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ALTERNATE METHOD FOR DERIVING GREEN'S FUNCTIONS IN LAYERED REGIONS

by

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I. INTRODUCTION

Planar transmission line structures such as microstrip line, coplanar line, and finline have been fundamental components of microwave integrated circuits for many years, [1]. More recently, there has been considerable effort devoted to the design and realization of monolithic microwave integrated circuits (MMIC's) for use in the f > 20 GHz region, [2]. Once fabricated, monolithic circuits are very difficult to tune for optimum performance and this is a major drawback, [2], [3]. Accurate theoretical models of MMIC components are required so that device performance can be predicted confidently, thus avoiding a time-consuming and costly production cycle. Such characterization requires a mathematically rigorous solution for the fields in a particular structure. The use of a Green's function is therefore appropriate. Research in this direction is ongoing, [4], [5], [6] and much has yet to be done. This report outlines an alternative method for deriving spatial domain Green's functions for multiple dielectric layered regions based on the principle of scattering superposition combined with appropriately chosen magnetic and electric vector potentials, [7].

To demonstrate this technique, the derivation of the electric field Green's function for a waveguide inhomogeneously filled with three dielectric layers is outlined.

II. ELECTROMAGNETIC VECTOR POTENTIALS

The electromagnetic fields in any region can be derived from appropriate choices of $\bar{A}$ and $\bar{F}$, the magnetic and electric vector potentials, respectively. For the horizontal electric dipole above a dielectric half-space, Sommerfeld [8], has demonstrated that two components of a magnetic vector potential are necessary to completely represent the electromagnetic fields of this problem. The argument presented in [8] is based on the fact that if only one component of $\bar{A}$ is used to generate the fields in each region, then continuity of tangential electromagnetic field components at the air-dielectric interface requires that the wavenumbers in each region be equal. This contradiction is resolved by considering a second component of $\bar{A}$. Instead of choosing two components of $\bar{A}$ to solve the above mentioned problem, one may use any two components of $\bar{A}$ and/or $\bar{F}$. And thus, although the fields which satisfy a particular boundary value problem are unique, the field generating potentials are not, [9].

In orthogonal coordinate systems it is conventional to denote fields as transverse electric (TE) or transverse magnetic (TM) with respect to coordinate axes. For example, fields derived from $\bar{F} = \hat{x}\, F_x$ are TE to x, or TE_x and so on.

When a rectangular waveguide is loaded with a dielectric layer, modes which are TE or TM with respect to the direction of propagation cannot exist. Instead, modes are designated as LSE and LSM. An LSE mode is said to be TE with respect to the direction that is normal to the air-dielectric interface in the guide. If this is normal is the $\hat{y}$ unit vector, then all waveguide modes can be generated from A_y and F_y. LSE_y and LSM_y modes are orthogonal, [10], and may be solved for separately. Thus, it is suggested that fields in layered regions be constructed using the components of $\bar{A}$ and $\bar{F}$ that are normal to the layer interfaces. This approach will give electromagnetic fields which decouple and substantially reduce the number of algebraic steps involved. The electromagnetic fields are generated from $\bar{A}$ and $\bar{F}$ as shown in equations (1).

$$\bar{E} = \frac{1}{\varepsilon} \bar{\nabla} \times \bar{F} - j\omega\bar{A} + \frac{1}{j\omega\mu\varepsilon} \bar{\nabla} \bar{\nabla} \bullet \bar{A} \quad (1a)$$

$$\bar{H} = \frac{1}{\mu} \bar{\nabla} \times \bar{A} + j\omega\bar{F} - \frac{1}{j\omega\mu\varepsilon} \bar{\nabla} \bar{\nabla} \bullet \bar{F} \quad (1b)$$

III. THE PRINCIPLE OF SCATTERING SUPERPOSITON

This method was first discussed by Tai, [11], and is conceptually simple. Figure (1a) shows a dipole source within a boundary S_1. A Green's function, $\bar{\bar{G}}_o$, due to this source is maintained. $\bar{\bar{G}}_o$ may be analogous to either an electromagnetic field or a vector potential, as long as it satisfies the proper boundary conditions. If another boundary, S_2, is introduced, as shown in Figure (1b), then $\bar{\bar{G}}_o$ will not satisfy the boundary conditions of this new problem. However, if a composite Green's function, $\bar{\bar{G}}_c$, given by

$$\bar{\bar{G}}_c = \bar{\bar{G}}_o + \bar{\bar{G}}_s \quad (2)$$

is assumed, then the "scattered" field, $\bar{\bar{G}}_s$, may be determined in such a way that $\bar{\bar{G}}_c$ would satisfy all boundary conditions. For most cases, scattering superposition requires solution in the spectral domain because usually only certain eigenvalues of the original $\bar{\bar{G}}_o$ are allowed after S_2 is introduced. Initially assuming that the eigenvalues of $\bar{\bar{G}}_c$ are a continuous spectrum (i.e. by representing $\bar{\bar{G}}_c$ as a fourier integral) allows them to take on their proper values in the spatial domain.

When constructing the scattered Green's function, $\bar{\bar{G}}_s$, for a layered structure fewest algebraic steps are required when the components of $\bar{A}$ and $\bar{F}$ which are normal to the boundary, S_2, are used. Consequently, this is the suggested approach.

