THE TRANSMISSION LINE MODEL ANALYSIS OF RECTANGULAR PATCH ANTENNAS*

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ABSTRACT

The transmission line model has been utilized to determine the input performance of a rectangular patch antenna excited by a coaxial probe. Approximate expressions have been developed for the resonant propagation constant (or the resonant frequency), quality factor and input resistance of such antennas which clearly indicate the effects of the various antenna parameters including the feed on these quantities under dynamic conditions. The results of computations based on the expressions developed are compared with available measured values. The present work establishes the transmission line model as a valid representation of rectangular patch antennas, and would facilitate the design of such antennas without elaborate numerical computation. The model has also been applied to determine the resonant parameters for a similar antenna tuned by passive metallic posts. Validity of an approximate expression derived for the resonant frequencies of such tunable antennas has also been established by comparison with available measured results.

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I. Introduction

Early analyses of rectangular microstrip or patch antennas has been based on the TEM or parallel plate transmission line model [1,2,3]. Although this simple model requires less computation [3] and provides results with sufficient engineering accuracy, it cannot account for the effects of the patch aspect ratio, substrate thickness and the feed parameters [4]; moreover, due to the nature of the assumed open-circuit termination the predicted resonant frequency is generally larger than the actual one. Later, the cavity model [5,6] and its variations [4,7] have been developed for such antennas; these models provide more accurate results but involve detailed numerical computation.

The present paper re-examines the transmission line model in greater detail and makes appropriate modifications to remove its abovementioned shortcomings, and to develop simple but accurate expressions so that the design parameters for rectangular microstrip antennas may be determined without elaborate numerical computation. The improved model is further modified to obtain the resonant parameters and the input performance of a tunable rectangular microstrip antenna [8,9,10] whose resonant or operating frequency is controlled by utilizing a number of passive conducting posts suitably located within the input region of the antenna.

II. Equivalent Transmission Line Representation of a Rectangular Microstrip Antenna

The transmission line model treats the rectangular patch radiator as a strip line resonator with no transverse field variation and assumes the radiation to occur from the two transverse open edges [2,3]. It is well known [11] that the dominant mode of propagation in a strip line is the TEM or quasi TEM mode having negligible variation of fields in the transverse direction. It has been reported in [12,13] that it is the parallel plate type of TEM mode which is the dominant mode propagating along the longitudinal direction of a rectangular patch configuration, and that waves incident at the two open side edges suffer mostly total reflection thereby causing negligible radiation. Therefore, when properly excited only the two open edges normal to the direction of propagation take part in the radiation. It then seems natural to treat such an antenna as a section of parallel plate or strip transmission line terminated at both ends by admittances appropriate for the two radiating edges. This concept is utilized to obtain the equivalent transmission line model, and subsequently develop an equivalent lumped circuit representation for the antenna.

Figure 1 shows a rectangular patch antenna of length ℓ, width a, and excited from the back by a coaxial probe of radius r_0, the exact location of which is usually governed from desired impedance and polarized considerations. It is assumed that propagation takes place along the $\pm$x-directions. The presence of two passive conducting posts, each of radius r_0, in Fig. 1 is ignored for the present discussion; they are used to tune or control the resonant or operating frequency of the antenna,and will be discussed later. Normally, the substrate of thickness d extends beyond the patch and may sustain surface waves for large d. However, as shown in [12], for sufficiently thin substrate, the surface wave effects may be neglected and we therefore neglect the effects of the substrate beyond the patch. The ground plane is assumed to be of infinite extent which is justified from the fact that in a practical antenna it is much larger than the patch.

The entire configuration of Fig. 1 with the exception of the two tuning posts may now be represented by an equivalent transmission line of length ℓ, characteristic admittance Y_c (impedance Z_c), propagation constant β and terminated at each end by a normalized admittance Y ($= Y_T/Y_c$, Y_T being the admittance of an open edge) as shown in Fig. 2 which also shows the source generator in series with a normalized (with respect to Z_c) internal inductive reactance jX_L located at a distance ℓ_f from one edge of the patch. With a strip line excitation, the source in Fig. 2 should be replaced by an approximately located generator having a capacitive reactance.

The characteristic admittance and the propagation constant of the transmission line are:

$$Y_c = \frac{1}{Z_c} = \frac{a\alpha\sqrt{\varepsilon_e}}{\eta_o d} \ , \quad \eta_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} \ , \tag{1}$$

with μ_o, ε_o being the permeability and permittivity of free space,

$$\beta = \frac{\omega}{c}\sqrt{\varepsilon_e} \quad , \quad c = \frac{1}{\sqrt{\mu_o \varepsilon_o}} \ , \tag{2}$$

where

$$\varepsilon_e = \frac{\varepsilon_r + 1}{2} + \frac{\varepsilon_r - 1}{2}\left(1 + \frac{10d}{a}\right)^{-1/2} \ , \tag{3}$$

$$\alpha = 1 + 1.393\left(\frac{d}{a}\right) + 0.667\left(\frac{d}{a}\right)\ln\left(\frac{a}{d} + 1.444\right) \quad , \tag{4}$$

and ε_r, $\omega(= 2\pi f)$, c are the relative permittivity of the substrate, the radian frequency and the velocity of light in free space, respectively. The above relations are well known [3,11,14] and have been found to provide an accurate representation for a strip line. It should be noted that for sufficiently small d/a (or $d/a \to 0$) the equivalent permittivity ε_e of the substrate and the form factor α given by (3) and (4), respectively, reduce to $\varepsilon_e \simeq \varepsilon_r$ and $\alpha \simeq 1$, respectively; in general, ε_e and α are slowly varying functions of (d/a) and for typical d/a used for patch antennas, ε_e and α are slightly smaller and larger than ε_r and 1, respectively.

The normalized terminating admittance in Fig. 2 is $Y = G + jB$ where G and B represent the (normalized) radiation conductance and capacitive susceptance appropriate for the reradiating aperture. Exact determination of G and B is a difficult boundary value problem. Although formal expressions for G and B have been derived in [12,13] using the Wiener-Hopf technique, they are too complicated, and meaningful results can only be obtained through detailed numerical computation. Various empirical approximations to G and B have also been found useful [2-4] in the previous analyses of such antennas. For the present purpose we shall obtain approximations to G and B from the known results for a semi-infinite parallel plate waveguide radiating into free space [15,16]. Using the image principle and the aperture admittance of the open-end of a parallel plate waveguide radiating into free space [17] it can be shown that

$$G = \frac{\beta d}{2\alpha\varepsilon_e} \quad , \tag{5}$$

$$B = \frac{\beta d}{\pi\alpha\varepsilon_e} \; \ln\left(\frac{\sqrt{\varepsilon_e}\; 2\pi e}{\gamma\beta d}\right) \quad , \tag{6}$$

where,

$$\gamma = 1.78107 \quad , \quad e = 2.71828 \quad .$$

It should be noted that (5) is the same as that obtained in [12] under thin substrate and no surface wave propagation. The nature of approximations involved to obtain (5) and (6) are such that the parameter a be sufficiently large.

