Differential Operators in Vector Analysis and the Laplacian of a Vector in the Curvilinear Orthogonal System

C.T. Tai, Radiation Laboratory, Department of Electrical Engineering and Computer Science, The University of Michigan, Ann Arbor, Michigan 48109-2122

Abstract

Some long existing misunderstandings of the meaning of the del operator in the curvilinear orthogonal system have been pointed out in this work. One misunderstanding results from a false manipulation of the notations for the divergence and the curl introduced by Gibbs. A proper analysis shows that there are three distinct differential operators in a curvilinear orthogonal system, and the Laplacian of a vector function is a well defined entity.

1 Introduction

Vector analysis is an indispensable tool in the teaching of electromagnetics, hydrodynamics and mechanics. Unfortunately, there have been some misunderstandings which have been in existence for a long time. In this work, we attempt to clarify them by a critical examination of these problems.

The del operator (or the Nabla operator, or Hamilton operator) in a

Cartesian system is defined by

$$\nabla = \sum_i \mathbf{a}_i \frac{\partial}{\partial x_i} \tag{1}$$

where x_i and $\mathbf{a}_i$ with $i = (1, 2, 3)$ denote, respectively, the coordinate variables and the unit vectors in that system. The gradient of a scalar function f can then be written as

$$\nabla f = \sum_i \mathbf{a}_i \frac{\partial f}{\partial x_i} \tag{2}$$

which has no ambiguity provided that we accept the distributive rule in such an operation. For the divergence and the curl of a vector function $\mathbf{f}$, Gibbs, one of the pioneers in vector analysis, introduced the <u>notations</u> $\nabla \cdot \mathbf{f}$ and $\nabla \times \mathbf{f}$ for these two functions and <u>defined</u> them as [Gibbs, 1881]

$$\nabla \cdot \mathbf{f} \;=\; \sum \mathbf{a}_i \cdot \frac{\partial \mathbf{f}}{\partial x_i} = \sum_i \frac{\partial f_i}{\partial x_i} \tag{3}$$

$$\nabla \times \mathbf{f} \;=\; \sum_i \mathbf{a}_i \times \frac{\partial \mathbf{f}}{\partial x_i} = \sum_i \left(\frac{\partial f_k}{\partial x_j} - \frac{\partial f_j}{\partial x_k} \right) \mathbf{a}_i \tag{4}$$

where $(i, j, k) = (1, 2, 3)$ in cyclic order. These are well known expressions.

It should be pointed out that if one treats $\nabla \cdot \mathbf{f}$, the notation for the divergence, as the 'scalar product' between ∇ and $\mathbf{f}$, then

$$\nabla \cdot \mathbf{f} = \left(\sum_i \mathbf{a}_i \frac{\partial}{\partial x_i} \right) \cdot \mathbf{f} \tag{5}$$

This is meaningless because the member at the right side of (5) consists of an assembly of functions and symbols, and is not a mathematically meaningful expression. We can not arbitrarily transport the dot to the front of the differential sign nor transport the vector $\mathbf{a}_i$ behind the differential sign in order to create a meaningful expression of our choice. This is not a matter of interpretation; it is a false manipulation. We are not allowed to do this in mathematics. The situation is as if we have an assembly of numbers and signs in the form of $2 + \times 3$ which has no meaning in arithmetic. But if we move the plus sign to the front we create a well defined number $+2 \times 3$, and if we move the plus sign to the back we create a numerical operator $(2 \times 3) + \; = 6+$. Neither of these expressions is equivalent to the original assembly. Unfortunately, many authors treat (5) to be equivalent to (3) and this creates a lot of confusion and misunderstandings. For example, Moon and Spencer [1965 a] state: "...A scalar product (between ∇ and $\mathbf{f}$) gives the divergence

$$\text{div } \mathbf{f} = \nabla \cdot \mathbf{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} \quad ."$$

