THE GREEN'S FUNCTION FOR A SLOT ON
THE GROUND OF A DIELECTRIC SLAB

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1. Introduction

Recently, a number of papers have been published [1]-[3], on the subject of waveguide slot arrays. These papers describe a procedure one can use to calculate self and mutual admittances between slots. This procedure can consist the basis of a design scheme where the lengths and offsets of individual slots can be computed to produce a specified pattern and input impedance in the presence of mutual coupling.

In many practical applications, the slot array is covered by a dielectric slab in order to protect the array as well as modify its characteristics. For this new array, a design scheme can be described only if the theoretical formulas for Y_{ij} are available.

Since analytically evaluating Y_{ij} is a very complicated boundary value problem, numerical evaluation of Y_{ij} may substitute for the needed theoretical formulas. A first step is to obtain the Green's function, that is, the field due to a suitably positioned Hertzian Magnetic Dipole (HMD), since each slot can be viewed as a collection of HMDs. The detailed derivation of the appropriate Green's function will be presented in this report.

Sommerfeld first analyzed the cases of both vertical and horizontal Hertzian electric dipoles at a distance h above a semi-infinite lossy dielectric [4]. Since that time, other investigators extended his work to grounded dielectric slabs [5]. A problem very similar to the one presented here is the derivation of the Green's function for a Hertzian electric dipole which has

been solved by R. S. Elliott [6]. Many details of that work will be followed here.

2. Wave Equation

Consider a horizontal HMD at the position $x'=0$, $y'=0$, $z'=-b_s$ (Fig. 1). The dielectric slab is assumed infinite in extent with a dielectric constant ε and a permeability μ. With a $e^{j\omega t}$ time dependance, Maxwell's equations take the form:

$$\vec{\nabla} \times \vec{E} = -j\omega\mu\vec{H} - \vec{M} \tag{1}$$

$$\vec{\nabla} \times \vec{H} = j\omega\varepsilon\vec{E} \tag{2}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{3}$$

$$\vec{\nabla} \cdot \vec{D} = 0 \tag{4}$$

where

$$\vec{B} = \mu\vec{H} \tag{5}$$

$$\vec{D} = \varepsilon\vec{E} \tag{6}$$

and

$$\mu = \mu'(1-j\tan\delta_\mu) \tag{7}$$

$$\varepsilon = \varepsilon'(1-j\tan\delta_\varepsilon). \tag{8}$$

Because of (4), the electric flux density $\vec{D}$ can be expressed as the curl of a vector potential $\vec{F}$ as following:

$$\vec{D} = \vec{\nabla} \times \vec{F}. \tag{9}$$

When we insert equation (9) into equation (2), the result is that

$$\vec{\nabla} \times (\vec{H} - j\omega\vec{F}) = 0 \tag{10}$$

or that

$$\vec{H} = j\omega\vec{F} + \vec{\nabla}\phi_m \tag{11}$$

where ϕ_m is an unknown scalar function. When equation (11) is

placed in equation (1), then we result in the following equation

$$-\nabla^2 \vec{F} + \vec{\nabla}\, \vec{\nabla} \cdot \vec{F} = \omega^2 \epsilon\mu\vec{F} - j\omega\epsilon\mu\vec{\nabla}\phi_m - \epsilon\vec{M} \tag{12}$$

Since ϕ_m is not yet defined, we may assume that

$$\phi_m = - \frac{1}{j\omega\epsilon\mu}\, \vec{\nabla} \cdot \vec{F}. \tag{13}$$

This relation is known as the Lorentz transformation. In view of

equation (13), equation (12) takes the form

$$\nabla^2 \vec{F} + k^2 \vec{F} = \epsilon\vec{M} \tag{14}$$

i.e., the inhomogeneous wave equation. At points away from the

source, the electric and magnetic fields are give by

$$\vec{E} = \frac{1}{\epsilon}\, \vec{\nabla} \times \vec{F} \tag{15}$$

and

$$\vec{H} = \frac{j}{\omega\epsilon\mu}\, \left[k^2\, \bar{\bar{I}} + \vec{\nabla}\,\vec{\nabla} \right] \cdot \vec{F}. \tag{16}$$

