Approximate Boundary Conditions, Part I

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Abstract

Approximate boundary conditions can be very helpful in simplifying the analytical and numerical solution of scattering problems. One of the simplest is the standard impedance boundary condition, but in an effort to improve the accuracy, more general boundary conditions are now being considered. To establish the general form of these new conditions and to explore the role played by the geometry of the surface, boundary conditions are developed for an inhomogeneous dielectric body whose surface is a coordinate surface in an orthogonal curvilinear coordinate system. The treatment is based on an asymptotic expansion of the interior fields in powers of $\frac{1}{N}$ where N is the complex refractive index of the dielectric. Boundary conditions through the second order are derived, and it is shown that beyond the zeroth order in $\frac{1}{N}$, the geometry of the surface affects the boundary conditions.

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1 Introduction

The impedance boundary condition attributed to Leontovich [1948] is widely used to simulate the material properties of a scatterer, and even for a non-planar surface of a material whose properties may vary laterally as well as in depth, it is customary to use the boundary condition derived from a consideration of a homogeneous half space. In other words, the effect of curvature is neglected, and the local value of the impedance is assumed at every point of the surface. As we shall see, this can be justified to the leading order, but in recent years more general boundary conditions have been proposed [Senior and Volakis, 1989] including derivatives of the fields and purporting to improve the simulation. Here again, the derivation is carried out for a laterally-uniform half space, and there is the presumption that the resulting conditions can be applied without modification to a curved surface whose properties may vary. This is not true, and to provide a simulation in these circumstances, we here examine the effects of surface curvature and material variations using the method employed by Rytov [1940] applied to the formulation developed by Leontovich [1948]. The results presented are more general than those given by either of these authors, and reveal errors in some of the formulas quoted by Leontovich.

2 Formulation

A lossy body is composed of a material whose complex permittivity ϵ and permeability μ may vary as functions of position. The body is immersed in free space and is illuminated by an electromagnetic field. On the assumption that the external field varies slowly over the surface S, we seek a boundary conditions that can be applied at S to simulate the effect of the material.

Inside the body Maxwell's equations are

$$\nabla \times \mathbf{E}^{\text{in}} = i\omega\mu\mathbf{H}^{\text{in}} \,, \quad \nabla \times \mathbf{H}^{\text{in}} = -i\omega\epsilon\mathbf{E}^{\text{in}}$$

where a time factor $e^{-i\omega t}$ has been assumed and suppressed. Since

$$\nabla \times \left(\sqrt{\epsilon}\,\mathbf{E}^{\text{in}}\right) = \nabla(\sqrt{\epsilon}) \times \mathbf{E}^{\text{in}} + \sqrt{\epsilon}\,\nabla \times \mathbf{E}^{\text{in}}$$

we have

$$\sqrt{\epsilon}\nabla \times \mathbf{E}^{\text{in}} = \nabla \times \left(\sqrt{\epsilon}\mathbf{E}^{\text{in}}\right) - \frac{1}{\sqrt{\epsilon}}\nabla(\sqrt{\epsilon}) \times \sqrt{\epsilon}\,\mathbf{E}^{\text{in}} = i\omega\sqrt{\epsilon}\,\mu\mathbf{H}^{\text{in}} \,.$$

and therefore

$$\nabla \times \left(\sqrt{\epsilon}\,\mathbf{E}^{\text{in}}\right) + \sqrt{\epsilon}\,\mathbf{E}^{\text{in}} \times \nabla(\ln\sqrt{\epsilon}) = ik_o N\sqrt{\mu}\,\mathbf{H}^{\text{in}}$$

where k_o is the propagation constant in free space and

$$N = \sqrt{\frac{\epsilon\mu}{\epsilon_o\mu_o}} \tag{1}$$

is the complex refractive index of the material. Similarly

$$\nabla \times \left(\sqrt{\mu}\,\mathbf{H}^{\text{in}}\right) + \sqrt{\mu}\,\mathbf{H}^{\text{in}} \times \nabla(\ln\sqrt{\mu}) = -ik_o N\sqrt{\epsilon}\,\mathbf{E}^{\text{in}} \,.$$

Assume $|N|$ is large everywhere inside the body, and on this basis set

$$N = \frac{v}{q} \tag{2}$$

where v is a functions of position and q is a small parameter. Writing

$$\mathcal{E} = \sqrt{\epsilon}\,\mathbf{E}^{\text{in}} \,, \quad \mathcal{H} = \sqrt{\mu}\,\mathbf{H}^{\text{in}} \,, \tag{3}$$

the defining equations can be written as

$$\begin{aligned} \nabla \times \mathcal{E} \;&+\; \mathcal{E} \times \nabla(\ln\sqrt{\epsilon}) = ik_o\frac{v}{q}\mathcal{H} \,, \\ \nabla \times \mathcal{H} \;&+\; \mathcal{H} \times \nabla(\ln\sqrt{\mu}) = -ik_o\frac{v}{q}\mathcal{E} \,. \end{aligned} \tag{4}$$

With geometrical optics as a guide, let

$$\mathcal{E} = \mathbf{A}e^{ik_o\psi/q} \,, \quad \mathcal{H} = \mathbf{B}e^{ik_o\psi/q} \,. \tag{5}$$

Then

$$\nabla \times \mathcal{E} = \left(\nabla \times \mathbf{A} + \frac{ik_o}{q}\nabla\psi \times \mathbf{A}\right)e^{ik_o\psi/q}$$

with a similar equation for $\nabla \times \mathcal{H}$, and (4) now become [Leontovich, 1948]

$$\begin{aligned} v\mathbf{A} + \nabla\psi \times \mathbf{B} &= -\frac{q}{ik_o}\left\{\nabla \times \mathbf{B} + \mathbf{B} \times \nabla(\ln\sqrt{\mu})\right\} , \\ v\mathbf{B} - \nabla\psi \times \mathbf{A} &= \frac{q}{ik_o}\left\{\nabla \times \mathbf{A} + \mathbf{A} \times \nabla(\ln\sqrt{\epsilon})\right\} . \end{aligned} \tag{6}$$

We seek a solution for **A** and **B** in the form of a power series in q, *viz.*

$$\begin{aligned} \mathbf{A} &= \mathbf{A}_o + q\mathbf{A}_1 + q^2\mathbf{A}_2 + \ldots \\ \mathbf{B} &= \mathbf{B}_o + q\mathbf{B}_1 + q^2\mathbf{B}_2 + \ldots \end{aligned} \tag{7}$$

For this purpose we introduce the orthogonal curvilinear coordinates α, β, γ with metrical coefficients $h_\alpha, h_\beta, h_\gamma$ such that the surface S of the body is $\gamma =$ constant with $\hat{\gamma}$ in the direction of the outward normal.

3 Zeroth Order Solution

Inserting (7) into (6) and retaining only the terms which are independent of q,

$$\begin{aligned} v\mathbf{A}_o + \nabla\psi \times \mathbf{B}_o = 0 \\ v\mathbf{B}_o - \nabla\psi \times \mathbf{A}_o = 0 \end{aligned} \tag{8}$$

showing that $\mathbf{A}_o$, $\mathbf{B}_o$ and $\nabla\psi$ are mutually perpendicular. Eliminating (say) $\mathbf{B}_o$

$$\left(v^2 - |\nabla\psi|^2\right)\mathbf{A}_o = 0$$

implying

$$|\nabla\psi|^2 = v^2$$

analogous to the eikonal equation of geometrical optics. Hence

$$\nabla\psi = v\hat{s}$$

where $\hat{s}$ is a unit vector in the direction of propagation in the body.

At the surface the tangential components of the electric and magnetic fields inside and outside the body must be equal and, as already noted, the fields outside are slowly varying over S. Hence, the fields inside must also vary slowly, and this is only possible if $\psi = 0$ on S. It follows that $\nabla\psi$ (and therefore $\hat{s}$) are normal to S, and consistent with propagation into the body,

$$\nabla\psi = -v\hat{\gamma} . \tag{9}$$

From (8) and (9)

$$\mathbf{A}_o = \hat{\gamma} \times \mathbf{B}_o , \quad \mathbf{B}_o = -\hat{\gamma} \times \mathbf{A}_o , \tag{10}$$

and to this approximation the local field inside the body looks like a plane wave propagating in the direction of the inward normal to S. In particular, just inside S,

$$A_{o\alpha} = -B_{o\beta} , \quad A_{o\beta} = B_{o\gamma} , \quad A_{o\gamma} = B_{o\gamma} = 0 . \tag{11}$$

From the continuity of the tangential field components at S we have

$$\mathbf{A}_o = \sqrt{\epsilon}\,\mathbf{E}^{\text{in}} = \sqrt{\epsilon}\,\mathbf{E} , \quad \mathbf{B}_o = \sqrt{\mu}\,\mathbf{H}^{\text{in}} = \sqrt{\mu}\,\mathbf{H}$$

where $\mathbf{E}$, $\mathbf{H}$ are the external fields, and (10) now implies the approximate boundary condition

$$\hat{\gamma} \times (\hat{\gamma} \times \mathbf{E}) = -Z\hat{\gamma} \times \mathbf{H} \tag{12}$$

on the external field where

$$Z = \sqrt{\frac{\mu}{\epsilon}} \tag{13}$$

is the intrinsic impedance of the material at the surface. This is the standard impedance boundary condition, customarily derived on the basis of a homogeneous half space. Thus, to the zeroth order,

$$E_\alpha = -ZH_\beta , \quad E_\beta = ZH_\alpha \tag{14}$$

with $E_\gamma = H_\gamma = 0$ on S.

