INTEGRAL EQUATIONS FOR THE SCATTERING OF A PLANE WAVE BY AN ELECTRICALLY AND MAGNETICALLY PERMEABLE BODY

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Abstract

The integral equations governing the scattering of a plane electromagnetic wave by an electrically and magnetically permeable body are derived with the aid of the free-space electric dyadic Green functions. In contrast to previous works, equivalent volume currents or surface currents are not introduced.

The General Formulation

The problem under consideration is illustrated in Fig. 1 where Region I is occupied by an electrically and magnetically permeable body with constitutive constants μ_1 and ε_1 where ε_1 equal to $\varepsilon[1 + i(\sigma/\omega\varepsilon)]$ for a lossy dielectric body. In general, μ_1 could be complex also for a lossy magnetized body. A plane wave is impending upon the body which is placed in air with constitutive constants μ_o and ε_o. This problem has previously been investigated by many authors. References [I] and [II] provided a quite complete bibliography on this problem.

In this work we shall formulate the problem without introducing the concepts of equivalent electrically and magnetically polarized currents, the result appears to be simpler and the derivation more expedient. The pertinent equations are:

Region (I):

$$\nabla \times \bar{E}_1 = i\omega\mu_1 \bar{H}_1$$

$$\nabla \times \bar{H}_1 = -i\omega\varepsilon_1 \bar{E}_1 ,$$

hence

$$\nabla \times \nabla \times \begin{pmatrix} \bar{E}_1 \\ \bar{H}_1 \end{pmatrix} - k_1^2 \begin{pmatrix} \bar{E}_1 \\ \bar{H}_1 \end{pmatrix} = 0 \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

where

$$k_1^2 = \omega^2\mu_1\varepsilon_1 = \omega^2\mu_1\varepsilon \left(1 + \frac{i\sigma}{\omega\varepsilon}\right)$$

Region (II)

$$\nabla \times \bar{E}_2 = i\omega\mu_o \bar{H}_2$$

$$\nabla \times \bar{H}_2 = -i\omega\varepsilon_o \bar{E}_2$$

hence

$$\nabla \times \nabla \times \begin{pmatrix} \bar{E}_2 \\ \bar{H}_2 \end{pmatrix} - k_0^2 \begin{pmatrix} \bar{E}_2 \\ \bar{H}_2 \end{pmatrix} = 0 \qquad \begin{matrix} (3) \\ (4) \end{matrix}$$

where $k_0^2 = \omega^2\mu_o\varepsilon_o$.

Equations (1-4) can now be integrated with the aid of the vector-dyadic Green's theorem and the free-space electric dyadic Green's function defined in air (μ_o, ε_o) which satisfies the equation

$$\nabla \times \nabla \times \bar{\bar{G}}_0 - k_0^2 \bar{\bar{G}}_0 = \bar{\bar{I}}\delta(\bar{R} - \bar{R}') \quad . \qquad (5)$$

The vector-dyadic Green's theorem states

$$\iiint_V [\bar{P} \cdot \nabla \times \nabla \times \bar{\bar{Q}} - (\nabla \times \nabla \times \bar{P}) \cdot \bar{\bar{Q}} \, dv = - \oiint_S \hat{n} \cdot [\bar{P} \times \nabla \times \bar{\bar{Q}} + (\nabla \times \bar{P}) \times \bar{\bar{Q}}] \, dS \quad , \qquad (6)$$

where $\hat{n}$ denotes the outward normal to S. Now let

$$\bar{P} = \bar{E}_1 \quad , \quad \bar{\bar{Q}} = \bar{\bar{G}}_0(\bar{R}/\bar{R}')$$

with $\bar{R}'$ located in Region (I). Substituting (1) and (5) into (6) we obtain

$$\bar{E}_1(\bar{R}') = (k_1^2 - k_0^2) \iiint_{V_1} \bar{E}_1(\bar{R}) \cdot \bar{\bar{G}}_0(\bar{R}/\bar{R}')\, dV$$

$$- \oiint_S \hat{n}_1 \cdot [\bar{E}_1(\bar{R}) \times \nabla \times \bar{\bar{G}}_0(\bar{R}/\bar{R}') + (\nabla \times \bar{E}_1(\bar{R})) \times \bar{\bar{G}}_0(\bar{R}/\bar{R}')]\, dS \qquad (7)$$