We have changed their notation for the vector function to $\mathbf{f}$. They did the same for $\nabla \times \mathbf{f}$, treating it as a 'vector product' between ∇ and $\mathbf{f}$. Later, they apply the same 'interpretation' to $\nabla \cdot \mathbf{f}$ in a curvilinear orthogonal system that leads them to a wrong conclusion. We should emphasize here that Gibbs introduced $\nabla \cdot \mathbf{f}$ and $\nabla \times \mathbf{f}$ merely as the <u>notations</u> for the divergence and the curl, and were not meant to be the 'scalar product' and the 'vector product' between ∇ and $\mathbf{f}$. The false manipulation was imposed upon these notations by later workers. There are dozens of authors who did the same as Moon and Spencer. Most of them are authors of books on electromagnetics, vector analysis, and calculus. The history behind this misunderstanding is long and interesting. The story will be covered in a separate article.

In a curvilinear orthogonal system with coordinate variable denoted by v_i, unit vector by $\mathbf{u}_i$ and metric coefficients by h_i, the del operator is defined by

$$\nabla = \sum_i \frac{\mathbf{u}_i}{h_i} \frac{\partial}{\partial v_i} \tag{6}$$

The gradient, still denoted by ∇f, can be written in the form

$$\nabla f = \sum_i \frac{\mathbf{u}_i}{h_i} \frac{\partial f}{\partial v_i} \tag{7}$$

It is understood that the distributive rule has been forced upon the operand of the del operator. The meaning of ∇ in (7) has no ambiguity. When this vector differential operator is applied to a scalar function the result yields the gradient of that function. It is the use of the del operator in divergence and curl that has created many problems.

For example, in the book by Morse and Feshbach [1953 a], we find the following statement: "...The vector operator must have different forms for its different uses:

$$\begin{aligned}\nabla &= \sum_i \frac{\mathbf{u}_i}{h_i}\frac{\partial}{\partial v_i}; \text{ for the gradient} \\ &= \frac{1}{\Omega}\sum_i \mathbf{u}_i \frac{\partial}{\partial v_i}\left(\frac{\Omega}{h_i}\right); \text{ for the divergence}\end{aligned}$$

and no form which can be written for the curl."

We have used Ω to represent $h_1h_2h_3$ and have changed their coordinate variables ξ_i to v_i and their notations $\mathbf{a}_i$ to $\mathbf{u}_i$. It is obvious that the 'operator' introduced by these two authors for the divergence can produce the correct differential expression for the divergence only if the operation is 'interpreted' as

$$\left[\frac{1}{\Omega}\sum_i \mathbf{u}_i \frac{\partial}{\partial v_i}\left(\frac{\Omega}{h_i}\right)\right]\cdot \mathbf{f} = \frac{1}{\Omega}\sum_i \frac{\partial}{\partial v_i}\left(\frac{\Omega}{h_i}\mathbf{u}_i \cdot \mathbf{f}\right) \tag{8}$$

Such an interpretation is quite arbitrary and it does not follow the accepted rule of a differential operator because their ∇ for the divergence is a differentiated function not an operator.

In the book by Moon and Spencer [1965 b] the two authors, presumably following their notion that $\nabla \cdot \mathbf{f}$ is the 'scalar product' between ∇ and $\mathbf{f}$, interpreted $\nabla \cdot \mathbf{f}$ as

$$\nabla \cdot \mathbf{f} = \sum_i \frac{1}{h_i}\frac{\partial}{\partial u_i}(\mathbf{u}_i \cdot \mathbf{f}) \tag{9}$$

and then concluded that it does not yield the correct result. Furthermore, in commenting on the work by Phillips [1933] they asserted that Phillips manages to use the del operator in curvilinear system for divergence and curl, but only by the trick of redefining the scalar and vector products for these particular applications. Actually, Phillips' method is not a trick at all; it is very ingenious and he did not redefine the scalar and vector products under consideration. He obtains the correct expressions for the divergence and the curl based on the differential expression for the gradient and some vector identities. By doing so, there is no need for him to discuss the role played by the del operator in $\nabla \cdot \mathbf{f}$ and $\nabla \times \mathbf{f}$ when they are expressed in the curvilinear system. The preceeding introduction clearly indicates that we need a better understanding of the role played by the del operator in a curvilinear system.