4 First Order Solution

Equating the coefficients of q on both sides of (6) we obtain

$$\mathbf{A}_1 - \hat{\gamma} \times \mathbf{B}_1 = -\frac{1}{ik_o v}\left\{\nabla \times \mathbf{B}_o + \mathbf{B}_o \times \nabla(\ln\sqrt{\mu})\right\} \tag{15}$$

$$\mathbf{B}_1 + \hat{\gamma} \times \mathbf{A}_1 = \frac{1}{ik_o v}\left\{\nabla \times \mathbf{A}_o + \mathbf{A}_o \times \nabla(\ln\sqrt{\epsilon})\right\}\ . \tag{16}$$

In terms of the chosen coordinates the components of (15) are

$$\begin{aligned}
A_{1\alpha} + B_{1\beta} &= \frac{1}{ik_o v}\left\{\frac{1}{h_\beta h_\gamma}\frac{\partial}{\partial\gamma}(h_\beta B_{o\beta}) - \frac{B_{o\beta}}{h_\gamma}\frac{\partial}{\partial\gamma}(\ln\sqrt{\mu})\right\}\\
A_{1\beta} - B_{1\alpha} &= -\frac{1}{ik_o v}\left\{\frac{1}{h_\alpha h_\gamma}\frac{\partial}{\partial\gamma}(h_\alpha B_{o\alpha}) - \frac{B_{o\alpha}}{h_\gamma}\frac{\partial}{\partial\gamma}(\ln\sqrt{\mu})\right\}\\
A_{1\gamma} &= -\frac{1}{ik_o v}\left\{\frac{1}{h_\alpha h_\beta}\left[\frac{\partial}{\partial\alpha}(h_\beta B_{o\beta}) - \frac{\partial}{\partial\beta}(h_\alpha B_{o\alpha})\right]\right.\\
&\qquad \left. + \frac{B_{o\alpha}}{h_\beta}\frac{\partial}{\partial\beta}(\ln\sqrt{\mu}) - \frac{B_{o\beta}}{h_\alpha}\frac{\partial}{\partial\alpha}(\ln\sqrt{\mu})\right\}
\end{aligned}$$

and in view of (11) the components of (16) are

$$\begin{aligned}
B_{1\alpha} - A_{1\beta} &= -\frac{1}{ik_o v}\left\{\frac{1}{h_\beta h_\gamma}\frac{\partial}{\partial\gamma}(h_\beta B_{o\alpha}) - \frac{B_{o\alpha}}{h_\gamma}\frac{\partial}{\partial\gamma}(\ln\sqrt{\epsilon})\right\}\\
B_{1\beta} + A_{1\alpha} &= -\frac{1}{ik_o v}\left\{\frac{1}{h_\alpha h_\gamma}\frac{\partial}{\partial\gamma}(h_\alpha B_{o\beta}) - \frac{B_{o\beta}}{h_\gamma}\frac{\partial}{\partial\gamma}(\ln\sqrt{\epsilon})\right\}\\
B_{1\gamma} &= \frac{1}{ik_o v}\left\{\frac{1}{h_\alpha h_\beta}\left[\frac{\partial}{\partial\alpha}(h_\beta B_{o\alpha}) + \frac{\partial}{\partial\beta}(h_\alpha B_{o\beta})\right]\right.\\
&\qquad \left. - \frac{B_{o\beta}}{h_\beta}\frac{\partial}{\partial\beta}(\ln\sqrt{\epsilon}) - \frac{B_{o\alpha}}{h_\alpha}\frac{\partial}{\partial\alpha}(\ln\sqrt{\epsilon})\right\}\ .
\end{aligned}$$

The expressions for $A_{1\alpha} + B_{1\beta}$ are identical if

$$h_\alpha\frac{\partial}{\partial\gamma}(h_\beta B_{o\beta}) + h_\beta\frac{\partial}{\partial\gamma}(h_\alpha B_{o\beta}) - h_\alpha h_\beta B_{o\beta}\frac{\partial}{\partial\gamma}(\ln\sqrt{\epsilon\mu}) = 0$$

implying

$$\frac{\partial}{\partial\gamma}\left\{\frac{h_\alpha h_\beta B_{o\beta}^2}{\sqrt{\epsilon\mu}}\right\} = 0$$

from which we obtain

$$\frac{\partial B_{o\beta}}{\partial\gamma} = -\frac{B_{o\beta}\sqrt{\epsilon\mu}}{2h_\alpha h_\beta}\frac{\partial}{\partial\gamma}\left(\frac{h_\alpha h_\beta}{\sqrt{\epsilon\mu}}\right)$$

i.e.

$$\frac{\partial B_{o\beta}}{\partial\gamma} = \frac{1}{2}B_{o\beta}\frac{\partial}{\partial\gamma}\left(\ln\frac{\sqrt{\epsilon\mu}}{h_\alpha h_\beta}\right) , \tag{17}$$

and therefore

$$\begin{aligned}\frac{\partial}{\partial\gamma}(h_\beta B_{o\beta}) &= B_{o\beta}\frac{\partial h_\beta}{\partial\gamma} + h_\beta\frac{\partial B_{o\beta}}{\partial\gamma} \\ &= \frac{1}{2}h_\beta B_{o\beta}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{h_\alpha}\sqrt{\epsilon\mu}\right) .\end{aligned} \tag{18}$$

Similarly, the expressions for $A_{1\beta} - B_{1\alpha}$ are identical if

$$h_\alpha\frac{\partial}{\partial\gamma}(h_\beta B_{o\alpha}) + h_\beta\frac{\partial}{\partial\gamma}(h_\alpha B_{o\alpha}) - h_\alpha h_\beta B_{o\alpha}\frac{\partial}{\partial\gamma}(\ln\sqrt{\epsilon\mu}) = 0$$

implying

$$\frac{\partial}{\partial\gamma}\left\{\frac{h_\alpha h_\beta B_{o\alpha}^2}{\sqrt{\epsilon\mu}}\right\} = 0 ,$$

so that

$$\frac{\partial B_{o\alpha}}{\partial\gamma} = \frac{1}{2}B_{o\alpha}\frac{\partial}{\partial\gamma}\left(\ln\frac{\sqrt{\epsilon\mu}}{h_\alpha h_\beta}\right) \tag{19}$$

giving

$$\frac{\partial}{\partial\gamma}(h_\alpha B_{o\alpha}) = \frac{1}{2}h_\alpha B_{o\alpha}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}\sqrt{\epsilon\mu}\right) . \tag{20}$$

Hence

$$A_{1\alpha} = -B_{1\beta} - \frac{B_{o\beta}}{2ik_o v h_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right) \tag{21}$$

$$A_{1\beta} = B_{1\alpha} + \frac{B_{o\alpha}}{2ik_o v h_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{h_\alpha}Z\right) \tag{22}$$

with

$$A_{1\gamma} = -\frac{\sqrt{\mu}}{ik_o v h_\alpha h_\beta}\left\{\frac{\partial}{\partial\alpha}\left(\frac{h_\beta B_{o\beta}}{\sqrt{\mu}}\right) - \frac{\partial}{\partial\beta}\left(\frac{h_\alpha B_{o\alpha}}{\sqrt{\mu}}\right)\right\} \tag{23}$$

$$B_{1\gamma} = \frac{\sqrt{\epsilon}}{ik_o v h_\alpha h_\beta}\left\{\frac{\partial}{\partial\alpha}\left(\frac{h_\beta B_{o\alpha}}{\sqrt{\epsilon}}\right) + \frac{\partial}{\partial\beta}\left(\frac{h_\alpha B_{o\beta}}{\sqrt{\epsilon}}\right)\right\}. \tag{24}$$

