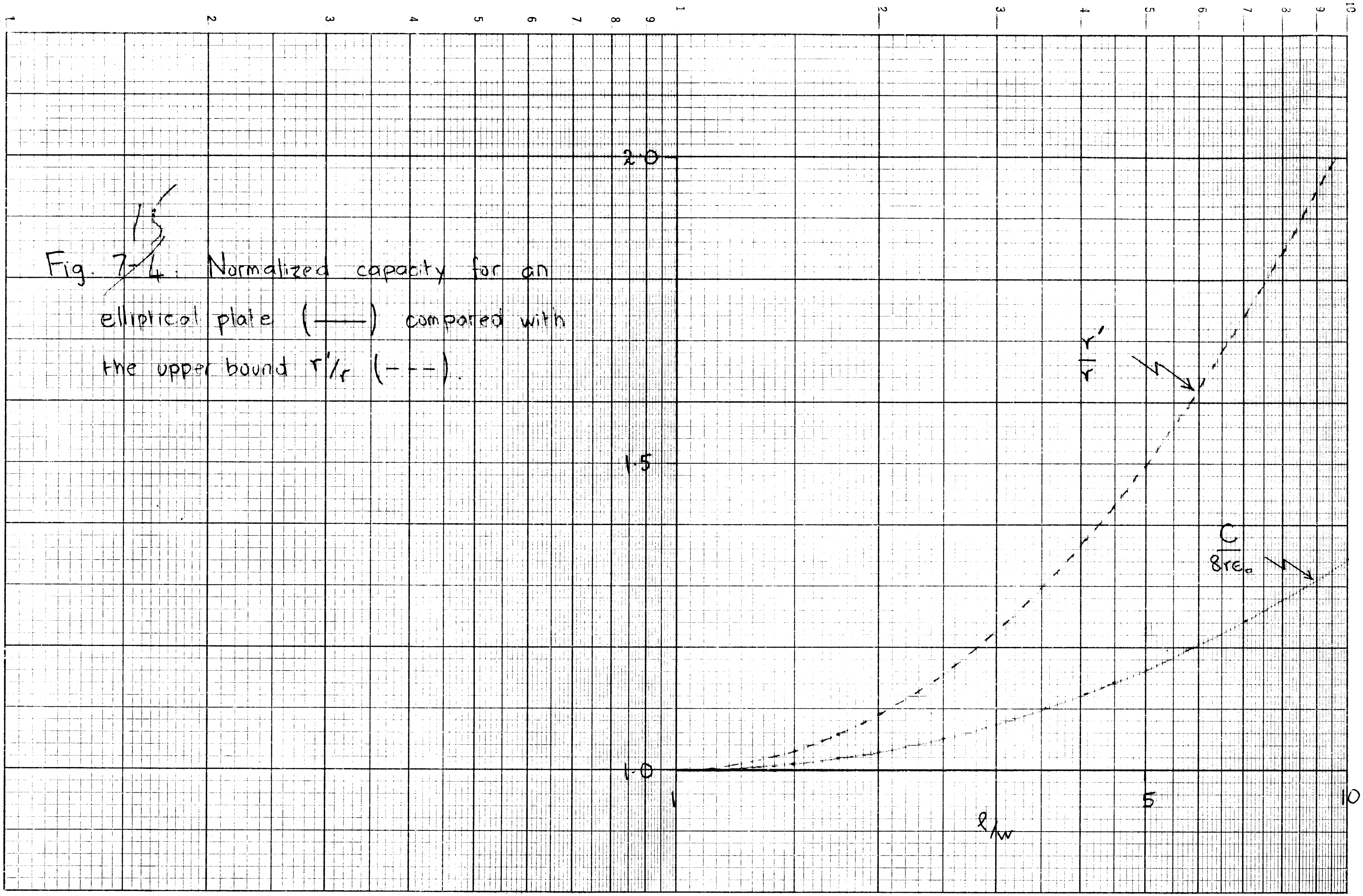


Fig. 7-4. Normalized capacity for an elliptical plate (——) compared with the upper bound r'/r (- - -).