The normalized inductive reactance of the probe may be obtained from considerations similar to the probe excitation of a parallel plate waveguide propagating only its dominant mode [17,18], and is given by

$$X_L = \frac{\beta a\alpha}{2\pi} \ln \frac{2}{\gamma\beta r_o} \quad . \tag{7}$$

This completes the description of the transmission line circuit equivalent to the coaxial probe fed untuned rectangular patch antenna. For a similarly excited rectangular patch antenna tuned by passive conducting posts (Fig. 1) the equivalent transmission line circuit is similar to Fig. 2. except that the susceptances appropriate for the posts would be inserted across the line at the corresponding locations of the posts. For example, with posts of radii r_o in Fig. 1 the required susceptance of each post to be inserted in Fig. 2 would be the reciprocal of that given by (7).

III. Resonance Parameters of an Untuned Rectangular Patch Antenna

In this section we at first obtain the general resonance conditions for a patch antenna in the absence of the exciting probe inductance. Later, the effects of the probe inductance are taken into account. Finally, various resonant parameters of the antenna are obtained and compared with available measured results.

A. Resonance Conditions

The input admittance (Y_{in}) of a section of transmission line of length x, characteristic admittance Y_c, propagation constant β and terminated by (normalized) Y is given by the following well known expression

$$\frac{Y_{in}}{Y_c} = \frac{Y + j \tan \beta x}{1 + jY \tan \beta x} . \tag{8}$$

It is found from the equivalent circuit of Fig. 2 that in the absence of the exciting probe inductance the admittance seen by the generator is the same as the total admittance Y_{in} at the terminals A-A. By using (8) it can be shown that

$$\frac{Y_{AA}}{Y_c} = (G_1 + G_2) + j(B_1 + B_2) \quad , \tag{9}$$

with

$$G_{1,2} = \frac{G}{\cos^2 \beta \ell_{1,2} + (B^2 + G^2)\sin^2 \beta \ell_{1,2} - B \sin 2\beta \ell_{1,2}} \quad , \tag{10}$$

$$B_{1,2} = \frac{B - (B^2 + G^2 - 1)\tan \beta\ell_{1,2} - B \tan^2 \beta\ell_{1,2}}{1 + (B^2 + G^2) \tan^2 \beta\ell_{1,2} - 2B \tan \beta\ell_{1,2}} \tag{11a}$$

$$= \frac{r \sin (\xi - 2\beta\ell_{1,2})}{(1 + B^2 + G^2) - r \cos(\xi - 2\beta\ell_{1,2})} , \tag{11b}$$

where

$$r^2 = 4B^2 + (B^2 + G^2 - 1)^2 , \tag{12}$$

$$\tan \xi = \frac{2B}{B^2 + G^2 - 1} , \tag{13}$$

and the subscripts 1,2 on the right-hand sides of (10) - (11) correspond to those on the left-hand sides.

It is now demanded that resonance is obtained when the input admittance of the antenna is real; i.e., $\mathrm{Im}(Y_{AA}/Y_c) = 0$ in (9). Using this condition along with (11) - (13) and after considerable algebra it can be shown that at the dominant (or lowest) resonance $\xi = \beta\ell$ which yields the following transcendental equation determining the values of β at the dominant resonance:

$$\tan \beta\ell = \frac{2B}{B^2 + G^2 - 1} , \tag{14}$$

which has been frequently quoted in the literature [2,13,19]. It can now be shown that under the assumption $X_L = 0$ the resonant conductance of an antenna fed at a distance $\ell_2 = \ell_f$ from one of the radiating edges is given by

$$\frac{G_r}{Y_c} = \frac{2G}{\cos^2 \beta\ell_f + (B^2 + G^2)\sin^2 \beta\ell_f - B \sin 2\beta\ell_f} , \tag{15}$$

from which we obtain the following expression for the resonant resistance R_r of an antenna excited from one edge ($\ell_f = 0$):

$$R_r = \frac{Z_c}{2G} . \tag{16}$$

B. Resonant Frequency

Solution of (14) yields the propagation constant β at resonance from which the resonant frequency is obtained by using (2). Substitution of (5) and (6) in (14) indicates that the latter is a complicated transcendental equation which is generally solved by numerical means [2,3,19]. Also, it is not obvious from (14) how the resonant frequency of an antenna varies with its characteristic parameters. In the following we develop an explicit expression for the resonant frequency from an approximate solution of (14).

Based on the observation B,G << 1 appropriate for most antennas of practical interest, we simplify (14) to

$$\tan \beta\ell = -2B = -\frac{2\beta d}{\varepsilon_e \alpha\pi} \ln\left(\frac{\sqrt{\varepsilon_e}\, 2\pi e}{\gamma\beta d}\right) , \tag{17}$$

where we have used (6) to obtain the last form. To solve (17) we make use of the experimentally observed fact that at the fundamental resonance the propagation constant $\approx\pi/\ell$, and assume that the solution β_r of (17) is such that

$$\beta_r\ell = \pi(1 - \delta/\pi) \tag{18}$$

where $\delta/\pi \ll 1$. Under the assumption of (18), (17) can be linearized in δ and we obtain

$$\frac{\delta}{\pi} \simeq \frac{\dfrac{2d}{\varepsilon_e \ell\pi\alpha}\ln\left[\dfrac{\sqrt{\varepsilon_e}\,2e\ell}{\gamma d}\right]}{1 + \dfrac{2d}{\varepsilon_e \ell\pi\alpha}\ln\left[\dfrac{\sqrt{\varepsilon_e}\,2\ell}{\gamma d}\right]} \quad . \tag{19}$$

Using (2), (18) and (19) we obtain the following expression for the resonant or operating frequency

$$f_r = f_o\left[\frac{1 - \dfrac{2d}{\varepsilon_e \ell\pi\alpha}}{1 + \dfrac{2d}{\varepsilon_e \ell\pi\alpha}\ln\left(\dfrac{\sqrt{\varepsilon_e}\,2\ell}{\gamma d}\right)}\right] \quad , \tag{20}$$

where

$$f_o = \frac{c}{2\ell\sqrt{\varepsilon_e}} \quad , \tag{21}$$

is the resonant frequency under the assumption B = 0. Equation (20) has been reported earlier [20,21]. Previously [1,3] the resonant frequency of a patch antenna as obtained from the transmission line model considerations has been given by f_0, with $\varepsilon_e = \varepsilon_r$, which is independent of the patch width and the substrate thickness [5,7]. The present expression (20) contains the correction factor to be applied to f_0, and it clearly indicates the dependence of the resonant frequency on all the basic parameters of the antenna including the feed.