2 The Differential Operators in Vector Analysis

To avoid repetition in writing down equations of similar form, we will introduce a unified definition of the three key functions in one formula [Gans, 1932; Javid and Brown, 1963a; Tai, 1986] which is independent of the coordinate system. The formula can be written in the form:

$$\nabla * \tilde{f} = \lim_{\Delta V \to 0} \frac{\sum_j \left(\mathbf{n}_j * \tilde{f}\right) \Delta S_j}{\Delta V} \tag{10}$$

The meaning of the asterisk "$*$" and the function $\tilde{f}$ with tilde is as follows:

$*$	$\tilde{f}$	$\nabla * \tilde{f}$	name
null	f	∇f	gradient
$\cdot$	$\mathbf{f}$	$\nabla \cdot \mathbf{f}$	divergence
$\times$	$\mathbf{f}$	$\nabla \times \mathbf{f}$	curl

In (10), $\mathbf{n}_j$ denotes a typical unit vector pointed outward from a surface $\Delta \mathbf{S}_j = \mathbf{n}_j \Delta S_j$ which is a part of the surface enclosing an elementary volume ΔV. By considering an elementary volume bounded by the constant coordinate surfaces in a curvilinear orthogonal system and taking the limit of (10) we obtain

$$\begin{aligned} \nabla * \tilde{f} &= \frac{1}{\Omega} \sum_i \frac{\partial}{\partial v_i} \left(\frac{\Omega}{h_i} \mathbf{u}_i * \tilde{f} \right) \\ &= \frac{1}{\Omega} \sum_i \left[\frac{\partial}{\partial v_i} \left(\frac{\Omega}{h_i} \mathbf{u}_i \right) * \tilde{f} + \frac{\Omega}{h_i} \mathbf{u}_i * \frac{\partial \tilde{f}}{\partial v_i} \right] \end{aligned} \tag{11}$$

where $\Omega = h_1 h_2 h_3$ as before.

It is known that the derivatives of the unit vectors in a curvilinear orthogonal system satisfy the following relations [Morse and Feshbach, 1953 b]: .

$$\frac{\partial \mathbf{u}_j}{\partial v_i} = \frac{\partial h_i}{h_j \partial v_j}\mathbf{u}_i, \quad i \neq j \tag{12}$$

$$\frac{\partial \mathbf{u}_i}{\partial v_i} = -\left(\frac{\partial h_i}{h_j \partial v_j}\mathbf{u}_j + \frac{\partial h_i}{h_k \partial v_k}\mathbf{u}_k\right) \tag{13}$$

with $(i, j, k) = (1, 2, 3)$ in cyclic order. Based on these relations it can be shown that

$$\sum_i \frac{\partial}{\partial v_i}\left(\frac{\Omega}{h_i}\mathbf{u}_i\right) = 0 \tag{14}$$

Equation (11) then reduces to

$$\nabla * \tilde{f} = \sum_i \frac{\mathbf{u}_i}{h_i} * \frac{\partial \tilde{f}}{\partial v_i} \tag{15}$$

The differential form of the three key functions, therefore, can be written as

$$\nabla f = \sum_i \frac{\mathbf{u}_i}{h_i}\frac{\partial f}{\partial v_i} \tag{16}$$

$$\nabla \cdot \mathbf{f} = \sum_i \frac{\mathbf{u}_i}{h_i} \cdot \frac{\partial \mathbf{f}}{\partial v_i} \tag{17}$$

$$\nabla \times \mathbf{f} = \sum_i \frac{\mathbf{u}_i}{h_i} \times \frac{\partial \mathbf{f}}{\partial v_i} \,. \tag{18}$$