It is now a trivial matter to construct approximate boundary conditions for the tangential components of the external field on S. From (11) and (21)

$$\begin{aligned}A_\alpha &= A_{o\alpha} + qA_{1\alpha} + O(q^2)\\ &= -B_{o\beta} - qB_{1\beta} - \frac{q}{2ik_o v h_\gamma}B_{o\beta}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right) + O(q^2)\\ &= -B_\beta\left\{1 + \frac{q}{2ik_o v h_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right)\right\} + O(q^2)\end{aligned}$$

and thus, to the first order in q,

$$A_\alpha = -B_\beta\left\{1 + \frac{1}{2ik_o N h_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right)\right\}. \tag{25}$$

Similarly, from (11) and (22)

$$A_\beta = B_\alpha\left\{1 + \frac{1}{2ik_o N h_\alpha}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{h_\gamma}Z\right)\right\} \tag{26}$$

and to the first order in $1/|N|$ the boundary conditions on S are

$$E_\alpha = -ZH_\beta \left\{1 + \frac{1}{2ik_oNh_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right)\right\}$$

$$E_\beta = ZH_\alpha \left\{1 + \frac{1}{2ik_oNh_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{h_\alpha}Z\right)\right\} . \tag{27}$$

where all quantities are evaluated at the surface. The factor 2 is missing from the formulas quoted by Leontovich [1948].

To the first order, only the normal variation of the impedance has any effect, consistent with the interpretation of the surface impedance as the local impedance looking in, and this provides justification for applying the standard boundary condition (12) at each point of a surface even when the properties of the material vary laterally. If $h_\alpha \neq h_\beta$ the effective surface impedance implied by (27) is anisotropic:

$$\bar{\bar{\eta}} = \Gamma\hat{\alpha}\hat{\alpha} + \Gamma_1\hat{\beta}\hat{\beta} \tag{28}$$

where

$$\Gamma = Z\left\{1 + \frac{1}{2ik_oNh_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right)\right\}$$

$$\Gamma_1 = Z\left\{1 + \frac{1}{2ik_oNh_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{h_\alpha}Z\right)\right\} \tag{29}$$

and in terms of $\bar{\bar{\eta}}$ a compact (vector) form for the boundary conditions (27) is

$$\hat{\gamma} \times (\hat{\gamma} \times \mathbf{E}) = -\bar{\bar{\eta}} \cdot \hat{\gamma} \times \mathbf{H} . \tag{30}$$

This should be compared with the zeroth order condition (12).

By taking the vector product of (30) with $\hat{\gamma}$ we obtain

$$\hat{\gamma} \times (\hat{\gamma} \times \mathbf{H}) = \left(\frac{1}{\Gamma_1}\hat{\alpha}\hat{\alpha} + \frac{1}{\Gamma}\hat{\beta}\hat{\beta}\right) \cdot \hat{\gamma} \times \mathbf{E}$$

and (30) therefore satisfies duality if

$$\frac{1}{\Gamma_1} = \Gamma^* , \quad \text{implying} \quad \frac{1}{\Gamma} = \Gamma_1^* ,$$

where an asterisk denotes the dual quantity. Since $N \to N$ and $Z \to 1/Z$ under the duality transformation,

$$\begin{aligned}
\frac{1}{\Gamma^*} &= Z\left\{1 + \frac{1}{2ik_o N h_\gamma}\frac{\partial}{\partial \gamma}\left(\ln\frac{h_\alpha}{h_\beta}\cdot\frac{1}{Z}\right)\right\}^{-1} \\
&= Z\left\{1 - \frac{1}{2ik_o N h_\gamma}\frac{\partial}{\partial \gamma}\left(\ln\frac{h_\beta}{h_\alpha}\cdot Z\right)\right\}^{-1} \\
&= Z\left\{1 + \frac{1}{2ik_o N h_\gamma}\frac{\partial}{\partial \gamma}\left(\ln\frac{h_\beta}{h_\alpha}\cdot Z\right)\right\} + O(|N|^{-2})
\end{aligned}$$

showing that duality is satisfied to the first order in $|N|^{-1}$. We can make this explicit by writing

$$\overline{\overline{\eta}} = \Gamma\hat{\alpha}\hat{\alpha} + \frac{1}{\Gamma^*}\hat{\beta}\hat{\beta} . \tag{31}$$

The anisotropy is a consequence of the curvature of S. In the special case when the coordinates α and β coincide with the directions of the principal curvatures at every point of S,

$$\frac{1}{h_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}\right) = \frac{1}{R_\alpha} - \frac{1}{R_\beta} \tag{32}$$

where R_α and R_β are the principal radii of curvature, and if $R_\alpha = R_\beta$ (including a planar surface as a particular example) the impedance becomes a scalar. Thus, for a planar surface ($\alpha = x, \beta = y, \gamma = z$ where x, y, z are Cartesian coorindates, implying $h_\alpha = h_\beta = h_\gamma = 1$)

$$\overline{\overline{\eta}} = Z\left\{1 + \frac{1}{2ik_o N}\frac{\partial}{\partial z}(\ln Z)\right\}\overline{\mathbf{I}} \tag{33}$$

where $\overline{\mathbf{I}}$ is the identity tensor in the α, β coordinates. Likewise for a spherical surface ($\alpha = \theta, \beta = \phi, \gamma = r$ where r, θ, ϕ are spherical polar coordinates, implying $h_\alpha = r, h_\beta = r\sin\theta, h_\gamma = 1$)

$$\bar{\bar{\eta}} = Z\left\{1 + \frac{1}{2ik_oN}\frac{\partial}{\partial r}(\ln Z)\right\}\bar{\mathbf{I}}\,, \tag{34}$$

but for a circular cylindrical surface ($\alpha = \phi, \beta = z, \gamma = \rho$ where ρ, θ, z are cylindrical polar coordinates, implying $h_\alpha = \rho, h_\beta = h_\gamma = 1$)

$$\bar{\bar{\eta}} = Z\left\{1 + \frac{1}{2ik_oN}\frac{\partial}{\partial \rho}(\ln Z)\right\}\bar{\mathbf{I}} + \frac{Z}{2ik_oN\rho}(\hat{\alpha}\hat{\alpha} - \hat{\beta}\hat{\beta})\,. \tag{35}$$

In all of these results the derivative is evaluated at the surface.

5 Second Order Solution

For the terms involving q^2 the analysis is more tedious, and to keep it as simple as possible, it is helpful to group the terms. To this end we note that

$$\nabla\times\mathbf{B} + \mathbf{B}\times\nabla(\ln\sqrt{\mu}) = \sqrt{\mu}\,\nabla\times\frac{\mathbf{B}}{\sqrt{\mu}}$$

and when this is inserted into (6), equating the coefficients of q^2 on both sides gives

$$\mathbf{A}_2 - \hat{\gamma}\times\mathbf{B}_2 = -\frac{\sqrt{\mu}}{ik_ov}\nabla\times\frac{\mathbf{B}_1}{\sqrt{\mu}} \tag{36}$$

$$\mathbf{B}_2 + \hat{\gamma}\times\mathbf{A}_2 = \frac{\sqrt{\epsilon}}{ik_ov}\nabla\times\frac{\mathbf{A}_1}{\sqrt{\epsilon}}\,. \tag{37}$$