Before discussing the accuracy of (20) we shall examine the validity of (17) to approximate (14). To this end, we have used the Newton-Raphson method to solve (14) numerically for rectangular patch antennas having dimensions reported in [4]; only two or three iterations were sufficient to obtain convergence solutions. The δ/π values obtained numerically from (14) and from (19) are shown in Table 1 where we have also indicated the relevant parameters of the antennas used. The results indicate that (17) is a good approximation to (14), and that (19) can be used to calculate the desired values of δ/π. The resonant frequencies of the antennas considered in Table I have been calculated by using (20) and the results along with the values of f_0 are shown in Table II where, for comparison, we also show the corresponding measured values obtained from [4]. For the antenna parameters used, the results of Table II indicate that (20) overestimates the measured resonant frequency by about 0.2 to 1.6 percent whereas the overestimation by f_0 (i.e., by (21) ranges from three to six percent.

C. Effects of the Probe Inductance

From resonant circuit considerations it is expected that an inductive probe (Fig. 2) would slightly increase the resonant frequency

Table I

δ/π Values for Rectangular Patch Antennas

(d = 0.1524 cm, ε_r = 2.5; a and ℓ variable)

a (cm)	ℓ (cm)	ε_e	α	δ/π Eq. (14)	δ/π Eq. (19)
4.100	4.140	2.390	1.135	0.0392	0.0404
6.858	4.140	2.428	1.088	0.0408	0.0417
10.800	4.140	2.452	1.060	0.0422	0.0422
11.049	6.909	2.453	1.059	0.0280	0.0283

Table II

Calculated and Measured [4] Values of Resonant Frequency for Rectangular Patch Antennas

(d = 0.1524 cm, ε_r = 2.5; a and ℓ variable)

a (cm)	ℓ (cm)	f_o (MHz) Eq. (21)	f_r (MHz) Eq. 20	f_r (MHz) Measured
4.100	4.140	2343	2248	2228
6.858	4.140	2325	2228	2200
10.800	4.140	2314	2216	2181
11.049	6.907	1386	1347	1344

given above. It is important to know this upward shift in the frequency because the measured resonant frequency is associated with the probe inductance.

To simplify the analysis we assume that the antenna is fed from one edge (i.e., $\ell_f = \ell_2 = 0$, $\ell_1 = \ell$ in Fig. 2) and observe that near resonance $\beta\ell \simeq \pi$. Under these conditions and using the approximation $B, G << 1$, it can be shown from (9) and (11a) that the admittance (without the probe) at the terminals A-A located at the edge is

$$\frac{Y_{AA}}{Y_c} \simeq 2G + j(2B + \tan \beta\ell) \quad . \tag{22}$$

Combining (22) with the probe reactance (Fig. 2) it is found that the total impedance seen by the generator of an edge-fed antenna is:

$$\frac{Z_{in}}{Z_c} = \frac{2G}{4G^2 + (2B + \tan \beta\ell)^2} + j\left[X_L - \frac{2B + \tan}{4G^2 + (2B + \tan \beta\ell)^2}\right] \quad . \tag{23}$$

Applying the resonance condition ($\mathrm{Im}\, Z_{in} = 0$) to (23) we now obtain the following

$$X_L \tan^2 \beta\ell - (1 - 4BY_L)\tan \beta\ell + [4(G^2 + B^2)X_L - 2B] = 0 \quad . \tag{24}$$

The appropriate solution of (24) which would yield the resonant β under the present conditions is

$$\tan \beta\ell = -2B + \frac{1}{2X_L} - \frac{1}{2X_L}\left(1 - 16G^2X_L^2\right)^{1/2} , \tag{25}$$

which should be compared with (17). We make further approximation $16G^2X_L << 1$ valid for practical antennas to obtain the following from (25).

$$\tan \beta\ell = -2B + 4G^2X_L , \tag{26}$$

with G, B and X_L given by (5), (6) and (7), respectively.

Equation (26) can be solved in a manner similar to the case with $X_L = 0$ (i.e., solution of (17) described earlier. It can now be shown from (26) that the relative upward shift in the resonant frequency caused by the probe inductance is

$$\frac{\Delta f_r}{f_r} = \left.\frac{4G^2X_L}{\pi - 2B}\right|_{\beta=\beta_r} \simeq \left.\frac{4G^2X_L}{\pi}\right|_{\beta=\beta_r} , \tag{27}$$

where f_r is given by (20) and in the last form of (27) we have used the approximation $\pi >> 2B$. After introducing (5), (7) in (27) we obtain the following explicit expression for the frequency shift:

$$\frac{\Delta f_r}{f_r} = \frac{\pi}{2}\left(\frac{d}{\ell}\right)^2\left(\frac{a}{\ell}\right)\frac{1}{\alpha\varepsilon_e^2}\ln\left(\frac{2\ell}{\gamma\pi r_o}\right) . \tag{28}$$

With a stripline excitation, a similar expression for the downward shift of the resonant frequency may be obtained after replacing X_L in (27) by the equivalent series reactance of the capactive junction.