Following the same procedure for $\bar{H}_1$ we obtain

$$\bar{H}_1(\bar{R}') = (k_1^2 - k_0^2) \iiint_V \bar{H}_1(\bar{R}) \cdot \bar{\bar{G}}_0(\bar{R}/\bar{R}')\, dV$$

$$- \oiint_S \hat{n}_1 \cdot [\bar{H}_1(\bar{R}) \times \nabla \times \bar{\bar{G}}_0(\bar{R}/\bar{R}') + (\nabla \times \bar{H}_1(\bar{R})) \times \bar{\bar{G}}_0(\bar{R}/\bar{R}')]\, dS\ . \qquad (8)$$

Equations (7) and (8) are compatible. In other words, Eq. (8) can be derived from Eq. (7) by using the relation $\nabla' \times \bar{E}_1(\bar{R}') = i\omega\mu_1 \bar{H}_1(\bar{R}')$. The proof is omitted here.

By integrating Eq. (3) in Region II using the same $\bar{\bar{G}}_0(\bar{R}/\bar{R}')$ with $\bar{R}'$ located in Region (I) we obtain

$$\iint_{S+S_\infty} \hat{n}_2 \cdot [\bar{E}_2 \times \nabla \times \bar{\bar{G}}_0(\bar{R}/\bar{R}') + (\nabla \times \bar{E}_2(\bar{R})) \times \bar{\bar{G}}_0(\bar{R}/\bar{R}')]dS = 0$$

$$(9)$$

The surface integral evaluated on S_∞ has a simple interpretation. If we consider a plane wave propagating in an empty space the integration of Eq. (3) with $\bar{\bar{G}}_0(\bar{R}/\bar{R}')$ would yield

$$\bar{E}^{(i)}(\bar{R}') = - \oiint_{S_\infty} \hat{n}_2 \cdot [\bar{E}^{(i)} \times \nabla \times \bar{\bar{G}}_0 + (\nabla \times \bar{E}^{(i)}) \times \bar{\bar{G}}_0] dS \quad (10)$$

because in the absence of a scattering body $\bar{E}_2 = \bar{E}^{(i)}$. In the presence of the scattered body we can write

$$\bar{E}_2 = \bar{E}^{(i)} + \bar{E}_2^{(s)} \quad (11)$$

$$\bar{E}_1 = \bar{E}^{(i)} + \bar{E}_1^{(s)} \quad (12)$$

The surface integral evaluated on S_∞ in (9) can be decomposed into two parts, i.e.,

$$\iint_{S_\infty} \hat{n}_2 \cdot [\bar{E}^{(i)} \times \nabla \times \bar{\bar{G}}_0 + (\nabla \times \bar{E}^{(i)}) \times \bar{\bar{G}}_0] dS$$

$$+ \iint_{S_\infty} \hat{n}_2 \cdot [\bar{E}_2^{(s)} \times \nabla \times \bar{\bar{G}}_0 + (\nabla \times \bar{E}_2^{(s)}) \times \bar{\bar{G}}_0] dS$$

Because of the radiation condition of $\bar{\bar{G}}_0$ and $\bar{E}^{(s)}$ at infinity the second integral involving $\bar{E}^{(s)}$ vanishes while the first integral, in view of Eq. (10), represents $-\bar{E}^{(i)}(\bar{R})$ hence Eq. (9) is equivalent to

$$\bar{E}^{(i)}(\bar{R}') = \oiint_{S} \hat{n}_2 \cdot [\bar{E}_2 \times \nabla \times \bar{\bar{G}}_0 + (\nabla \times \bar{E}_2) \times \bar{\bar{G}}_0] dS \quad (13)$$

where we have omitted the dependent variables pertaining to various terms in the integrand. Similarly it can be shown

$$\bar{H}^{(i)}(\bar{R}') = \oiint_S \hat{n}_2 \cdot [\bar{H}_2 \times \nabla \times \bar{\bar{G}}_0 + (\nabla \times \bar{H}_2) \times \bar{\bar{G}}_0]dS \tag{14}$$