The components of (36) are

$$\begin{aligned}
A_{2\alpha} + B_{2\beta} &= -\frac{\sqrt{\mu}}{ik_ovh_\beta h_\gamma}\left\{\frac{\partial}{\partial\beta}\left(\frac{h_\gamma B_{1\gamma}}{\sqrt{\mu}}\right) - \frac{\partial}{\partial\gamma}\left(\frac{h_\beta B_{1\beta}}{\sqrt{\mu}}\right)\right\}\\
A_{2\beta} - B_{2\alpha} &= -\frac{\sqrt{\mu}}{ik_ovh_\alpha h_\gamma}\left\{\frac{\partial}{\partial\gamma}\left(\frac{h_\alpha B_{1\alpha}}{\sqrt{\mu}}\right) - \frac{\partial}{\partial\alpha}\left(\frac{h_\gamma B_{1\gamma}}{\sqrt{\mu}}\right)\right\}\\
A_{2\gamma} &= -\frac{\sqrt{\mu}}{ik_ovh_\alpha h_\beta}\left\{\frac{\partial}{\partial\alpha}\left(\frac{h_\beta B_{1\beta}}{\sqrt{\mu}}\right) - \frac{\partial}{\partial\beta}\left(\frac{h_\alpha B_{1\alpha}}{\sqrt{\mu}}\right)\right\}
\end{aligned} \tag{38}$$

and similarly, from (37),

$$\begin{aligned} B_{2\alpha} - A_{2\beta} &= \frac{\sqrt{\epsilon}}{ik_o v h_\beta h_\gamma}\left\{\frac{\partial}{\partial\beta}\left(\frac{h_\gamma A_{1\gamma}}{\sqrt{\epsilon}}\right) - \frac{\partial}{\partial\gamma}\left(\frac{h_\beta A_{1\beta}}{\sqrt{\epsilon}}\right)\right\} \\ B_{2\beta} + A_{2\alpha} &= \frac{\sqrt{\epsilon}}{ik_o v h_\alpha h_\gamma}\left\{\frac{\partial}{\partial\gamma}\left(\frac{h_\alpha A_{1\alpha}}{\sqrt{\epsilon}}\right) - \frac{\partial}{\partial\alpha}\left(\frac{h_\gamma A_{1\gamma}}{\sqrt{\epsilon}}\right)\right\} \\ B_{2\gamma} &= \frac{\sqrt{\epsilon}}{ik_o v h_\alpha h_\beta}\left\{\frac{\partial}{\partial\alpha}\left(\frac{h_\beta A_{1\beta}}{\sqrt{\epsilon}}\right) - \frac{\partial}{\partial\beta}\left(\frac{h_\alpha A_{1\alpha}}{\sqrt{\epsilon}}\right)\right\} . \end{aligned} \tag{39}$$

Consider first the expressions for $A_{2\alpha} + B_{2\beta}$. These can be written as

$$A_{2\alpha} + B_{2\beta} = \frac{1}{ik_o v h_\gamma}\left\{\frac{\partial B_{1\beta}}{\partial\gamma} + B_{1\beta}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{\sqrt{\mu}}\right) - \frac{\sqrt{\mu}}{h_\beta}\frac{\partial}{\partial\beta}\left(\frac{h_\gamma B_{1\gamma}}{\sqrt{\mu}}\right)\right\} \tag{40}$$

and

$$A_{2\alpha} + B_{2\beta} = \frac{1}{ik_o v h_\gamma}\left\{\frac{\partial A_{1\alpha}}{\partial\gamma} + A_{1\alpha}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{\sqrt{\epsilon}}\right) - \frac{\sqrt{\epsilon}}{h_\alpha}\frac{\partial}{\partial\alpha}\left(\frac{h_\gamma A_{1\gamma}}{\sqrt{\epsilon}}\right)\right\} , \tag{41}$$

and are identical if

$$\begin{aligned} \frac{\partial B_{1\beta}}{\partial\gamma} = \frac{\partial A_{1\alpha}}{\partial\gamma} &+ A_{1\alpha}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{\sqrt{\epsilon}}\right) - B_{1\beta}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{\sqrt{\mu}}\right) \\ &- \frac{\sqrt{\epsilon}}{h_\alpha}\frac{\partial}{\partial\alpha}\left(\frac{h_\gamma A_{1\gamma}}{\sqrt{\epsilon}}\right) + \frac{\sqrt{\mu}}{h_\beta}\frac{\partial}{\partial\beta}\left(\frac{h_\gamma B_{1\gamma}}{\sqrt{\mu}}\right) . \end{aligned}$$

In view of (21), this serves to determine $\frac{\partial B_{1\beta}}{\partial\gamma}$ as

$$\begin{aligned} \frac{\partial B_{1\beta}}{\partial\gamma} &= -\frac{1}{2}B_{1\beta}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha h_\beta}{\sqrt{\epsilon\mu}}\right) - \frac{\sqrt{\epsilon}}{2h_\alpha}\frac{\partial}{\partial\alpha}\left(\frac{h_\gamma A_{1\gamma}}{\sqrt{\epsilon}}\right) + \frac{\sqrt{\mu}}{2h_\beta}\frac{\partial}{\partial\beta}\left(\frac{h_\gamma B_{1\gamma}}{\sqrt{\mu}}\right) \\ &- \frac{B_{o\beta}}{4ik_o v h_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right)\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{\sqrt{\epsilon}}\right) - \frac{1}{4ik_o}\frac{\partial}{\partial\gamma}\left\{\frac{B_{o\beta}}{v h_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right)\right\} \end{aligned} \tag{42}$$

and substitution into (40) then produces a unique expression for $A_{2\alpha} + B_{2\beta}$. Alternatively, since a knowledge of $\frac{\partial B_{1\beta}}{\partial\gamma}$ is not needed for the second order boundary condition, a simpler (but equivalent) approach is to average (40) and (41), giving

$$A_{2\alpha}+B_{2\beta}=\frac{1}{2ik_ovh_\gamma}\left\{A_{1\alpha}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{\sqrt{\epsilon}}\right)+B_{1\beta}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{\sqrt{\mu}}\right)\right.$$
$$\left.-\frac{\sqrt{\epsilon}}{h_\alpha}\frac{\partial}{\partial\alpha}\left(\frac{h_\gamma A_{1\gamma}}{\sqrt{\epsilon}}\right)-\frac{\sqrt{\mu}}{h_\beta}\frac{\partial}{\partial\beta}\left(\frac{h_\gamma B_{1\gamma}}{\sqrt{\mu}}\right)-\frac{1}{2ik_o}\frac{\partial}{\partial\gamma}\left[\frac{B_{o\beta}}{vh_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right)\right]\right\}.$$

and on using (21), (23) and (24) we obtain

$$\begin{aligned}A_{2\alpha}+B_{2\beta} &= \frac{1}{2ik_ovh_\gamma}\left\{-B_{1\beta}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right)\right.\\
&- \frac{B_{o\beta}}{2ik_o}\left[\frac{1}{2vh_\gamma}\left\{\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right)\right\}^2+\frac{\partial}{\partial\gamma}\left\{\frac{1}{vh_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right)\right\}\right]\\
&+ \frac{\sqrt{\epsilon}}{ik_oh_\alpha}\frac{\partial}{\partial\alpha}\left[\frac{Z}{v}\frac{h_\gamma}{h_\alpha h_\beta}\left\{\frac{\partial}{\partial\alpha}\left(\frac{h_\beta B_{o\beta}}{\sqrt{\mu}}\right)-\frac{\partial}{\partial\beta}\left(\frac{h_\alpha B_{o\alpha}}{\sqrt{\mu}}\right)\right\}\right]\\
&- \left.\frac{\sqrt{\mu}}{ik_oh_\beta}\frac{\partial}{\partial\beta}\left[\frac{1}{Zv}\frac{h_\gamma}{h_\alpha h_\beta}\left\{\frac{\partial}{\partial\alpha}\left(\frac{h_\beta B_{o\alpha}}{\sqrt{\epsilon}}\right)+\frac{\partial}{\partial\beta}\left(\frac{h_\alpha B_{o\beta}}{\sqrt{\epsilon}}\right)\right\}\right]\right\}.\end{aligned} \tag{43}$$

A similar procedure can be applied to $A_{2\beta}-B_{2\alpha}$. From (38)

$$A_{2\beta}-B_{2\alpha}=\frac{1}{ik_ovh_\gamma}\left\{-\frac{\partial B_{1\alpha}}{\partial\gamma}-B_{1\alpha}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{\sqrt{\mu}}\right)+\frac{\sqrt{\mu}}{h_\alpha}\frac{\partial}{\partial\alpha}\left(\frac{h_\gamma B_{1\gamma}}{\sqrt{\mu}}\right)\right\} \tag{44}$$

and from (39)

$$A_{2\beta}-B_{2\alpha}=\frac{1}{ik_ovh_\gamma}\left\{\frac{\partial A_{1\beta}}{\partial\gamma}+A_{1\beta}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{\sqrt{\epsilon}}\right)-\frac{\sqrt{\epsilon}}{h_\beta}\frac{\partial}{\partial\beta}\left(\frac{h_\gamma A_{1\gamma}}{\sqrt{\epsilon}}\right)\right\}. \tag{45}$$

These are identical if

$$\frac{\partial B_{1\alpha}}{\partial\gamma}=-\frac{1}{2}B_{1\alpha}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha h_\beta}{\sqrt{\epsilon\mu}}\right)+\frac{\sqrt{\epsilon}}{2h_\beta}\frac{\partial}{\partial\beta}\left(\frac{h_\gamma A_{1\gamma}}{\sqrt{\epsilon}}\right)+\frac{\sqrt{\mu}}{2h_\alpha}\frac{\partial}{\partial\alpha}\left(\frac{h_\gamma B_{1\gamma}}{\sqrt{\mu}}\right)$$
$$-\frac{B_{o\alpha}}{4ik_ovh_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{h_\alpha}Z\right)\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{\sqrt{\epsilon}}\right)-\frac{1}{4ik_o}\frac{\partial}{\partial\gamma}\left\{\frac{B_{o\alpha}}{vh_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{h_\alpha}Z\right)\right\}. \tag{46}$$