For an edge fed antenna having d = 0.1524 cm, r_o = 0.064 cm, a = 4.10 cm, ℓ = 4.140 cm and ε_r = 2.5, the calculated resonant frequency f_r = 2248 MHz with X_L = 0 (Table 2). The frequency shift due to the probe inductance as calculated from (28) is

$$\frac{\Delta f_r}{f_r} = 1.02 \times 10^{-3} \ (\sim 0.1 \text{ percent}) \quad .$$

D. Resonant Resistance

For matching purposes it is of practical interest to know the (input) resistance of the antenna at resonance as seen by the generator. This resistance consists of the antenna radiation resistance and the resistances associated with the copper and dielectric losses. The latter two are usually very small [4] and will be neglected. For an edge-fed antenna the desired resonant resistance R_r can be obtained easily by using (26) and (23) and is given by

$$R_r = \frac{Z_c}{2G} \left. \frac{1}{1 + 4G^2X_L^2} \right|_{\beta=\beta_r} \tag{29}$$

Note that with X_L = 0, (29) reduces to the value given earlier by (16). Neglecting the effects of the probe the input resistance of the antenna at resonance may be obtained from

$$R_r \simeq \left. \frac{Z_c}{2G} \right|_{\beta=\beta_r} = \frac{120\sqrt{\varepsilon_e}}{(1 - \delta/\pi)} \left(\frac{\ell}{a}\right) \text{ ohms} \quad . \tag{30}$$

where δ/π is given by (19). For the edge-fed patch antennas discussed earlier, the resonant resistance has been computed by using (30), and the results are shown in Table III where we also show the corresponding measured values obtained from [4]. The results indicate that except for the lowest value of a, the agreement between the calculated and measured values of resistance may be considered to be fair.

E. Q of the Patch Antenna

The bandwidth of the patch antenna is intimately related to its quality or Q-factor. Hence, it is of considerable interest to obtain an explicit expression for the Q-factor. Again, we consider an edge-fed antenna and ignore the effects of the probe and the dielectric and copper losses associated with the antenna, i.e., we shall obtain an expression for the radiation Q or Q_r of the antenna.

To derive the desired expression for Q_r we at first develop an equivalent lumped circuit parameter for the rectangular patch antenna near resonance which may be found useful for other applications as well. Referring to Fig. 2 and assuming $Y_L = 0$, we rewrite (22) for the input admittance near resonance as:

$$\frac{Y_{AA}}{Y_c} = 2G(\beta) + j[2B(\beta) + \tan \beta\ell] \quad , \tag{31}$$

where we have explicitly shown the dependence of various parameters on β. Note that at resonance

$$\tan \beta_r\ell + 2B(\beta_r) = 0 \quad , \tag{32a}$$

Table III

Resonant Resistance of Edge-Fed Rectangular Patch Antennas

($d = 0.1524$ cm, $\varepsilon_r = 2.5$, $r_o = 0.064$ cm; a and ℓ variable)

a (cm)	ℓ (cm)	ε_e	δ/π	R_r (ohms) Eq. (30)	Measured [4]
4.100	4.140	2.390	0.0404	195	280
6.858	4.140	2.428	0.0417	118	115
10.800	4.140	2.452	0.0422	75	65
11.049	6.909	2.453	0.0283	120	102

and near resonance

$$\beta = \beta_r + \Delta\beta \quad . \tag{33b}$$

Using (32) we obtain from (31)

$$\frac{Y_{AA}(\beta_r + \Delta\beta)}{Y_c} \simeq 2G(\beta_r) + j\left[\frac{2d}{\varepsilon_e \alpha\pi} \ln\left(\frac{\sqrt{\varepsilon_e}\, 2\pi}{\gamma\beta_r d}\right) \Delta\beta + \ell\Delta\beta\right] \quad , \tag{33c}$$

where it has been assumed that $G(\beta_r + \Delta\beta) \simeq G(\beta_r)$.

Let the inductance and capacitance per unit length of the strip line comprising the patch be denoted by L_e and C_e, respectively. Then from the considerations of the TEM mode characteristics it can be shown that:

$$\left.\begin{aligned} Y_c &= \sqrt{C_e/L_e} \quad , \quad \beta = \omega\sqrt{\mu_o \mu_o \varepsilon_e} = \omega\sqrt{L_e C_e} = \omega C_e/Y_c \\ \Delta\beta &= C_e \Delta\omega / Y_c \quad , \\ C_e &= \varepsilon_o \varepsilon_e \alpha a/d \quad ; \end{aligned}\right\} . \tag{34}$$

Using (34) we obtain the following from (33)

$$Y_{AA}(\omega_r + \Delta\omega) \simeq 2Y_c G(\beta_r) + j\left[\frac{C_e 2d}{\varepsilon_e \alpha\pi} \ln\left(\frac{\sqrt{\varepsilon_e}\, 2\pi}{r\beta_r d}\right) + \ell C_e\right] \Delta\omega \quad , \tag{35}$$

Comparing (35) with the admittance near resonance of a parallel combination of R,L,C having a resonant frequency $\omega_r = 1/\sqrt{LC}$, it can be shown that the equivalent R and C of the patch antenna are:

$$R = R_{AA} = \frac{Z_c}{2G(\beta_r)} = \frac{\alpha\varepsilon_e}{\beta_r d} \tag{36a}$$

$$C = \frac{\ell C_e}{2} + \frac{C_e d}{\varepsilon_e \alpha\pi} \ln\left(\frac{\sqrt{\varepsilon_e}\, 2\pi}{\gamma\beta_r d}\right)$$

$$\simeq \frac{\varepsilon_o \varepsilon_e \alpha a \ell}{2d} + \frac{\varepsilon_o a}{\pi} \ln\left(\frac{\sqrt{\varepsilon_e}\, 2\ell}{\gamma d}\right) . \tag{36b}$$

It is interesting to note that the first term in (36b) is one half the equivalent static capacitance of the patch and the second term is the dynamic contribution to the capacitance due to the two radiating apertures. Using the basic definition of Q_r as

$$Q_r = \omega_r CR \quad , \tag{37}$$

we obtain the following expression for the radiation Q of the patch antenna

$$Q_r = \left.\frac{\beta_r \ell}{4G}\right|_{\beta_r} + \frac{1}{\pi} \ln\left(\frac{\sqrt{\varepsilon_e}\, 2\pi}{\gamma\beta_r d}\right)_r$$

$$\simeq \frac{\varepsilon_e \alpha\ell}{2d} + \frac{1}{\pi} \ln\left(\frac{\sqrt{\varepsilon_e}\, 2\ell}{\gamma d}\right) . \tag{37}$$

It is interesting to observe that if the dynamic contribution to C is ignored,

$$Q_r \simeq \frac{\beta_r \ell}{4G} = \frac{\pi - \delta}{4G} \simeq \left.\frac{\pi - 2B}{4G}\right|_{\beta=\beta_r} \tag{38}$$

the last form of which is exactly the same as given in [12]. Equation (37) shows explicitly the dependence of Q_r on the various parameters of the antenna.