IV. DERIVATION OF THE SPATIAL DOMAIN GREEN'S FUNCTION FOR A RECTANGULAR WAVEGUIDE INHOMOGENEOUSLY FILLED WITH THREE DIELECTRICS

Consider the rectangular waveguide inhomogeneously filled with three dielectrics shown in Figure 2. The dyadic Green's function for an arbitrary current, $\bar{J}(\bar{r}')$, in this waveguide has the form:

$$\begin{aligned}\bar{\bar{G}}_{\bar{J}}(\bar{r}/\bar{r}') = {} & G_{xx}(\bar{r}/\bar{r}')\,\hat{x}\hat{x} + G_{xy}(\bar{r}/\bar{r}')\,\hat{x}\hat{y} + G_{xz}(\bar{r}/\bar{r}')\,\hat{x}\hat{z} \\ & + G_{yx}(\bar{r}/\bar{r}')\,\hat{y}\hat{x} + G_{yy}(\bar{r}/\bar{r}')\,\hat{y}\hat{y} + G_{yz}(\bar{r}/\bar{r}')\,\hat{y}\hat{z} \\ & + G_{zx}(\bar{r}/\bar{r}')\,\hat{z}\hat{x} + G_{zy}(\bar{r}/\bar{r}')\,\hat{z}\hat{y} + G_{zz}(\bar{r}/\bar{r}')\,\hat{z}\hat{z}\end{aligned} \tag{3}$$

where the electric field is obtained from

$$\bar{E}\ (\bar{r}) = \iiint_{V'} \bar{\bar{G}}_{\bar{J}}\ (\bar{r}/\bar{r}') \cdot \bar{J}\ (\bar{r}')\ dv' \tag{4}$$

To obtain all nine terms of (3) is an extremely tedious process. In this section, the derivation of $G_{zz}\ (\bar{r}/\bar{r}')$ is outlined. Note that $G_{zz}\ (\bar{r}/\bar{r}')$ is merely the $\hat{z}$-component of the electric field due to a $\hat{z}$-directed dipole located at (x',y',z'). We begin by considering the total field generated by the dipole as a superposition of primary and scattered components. The primary fields are generated directly by the source and the scattered fields result when the dielectric boundary layers are introduced. Consequently, the waveguide problem may be considered as a parallel-plate waveguide shorted at x = a which contains a primary field and a scattered field, combined with another parallel-plate waveguide shorted at x = 0, with three dielectric layers stacked on the shorted end, and which contains only scattered fields. These situations are illustrated in Figures (3a) and (3b).

The eigenfunction expansions for the primary fields are obtained from the magnetic vector potential $\bar{A}$, which satisfies the equation:

$$\nabla^2 \bar{A} + k^2 \bar{A} = -\mu_o \bar{J}\ (\bar{r}') \tag{5}$$

We are looking for $E_z(\bar{r}/\bar{r}')$ due to an infinitesimal $\hat{z}$-directed dipole.

Therefore, the appropriate primary field generating function is a solution of

$$\nabla^2 \bar{A} + k^2 \bar{A} = -\hat{z}\, \mu_o\, \delta(x-x')\, \delta(y-y')\, \delta(z-z') \tag{6}$$

subject to the boundary conditions of the parallel plate structure in Figure (3a). It should be pointed out that the primary field will have different x-dependence above and below the source. Above the source is designated as region (0) and below, region (0'). The boundary conditions on $\bar{A}_p$ are obtained from those on the electric field. The necessary relationships are given in equation (1). The expression for $A_{zp}^{(0')}$ for this problem is:

$$A_{zp}^{(0')} = \int_{-\infty}^{\infty} dk_z \sum_m \frac{\varepsilon_{om}\, \mu_o}{2\pi\, k_x^{(0)}\, b} \sin k_x^{(0)} (a-x')\, e^{-jk_x^{(0)}(a-x)} \cdot \left[\sin \left(\frac{m\pi y'}{b}\right) \sin \left(\frac{m\pi y}{b}\right) \right] e^{-j k_z (z-z')} \tag{7}$$

where $\left(k_x^{(0')}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + k_z^2 = \omega^2 \mu_o \varepsilon_o$

The primary electromagnetic fields are obtained from equations (1) with $\bar{F} = 0$. Scattered fields are generated from magnetic and electric vector potentials $\bar{A}$ and $\bar{F}$, respectively. The proper choices are $\bar{A} = \hat{x}\, A_{xs}$ and $\bar{F} = \hat{x}\, F_{xs}$, for reasons discussed earlier. The scattered electromagnetic fields are obtained from equations (1).