Also, by averaging (44) and (45),

$$A_{2\beta} - B_{2\alpha} = \frac{1}{2ik_o v h_\gamma}\left\{A_{1\beta}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{\sqrt{\epsilon}}\right) - B_{1\beta}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{\sqrt{\mu}}\right)\right.$$

$$\left. -\frac{\sqrt{\epsilon}}{h_\beta}\frac{\partial}{\partial\beta}\left(\frac{h_\gamma A_{1\gamma}}{\sqrt{\epsilon}}\right) + \frac{\sqrt{\mu}}{h_\alpha}\frac{\partial}{\partial\alpha}\left(\frac{h_\gamma B_{1\gamma}}{\sqrt{\mu}}\right) + \frac{1}{2ik_o}\frac{\partial}{\partial\gamma}\left[\frac{B_{o\alpha}}{vh_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{h_\alpha}Z\right)\right]\right\}$$

and on using (22), (23) and (24) we then obtain

$$\begin{aligned} A_{2\beta} - B_{2\alpha} &= \frac{1}{2ik_o v h_\gamma}\left\{B_{1\alpha}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{h_\alpha}Z\right)\right. \\ &+ \frac{B_{o\alpha}}{2ik_o}\left[\frac{1}{2vh_\gamma}\left\{\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{h_\alpha}Z\right)\right\}^2 + \frac{\partial}{\partial\gamma}\left\{\frac{1}{vh_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{h_\alpha}Z\right)\right\}\right] \\ &+ \frac{\sqrt{\epsilon}}{ik_o h_\beta}\frac{\partial}{\partial\beta}\left[\frac{Z}{v}\frac{h_\gamma}{h_\alpha h_\beta}\left\{\frac{\partial}{\partial\alpha}\left(\frac{h_\beta B_{o\beta}}{\sqrt{\mu}}\right) - \frac{\partial}{\partial\beta}\left(\frac{h_\alpha B_{o\alpha}}{\sqrt{\mu}}\right)\right\}\right] \\ &+ \left.\frac{\sqrt{\mu}}{ik_o h_\alpha}\frac{\partial}{\partial\alpha}\left[\frac{1}{Zv}\frac{h_\gamma}{h_\alpha h_\beta}\left\{\frac{\partial}{\partial\alpha}\left(\frac{h_\beta B_{o\alpha}}{\sqrt{\epsilon}}\right) + \frac{\partial}{\partial\beta}\left(\frac{h_\alpha B_{o\beta}}{\sqrt{\epsilon}}\right)\right\}\right]\right\}. \end{aligned} \tag{47}$$

We are now in a position to construct the second order boundary conditions for the tangential components of the external field on S. From (11), (21) and (43)

$$\begin{aligned} A_\alpha &= A_{o\alpha} + qA_{1\alpha} + q^2 A_{2\alpha} + O(q^3) \\ &= -(B_{o\beta} + qB_{1\beta} + q^2 B_{2\beta}) - q(B_{o\beta} + qB_{1\beta})\frac{1}{2ik_o v h_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right) \\ &+ q^2 B_{o\beta}\frac{1}{(2k_o v h_\gamma)^2}\left[\frac{1}{2}\left\{\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right)\right\}^2 + \frac{\partial^2}{\partial\gamma^2}\left(\ln\frac{h_\alpha}{h_\beta}Z\right)\right. \\ &- \left.\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right)\frac{\partial}{\partial\gamma}(\ln vh_\gamma)\right] - \frac{q^2\sqrt{\epsilon}}{2k_o^2 v h_\gamma h_\alpha}\frac{\partial}{\partial\alpha}\left[\frac{Z}{v}\frac{h_\gamma}{h_\alpha h_\beta}\left\{\frac{\partial}{\partial\alpha}\left(\frac{h_\beta B_{o\beta}}{\sqrt{\mu}}\right)\right.\right. \\ &- \left.\left.\frac{\partial}{\partial\beta}\left(\frac{h_\alpha B_{o\alpha}}{\sqrt{\mu}}\right)\right\}\right] + \frac{q^2\sqrt{\mu}}{2k_o^2 v h_\gamma h_\beta}\frac{\partial}{\partial\beta}\left[\frac{1}{Zv}\frac{h_\gamma}{h_\alpha h_\beta}\left\{\frac{\partial}{\partial\alpha}\left(\frac{h_\beta B_{o\alpha}}{\sqrt{\epsilon}}\right)\right.\right. \end{aligned}$$

$$+ \quad \frac{\partial}{\partial\beta}\left(\frac{h_\alpha B_{o\beta}}{\sqrt{\epsilon}}\right)\Bigg\}\Bigg] + O(q^3)$$

and thus, to the second order in q,

$$\begin{aligned}
A_\alpha &= -\Bigg\{1 + \frac{q}{2ik_o v h_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right) - \frac{q^2}{(2k_o v h_\gamma)^2}\left[\frac{1}{2}\left\{\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right)\right\}^2\right.\\
&\quad \left. + \frac{\partial^2}{\partial\gamma^2}\left(\ln\frac{h_\alpha}{h_\beta}Z\right) - \frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right)\frac{\partial}{\partial\gamma}(\ln v h_\gamma)\right]\Bigg\} B_\beta\\
&\quad - \frac{q^2\sqrt{\epsilon}}{2k_o^2 v h_\gamma h_\alpha}\frac{\partial}{\partial\alpha}\left[\frac{Z}{v}\frac{h_\gamma}{h_\alpha h_\beta}\left\{\frac{\partial}{\partial\alpha}\left(\frac{h_\beta B_\beta}{\sqrt{\mu}}\right) - \frac{\partial}{\partial\beta}\left(\frac{h_\alpha B_\alpha}{\sqrt{\mu}}\right)\right\}\right]\\
&\quad + \frac{q^2\sqrt{\mu}}{2k_o^2 v h_\gamma h_\beta}\frac{\partial}{\partial\beta}\left[\frac{1}{Zv}\frac{h_\gamma}{h_\alpha h_\beta}\left\{\frac{\partial}{\partial\alpha}\left(\frac{h_\beta B_\alpha}{\sqrt{\epsilon}}\right) + \frac{\partial}{\partial\beta}\left(\frac{h_\alpha B\beta}{\sqrt{\epsilon}}\right)\right\}\right] . \qquad (48)
\end{aligned}$$

In terms of the external fields[1]

$$\begin{aligned}
\frac{1}{h_\alpha h_\beta}\left\{\frac{\partial}{\partial\alpha}\left(\frac{h_\beta B_\beta}{\sqrt{\mu}}\right) - \frac{\partial}{\partial\beta}\left(\frac{h_\alpha B_\alpha}{\sqrt{\mu}}\right)\right\} &= \frac{1}{h_\alpha h_\beta}\left\{\frac{\partial}{\partial\alpha}(h_\beta H_\beta) - \frac{\partial}{\partial\beta}(h_\alpha H_\alpha)\right\}\\
&= \hat{\gamma}\cdot\nabla\times\mathbf{H} = -ik_o Y_o E_\gamma
\end{aligned}$$

and therefore

$$\frac{1}{h_\alpha}\frac{\partial}{\partial\alpha}\left[\frac{Z}{v}\frac{h_\gamma}{h_\alpha h_\beta}\left\{\frac{\partial}{\partial\alpha}\left(\frac{h_\beta B_\beta}{\sqrt{\mu}}\right) - \frac{\partial}{\partial\beta}\left(\frac{h_\alpha B_\alpha}{\sqrt{\mu}}\right)\right\}\right] = -\frac{ik_o}{q}\hat{\alpha}\cdot\nabla\left(\frac{\epsilon_o}{\epsilon}h_\gamma E_\gamma\right) .$$

Similarly, on using (11),

$$\frac{1}{h_\beta}\frac{\partial}{\partial\beta}\left[\frac{1}{Zv}\frac{h_\gamma}{h_\alpha h_\beta}\left\{\frac{\partial}{\partial\alpha}\left(\frac{h_\beta B_\alpha}{\sqrt{\epsilon}}\right) + \frac{\partial}{\partial\beta}\left(\frac{h_\alpha B_\beta}{\sqrt{\epsilon}}\right)\right\}\right] = \frac{ik_o}{q}\hat{\beta}\cdot\nabla\left(\frac{\mu_o}{\mu}h_\gamma H_\gamma\right)$$