As an application of (37), consider the patch antenna having $a = 6.858$ cm, $\ell = 4.14$ cm, $d = 0.1524$ cm and $\varepsilon_r = 2.50$. From Table I we have $\varepsilon_e = 2.428$ cm and $\alpha = 1.088$. This, from (37) it is found that $Q_r = 35.88 + 1.23 = 37$, which indicates that the contribution to the dynamic term (i.e., second tern in (37) is negligible. The resonant frequency of the antenna is from Table 2: $f_r = 2228$ MHz. Thus, the theoretical bandwidth of the antenna is

$$\Delta f = \frac{f_r}{Q_r} \simeq 60 \text{ MHz} \quad .$$

No experimental results are available for comparison. However, corresponding theoretical values obtained by detailed numerical computation using the cavity model [4] for the same antenna are Q_r = 30 and Δf = 73 MHz.

IV. Tunable Rectangular Patch Antenna

It is known [8,9,10] that the addition of passive metallic posts within the input region of a rectangular patch antenna increases its resonant or operating frequency without significantly altering its input impedance or radiation pattern and without increasing the complexity of the external feed network. A variety of such tuning post arrangements have been developed; for example, it has been reported [9], that the range of operating frequency can be increased to 20 percent with a pair of posts placed along the center line of the antenna and in excess of 50 percent with more posts located along and offset from the center line. The present section utilizes the transmission line model discussed in the previous section to determine the resonant characteristics of tunable rectangular patch antennas.

A. Numerical Evaluation of the Resonant Parameters

We consider a rectangular patch antenna fed by a coaxial line at one edge and tuned by a single post located ($y = a/2$ in Fig. 1) at a distance ℓ_2 from the feed edge. Under these conditions the equivalent transmission line circuit is as shown in Fig. 3 where $-jB_L$ represents the inductive susceptance of the post, and the other parameters are as explained before. For simplicity, it is assumed that the radius of the post is the same as that of the probe; therefore, $B_L = 1/X_L$ where X_L is given by (7). For more than one post located along the transverse direction at ℓ_2, B_L should be changed appropriately. With more set or sets of similar posts placed at various distances, the appropriate susceptance be placed at the corresponding locations in Fig. 3. The resonant frequency or propagation constant of a tunable antenna may now be found by demanding the vanishing of the imaginary part of the admittance seen by the generator obtained by multiple application of (8) in conjunction with Fig. 3. From a knowledge of β_r all other resonant parameters of the antenna can be obtained in a manner described in the previous section. In general, this is best accomplished by numerical computation. In the following we shall obtain an approximate expression for the resonant frequency which may be found useful for the design of such antennas.

B. Approximate Expression for the Resonant Frequency

For simplicity we neglect the effects of the probe inductance, and obtain the following for the total admittance at the AA terminals in Fig. 3:

$$\frac{Y_{AA}}{Y_c} = (G_1 + G_2) + j(B_1 + B_2 - B_L) \quad , \tag{39}$$

where all the notations are as explained before. Applying the condition $\mathrm{Im}(Y_{AA}/Y_c) = 0$ to (39), and after considerable algebra using (10) through (13) it can be shown that the resonant β satisfies the following equation

$$\sin(\xi-\beta\ell) = B_L \frac{[1+B^2+G^2-r\cos\{(\xi-\beta\ell)+\beta(\ell_1-\ell_2)\}][1+B^2+G^2-r\cos\{(\xi-\beta\ell)-\beta(\ell_1-\ell_2)\}]}{2r(1+B^2+G^2)\cos\beta(\ell_1-\ell_2)-2r^2\cos(\xi-\beta\ell)} \, ,$$

$$\ell_1 + \ell_2 = \ell \, . \tag{40}$$

Since $B_L << 1$, close examination of (40) indicates that at the desired solution $\xi - \beta\ell \simeq 0$. It is also known that $B,G << 1$, $B^2 + G^2 << 1$ which imply that $r \simeq 1$, $\xi \simeq \pi - 2B$. Under these approximations it can be shown that (40) simplifies to

$$\sin(\pi - 2B - \beta\ell) \;\simeq\; -B_L \cos^2\left(\frac{\pi\ell_2}{e}\right) \quad . \tag{41}$$

Using the small argument approximation on the left side of (41) it can be shown that

$$\beta\ell \;\simeq\; (\pi - 2B) + B_L \cos^2\left(\frac{\pi\ell_2}{e}\right) \quad . \tag{41}$$

Finally, identifying $\beta_r\ell \simeq \pi - 2B$ (Eqs. (17) and (18)), where β_r being the resonant propagation constant in the absence of the post we obtain the following expression for the resonant propagation constant from (41)

$$\beta\ell \;=\; \beta_r\ell + B_L\Big|_{\beta=\pi/\ell} \cos^2\left(\frac{\pi\ell_2}{\ell}\right) \quad . \tag{42}$$

Introducing the value of B_L (= $1/X_L$: Eq. (7)) we obtain the following expression for the operating frequency of a rectangular patch antenna tuned by a single post

$$f_{op} \;\simeq\; f_r\left[1 + \frac{2\ell}{\pi a\alpha}\,\frac{1}{\ln\dfrac{2\ell}{\gamma\pi r_o}}\,\cos^2\left(\frac{\pi\ell_2}{\ell}\right)\right] \quad , \tag{43}$$

where f_r is given by (20). Note that (43) indicates that $f = f_r$ when $\ell_2 = \ell/2$, a fact confirmed by experiment [9]. For N similar posts located at $\ell = \ell_n$, and ignoring the mutual effects between the posts (43) can be generalized and it can be shown that the operating frequency of rectangular patch antenna tuned by N similar posts located at ℓ_{2n} is

$$f_{op} \simeq f_r \left\{ 1 + \frac{1}{\pi} \sum_{1}^{N} \sin^{-1}\left[\frac{2\ell}{a\alpha} \frac{1}{\ln \dfrac{2\ell}{\gamma\pi r_0}} \cos^2\left(\frac{\pi\ell_n}{\ell}\right)\right]\right\} \tag{44}$$

C. Results

Figure 4 shows the measured [9] operating frequency and input VSWR vs the tuning post location for a rectangular patch antenna fed at one edge by a coaxial probe and tuned with a single post (having the same radius as that of the probe) located along the central line of the antenna. Corresponding theoretical values shown in Fig. 4 were obtained numerically by using (8) in conjunction with the equivalent transmission line circuit shown in Fig. 3. We have also calculated the operating frequencies using (43), and the results were almost identical with the theoretical values shown in Fig. 4 indicating thereby that (43) can be used with advantage for such purposes. The theoretical f_{op} vs ℓ_2/ℓ curve is symmetrical about $\ell_2/\ell = 0.5$ where the operating frequency is unaffected by the post (i.e., $f_{op} = f_r$); the latter fact is also borne out by the measured results. The slight asymmetry in the measured frequency variation is attributed to that caused by the feed.