By considering the boundary conditions on $\bar{E}_s$ in the parallel-plate structures of Figure 3 we obtain the eigenfunction expansions for the vector potentials $A_{xs}^{(i)}$ and $F_{xs}^{(i)}$ as:

$$A_{xs}^{(0)} = \int_{-\infty}^{\infty} dk_z \sum_m D_m^{(0)} \cos k_x^{(0)} (a-x) \sin \left(\frac{m\pi y}{b} \right) e^{-jk_z z} \tag{8a}$$

$$A_{xs}^{(1)} = \int_{-\infty}^{\infty} dk_z \sum_m \left[F_m^{(1)} \sin k_x^{(1)} x + G_m^{(1)} \cos k_x^{(1)} x \right] \sin \left(\frac{m\pi y}{b} \right) e^{-jk_z z} \tag{8b}$$

$$A_{xs}^{(2)} = \int_{-\infty}^{\infty} dk_z \sum_m \left[F_m^{(2)} \sin k_x^{(2)} x + G_m^{(2)} \cos k_x^{(2)} x \right] \sin \left(\frac{m\pi y}{b} \right) e^{-jk_z z} \tag{8c}$$

$$A_{xs}^{(3)} = \int_{-\infty}^{\infty} dk_z \sum_m D_m^{(3)} \cos k_x^{(3)} x \sin \left(\frac{m\pi y}{b} \right) e^{-jk_z z} \tag{8d}$$

$$F_{xs}^{(0)} = \int_{-\infty}^{\infty} dk_z \sum_m A_m^{(0)} \sin k_x^{(0)} (a-x) \cos \left(\frac{m\pi y}{b} \right) e^{-jk_z z} \tag{9a}$$

$$F_{xs}^{(1)} = \int_{-\infty}^{\infty} dk_z \sum_m \left[B_m^{(1)} \sin k_x^{(1)} x + C_m^{(1)} \cos k_x^{(1)} x \right] \cos \left(\frac{m\pi y}{b} \right) e^{-jk_z z} \tag{9b}$$

$$F_{xs}^{(2)} = \int_{-\infty}^{\infty} dk_z \sum_m \left[B_m^{(2)} \sin k_x^{(2)} x + C_m^{(2)} \cos k_x^{(2)} x \right] \cos \left(\frac{m\pi y}{b} \right) e^{-jk_z z} \tag{9c}$$

$$F_{xs}^{(3)} = \int_{-\infty}^{\infty} dk_z \sum_m A_m^{(3)} \sin k_x^{(3)} x \cos \left(\frac{m\pi y}{b} \right) e^{-jk_z z} \tag{9d}$$

$$\text{where} \quad \left(k_x^{(i)} \right)^2 + \left(\frac{m\pi}{b} \right)^2 + k_z^2 = k_i^2 \tag{10}$$

The scattered fields in regions (0) and (0') are identical.

The electromagnetic fields obtained from (1), (7), (8) and (9) satisfy all boundary conditions in the inhomogeneously filled waveguide except continuity of E_y, E_z, H_y and H_z at $x = x_{01}$. Imposing these boundary conditions allows us to find exact expressions for the scattered fields. $G_{zz}(\bar{r}/\bar{r}')$ is then obtained. Since only scattered fields exist in the dielectric layers, they must be continuous at each of the interfaces. Consequently, $F_m^{(1)}$ and $G_m^{(1)}$ can both be expressed in terms of $D_m^{(3)}$. Also, $B_m^{(1)}$ and $C_m^{(1)}$ can be expressed in terms of $A_m^{(3)}$. The boundary conditions at $x = x_{01}$ are:

$$
\begin{aligned}
E_{yp}^{(0')} + E_{yA}^{(0)} + E_{yF}^{(0)} &= E_{yA}^{(1)} + E_{yF}^{(1)} \\
E_{zp}^{(0')} + E_{zA}^{(0)} + E_{zF}^{(0)} &= E_{zA}^{(1)} + E_{zF}^{(1)} \\
H_{yp}^{(0')} + H_{yA}^{(0)} + H_{yF}^{(0)} &= H_{yA}^{(1)} + H_{yF}^{(1)} \\
H_{zp}^{(0')} + H_{zA}^{(0)} + H_{zF}^{(0)} &= H_{zA}^{(1)} + H_{zF}^{(1)}
\end{aligned}
\qquad (11)
$$

Equations (11) yield two sets of 2x2 equations:

$$
\begin{bmatrix} M_{11}^{A} & M_{12}^{A} \\ M_{21}^{A} & M_{22}^{A} \end{bmatrix} \begin{bmatrix} D_m^{(0)} \\ D_m^{(3)} \end{bmatrix} = \begin{bmatrix} S_1^{A} \\ S_2^{A} \end{bmatrix} \qquad (12)
$$

$$
\begin{bmatrix} N_{11}^{F} & N_{12}^{F} \\ N_{21}^{F} & N_{22}^{F} \end{bmatrix} \begin{bmatrix} A_m^{(0)} \\ A_m^{(3)} \end{bmatrix} = \begin{bmatrix} S_1^{F} \\ S_2^{F} \end{bmatrix} \qquad (13)
$$

Solution of (12) and (13) provides the unknown amplitude coefficients for, respectively, $A_x^{(i)}$ and $F_x^{(i)}$.