[1]The process is valid only because $\psi = 0$ on S and the derivatives are tangential ones. Hence

$$\frac{h_\alpha B_\alpha}{\sqrt{\mu}} = h_\alpha H_\alpha^{\text{in}} e^{-ik_o\psi/q} = h_\alpha H_\alpha^{\text{in}} = h_\alpha H_\alpha .$$

and since $v = Nq$, implying

$$\frac{\partial}{\partial\gamma}(\ln v) = \frac{\partial}{\partial\gamma}(\ln N) \, ,$$

the boundary condition becomes

$$E_\alpha + \frac{1}{2ik_oNh_\gamma}\hat{\alpha}\cdot\nabla\left(\frac{\epsilon_o}{\epsilon}h_\gamma E_\gamma\right) = -\left\{\Gamma H_\beta + \frac{Z}{2ik_oNh_\gamma}\hat{\beta}\cdot\nabla\left(\frac{\mu_o}{\mu}h_\gamma H_\gamma\right)\right\} \tag{49}$$

with

$$\Gamma = Z\left\{1 + \frac{1}{2ik_oNh_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right) - \frac{1}{(2k_oNh_\gamma)^2}\left[\frac{\partial^2}{\partial\gamma^2}\left(\ln\frac{h_\alpha}{h_\beta}Z\right)\right.\right.$$
$$\left.\left. + \frac{1}{2}\left\{\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right)\right\}^2 - \frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right)\frac{\partial}{\partial\gamma}(\ln h_\gamma N)\right]\right\} \tag{50}$$

We note that to the second order in q (49) can also be written as

$$E_\alpha + \frac{1}{2ik_oNh_\gamma}\hat{\alpha}\cdot\nabla\left(\frac{\epsilon_o}{\epsilon}h_\gamma E_\gamma\right) = -\Gamma\left\{H_\beta + \frac{1}{2ik_oNh_\gamma}\hat{\beta}\cdot\nabla\left(\frac{\mu_o}{\mu}h_\gamma H_\gamma\right)\right\} . \tag{51}$$

The analysis for the second boundary condition involving the components E_β and H_α is similar in all respects, and using (11), (22) and (47) we obtain

$$E_\beta + \frac{1}{2ik_oNh_\gamma}\hat{\beta}\cdot\nabla\left(\frac{\epsilon_o}{\epsilon}h_\gamma E_\gamma\right) = \Gamma_1\left\{H_\alpha + \frac{1}{2ik_oNh_\gamma}\hat{\alpha}\cdot\nabla\left(\frac{\mu_o}{\mu}h_\gamma H_\gamma\right)\right\} \tag{52}$$

where Γ_1 differs from Γ only in having h_α and h_β interchanged. Accordingly,

$$\Gamma_1 = \frac{1}{\Gamma^*}$$

to the second order in q, where the asterisk denotes the dual quantity, and a compact (vector) form for the boundary conditions which makes the duality self-evident is

$$\hat{\gamma} \times \left(\hat{\gamma} \times \left\{\mathbf{E} + \frac{1}{2ik_oNh_\gamma}\nabla\left(\frac{\epsilon_o}{\epsilon}h_\gamma E_\gamma\right)\right\}\right) = -\bar{\bar{\eta}} \cdot \hat{\gamma} \times \left\{\mathbf{H} + \frac{1}{2ik_oNh_\gamma}\nabla\left(\frac{\mu_o}{\mu}h_\gamma H_\gamma\right)\right\} \tag{53}$$

with

$$\bar{\bar{\eta}} = \Gamma\hat{\alpha}\hat{\alpha} + \frac{1}{\Gamma^*}\hat{\beta}\hat{\beta}\,. \tag{54}$$

To the required order

$$\Gamma = Z\left(1 + \frac{1}{2ik_oNh_\gamma}\frac{\partial}{\partial\gamma}\right)e^a \tag{55}$$

where

$$a = \frac{1}{2ik_oNh_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right) \tag{56}$$

6 Examination of the Solution

The second order boundary condition (53) has several interesting features and it is worthwhile examining these as well as the forms taken in special cases.

The most intriguing feature of all is that the *form* of the boundary condition is independent of any variation in the material properties of the surface, and in this respect (53) is similar to the zeroth and first order conditions. Any lateral variation of the properties is taken care of by the gradient operations, and a variation in the normal direction affects only the second order terms in the expression for the effective surface impedance $\bar{\bar{\eta}}$. Nevertheless, for simplicity we shall henceforth assume that the material is homogeneous.

For the special case of the planar surface $z = 0$ we choose $\alpha = x, \beta = y, \gamma = z$ implying $h_\alpha = h_\beta = h_\gamma = 1$. Then $\Gamma = \frac{1}{\Gamma^*} = Z$ so that $\bar{\bar{\eta}} = Z\bar{\mathbf{I}}$, and the boundary condition becomes

$$\hat{z} \times \left(\hat{z} \times \left\{\mathbf{E} + \frac{1}{2ik_oN}\frac{\epsilon_o}{\epsilon}\nabla E_z\right\}\right) = -Z\hat{z} \times \left\{\mathbf{H} + \frac{1}{2ik_oN}\frac{\mu_o}{\mu}\nabla H_z\right\}\,. \tag{57}$$

The components of this are

$$E_x + \frac{1}{2ik_oN}\frac{\epsilon_o}{\epsilon}\frac{\partial E_z}{\partial x} = -Z\left\{H_y + \frac{1}{2ik_oN}\frac{\mu_o}{\mu}\frac{\partial H_z}{\partial y}\right\}$$

$$E_y + \frac{1}{2ik_oN}\frac{\epsilon_o}{\epsilon}\frac{\partial E_z}{\partial y} = Z\left\{H_x + \frac{1}{2ik_oN}\frac{\mu_o}{\mu}\frac{\partial H_z}{\partial x}\right\} , \tag{58}$$

and since

$$E_z = \frac{iZ_o}{k_o}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) , \quad H_z = -\frac{iY_o}{k_o}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) ,$$

(58) can be written as

$$E_x = -Z\left\{H_y + \frac{1}{2(k_oN)^2}\left(\frac{\partial^2 H_y}{\partial x^2} - \frac{\partial^2 H_x}{\partial x \partial y}\right) - \frac{Y}{2(k_oN)^2}\left(\frac{\partial^2 E_y}{\partial x \partial y} - \frac{\partial^2 E_x}{\partial y^2}\right)\right\}$$

$$E_y = Z\left\{H_x + \frac{1}{2(k_oN)^2}\left(\frac{\partial^2 H_x}{\partial y^2} - \frac{\partial^2 H_y}{\partial x \partial y}\right) - \frac{Y}{2(k_oN)^2}\left(\frac{\partial^2 E_y}{\partial x^2} - \frac{\partial^2 E_x}{\partial x \partial y}\right)\right\} . \tag{59}$$

On inserting the zeroth order approximations to the derivatives of the electric field on the right hand side, we then obtain

$$E_x = -Z\left\{H_y + \frac{1}{2(k_oN)^2}\left[\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)H_y - 2\frac{\partial^2 H_x}{\partial x \partial y}\right]\right\}$$

$$E_y = Z\left\{H_x + \frac{1}{2(k_oN)^2}\left[\left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2}\right)H_x - 2\frac{\partial^2 H_y}{\partial x \partial y}\right]\right\} , \tag{60}$$

and these are analogous to the results cited by Leontovich [1948]. However, the coefficient of the terms $O(|N|^{-2})$ differs by a factor $i/2$ from his, and it appears that the error was made in extending the correct but specialized results of Rytov [1940].

Using Maxwell's equations and the divergence conditions, (60) can be expressed in terms of the normal field components:

$$\left\{\frac{\partial^2}{\partial z^2} - 2ik_oN\frac{\partial}{\partial z} - k_o^2(2N^2-1)\right\}E_z = 0$$

$$\left\{\frac{\partial^3}{\partial z^3} + k_o^2(2N^2+1)\frac{\partial}{\partial z} - 2i(k_oN)^3\right\}H_z = 0 \qquad (61)$$

and from these it is clear that (60) do not satisfy the duality condition. On the other hand, from (58) or (59) without any approximation at all,

$$\left\{\frac{\partial^2}{\partial z^2} + 2ik_oN\frac{\epsilon}{\epsilon_o}\frac{\partial}{\partial z} - k_o^2(2N^2-1)\right\}E_z = 0$$

$$\left\{\frac{\partial^2}{\partial z^2} + 2ik_oN\frac{\mu}{\mu_o}\frac{\partial}{\partial z} - k_o^2(2N^2-1)\right\}H_z = 0 \qquad (62)$$

which are quite different from (61). Equations (62) are second order generalized impedance boundary conditions in the form generally adopted [Senior and Volakis, 1989] for a planar surface and *do* satisfy duality. In view of the discrepancy between (61) and (62) it would seem that boundary conditions like this are very sensitive to the precise form of conditions such as (57) from which they are often derived.