The disagreement between the measured and calculated frequencies appears to be maximum when the post is located at the radiating edges. This may be due to the fact that edge effects have been ignored in obtaining the effective susceptance of the post.

Similar results are shown in Fig. 5 for the same antenna tuned by a pair of similar posts located symmetrically along the center line of the antenna. Here again (44) has been found to reproduce the numerically computed values without significant error.

Resonant parameters have been determined for the same antenna with four similar tuning posts symmetrically located along the two radiating edges each pair being located at a distance of 2.5 cm from its nearest side edge, and with five pairs of similar posts spaced symmetrically 1.5 cm apart along the two radiating edges. The results are shown in Table IV where some of the available measured [9] results are also given for comparison. Further calculations indicated that (44) can be used with advantage to determine the operating frequencies of antennas using up to six tuning posts.

V. Conclusions

The input performance of a rectangular patch antenna fed by a coaxial probe has been investigated by using the transmission line model. Approximate expressions have been developed for the resonant frequency (or propagation constant), quality factor and input resistance of the antenna which indicates clearly the influence of various antenna parameters including the feed under dynamic conditions. The validity

Table IV

Resonant Parameters for a Rectangular Patch Antenna Tuned by Four and Ten Posts

ℓ = 6.2 cm, a = 9.0 cm, d = 0.16 cm, r_0 = 0.064 cm, ε_r = 2.55

No. of Posts	Operating Frequency f_{op} (GHz)			VSWR
	Numerical	Eq. (44)	Measured	(Numerical)
4	2.187	2.013	1.995	1.95
10	2.409	3.238	2.240	3.9

of theoretical values obtained has been confirmed by comparison with available experimental results. The present work establishes the transmission line model as a valid representation of rectangular patch antennas, and the expressions derived would facilitate the design of such antennas without elaborate numerical computation. The model has also been used to determine the resonant parameters of a similar antenna tuned by a number of passive metallic posts. The results based on the theory developed were compared with available experimental results for antennas using one, two, four and ten tuning posts. Aprroximate expression given can be used to obtain the operating frequencies of such tunable antennas using up to six tuning posts; numerical method is preferable for antennas using more than six similar tuning posts.

VI. Acknowledgements

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Figure Captions

<u>Figure</u>

1 Rectangular microstrip antenna excited by a coaxial probe and tuned by two metallic posts.

2 Equivalent transmission line circuit for a rectangular patch antenna.

3 Equivalent transmission line circuit for an edge-fed rectangular patch antenna tuned by a single post.

4 Operating frequency (upper curves) and VSWR (lower curves) vs. post locations for a single post tuned rectangular patch antenna. ——— Calculated; ----- Measured.

5 Operating frequency (upper curves) and VSWR (lower curves) vs. post location for a double post tuned rectangular patch antenna. ——— Calculated; ----- Measured.

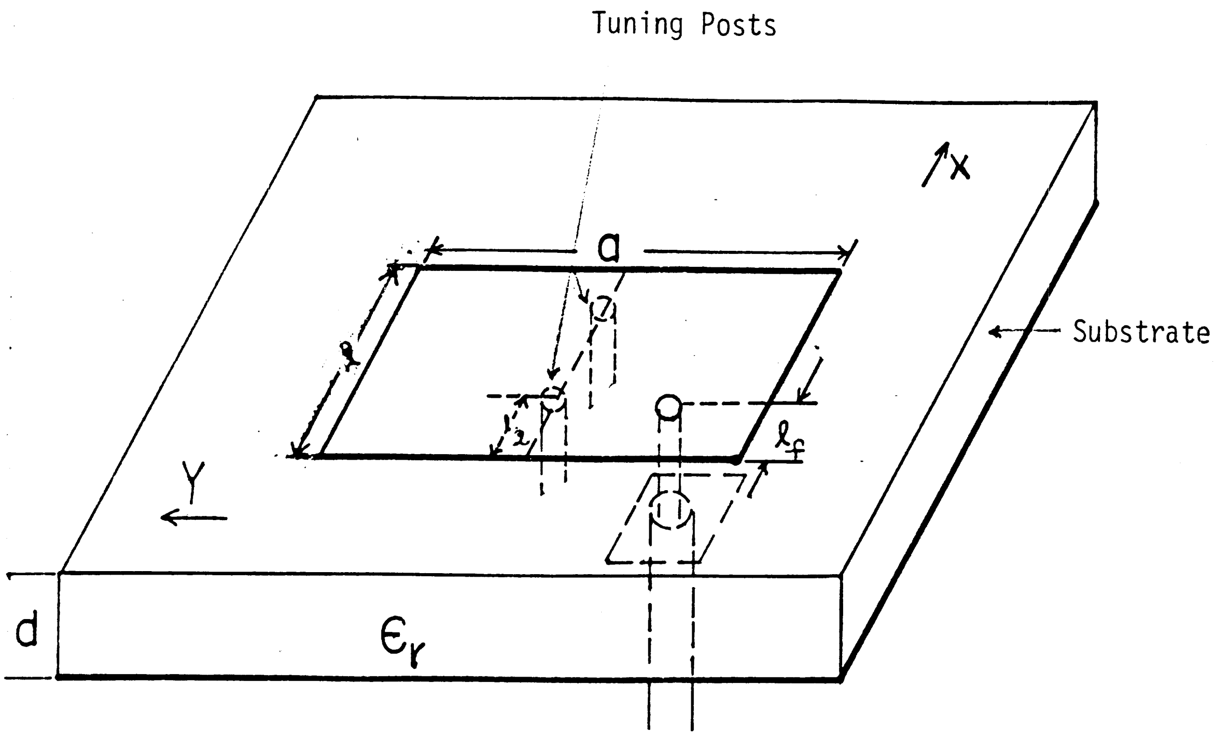

Figure 1. Rectangular microstrip antenna excited by a coaxial probe and tuned by two metallic posts.