3. Solutions to Wave Equation

The solutions to Wave Equation can be classified to primary
and secondary. Primary is the solution to the inhomogeneous
equation

$$\nabla^2 \vec{F}^P + k^2 \vec{F}^P = \varepsilon\delta(x-x')\ \delta(y-y')\ \delta(z-z')\hat{x} \tag{17}$$

while secondary is the solution to the homogeneous one

$$\nabla^2 \vec{F}^S + k^2 \vec{F}^S = 0. \tag{18}$$

It is known that the primary solution is given by

$$\vec{F}^P = \hat{x}\left(-\frac{\varepsilon}{4\pi}\right)\frac{e^{-jkr}}{r}. \tag{19}$$

In equation (19), the origin of the spherical coordinate system is
located at the position of the magnetic dipole.

As it has been shown in [4], [6], the function $\dfrac{e^{-jkr}}{r}$ can be

written in the form

$$\frac{e^{-jkr}}{r} = \int_0^\infty J_0(\lambda\rho)\ e^{-u|z+b_s|}\frac{\lambda}{u}\ d\lambda. \tag{20}$$

In equation (20), $u = \sqrt{\lambda^2 - k^2}$.

Next, we need to turn our attention to the secondary solution. As
Sommerfeld has shown (his proof is reproduced in Appendix A), the
secondary solution must have an x and a z component which can be
put in the following form:

$$F_x^s = -\frac{\varepsilon}{4\pi}\int_0^\infty A_x(\lambda)\ J_0(\lambda\rho)\ e^{-u|z+b_s|}d\lambda \tag{21}$$

and

$$F_z^s = - \frac{\varepsilon}{4\pi} \cos\phi \int_0^\infty A_z(\lambda) \ J_1(\lambda\rho) \ e^{-u|z+b_s|} d\lambda \qquad (22)$$

in which $A_x(\lambda)$ and $A_z(\lambda)$ are functions to be determined from the boundary conditions. In this manner, the solution was transform from the spherical coordinate system to a cylindrical one shown in figure 1.

In view of the above, the solutions to the wave equation in regions I, II and III can be put into the form:

Region I

$$\vec{F}_1 = F_{1x}^s \ \hat{x} + F_{1z}^s \ \hat{z} \qquad (23)$$

Region II

$$\vec{F}_2 = \left(F_{2x}^p + F_{2x}^s \right) \hat{x} + F_{2z}^s \ \hat{z} \qquad (24)$$

Region III

$$\vec{F}_3 = \left(F_{3x}^p + F_{3x}^s \right) \hat{x} + F_{3z}^s \ \hat{z} \qquad (25)$$

Equations (23)-(25) in integral form may be written as:

$$\vec{F}_1 = \hat{x} \left(- \frac{\varepsilon_o}{4\pi} \right) \int_o^\infty A_{1x}(\lambda) \ J_o(\lambda\rho) e^{-u_o z} d\lambda +$$

$$+ \hat{z} \left(- \frac{\varepsilon_o}{4\pi} \right) \cos\phi \int_o^\infty A_{1z}(\lambda) \ J_1(\lambda\rho) e^{-u_o z} d\lambda \qquad (26)$$

$$\vec{F}_2 = \hat{x} \left(- \frac{\varepsilon}{4\pi} \right) \int_o^\infty J_o(\lambda) \left[e^{-u(z+b_s)} \frac{\lambda}{u} + A_{2x}(\lambda) e^{-uz} + \right.$$

$$\left. + B_{2x}(\lambda) e^{uz} \right] d\lambda +$$

$$+ \hat{z} \left(- \frac{\varepsilon}{4\pi} \right) \cos\phi \int_o^\infty J_1(\lambda\rho) \left[A_{2z}(\lambda) e^{-uz} + B_{2z}(\lambda) e^{uz} \right] d\lambda \qquad (27)$$

$$\vec{F}_3 = \hat{x} \left(- \frac{\varepsilon}{4\pi} \right) \int_o^\infty J_o(\lambda\rho) \left[e^{u(z+b_s)} \frac{\lambda}{u} + A_{3x}(\lambda) e^{-uz} + \right.$$

$$\left. + B_{3x}(\lambda) e^{uz} \right] d\lambda +$$

$$+ \hat{z} \left(- \frac{\varepsilon}{4\pi} \right) \cos\phi \int_o^\infty J_1(\lambda\rho) \left[A_{3z}(\lambda) e^{-uz} + B_{3z}(\lambda) e^{uz} \right] d\lambda. \qquad (28)$$