When applying these formulas to a Cartesian system with $h_i = 1, v_i = x_i, \mathbf{u}_i = \mathbf{a}_i$, we obtain the expressions used by Gibbs to <u>define</u> these functions. The proper meaning of the operators in the curvilinear system is now shown explicitly in (16) to (18). There are three distinct differential operators involved. For the gradient, we have the ordinary del operator. For the divergence, we have a differential operator which will be denoted by $\nabla\!\!\!\!\cdot\;$ and designated as the divergence operator or the Dot-del operator. It is defined by

$$\nabla\!\!\!\!\cdot\; = \sum_i \frac{\mathbf{u}_i}{h_i} \cdot \frac{\partial}{\partial v_i} \tag{19}$$

and for the curl we have another differential operator which will be denoted by $\nabla\!\!\!\!\times$ and designated as the curl operator or the Cross-del operator. It is defined by

$$\nabla\!\!\!\!\times = \sum_i \frac{\mathbf{u}_i}{h_i} \times \frac{\partial}{\partial v_i} \tag{20}$$

The three key functions in vector analysis can now be written as $\nabla f, \nabla\!\!\!\!\cdot\;\mathbf{f}$, and $\nabla\!\!\!\!\times\mathbf{f}$. These notations are very descriptive. There is, of course, very little hope that we can change the long established notation of Gibbs. We shall still use Gibbs' notations in this work.

The derivatives of the vector function $\mathbf{f}$ in (17) and (18) can be evaluated explicitly to obtain the well known differential expressions for these two functions. Thus,

$$\begin{aligned} \frac{\partial \mathbf{f}}{\partial v_i} &= \frac{\partial}{\partial v_i} \sum_j f_j \mathbf{u}_j \\ &= \sum_j \left(\frac{\partial f_i}{\partial v_i} \mathbf{u}_j + f_j \frac{\partial \mathbf{u}_j}{\partial v_i} \right) \end{aligned} \tag{21}$$

With the aid of (12) and (13) the derivatives of the unit vectors in (21) can be expressed in terms of the unit vectors themselves. After some straight forward reductions we obtain

$$\nabla \cdot \mathbf{f} = \frac{1}{\Omega} \sum \frac{\partial}{\partial v_i} \left(\frac{\Omega}{h_i} f_i \right) \tag{22}$$

and

$$\nabla \times \mathbf{f} = \frac{1}{\Omega} \sum \left[\frac{\partial (h_k f_k)}{\partial v_j} - \frac{\partial (h_j f_j)}{\partial v_k} \right] h_i \mathbf{u}_i \tag{23}$$

where

$$(i, j, k) = (1, 2, 3) \text{ in cyclic order.}$$

The expression for both $\nabla \cdot \mathbf{f}$ and $\nabla \times \mathbf{f}$ can be obtained more conveniently from the second term of (11) before its decomposition [Javid and Brown, 1963a; Tai, 1986]. With this much discussion of the meaning of the operators in curvilinear system we turn to another related subject dealing with the Laplacian of a vector function and its identity.

3 The Laplacian of a Vector Function

In Cartesian system, it is well known that the following identity exists

$$\nabla \cdot \nabla \mathbf{F} = \nabla (\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F}) \tag{24}$$

where

$$\nabla \cdot \nabla \mathbf{F} = \sum_i \left(\frac{\partial^2 F_i}{\partial x_i^2} \right) \mathbf{a}_i = \sum_i \left(\nabla^2 F_i \right) \mathbf{a}_i \tag{25}$$

The Laplacian of the scalar functions F_i has the meaning of

$$\nabla^2 F_i = \nabla \cdot (\nabla F_i) = \text{ div grad } F_i \tag{26}$$

The fact that (24) is an identity independent of the choice of the coordinate system, including oblique system, has been proved by Ignatowsky [1925a] and Javid and Brown [1963b] based on the integral representation of the del operator [Ignatowsky 1925b, Gans 1932] corresponding to our equation (10). Ignatowsky's proof requires some additional interpretation for curvilinear system while the proof by Javid and Brown in clear-cut.