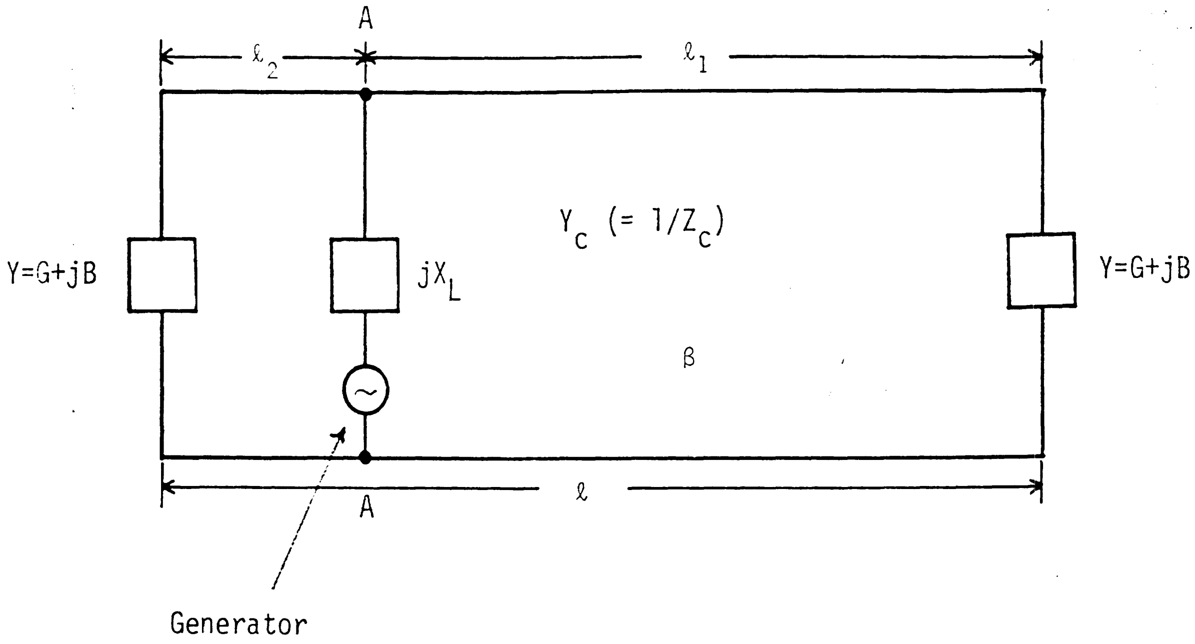

Figure 2. Equivalent transmission line circuit for a rectangular patch antenna.

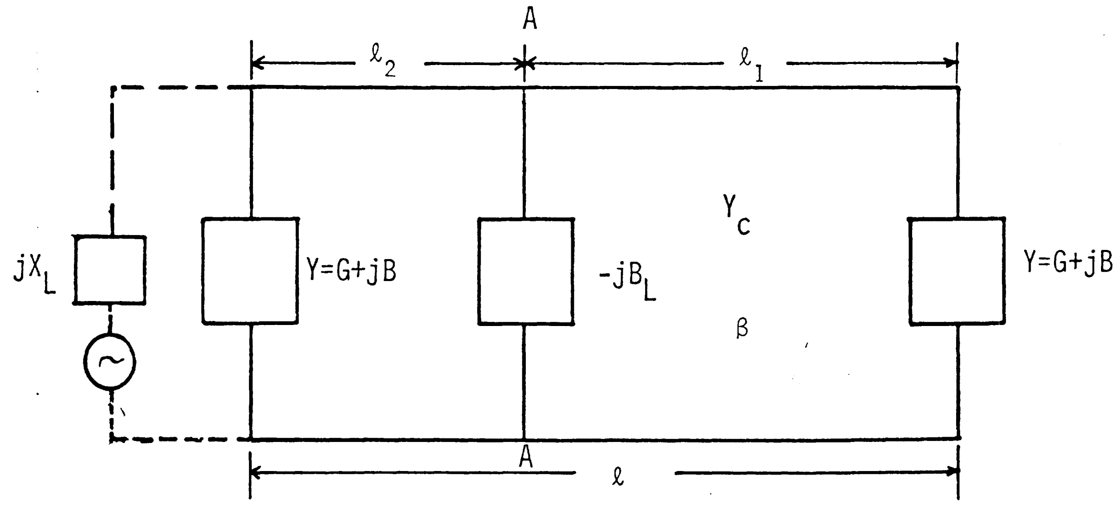

Figure 3. Equivalent transmission line circuit for an edge-fed rectangular patch antenna tuned by a single post.