In equations (26)-(28) $u_o = \sqrt{\lambda^2 - k_o^2}$, $u = \sqrt{\lambda^2 - k^2}$ with $k_o^2 = \omega^2 \varepsilon_o \mu_o$ and $k^2 = \omega^2 \varepsilon \mu$.

4. Boundary Conditions

In this boundary value problem, there are three interfaces to be considered. The first is the ground plane ($z=-h$), the second is an imaginary horizontal plane going through the center of the magnetic dipole and the third one coincides with the air-dielectric interface. Continuity of the tangential electromagnetic field through each of these interfaces give the following relations:

$$
\left.
\begin{aligned}
E_{3x} &= 0 \\
E_{3y} &= 0
\end{aligned}
\right\} \; z=-h
\tag{29}
$$

$$
\left.
\begin{aligned}
E_{2x}^{s} &= E_{3x}^{s} \\
E_{2y}^{s} &= E_{3y}^{s} \\
H_{2x}^{s} &= H_{3x}^{s} \\
H_{2y}^{s} &= H_{3y}^{s}
\end{aligned}
\right\} \; z=-bs
\tag{30}
$$

$$
\left.
\begin{aligned}
E_{2x} &= E_{1x} \\
E_{2y} &= E_{1y} \\
H_{2x} &= H_{1x} \\
H_{2y} &= H_{1y}
\end{aligned}
\right\} \; z=0
\tag{31}
$$

In order to translate these equations into boundary conditions for the electric vector potential $\vec{F}$, equations (15) and (16) have to be considered. From equation (15), the following relations can be derived

$$E_x = \frac{1}{\varepsilon} \frac{\partial F_z}{\partial y} \tag{32}$$

$$E_y = \frac{1}{\varepsilon} \left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z} \right) \tag{33}$$

$$E_z = - \frac{1}{\varepsilon} \frac{\partial F_x}{\partial y} \tag{34}$$

Also from equation (16), the components of the magnetic field are given by

$$H_x = \frac{j}{\omega\varepsilon\mu} \left(k^2 F_x + \frac{\partial}{\partial x} \vec{\nabla} \cdot \vec{F} \right) \tag{35}$$

$$H_y = \frac{j}{\omega\varepsilon\mu} \frac{\partial}{\partial y} \vec{\nabla} \cdot \vec{F} \tag{36}$$

$$H_z = \frac{j}{\omega\varepsilon\mu} \left(k^2 F_z + \frac{\partial}{\partial z} \vec{\nabla} \cdot \vec{F} \right) \tag{37}$$

In view of equations (32)-(37), equations (29)-(31) give the following relations

$$\frac{\partial F_{3z}}{\partial \rho} = 0 \qquad\qquad z = -h \tag{38}$$

$$\frac{\partial F_{3x}}{\partial z} = 0 \qquad\qquad z = -h \tag{39}$$

$$\frac{\partial F_{2z}^{s}}{\partial \rho} = \frac{\partial F_{3z}^{s}}{\partial \rho} \qquad\qquad z = -b_s \tag{40}$$

$$F^s_{2x} = F^s_{3x} \qquad z=-b_s \qquad (41)$$

$$\frac{\partial F^s_{2x}}{\partial z} = \frac{\partial F^s_{3x}}{\partial z} \qquad z=-b_s \qquad (42)$$

$$\frac{\partial}{\partial \rho} \vec{\nabla} \cdot \vec{F}^s_2 = \frac{\partial}{\partial \rho} \vec{\nabla} \cdot \vec{F}^s_3 \qquad z=-b_s \qquad (43)$$

$$F_{1x} = F_{2x} \qquad z=0 \qquad (44)$$

$$\frac{1}{\varepsilon_o} \frac{\partial F_{1z}}{\partial \rho} = \frac{1}{\varepsilon} \frac{\partial F_{2z}}{\partial \rho} \qquad z=0 \qquad (45)$$