In this work, we will prove (24) more specifically in a curvilinear orthogonal system based on a functional analysis.

Let us first examine the structure of $\nabla \cdot \nabla \mathbf{F}$ based on the differential forms of the gradient, the divergence and the curl as described by (16 - 18). Thus,

$$\nabla \cdot \nabla \mathbf{F} = \sum_i \frac{\mathbf{u}_i}{h_i} \cdot \frac{\partial}{\partial v_i} \left(\sum_j \frac{\mathbf{u}_j}{h_j} \frac{\partial \mathbf{F}}{\partial v_j} \right) \tag{27}$$

It should be observed that $\nabla \mathbf{F}$ is not a vector; it is a dyadic defined by

$$\nabla \mathbf{F} = \sum_j \frac{\mathbf{u}_j}{h_j} \frac{\partial \mathbf{F}}{\partial v_j} \tag{28}$$

The positioning of the two terms in (28) must be kept in that order. Then

$$\frac{\partial}{\partial v_i} \nabla \mathbf{F} = \sum_j \left[\frac{\mathbf{u}_j}{h_j} \frac{\partial^2 \mathbf{F}}{\partial v_i \partial v_j} + \frac{\partial}{\partial v_i} \left(\frac{\mathbf{u}_j}{h_j} \right) \frac{\partial \mathbf{F}}{\partial v_j} \right] \tag{29}$$

hence

$$
\begin{aligned}
\nabla \cdot \nabla \mathbf{F} &= \sum_i \sum_j \frac{\mathbf{u}_i}{h_i} \cdot \left[\frac{\mathbf{u}_j}{h_j} \frac{\partial^2 \mathbf{F}}{\partial v_i \partial v_j} + \frac{\partial}{\partial v_i} \left(\frac{\mathbf{u}_j}{h_j} \right) \frac{\partial \mathbf{F}}{\partial v_j} \right] \\
&= \sum_i \sum_j \left\{ \left(\frac{\mathbf{u}_i}{h_i} \cdot \frac{\mathbf{u}_j}{h_j} \right) \frac{\partial^2 \mathbf{F}}{\partial v_i \partial v_j} + \left[\frac{\mathbf{u}_i}{h_i} \cdot \frac{\partial}{\partial v_i} \left(\frac{\mathbf{u}_j}{h_j} \right) \right] \frac{\partial \mathbf{F}}{\partial v_j} \right\} \\
&= \sum_i \frac{1}{h_i^2} \frac{\partial^2 \mathbf{F}}{\partial v_i^2} + \sum_i \sum_j \left[\frac{\mathbf{u}_i}{h_i} \cdot \frac{\partial}{\partial v_i} \left(\frac{\mathbf{u}_j}{h_j} \right) \right] \frac{\partial \mathbf{F}}{\partial v_j} \qquad (30)
\end{aligned}
$$

By interchanging the roles of i and j in the last term of (30) and with the aid of (12-13) we have

$$
\sum_i \sum_j \left[\frac{\mathbf{u}_j}{h_j} \cdot \frac{\partial}{\partial v_j} \left(\frac{\mathbf{u}_i}{h_i} \right) \right] = \sum_i \frac{1}{\Omega} \frac{\partial}{\partial v_i} \left(\frac{\Omega}{h_i^2} \right), \quad \Omega = h_1 \, h_2 \, h_3
$$

hence

$$
\begin{aligned}
\nabla \cdot \nabla \mathbf{F} &= \sum_i \left[\frac{1}{h_i^2} \frac{\partial^2 \mathbf{F}}{\partial v_i^2} + \frac{1}{\Omega} \frac{\partial}{\partial v_i} \left(\frac{\Omega}{h_i^2} \right) \frac{\partial \mathbf{F}}{\partial v_i} \right] \\
&= \sum_i \frac{1}{\Omega} \frac{\partial}{\partial v_i} \left(\frac{\Omega}{h_i^2} \frac{\partial \mathbf{F}}{\partial v_i} \right) \qquad (31)
\end{aligned}
$$

which is the Laplacian of $\mathbf{F}$ and can be written in the form of $\nabla^2 \mathbf{F}$ with the Laplacian operator defined by

$$
\nabla^2 = \sum_i \frac{1}{\Omega} \frac{\partial}{\partial v_i} \left(\frac{\Omega}{h_i^2} \frac{\partial}{\partial v_i} \right) \qquad (32)
$$