Using these two relations Eqs. (7) and (8) can be changed to an alternative form involving the incident field. We consider the surface term in Eq. (7). On S the boundary conditions are

$$\hat{n}_1 \times (\bar{E}_1 - \bar{E}_2) = 0 \tag{15}$$

$$\hat{n}_1 \times \left[\frac{\nabla \times \bar{E}_1}{\mu_1} - \frac{\nabla \times \bar{E}_2}{\mu_o}\right] = 0 \tag{16}$$

hence

$$\hat{n}_1 \times \nabla \times \bar{E}_1 = \left(\frac{\mu_1}{\mu_o}\right) \hat{n}_1 \times \nabla \times \bar{E}_2$$

$$= \left[1 + \left(\frac{\mu_1 - \mu_o}{\mu_o}\right)\right] \hat{n}_1 \times \nabla \times \bar{E}_2$$

The surface term in Eq. (7) therefore can be written in the form

$$- \oiint_S \hat{n}_1 \cdot [\bar{E}_2 \times \nabla \times \bar{\bar{G}}_0 + (\nabla \times \bar{E}_2) \times \bar{\bar{G}}_0]dS$$

$$- \left(\frac{\mu_1 - \mu_o}{\mu_o}\right) \oiint_S \hat{n}_1 \cdot [(\nabla \times \bar{E}_2) \times \bar{\bar{G}}_0]dS$$

$$= \bar{E}^{(i)}(\bar{R}') - \left(\frac{\mu_1 - \mu_o}{\mu_1}\right) \oiint_S \hat{n}_1 \cdot [(\nabla \times \bar{E}_1) \times \bar{\bar{G}}_0]dS \tag{17}$$

Equation (7) thus can be transformed to

$$\bar{E}_1(\bar{R}') - \bar{E}^{(i)}(\bar{R}') = (k_1^2 - k_0^2) \iiint_{V_1} \bar{E}_1(\bar{R}) \cdot \bar{\bar{G}}_0(\bar{R}/\bar{R}')dV$$

$$-\left(\frac{\mu_1 - \mu_0}{\mu_1}\right) \oiint_S \hat{n}_1 \cdot [(\nabla \times \bar{E}_1) \times \bar{\bar{G}}_0(\bar{R}/\bar{R}')]dS \quad (18)$$

By switching the primed and unprimed variables and making use of the symmetrical property of dyadic Green's function Eq. (18) can finally be written in the form

$$\bar{E}_1^{(s)}(\bar{R}) = (k_1^2 - k_0^2) \iiint_{V_1} \bar{\bar{G}}_0(\bar{R}/\bar{R}') \cdot \bar{E}_1(\bar{R}')dV'$$

$$-\left(\frac{\mu_1 - \mu_0}{\mu_1}\right) \oiint_S \bar{\bar{G}}_0(\bar{R}/\bar{R}') \cdot [\hat{n}_1 \times \nabla' \times \bar{E}_1(\bar{R}')]dS' \quad (19)$$

where $\bar{E}_1^{(s)} = \bar{E}_1 - \bar{E}^{(i)}$. Carrying out the same manipulation for $\bar{H}_1$ one finds

$$\bar{H}_1^{(s)}(\bar{R}) = (k_1^2 - k_0^2) \iiint_{V_1} \bar{\bar{G}}_0(\bar{R}/\bar{R}') \cdot \bar{H}_1(\bar{R}')dV'$$

$$-\left(\frac{\varepsilon_1 - \varepsilon_0}{\varepsilon_1}\right) \oiint_S \bar{\bar{G}}_0(\bar{R}/\bar{R}') \cdot [\hat{n}_1 \times \nabla' \times \bar{H}_1(\bar{R}')]dS' \quad (20)$$

where $H_1^{(s)} = \bar{H}_1 - \bar{H}^{(i)}$.

For a purely permeable dielectric body the surface integral disappears in Eq. (19) and for a purely permeable magnetized body the surface integral disappears in Eq. (20). To solve these equations numerically one can split the volume integral into an indented part and a principal part.