To analyze microstrip circuitry enclosed in the structure of Figure 2, the Green's function must be known at the air-dielectric interface because this is where the current carrying strip lies, [5]. From (11) we know that the tangential fields must be continuous at this boundary and, therefore, so are the tangential components of the Green's dyad. Consequently, we can obtain $G_{zz}\ (\bar{r}/\bar{r}')\ \Big|_{x=x_{01}}$ from E_z in either region (0') or region (1). In the first dielectric region, region (1), the $\hat{z}$-component of the electric field due to a $\hat{z}$-directed current source, from (1a), (8) and (9), is given by:

$$E_z^{(1)} = E_{zA}^{(1)} + E_{zF} \quad \text{where}$$

$$E_{zA}^{(1)} = \frac{1}{j\omega\mu_o\varepsilon_1} \frac{\partial^2 A_{xs}^{(1)}}{\partial z \partial x} \tag{14}$$

$$E_{zF}^{(1)} = -\frac{1}{\varepsilon_1} \frac{\partial F_{xs}^{(1)}}{\partial y} \tag{15}$$

Substituting (8b) and (9b) into (14) and (15), with the respective amplitude coefficients derived from (12) and (13), yields:

$$E_{zA}^{(1)} = \frac{j\omega\mu_o}{2\pi} \int_{-\infty}^{\infty} dk_z \sum_m \frac{2 \sin k_x^{(0)} (a-x') \sin (\frac{m\pi y'}{b}) \sin (\frac{m\pi y}{b}) e^{-jk_z (z-z')}}{b\left[(\frac{m\pi}{b})^2 + k_z^2 \right]}$$

$$\bullet \left\{ \frac{k_z^2 k_x^{(0)} k_x^{(1)} \left[\tilde{K}_1 \cos k_x^{(1)} x - \tilde{K}_2 \sin k_x^{(1)} x \right]}{k_o^2 \tilde{\Lambda}(k_z^{(1)}, k_z)} \right\} \quad (16)$$

$$E_{zF}^{(1)} = \frac{-j\omega\mu_o}{2\pi} \int_{-\infty}^{\infty} dk_z \sum_m \frac{2 \sin k_x^{(0)} (a-x') \sin (\frac{m\pi y'}{b}) \sin (\frac{m\pi y}{b}) e^{-jk_z (z-z')}}{b\left[(\frac{m\pi}{b})^2 + k_z^2 \right]}$$

$$\bullet \left\{ \frac{(\frac{m\pi}{b})^2 \left[\tilde{R}_1 \sin k_x^{(1)} x + \tilde{R}_2 \cos k_x^{(1)} x \right]}{\tilde{\Gamma}(k_x^{(1)}, k_z)} \right\} \quad (17)$$

Specific terms in (16) and (17) will be discussed later. Notice that $E_{zA}^{(1)}$ is associated with a total electromagnetic field which is LSM_x and $E_{zF}^{(1)}$ is associated with a total electromagnetic field which is LSE_x. The remaining task is to complete the inverse fourier transforms of (16) and (17). This will provide us with the desired spatial domain component of the Green's dyad. Both integrals may be evaluated via the calculus of residues since no branch points exist in their respective integrands. For both LSM_x and LSE_x modes, the inversion contour in the k_z-plane is closed in the lower half for $z > z'$, and in the upper half for $z < z'$. Of course, the distribution of poles in the k_z-plane is symmetric about the origin. Completing the inverse transforms yields the electric field as

$$E_z^{(1)} = \sum_n \sum_m \frac{U_{nm}\ (x,y,x',y',\ k_{xn}^{(i)}\ k_{znm})}{\tilde{\Lambda}'\ (k_{znm})}\ e^{-jk_{znm}|z-z'|}$$

$$+ \sum_p \sum_m \frac{V_{pm}\ (x,y,x',y',\ k_{xp}^{(i)},\ k_{zpm})}{\tilde{\Gamma}'\ (k_{zpm})}\ e^{-jk_{zpm}|z-z'|} \qquad (18)$$

where U_{nm} is easily deduced from (16) to be

$$U_{nm} = -2\omega\mu_o \sin k_{xn}^{(0)}\ (a-x') \sin\left(\frac{m\pi y'}{b}\right) \sin\left(\frac{m\pi y}{b}\right)$$

$$\cdot \left\{ \frac{k_{znm}^2\ k_{xn}^{(0)}\ k_{xn}^{(1)}\ (\tilde{K}_1 \cos k_{xn}^{(1)}\ x - \tilde{K}_2 \sin k_{xn}^{(1)}\ x)}{k_o^2\ b \left[\left(\frac{m\pi}{b}\right)^2 + k_{znm}^2 \right]} \right\} \qquad (19)$$

and V_{pm} is similarly deduced from (17) to be

$$V_{pm} = 2\omega\mu_o \sin k_{xp}^{(0)}\ (a-x') \sin\left(\frac{m\pi y'}{b}\right) \sin\left(\frac{m\pi y}{b}\right)$$

$$\cdot \left\{ \frac{\left(\frac{m\pi}{b}\right)^2 (\tilde{R}_1 \sin k_{xp}^{(1)}\ x + \tilde{R}_2 \cos k_{xp}^{(1)}\ x)}{b \left[\left(\frac{m\pi}{b}\right)^2 + k_{zpm}^2 \right]} \right\} \qquad (20)$$