The operator applies to the entire function of $\mathbf{F}$, including its scalar components and the unit vectors. Actually, this form can be obtained in a very simple way by starting with the differential form of $\nabla^2 \mathbf{F}$ in Cartesian system and then convert the Laplacian operator to its form in a curvilinear system as follows:

$$\nabla^2 \mathbf{F} = \sum_j \left(\nabla^2 F_{x_j}\right) \mathbf{a}_j = \sum_j \sum_i \frac{1}{\Omega} \frac{\partial}{\partial v_i} \left(\frac{\Omega}{h_i^2} \frac{\partial F_{x_j}}{\partial v_i} \right) \mathbf{a}_j$$

$$= \sum \frac{1}{\Omega} \frac{\partial}{\partial v_i} \left(\frac{\Omega}{h_i^2} \frac{\partial \mathbf{F}}{\partial v_i} \right) \tag{33}$$

where

$$\mathbf{F} = \sum_j F_{x_j} \mathbf{a}_j = \sum_j F_j \mathbf{u}_j$$

The structure of $\nabla \cdot \nabla \mathbf{F}$ in the form of (30), however, is needed later to prove identity (24) in a curvilinear system.

The two functions at the right side of (24) can be written in the form:

$$\nabla (\nabla \cdot \mathbf{F}) = \sum_i \frac{\mathbf{u}_i}{h_i} \frac{\partial}{\partial v_i} \sum_j \frac{\mathbf{u}_j}{h_j} \cdot \frac{\partial \mathbf{F}}{\partial v_j}$$

$$= \sum_i \sum_j \frac{\mathbf{u}_i}{h_i} \left[\frac{\partial}{\partial v_i} \left(\frac{\mathbf{u}_j}{h_j} \right) \cdot \frac{\partial \mathbf{F}}{\partial v_j} + \frac{\mathbf{u}_j}{h_j} \cdot \frac{\partial^2 \mathbf{F}}{\partial v_i \partial v_j} \right] \tag{34}$$

$$\nabla \times (\nabla \times \mathbf{F}) = \sum_i \frac{\mathbf{u}_i}{h_i} \times \frac{\partial}{\partial v_i} \left(\sum_j \frac{\mathbf{u}_j}{h_j} \times \frac{\partial \mathbf{F}}{\partial v_j} \right)$$

$$= \sum_i \sum_j \frac{\mathbf{u}_i}{h_i} \times \left[\frac{\partial}{\partial v_i} \left(\frac{\mathbf{u}_j}{h_j} \right) \times \frac{\partial \mathbf{F}}{\partial v_j} + \frac{\mathbf{u}_j}{h_j} \times \frac{\partial^2 \mathbf{F}}{\partial v_i \partial v_j} \right] \tag{35}$$

The triple products in (35) can be decomposed into vectors using the identity

$$\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{c} \cdot \mathbf{b}) \mathbf{a} - (\mathbf{c} \cdot \mathbf{a}) \mathbf{b} \tag{36}$$

The subtraction of (37) from (36) yields

$$\nabla(\nabla\cdot\mathbf{F}) - \nabla\times(\nabla\times\mathbf{F}) = \nabla\cdot\nabla\mathbf{F} + \sum_i\sum_j \frac{\partial\mathbf{F}}{\partial v_j}\times\left[\frac{\mathbf{u}_i}{h_i}\times\frac{\partial}{\partial v_i}\left(\frac{\mathbf{u}_j}{h_j}\right)\right] \qquad (37)$$

where we have used (30) to represent two of the terms in that resultant equation, and the last term in (37) results from a recombination of another two terms. With the aid of (12-13) it can be shown that

$$\sum_i\left[\frac{\mathbf{u}_i}{h_i}\times\frac{\partial}{\partial v_i}\left(\frac{\mathbf{u}_j}{h_j}\right)\right] = 0 \qquad (38)$$