Once $\bar{E}_1$ and $\bar{H}_1$ are determined the scattered field in Region (II) can be calculated using the formula

$$\bar{E}_2^{(s)}(\bar{R}) = - \oint\!\!\oint_S \nabla \times \bar{\bar{G}}_0 \cdot [\hat{n}_1 \times \bar{E}_1] + \frac{\mu_0}{\mu_1} \bar{\bar{G}}_0 \cdot [\hat{n}_1 \times \nabla \times \bar{E}_1] dS \tag{21}$$

Inhomogeneous Dielectric Body

For an inhomogeneous dielectric body, nonmagnetic, the governing equations are

$$\nabla \times \bar{E}_1 = i\omega\mu_o \bar{H}_1 \tag{22}$$

$$\nabla \times \bar{H}_1 = -i\omega\varepsilon(\bar{R})\bar{E}_1 \tag{23}$$

hence

$$\nabla \times \nabla \times \bar{E}_1 - k^2(\bar{R})\bar{E}_1 = 0 \tag{24}$$

where

$$k^2(\bar{R}) = \omega^2\mu_o\varepsilon(\bar{R}) \quad .$$

The integration of (24) with the aid of the free space electric dyadic Green's function yields

$$\bar{E}_1^{(s)}(\bar{R}) = \bar{E}_1(\bar{R}) - \bar{E}^{(i)}(\bar{R}) = \iiint_{V_1} [k^2(\bar{R}) - k_o^2]\bar{\bar{G}}_o(\bar{R}/\bar{R}') \cdot \bar{E}_1(\bar{R}')dV' \tag{25}$$

which is an exact integral equation for $\bar{E}_1(\bar{R})$ inside an inhomogeneous dielectric body.

References

(I) Research Topics in Electromagnetic Wave Theory, Edited by J.A. Kong, Chap. 13: K. M. Chen, Interaction of Electromagnetic Fields with Biological Bodies, John Wiley and Sons, NY, 1980.

(II) A. W. Glissen and D. R. Wilton, "Simple and Efficient Numerical Technique for Treating Bodies of Revolution," Tech. Report 105, Engineering Experimental Station, University of Mississippi, May, 1982.

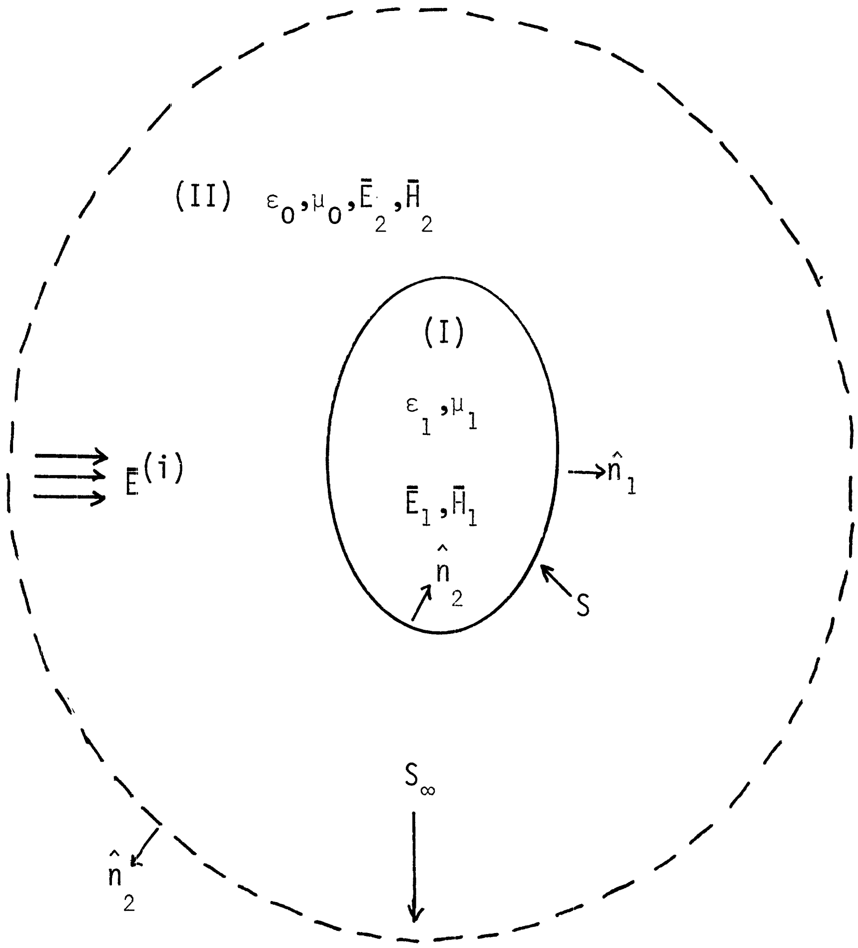

Fig. 1: Scattering of a Plane Wave by a Permeable Body.