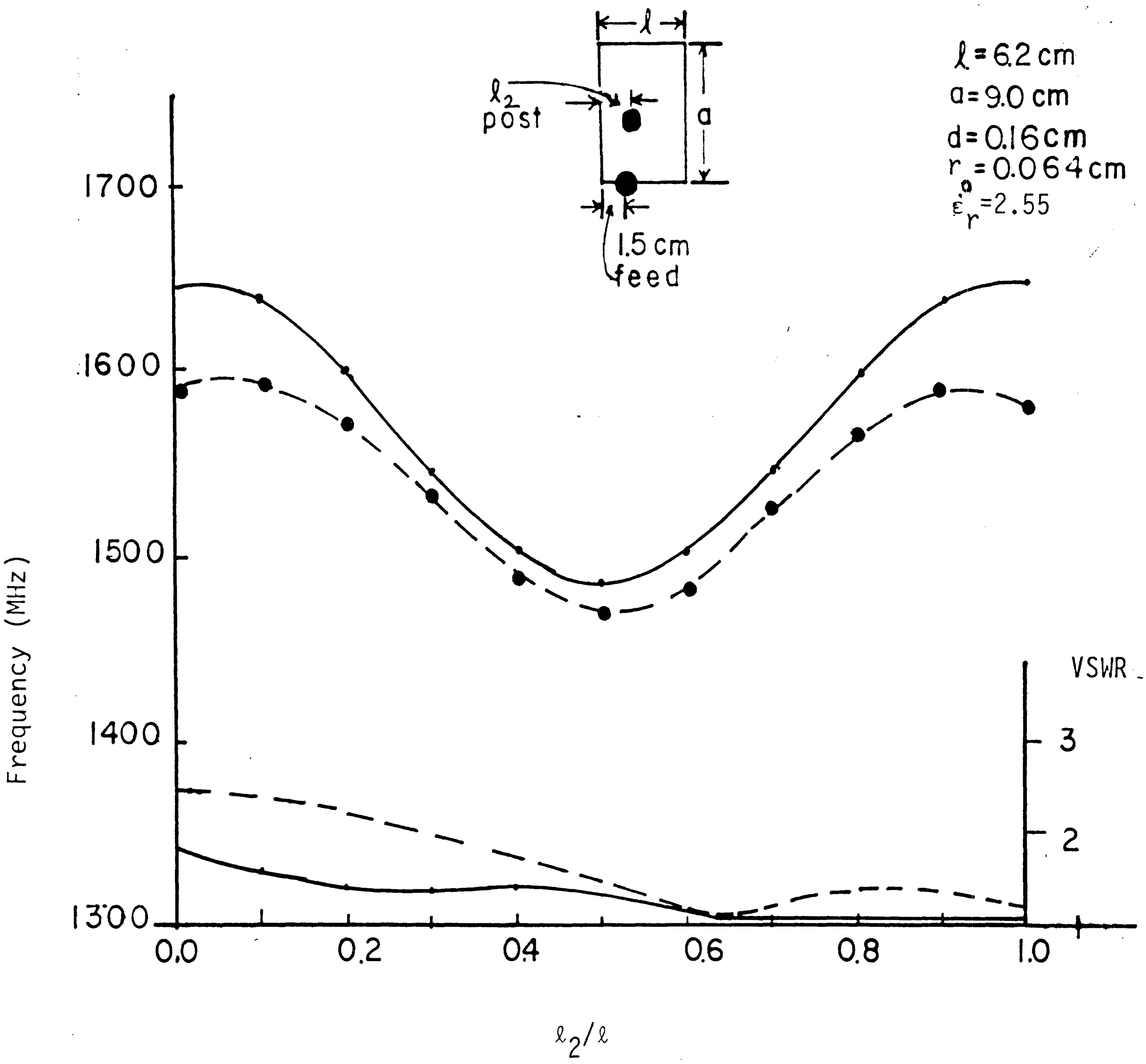

Figure 4. Operating frequency (upper curves) and VSWR (lower curves) vs. post locations for a single post tuned rectangular patch antenna. ——— Calculated; ----- Measured.

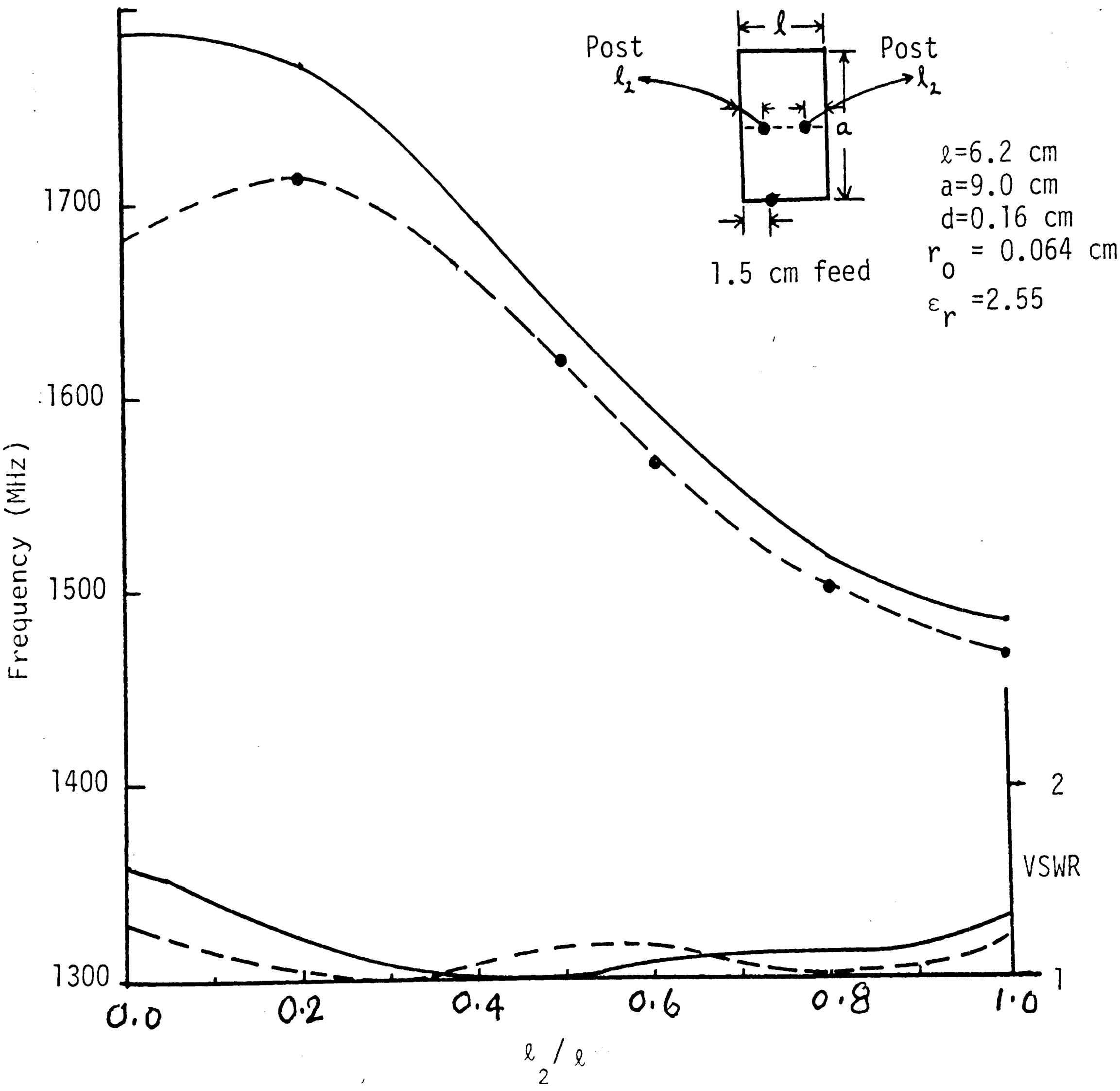

Figure 5. Operating frequency (upper curves) and VSWR (lower curves) vs. post location for a double post tuned rectangular patch antenna. ——— Calculated; ----- Measured.