Explicit expressions for $\tilde{K}_1$ and $\tilde{K}_2$ are given in Appendix A and explicit expressions for $\tilde{R}_1$ and $\tilde{R}_2$ in Appendix B. From the residue calculus we know that k_{znm} is a root of $\tilde{\Lambda}(k_x^{(i)},\ k_z)$ and corresponds to an allowed LSM_x eigenvalue in the guide. Similarly, we know that k_{zpm} is a root of $\tilde{\Gamma}(k_x^{(i)}, k_z)$ and is an allowed LSE_x eigenvalue of this structure. These eigenvalues may also be determined by the transverse resonance technique, [10]. In our assumed model it was required that $x' \geq x_{01}$ and this restriction applies to the field given in equation (18). However, it is not a problem because we are interested in the fields generated in the waveguide due to a source at $x' = x_{01}$. The functions $\tilde{\Lambda}'(k_{znm})$ and $\tilde{\Gamma}'(k_{zpm})$ result from the Taylor series expansions of the denominators of (16) and (17), respectively, and are defined as:

$$\tilde{\Lambda}'(k_{znm}) = \left. \frac{d\tilde{\Lambda}(k_x^{(i)}, k_z)}{dk_z} \right|_{k_z = k_{zm}} \tag{21}$$

$$\tilde{\Gamma}'(k_{zpm}) = \left. \frac{d\tilde{\Gamma}(k_x^{(i)}, k_z)}{dk_z} \right|_{k_z = k_{zpm}} \tag{22}$$

Expressions for $\frac{d\tilde{\Lambda}(k_x^{(i)}, k_z)}{dk_z}$ and $\frac{d\tilde{\Gamma}(k_x^{(i)}, k_z)}{dk_z}$ are given in Appendix A and B respectively.

Equations (16) and (17) also show poles that appear when $k_z = \pm j\left(\frac{m\pi}{b}\right)$. These are non-physical spurious modes which are not orthogonal to the LSM_x and LSE_x modes. Consequently, they need not be discussed any further. The final expression for $G_{zz}(x_{01}, y, z \mid x_{01}, y', z')$ is given in equation (23)

$$G_{zz}\ (x, x' = x_{01}) = \sum_n \sum_m \frac{U_{nm}\ (x_{01}, y, x_{01}, y', k_{xr}^{(i)}, k_{znm})}{\tilde{\Lambda}'\ (k_{znm})}\ e^{-jk_{znm}|z-z'|}$$

$$+ \sum_p \sum_m \frac{V_{pm}\ (x_{01}, y, x_{01}, y', k_{xp}^{(i)}, k_{zpm})}{\tilde{\Gamma}'\ (k_{zpm})}\ e^{-jk_{zpm}|z-z'|} \qquad (23)$$

The remaining components of the Green's function may be obtained by applying the technique used to find G_{zz}.

V. CONCLUSION

This report has discussed an alternative method for deriving Green's functions in layered regions. It has been shown that the usual tedious algebra encountered when working with many layer structures can be reduced to having to solve two 2x2 sets of equations for unknown vector potential amplitude coefficients. To demonstrate the usefulness of this technique, a component of the elecric field Green's function was derived for a rectangular waveguide loaded with three isotropic, lossy dielectric slabs. This research was conducted independently from the work of Professor Tai, [12]. Our techniques are similar in that scattering superposition is used, but different in the sense that we use different generating functions to obtain the scattered fields.

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APPENDIX A

Expressions for $\tilde{K}_1$, $\tilde{K}_2$ and $\dfrac{d\,\tilde{\Lambda}(k_x^{(i)},\ k_z)}{dk_z}$

In Appendix A, primed notation represents the total derivative with respect to k_z. It is convenient to designate the following functions:

$$\lambda_{01} = \varepsilon_2^2\ k_x^{(1)}\ k_x^{(3)}\ \sin\ (k_x^{(3)}\ x_{23})\ \cos\ (k_x^{(2)}\ x_{23})$$

$$\lambda_{02} = \varepsilon_2\ \varepsilon_3\ k_x^{(1)}\ k_x^{(2)}\ \cos\ (k_x^{(3)}\ x_{23})\ \sin\ (k_x^{(2)}\ x_{23})$$

$$\lambda_{03} = \varepsilon_1\ \varepsilon_3\ (k_x^{(2)})^2\ \cos\ (k_x^{(2)}\ x_{12})\ \cos\ (k_x^{(1)}\ x_{12})$$

$$\lambda_{04} = \varepsilon_2\ \varepsilon_3\ k_x^{(1)}\ k_x^{(2)}\ \sin\ (k_x^{(2)}\ x_{12})\ \sin\ (k_x^{(1)}\ x_{12})$$

$$\lambda_{05} = \varepsilon_2\ \varepsilon_3\ k_x^{(1)}\ k_x^{(2)}\ \cos\ (k_x^{(3)}\ x_{23})\ \cos\ (k_x^{(2)}\ x_{23})$$

$$\lambda_{06} = \varepsilon_2^2\ k_x^{(1)}\ k_x^{(3)}\ \sin\ (k_x^{(3)}\ x_{23})\ \sin\ (k_x^{(2)}\ x_{23})$$

$$\lambda_{07} = \varepsilon_2\ \varepsilon_3\ k_x^{(1)}\ k_x^{(2)}\ \cos\ (k_x^{(2)}\ x_{12})\ \sin\ (k_x^{(1)}\ x_{12})$$

$$\lambda_{08} = \varepsilon_1\ \varepsilon_3\ (k_x^{(2)})^2\ \sin\ (k_x^{(2)}\ x_{12})\ \cos\ (k_x^{(1)}\ x_{12})$$

$$\lambda_{09} = \varepsilon_2\ \varepsilon_3\ k_x^{(1)}\ k_x^{(2)}\ \sin\ (k_x^{(2)}\ x_{12})\ \cos\ (k_x^{(1)}\ x_{12})$$