$$\frac{1}{\varepsilon_o} \frac{\partial}{\partial \rho} \left(\vec{\nabla} \cdot \vec{F}_1 \right) = \frac{1}{\varepsilon} \frac{\partial}{\partial \rho} \left(\vec{\nabla} \cdot \vec{F}_2 \right) \qquad z=0 \qquad (46)$$

$$\frac{1}{\varepsilon_o} \frac{\partial F_{1x}}{\partial z} = \frac{1}{\varepsilon} \frac{\partial F_{2x}}{\partial z} \qquad z=0 \qquad (47)$$

Equations (38)-(47) provide ten necessary conditions for the ten unknown functions appearing in the expressions for the electric vector potential $\vec{F}$ which is also called the Green's function for the problem. Specifically we have:

$$A_{3z}(\lambda) e^{uh} + B_{3z}(\lambda) e^{-uh} = 0 \qquad (48)$$

$$A_{3x}(\lambda) e^{uh} - \left[e^{ub_s} \frac{\lambda}{u} + B_{3x}(\lambda) \right] e^{-uh} = 0 \qquad (49)$$

$$A_{2z}(\lambda) e^{ub_s} + B_{2z}(\lambda) e^{-ub_s} = A_{3z}(\lambda) e^{ub_s} + B_{3z}(\lambda) e^{-ub_s} \qquad (50)$$

$$A_{2x}(\lambda) e^{ub_s} + B_{2x}(\lambda) e^{-ub_s} = A_{3x}(\lambda) e^{ub_s} + B_{3x}(\lambda) e^{-ub_s} \qquad (51)$$

$$A_{2x}(\lambda) e^{ub_s} - B_{2x}(\lambda) e^{-ub_s} = A_{3x}(\lambda) e^{ub_s} - B_{3x}(\lambda) e^{-ub_s} \qquad (52)$$

$$\lambda \left[A_{2x}(\lambda) e^{ub_s} + B_{2x}(\lambda) e^{-ub_s} \right] + uA_{2z}(\lambda) e^{ub_s} - uB_{2z}(\lambda) e^{-ub_s} =$$

$$= \lambda \left[A_{3x}(\lambda) e^{ub_s} + B_{3x}(\lambda) e^{-ub_s} \right] + uA_{3z}(\lambda) e^{ub_s} - uB_{3z}(\lambda) e^{-ub_s} \qquad (53)$$

$$\varepsilon_o A_{1x}(\lambda) = \varepsilon \left[e^{-ub_s} \frac{\lambda}{u} + A_{2x}(\lambda) \right] + \varepsilon B_{2x}(\lambda) \qquad (54)$$

$$A_{1z}(\lambda) = A_{2z}(\lambda) + B_{2z}(\lambda) \qquad (55)$$

$$\lambda A_{1x}(\lambda) + u_o A_{1z}(\lambda) = \lambda \left[e^{-ub_s} \frac{\lambda}{u} + A_{2x}(\lambda) + B_{2x}(\lambda) \right] +$$

$$+ uA_{2z}(\lambda) - uB_{2z}(\lambda) \qquad (56)$$

$$u_o A_{1x}(\lambda) = u \left[e^{-ub_s} \frac{\lambda}{u} + A_{2x}(\lambda) \right] - u B_{2x}(\lambda) \tag{57}$$

5. Green's Function

The simultaneous satisfaction of equations (48) to (57) gives

$$A_{1x} = 2\varepsilon_r \lambda \frac{\cosh[u(b_s-h)]}{f_1(\lambda,\varepsilon_r,h)} \tag{58}$$

$$A_{1z} = 2(1-\varepsilon_r)\lambda^2 \frac{\cosh[u(b_s-h)]}{f_1(\lambda,\varepsilon_r,h)} \cdot \frac{\sinh(uh)}{f_2(\lambda,\varepsilon_r,h)} \tag{59}$$

$$A_{2x} = \frac{\lambda}{u} e^{-uh} \frac{\varepsilon_r u_0 \sinh(ub_s) + u\cosh(ub_s)}{f_1(\lambda,\varepsilon_r,h)} \tag{60}$$