Alternatively, we can treat (38) as a vector identity, viz.,

$$\sum_i \frac{\mathbf{u}_i}{h_i}\times\frac{\partial}{\partial v_i}\left(\frac{\mathbf{u}_j}{h_j}\right) = \nabla\times\left(\frac{\mathbf{u}_j}{h_j}\right) = \nabla\times\nabla v_j = 0 \qquad (39)$$

Hence (37) reduces to the identity

$$\nabla\cdot\nabla\mathbf{F} = \nabla(\nabla\cdot\mathbf{F}) - \nabla\times(\nabla\times\mathbf{F}) \qquad (40)$$

which is valid in a curvilinear orthogonal system including the Cartesian system as a special case. In view of this analysis there should be no more misunderstanding about the meaning of $\nabla\cdot\nabla\mathbf{F}$ in a curvilinear orthogonal system [Moon and Spencer 1955]. If one accepts our newly suggested notations for the divergence and the curl, (40) can be presented in a rather compact form, namely,

$$\nabla\nabla\mathbf{F} = \nabla\nabla\mathbf{F} - \boldsymbol{\nabla}\boldsymbol{\nabla}\mathbf{F} \qquad (41)$$

In conclusion, we have clarified several misunderstandings in vector analysis which have been in existence for a long time without a critical examination. It is hoped that the analysis given in this paper will facilitate the teaching of vector analysis in the future. It should be mentioned that this work is motivated by a recent study of vector analysis based on a symbolic operational method. This new work will appear elsewhere.

The valuable comment and suggestion by Prof. H.C. Ko, of Ohio State University, is very much appreciated. The assistance of Professor Nenheng Fang in the course of this study is gratefully acknowledged. The author is indebted to Ms. Bonnie Kidd for typing this manuscript in a very professional and skilful manner.

REFERENCES

[1] Javid, M., and P.M. Brown, *Field Analysis and Electromagnetics*, McGraw-Hill, New York, Appendix II, p. 477, 1963a.

[2] Javid, M., and P.M. Brown, ibid, Appendix IV, 483-484, 1963b.

[3] Gans, R., *Vector Analysis*, English translation of the Sixth German Edition by W.M. Deans, Blackie & Sons, London, 47, 1932.

[4] Gibbs, J.W., Elements of Vector Analysis, [Privately printed, New Haven, 17-50, 1881; 50-90, 1884], published in *The Scientific Papers of J. Willard Gibbs*, New York, NY, Dover Publications, *3*, 31, 1961.

[5] Ignatowsky, W.v., *Die Vektoranalysis*, Dritten Auflage, G.G. Teubner, Leipzig, 29, 1925a.

[6] Ignatowsky, W.v., ibid, 16, 1925b.

[7] Moon, P., D.E. Spencer, *Vectors*, Princeton, NJ, D. van Nostrand, Appendix C, 324, 1965 a.

[8] Moon, P., D.E. Spencer, ibid, 325-326, 1965b.

[9] Moon, P., D.E. Spencer, The Meaning of the Vector Laplacian, *J. Franklin Institute*, *256*, 551-558, 1955.

[10] -Morse, P.M., H. Feshbach, *Methods of Theoretical Physics*, New York, NY, McGraw-Hill, Part 1, 44, 1953a.

[11] Morse, P.M., H. Feshbach, ibid, 25-26, 1953b.

[12] Phillips, H.B., *Vector Analysis*, John Wiley, 87-90, 1933.

[13] Tai, C.T., Unified Definition for Divergence, Rotation, and Gradient, *Appl. Math. and Mech.* (Beijing, China), *7*, *1*, 1-6, 1986.