$$\lambda_{10} = \varepsilon_1\ \varepsilon_3\ (k_x^{(2)})^2\ \cos\ (k_x^{(2)}\ x_{12})\ \sin\ (k_x^{(1)}\ x_{12})$$

$$\lambda_{11} = \varepsilon_1\ \varepsilon_3\ (k_x^{(2)})^2\ \sin\ (k_x^{(2)}\ x_{12})\ \sin\ (k_x^{(1)}\ x_{12})$$

$$\lambda_{12} = \varepsilon_2\ \varepsilon_3\ k_x^{(1)}\ k_x^{(2)}\ \cos\ (k_x^{(2)}\ x_{12})\ \cos\ (k_x^{(1)}\ x_{12})$$

$$\lambda_{13} = \varepsilon_{r1}\ k_x^{(0)}\ \sin\ k_x^{(0)}\ (a-x_{01})\ \sin\ (k_x^{(1)}\ x_{01})$$

$$\lambda_{14} = k_x^{(1)}\ \cos\ k_x^{(0)}\ (a-x_{01})\ \cos\ (k_x^{(1)}\ x_{01})$$

$$\lambda_{15} = \varepsilon_{r1}\ k_x^{(0)}\ \sin\ k_x^{(0)}\ (a-x_{01})\ \cos\ (k_x^{(1)}\ x_{01})$$

$$\lambda_{16} = k_x^{(1)}\ \cos\ k_x^{(0)}\ (a-x_{01})\ \sin\ (k_x^{(1)}\ x_{01})$$

Expressions for $\tilde{K}_1$, $\tilde{K}_2$ and $\tilde{\Lambda}'$ are:

$$\tilde{K}_1 = (\lambda_{01} + \lambda_{02}) (\lambda_{03} + \lambda_{04}) + (\lambda_{05} - \lambda_{06}) (\lambda_{07} - \lambda_{08})$$

$$\tilde{K}_2 = (\lambda_{01} + \lambda_{02}) (\lambda_{09} - \lambda_{10}) + (\lambda_{05} - \lambda_{06}) (\lambda_{11} + \lambda_{12})$$

$$\begin{aligned}\tilde{\Lambda}' = & (\lambda'_{13} - \lambda'_{14}) \tilde{K}_1 + (\lambda'_{15} + \lambda'_{16}) \tilde{K}_2 \\ & + (\lambda_{13} - \lambda_{14}) \Big[(\lambda'_{01} + \lambda'_{02}) (\lambda_{03} + \lambda_{04}) + (\lambda_{01} + \lambda_{02}) (\lambda'_{03} + \lambda'_{04}) \\ & \qquad + (\lambda'_{05} - \lambda'_{06}) (\lambda_{07} - \lambda_{08}) + (\lambda_{05} - \lambda_{06}) (\lambda'_{07} - \lambda'_{08}) \Big] \\ & + (\lambda_{15} + \lambda_{16}) \Big[(\lambda'_{01} + \lambda'_{02}) (\lambda_{09} - \lambda_{10}) + (\lambda_{01} + \lambda_{02}) (\lambda'_{09} - \lambda'_{10}) \\ & \qquad + (\lambda'_{05} - \lambda'_{06}) (\lambda_{11} + \lambda_{12}) + (\lambda_{05} - \lambda_{06}) (\lambda'_{11} + \lambda'_{12}) \Big]\end{aligned}$$

Appendix B

Expressions for $\tilde{R}_1$, $\tilde{R}_2$ and $\dfrac{d\,\tilde{\Gamma}(k_x^{(i)},\ k_z)}{dk_z}$

In Appendix B primed notation represents the total derivative with respect to k_z. It is convenient to designate the following functions:

$$\delta_{01} = k_x^{(1)}\ k_x^{(2)}\ \sin\ (k_x^{(3)}\ x_{23})\ \sin\ (k_x^{(2)}\ x_{23})$$

$$\delta_{02} = k_x^{(1)}\ k_x^{(3)}\ \cos\ (k_x^{(3)}\ x_{23})\ \cos\ (k_x^{(2)}\ x_{23})$$

$$\delta_{03} = k_x^{(1)}\ k_x^{(2)}\ \sin\ (k_x^{(2)}\ x_{12})\ \sin\ (k_x^{(1)}\ x_{12})$$

$$\delta_{04} = (k_x^{(2)})^2\ \cos\ (k_x^{(2)}\ x_{12})\ \cos\ (k_x^{(1)}\ x_{12})$$

$$\delta_{05} = k_x^{(1)}\ k_x^{(2)}\ \sin\ (k_x^{(3)}\ x_{23})\ \cos\ (k_x^{(2)}\ x_{23})$$

$$\delta_{06} = k_x^{(1)}\ k_x^{(3)}\ \cos\ (k_x^{(3)}\ x_{23})\ \sin\ (k_x^{(2)}\ x_{23})$$

$$\delta_{07} = k_x^{(1)}\ k_x^{(2)}\ \cos\ (k_x^{(2)}\ x_{12})\ \sin\ (k_x^{(1)}\ x_{12})$$

$$\delta_{08} = (k_x^{(2)})^2\ \sin\ (k_x^{(2)}\ x_{12})\ \cos\ (k_x^{(1)}\ x_{12})$$

$$\delta_{09} = k_x^{(1)}\ k_x^{(2)}\ \cos\ (k_x^{(1)}\ x_{12})\ \sin\ (k_x^{(2)}\ x_{12})$$