$$A_{2z} = (\varepsilon_r-1)e^{-uh} \lambda^2 \frac{\cosh[u(b_s-h)]}{f_1(\lambda,\varepsilon_r,h)} \cdot \frac{1}{f_2(\lambda,\varepsilon_r,h)} \tag{61}$$

$$B_{2x} = (u-\varepsilon_r u_0) \frac{\lambda}{u} \frac{\cosh[u(b_s-h)]}{f_1(\lambda,\varepsilon_r,h)} \tag{62}$$

$$B_{2z} = (1-\varepsilon_r)e^{uh} \lambda^2 \frac{\cosh[u(b_s-h)]}{f_1(\lambda,\varepsilon_r,h)} \cdot \frac{1}{f_2(\lambda,\varepsilon_r,h)} \tag{63}$$

$$A_{3x}(\lambda) = A_{2x}(\lambda) \tag{64}$$

$$B_{3x}(\lambda) = B_{2x}(\lambda) \tag{65}$$

$$A_{3z}(\lambda) = A_{2z}(\lambda) \tag{66}$$

$$B_{3z}(\lambda) = B_{2z}(\lambda) \tag{67}$$

In the above equations

$$f_1(\lambda, \varepsilon_r, h) = \varepsilon_r u_o \cosh(uh) + u\sinh(uh) \tag{68}$$

$$f_2(\lambda, \varepsilon_r, h) = u\cosh(uh) + u_o \sinh(uh) \tag{69}$$

When these results are placed in equations (26)-(28), one has the Green's function for an HMD a distance b_s below the air-dielectric interface.

$$F_{1x} = -\frac{\varepsilon}{2\pi} \int_0^\infty \frac{\lambda\cosh[u(b_s-h)]}{f_1(\lambda, \varepsilon_r, h)} J_o(\lambda\rho) e^{-u_o z} \, d\lambda \tag{70}$$

$$F_{1z} = -\frac{\varepsilon_o(1-\varepsilon_r)}{2\pi} \cos\phi \int_0^\infty \frac{\cosh[u(b_s-h)]}{f_1(\lambda, \varepsilon_r, h)} \cdot \frac{\sinh(uh)}{f_2(\lambda, \varepsilon_r, h)} e^{-u_o z} J_1(\lambda\rho) \lambda^2 d\lambda \tag{71}$$

$$F_{2x} = -\frac{\varepsilon}{2\pi} \int_0^\infty \frac{\lambda}{u} \cosh[u(b_s-h)] \frac{u\cosh(uz)-\varepsilon_r u_o \sinh(uz)}{f_1(\lambda, \varepsilon_r, h)} J_o(\lambda\rho) \, d\lambda \tag{72}$$

$$F_{2z} = -\frac{\varepsilon}{2\pi}(1-\varepsilon_r) \cos\phi \int_0^\infty \frac{\cosh[u(b_s-h)]}{f_1(\lambda, \varepsilon_r, h)} \cdot \frac{\sinh[u(h+z)]}{f_2(\lambda, \varepsilon_r, h}J_1(\lambda\rho)\lambda^2 d\lambda \tag{73}$$

$$\left\{\begin{array}{l} F_{3x} = - \dfrac{\varepsilon}{2\pi} \displaystyle\int_{0}^{\infty} \dfrac{\lambda}{u} \cosh[u(h+z)] \ \dfrac{\varepsilon_r u_0 \sinh(ub_s) + u\cosh(ub_s)}{f_1(\lambda, \varepsilon_r, h)} \ J_0(\lambda\rho)\,d\lambda \qquad (74) \\[4mm] F_{3z} = - \dfrac{\varepsilon}{2\pi} (1-\varepsilon_r)\cos\phi \displaystyle\int_{0}^{\infty} \dfrac{\cosh[u(b_s-h)]}{f_1(\lambda, \varepsilon_r, h)} \cdot \dfrac{\sinh[u(h+z)]}{f_2(\lambda, \varepsilon_r, h)} J_1(\lambda\rho)\lambda^2\,d\lambda \qquad (75) \end{array}\right.$$

Conclusions

The Green's function for a horizontal Hertzian magnetic dipole has been derived by solving the appropriate boundary value problem.