For the circular cylindrical surface $\rho =$ constant we choose $\alpha = \phi, \beta = z, \gamma = \rho$, implying $h_\alpha = \rho, h_\beta = h_\gamma = 1$. Then

$$\hat{\rho}\times\left(\hat{\rho}\times\left\{\mathbf{E} + \frac{1}{2ik_oN}\frac{\epsilon_o}{\epsilon}\nabla E_\rho\right\}\right) = -\overline{\overline{\eta}}\cdot\hat{\rho}\times\left\{\mathbf{H} + \frac{1}{2ik_oN}\frac{\mu_o}{\mu}\nabla H_\rho\right\} \qquad (63)$$

with

$$\overline{\overline{\eta}} = \Gamma\hat{\phi}\hat{\phi} + \frac{1}{\Gamma^*}\hat{z}\hat{z}$$

and

$$\Gamma = Z\left\{1 + \frac{1}{2ik_oN\rho} + \frac{1}{2(2k_oN\rho)^2}\right\}$$

so that

$$\frac{1}{\Gamma^*} = Z\left\{1 - \frac{1}{2ik_oN\rho} - \frac{3}{2(2k_oN\rho)^2}\right\} .$$

The components of (63) are

$$E_\phi + \frac{1}{2ik_oN}\frac{\epsilon_o}{\epsilon}\frac{1}{\rho}\frac{\partial E_\rho}{\partial\phi} = -\Gamma\left\{H_z + \frac{1}{2ik_oN}\frac{\mu_o}{\mu}\frac{\partial H_\rho}{\partial z}\right\}$$

$$E_z + \frac{1}{2ik_oN}\frac{\epsilon_o}{\epsilon}\frac{\partial E_\rho}{\partial z} = \frac{1}{\Gamma^*}\left\{H_\phi + \frac{1}{2ik_oN}\frac{\mu_o}{\mu}\frac{1}{\rho}\frac{\partial H_\rho}{\partial\phi}\right\} \qquad (64)$$

in agreement with the results in Appendix A, but without some approximation it is not possible to write these in terms of normal derivatives of the normal field components alone. However, they can be written in terms of tangential components. Since

$$E_\rho = \frac{iZ_o}{k_o}\left(\frac{1}{\rho}\frac{\partial H_z}{\partial\phi} - \frac{\partial H_\phi}{\partial z}\right) , \quad H_\rho = -\frac{iY_o}{k_o}\left(\frac{1}{\rho}\frac{\partial E_z}{\partial\phi} - \frac{\partial E_\phi}{\partial z}\right) ,$$

we have

$$E_\phi = -Z\left\{\frac{\Gamma}{Z}H_z + \frac{1}{2(k_oN)^2}\left(\frac{1}{\rho^2}\frac{\partial^2 H_z}{\partial\phi^2} - \frac{1}{\rho}\frac{\partial^2 H_\phi}{\partial\phi\partial z}\right)\right.$$
$$\left. - \frac{1}{2(k_oN)^2}Y^2\Gamma\left(\frac{1}{\rho}\frac{\partial^2 E_z}{\partial\phi\partial z} - \frac{\partial^2 E_\phi}{\partial z^2}\right)\right\} .$$

As regards the last term on the right hand side it is sufficient to insert the zeroth order approximations

$$E_\phi = -\Gamma H_z , \quad E_z = \frac{1}{\Gamma^*}H_\phi \quad \text{with } \Gamma = Z$$

and hence

$$E_\phi = -Z\left\{\left[1 + \frac{1}{2ik_oN\rho} + \frac{1}{2(2k_oN\rho)^2} - \frac{1}{2(k_oN)^2}\left(\frac{\partial^2}{\partial z^2} - \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)\right]H_z\right.$$
$$\left. - \frac{1}{(k_oN)^2\rho}\frac{\partial^2 H_\phi}{\partial\phi\partial z}\right\} . \qquad (65)$$

Similarly

$$E_z = Z\left\{\left[1 - \frac{1}{2ik_oN_\rho} - \frac{3}{2(2k_oN\rho)^2} + \frac{1}{2(k_oN)^2}\left(\frac{\partial^2}{\partial z^2} - \frac{1}{\partial^2}\frac{\partial^2}{\partial \phi^2}\right)\right]H_\phi \right.$$

$$\left. - \frac{1}{(k_oN)^2\rho}\frac{\partial^2 H_z}{\partial\phi\partial z}\right\} . \qquad (66)$$

For the spherical surface r = constant we choose $\alpha = \theta, \beta = \phi, \gamma = r$ so that $h_\alpha = r, h_\beta = r\sin\theta, h_\gamma = 1$. The boundary condition (53) then is

$$\hat{r}\times\left(\hat{r}\times\left\{\mathbf{E} + \frac{1}{2ik_oN}\frac{\epsilon_o}{\epsilon}\nabla E_r\right\}\right) = -\bar{\bar{\eta}}\cdot\hat{r}\times\left\{\mathbf{H} + \frac{1}{2ik_oN}\frac{\mu_o}{\mu}\nabla H_r\right\} \qquad (67)$$

with

$$\bar{\bar{\eta}} = \Gamma\hat{\theta}\hat{\theta} + \frac{1}{\Gamma^*}\hat{\phi}\hat{\phi}$$

and (see (50))

$$\Gamma = Z = \frac{1}{\Gamma^*} .$$

By a method similar to that used to derive (65) and (66), it can be shown that the boundary conditions expressed in terms of the tangential field components are

$$E_\theta = -Z\left\{\left[1 + \frac{1}{2(k_oNr)^2}\left(\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\right]H_\phi \right.$$

$$\left. - \frac{1}{(k_oNr)^2\sin\theta}\frac{\partial^2 H_\theta}{\partial\theta\partial\phi}\right\} \qquad (68)$$

$$E_\phi = Z\left\{\left[1 - \frac{1}{2(k_oNr)^2}\left(\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\right]H_\theta \right.$$

$$\left. - \frac{1}{(k_oNr)^2\sin\theta}\frac{\partial^2 H_\phi}{\partial\theta\partial\phi}\right\} . \qquad (69)$$

In general form at least, (65) and (66) and (68) and (69) are similar to the boundary conditions (60) for a planar surface, and it is tempting to ask whether, for a curved surface, we can derive the boundary conditions by treating x, y, z as local coordinates. To test this, consider the cylindrical surface ρ = constant. Replacing z by ρ, x by $\rho\theta$ and y by z in (60) we do indeed reproduce the differentiated terms in (65) and (66), but do not obtain the undifferentiated terms associated with Γ/Z and Y/Γ^*. This is not unexpected in view of the first order condition (35), and since the dominant (for large $|N|$) terms then differ from those in (65) and (66), it is clear that the procedure cannot be justified.

Our final point concerns the sequence of boundary conditions through the second order for an inhomogeneous curved surface, and the implications for simulating other surfaces using boundary conditions of this type. In its most general form the zeroth order condition is

$$\hat{\gamma} \times (\hat{\gamma} \times \mathbf{E}) = -\eta \hat{\gamma} \times \mathbf{H} \tag{70}$$

and involves only a single scalar parameter η which is independent of any material inhomogeneity and the shape of the surface. The first order condition is

$$\hat{\gamma} \times (\hat{\gamma} \times \mathbf{E}) = -\bar{\bar{\eta}} \cdot \hat{\gamma} \times \mathbf{H} \tag{71}$$

with

$$\bar{\bar{\eta}} = \Gamma \hat{\alpha}\hat{\alpha} + \Gamma_1 \hat{\beta}\hat{\beta} \, ,$$

but if duality is satisfied, $\Gamma_1 = 1/\Gamma^*$. There is still only one parameter (Γ) involved, but this may be a function of the variation of the material properties in the normal direction, as well as the curvature of the surface. Finally, the second order condition is

$$\hat{\gamma} \times \left(\hat{\gamma} \times \left\{\mathbf{E} + \frac{A}{2ik_oNh_\gamma} \nabla \left(\frac{\epsilon_o}{\epsilon} h_\gamma E_\gamma\right)\right\}\right) = -\bar{\bar{\eta}} \cdot \hat{\gamma} \times \left\{\mathbf{H} + \frac{A^*}{2ik_oNh_\gamma} \nabla \left(\frac{\mu_o}{\mu} h_\gamma H_\gamma\right)\right\} \tag{72}$$

involving the two parameters Γ and A, and all material variations in the normal direction are embedded in Γ.