$$\delta_{10} = (k_x^{(2)})^2\ \cos\ (k_x^{(2)}\ x_{12})\ \sin\ (k_x^{(1)}\ x_{12})$$

$$\delta_{11} = k_x^{(1)}\ k_x^{(2)}\ \cos\ (k_x^{(1)}\ x_{12})\ \cos\ (k_x^{(2)}\ x_{12})$$

$$\delta_{12} = (k_x^{(2)})^2\ \sin\ (k_x^{(1)}\ x_{12})\ \sin\ (k_x^{(2)}\ x_{12})$$

$$\delta_{13} = k_x^{(0)}\ \cos\ k_x^{(0)}\ (a-x_{01})\ \sin\ (k_x^{(1)}\ x_{01})$$

$$\delta_{14} = k_x^{(1)}\ \sin\ k_x^{(0)}\ (a-x_{01})\ \cos\ (k_x^{(1)}\ x_{01})$$

$$\delta_{15} = k_x^{(0)}\ \cos\ k_x^{(0)}\ (a-x_{01})\ \cos\ (k_x^{(1)}\ x_{01})$$

$$\delta_{16} = k_x^{(1)}\ \sin\ k_x^{(0)}\ (a-x_{01})\ \sin\ (k_x^{(1)}\ x_{01})$$

Expressions for $\tilde{R}_1$, $\tilde{R}_2$ and $\tilde{\Gamma}'$ are:

$$\tilde{R}_1 = (\delta_{01} + \delta_{02})\ (\delta_{03} + \delta_{04}) + (\delta_{05} - \delta_{06})\ (\delta_{07} - \delta_{08})$$

$$\tilde{R}_2 = (\delta_{01} + \delta_{02})\ (\delta_{09} - \delta_{10}) + (\delta_{05} - \delta_{06})\ (\delta_{11} + \delta_{12})$$

$$\begin{aligned}\tilde{\Gamma}' = {} & (\delta'_{13} + \delta'_{14})\ \tilde{R}_1 + (\delta'_{15} - \delta'_{16})\ \tilde{R}_2 \\ & + (\delta_{13} + \delta_{14})\ \Big[(\delta'_{01} + \delta'_{02})\ (\delta_{03} + \delta_{04}) + (\delta_{01} + \delta_{02})\ (\delta'_{03} + \delta'_{04}) \\ & \qquad + (\delta'_{05} - \delta'_{06})\ (\delta_{07} - \delta_{08}) + (\delta_{05} - \delta_{06})\ (\delta'_{07} - \delta'_{08}) \Big] \\ & + (\delta_{15} - \delta_{16})\ \Big[(\delta'_{01} + \delta'_{02})\ (\delta_{09} - \delta_{10}) + (\delta_{01} + \delta_{02})\ (\delta'_{09} - \delta'_{10}) \\ & \qquad + (\delta'_{05} - \delta'_{06})\ (\delta_{11} + \delta_{12}) + (\delta_{05} - \delta_{06})\ (\delta'_{11} + \delta'_{12}) \Big]\end{aligned}$$

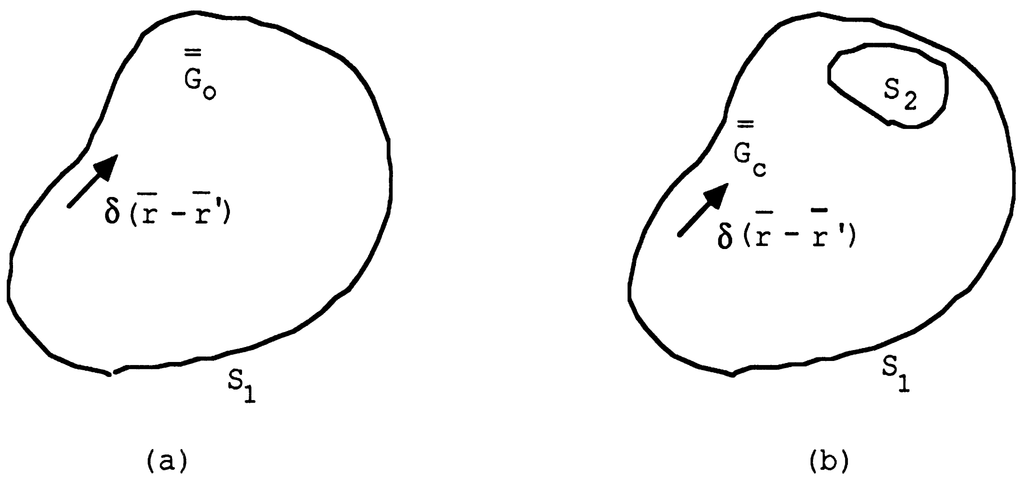

Figure 1. (a) Dipole maintaining $\bar{\bar{G}}_o$ within boundary S_1.

(b) Dipole maintaining $\bar{\bar{G}}_c$ and boundary S_2 within boundary S_1.