This Green's function can be used in formulating solutions to the problems of slots on the ground of a dielectric substrate, cavity-backed slots and dielectric covered waveguide slot arrays.

References

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Appendix A

Proof of the Need for a F_z^s Component

Assume that there is only a F_x^s component in all three regions.

Then from equations (35)-(37), we have

$$H_x = \frac{j}{\omega\epsilon\mu} \left(k^2 F_x + \frac{\partial^2}{\partial x^2} F_x \right) \tag{A.1}$$

$$H_y = \frac{j}{\omega\epsilon\mu} \frac{\partial^2}{\partial y \partial x} F_x \tag{A.2}$$

$$H_z = \frac{j}{\omega\epsilon\mu} \frac{\partial^2 F_x}{\partial z \partial x} \tag{A.3}$$

Continuity of H_y along the air-dielectric interface results in the following equation

$$\frac{1}{\epsilon_o} \frac{\partial^2 F_{1x}}{\partial y \partial x} = \frac{1}{\epsilon} \frac{\partial^2 F_{2x}}{\partial y \partial x} \tag{A.4}$$

Integration of equation (A.4) with respect to y and then differentiation with respect to x gives

$$\frac{1}{\varepsilon_o} \frac{\partial^2 F_{1x}}{\partial x^2} = \frac{1}{\varepsilon} \frac{\partial^2 F_{2x}}{\partial x^2} \qquad (A.5)$$

From continuity of H_x along the same interface, we get

$$F_{1x} = F_{2x} \qquad (A.6)$$

However, differentiation of equation (A.6) twice with respect to x will give

$$\frac{\partial^2 F_{1x}}{\partial x^2} = \frac{\partial^2 F_{2x}}{\partial x^2} \qquad (A.7)$$

Equations (A.5) and (A.7) agree only if $\varepsilon = \varepsilon_o$ which is not possible. Therefore F^S should have one more component along the z direction.
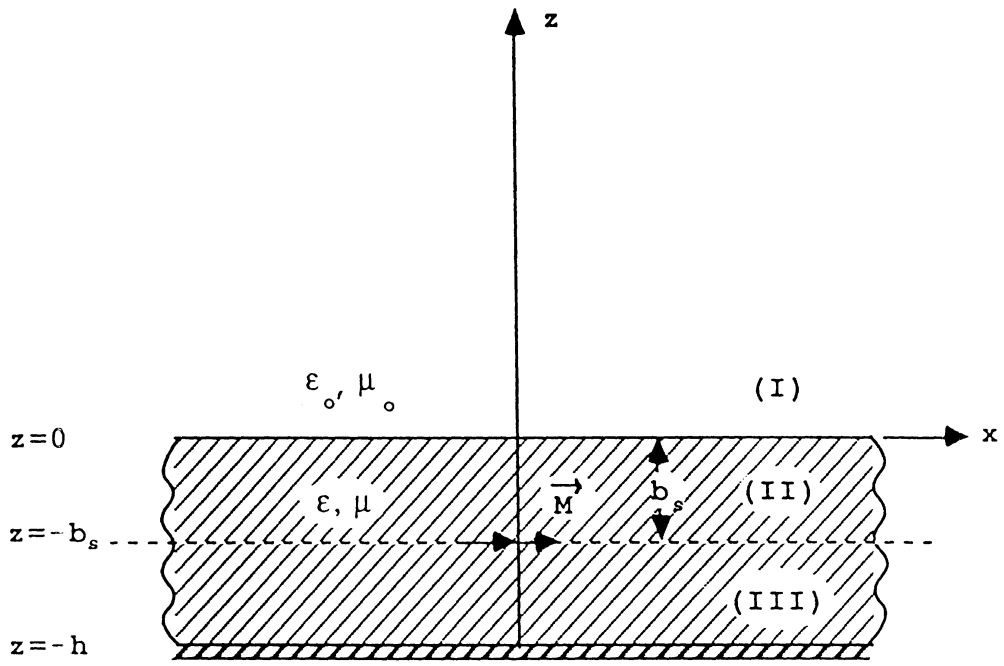
Figure 1