7 Conclusions

The method developed by Rytov [1940] has been used to derive approximate boundary conditions at a surface which may be non-planar and have material properties that vary laterally as well as in depth. The approximations are based on the assumption that $|N|$ is large, where N is the complex refractive index of the material, and boundary conditions through the second order have been obtained. The lowest (zeroth) order one is the standard Leontovich impedance condition whose form is the same for any surface, planar or curved, and is independent of any variation of the material properties. This is not true of a higher order condition and, in particular, the surface curvature now affects the boundary condition. Because of this it should not be assumed that a generalized boundary condition developed for a planar surface is immediately applicable to a non-planar surface.

References

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Appendix A: Homogeneous Circular Cylinder

When the exact solution of a scattering problem is known, it can be used to derive an approximate boundary condition applicable at the surface, and a case in point is a right circular homogeneous dielectric cylinder.

In terms of the cylindrical polar coordinates ρ, ϕ, z the cylinder is defined as $\rho = \rho_o$ and is illuminated by an H-polarized plane wave incident in a plane perpendicular to the axis. The only components of the field are then H_z, E_ρ and E_ϕ, and their expessions are

$\rho \geq \rho_o$:

$$\begin{aligned} H_z &= \sum_{n=0}^{\infty} \epsilon_n(-i)^n \left\{J_n(k_o\rho) + R_n H_n^{(1)}(k_o\rho)\right\} \cos n\phi \\ E_\rho &= -i\frac{Z_o}{k_o\rho} \sum_{n=0}^{\infty} \epsilon_n(-i)^n \left\{J_n(k_o\rho) + R_n H_n^{(1)}(k_o\rho)\right\} \sin n\phi \\ E_\phi &= -iZ_o \sum_{n=0}^{\infty} \epsilon_n(-i)^n \left\{J_n'(k_o\rho) + R_n H_n'^{(1)}(k_o\rho)\right\} \cos n\phi \end{aligned}$$

$\rho \leq \rho_o$:

$$\begin{aligned} H_z &= \sum_{n=0}^{\infty} \epsilon_n(-i)^n a_n J_n(Nk_o\rho) \cos n\phi \\ E_\rho &= -i\frac{Z}{Nk_o\rho} \sum_{n=0}^{\infty} \epsilon_n(-i)^n n a_n J_n(Nk_o\rho) \sin n\phi \\ E_\phi &= -iZ \sum_{n=0}^{\infty} \epsilon_n(-i)^n a_n J_n'(Nk_o\rho) \cos n\phi \end{aligned}$$

where the prime denotes the derivative and R_n, a_n are coefficients to be determined. From the continuity of H_z and E_ϕ at $\rho = \rho_o$ we find

$$R_n = -\frac{J_n'(k_o\rho_o) + iY_oPJ_n(k_o\rho_o)}{H_n^{(1)\prime}(k_o\rho_o) + iY_oPH_n^{(1)}(k_o\rho_o)} \tag{A.1}$$

with

$$P = iZ\frac{J_n'(Nk_o\rho_o)}{J_n(Nk_o\rho_o)} \,. \tag{A.2}$$

On the assumption that $|\mathcal{N}|k_o\rho_o \gg 1$ with Im.$\mathcal{N} > 0$ to prevent any penetration through the cylinder, P can be expanded in an asymptotic series for large $t = \mathcal{N}k_o\rho_o$. Since

$$J_n(t) = \frac{1}{2}\left\{H_n^{(1)}(t) + H_n^{(2)}(t)\right\}$$

it follows that

$$\begin{aligned}
J_n(t) &\sim \frac{e^{-i(t-n\frac{\pi}{2}-\frac{\pi}{4})}}{\sqrt{2\pi t}} \sum_{m=0}^{\infty} \frac{(n,m)}{(2it)^m} \\
&= \frac{e^{-i(t-n\frac{\pi}{2}-\frac{\pi}{4})}}{\sqrt{2\pi t}} \left\{1 + \frac{4n^2-1}{8it} - \frac{(4n^2-1)(4n^2-9)}{128t^2}\right. \\
&- \frac{(4n^2-1)(4n^2-9)(4n^2-25)}{128\cdot 24it^3} \\
&+ \left.\frac{(4n^2-1)(4n^2-9)(4n^2-25)(4n^2-49)}{(128)^2\cdot 6t^4} + O(t^{-5})\right\} .
\end{aligned}$$

Hence

$$\begin{aligned}
J_n'(t) &= -iJ_n(t) + \frac{e^{-i(t-n\frac{\pi}{2}-\frac{\pi}{4})}}{\sqrt{2\pi t}} \left\{-\frac{1}{2t} - \frac{3}{2t}\cdot\frac{4n^2-1}{8it}\right. \\
&+ \left.\frac{5}{2t}\cdot\frac{(4n^2-1)(4n^2-9)}{128t^2} + \frac{7}{2t}\frac{(4n^2-1)(4n^2-9)(4n^2-25)}{128\cdot 24it^3} + O(t^{-5})\right\} \\
&= -i\frac{e^{-i(t-n\frac{\pi}{2}-\frac{\pi}{4})}}{\sqrt{2\pi t}} \left\{1 + \frac{4n^2+3}{8it} - \frac{(4n^2-1)(4n^2+15)}{128t^2}\right. \\
&- \frac{(4n^2-1)(4n^2-9)(4n^2+35)}{128\cdot 24it^3} \\
&+ \left.\frac{(4n^2-1)(4n^2-9)(4n^2-25)(4n^2+63)}{(128)^2\cdot 6t^4} + O(t^{-5})\right\}
\end{aligned}$$

giving

$$P = Z\left\{1 + \frac{1}{2it} - \frac{4n^2-1}{8t^2} + \frac{4n^2-1}{8it^3}\right.$$

$$\left. - \frac{(4n^2-1)(4n^2-25)}{128t^4} + O(t^{-5})\right\} . \qquad (A.3)$$

An approximate boundary condition must reproduce P (and therefore R_n) to a specified order in $1/t$. A standard impedance boundary condition has the form

$$E_\phi = -\Gamma H_z , \qquad (A.4)$$

and using the previous expressions for the field components in $\rho \geq \rho_o$ we obtain (A.1) with $P = \Gamma$. Thus, to the zeroth order in $1/t$,

$$\Gamma = Z , \qquad (A.5)$$

but (A.4) is also sufficient to reproduce the exact solution to the first order in $1/t$ if Γ is chosen as

$$\Gamma = Z\left\{1 + \frac{1}{2iNk_o\rho_o}\right\} , \qquad (A.6)$$

and (A.4) is then a first order impedance boundary condition.

To reproduce terms of higher order in $1/t$, it is necessary to generalize the boundary condition (A.4). In line with (72) we now consider the second order boundary condition

$$E_\phi + \frac{A}{2iNk_o\rho_o}\frac{\partial}{\partial\phi}\left(\frac{\epsilon_o}{\epsilon}E_\rho\right) = -\Gamma H_z , \qquad (A.7)$$

and using again the field expressions in $\rho \geq \rho_o$ we find

$$P = \Gamma - \frac{A}{2N}\frac{\epsilon_o}{\epsilon}Z_o\left(\frac{n}{k_o\rho_o}\right)^2 = \Gamma - \frac{A}{2}Z\left(\frac{n}{t}\right)^2 . \qquad (A.8)$$

Comparison with (A.3) now shows that to the second order in $1/t$

$$\Gamma = Z\left\{1 + \frac{1}{2iNk_o\rho_o} + \frac{1}{2(2Nk_o\rho_o)^2}\right\} \qquad (A.9)$$

and

$$A = 1\,, \tag{A.10}$$

in agreement with (63). As a matter of fact, by choosing

$$\Gamma = Z\left\{1 + \frac{1}{2iNk_o\rho_o} + \frac{1}{2(2Nk_o\rho_o)^2} - \frac{1}{i(2Nk_o\rho_o)^3}\right\} \tag{A.11}$$

and

$$A = 1 - \frac{1}{iNk_o\rho_o}\,, \tag{A.12}$$

the boundary condition (A.7) is sufficient to match the third order terms as well, but this is peculiar to the geometry. In general, a third order boundary condition is necessary to achieve this accuracy, and as evident from the term in n^4 in (A.3), a still higher order boundary condition is required to match the terms in $1/t^4$.