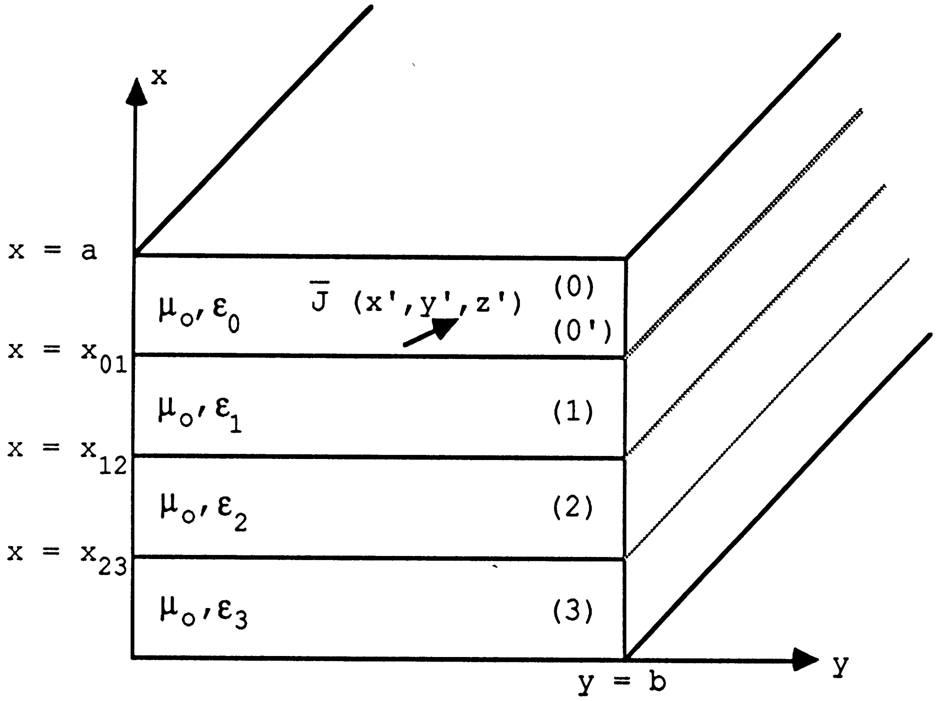

Figure 2. Rectangular waveguide inhomogeneously filled with three dielectrics, excited by current source $\bar{J}$ (x', y', z').

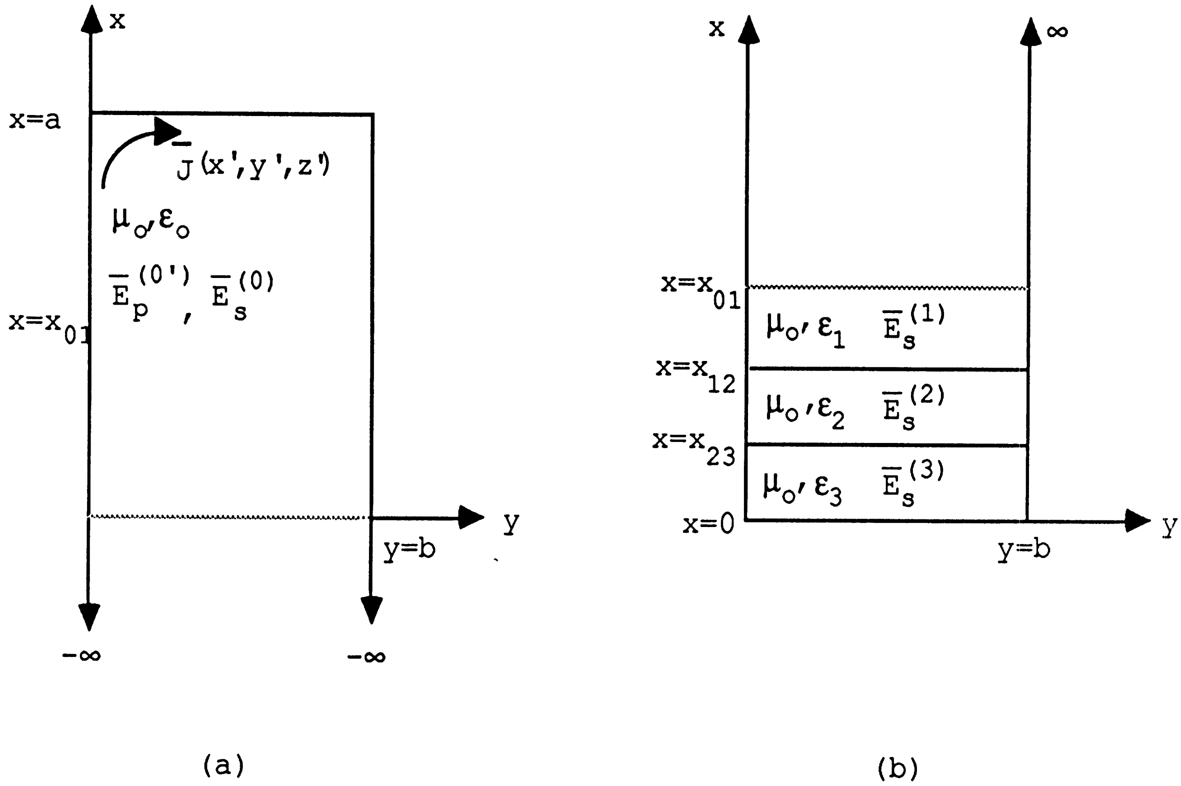

Figure 3. Decomposition of dielectric-loaded waveguide of Figure 2 into equivalent superposition of parallel-plate structures.