APPROXIMATE BOUNDARY CONDITIONS, PART II

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Abstract

Approximate boundary conditions are constructed to simulate the scattering from an opaque body covered with a layer of inhomogeneous dielectric whose inner and outer boundaries are coordinate surfaces in an orthogonal curvilinear coordinate system. Two different situations are considered. The first is a low contrast material for which an expansion in powers of the layer thickness τ is appropriate. Approximate boundary conditions through the third order are derived and these illustrate the general form of higher order boundary conditions for a non-planar surface. The second is a high contrast material, and the technique developed in Part I of this report is used to obtain boundary conditions accurate to the second order in $\frac{1}{N}$ where N is the complex refractive index of the material. In the special case of a circular cylinder covered with a layer of homogeneous dielectric of uniform thickness, the results are confirmed by starting with the known modal expansion for the scattered field.

1 Introduction

Approximate boundary conditions are frequently employed to simulate the material properties of a scattering object, and the standard impedance boundary condition has been in use for many years. In principle at least, the accuracy can be improved by including higher derivatives of the field components, and the application of such generalized impedance boundary conditions (GIBCs) is now being considered [Volakis and Senior, 1989]. For a planar surface it is relatively easy to construct a hierarchy of conditions in terms of the normal derivatives of the normal field components, and methods are available [Senior and Volakis, 1989] to determine the coefficients. By tangential integration, the conditions can also be expressed in terms of the tangential field components. For a curved surface, however, a natural hierarchy is less apparent.

In Part I of this report, hereafter referred to as I, a method developed by Rytov [1940] was used to derive a series of approximate boundary conditions applicable at the curved surface of a body whose material properties may vary continuously, both laterally and in depth. The approximations are based on the assumption that $|N| \gg 1$ where N is the complex refractive index of the material, and the order of the condition is determined by the highest power of $|N|^{-1}$ which is retained. For a body whose surface is $\gamma = \gamma_o$ where α, β, γ are orthogonal curvilinear coordinates, boundary conditions through the second order were derived.

We now turn our attention to the practically important case of a coated body, and seek approximate boundary conditions to simulate the scattering properties of an opaque (typically metallic) body covered with a layer of dielectric material. In Sections 2 and 3 a method proposed by Weinstein [1969] is used to obtain boundary conditions for a thin layer of "low contrast" material whose complex refractive index is not large in magnitude. Although this is not the situation of most practical interest, the procedure is very simple and leads to a natural hierarchy of conditions. The analogous problem of a "high contrast" material is treated in Sections 4-6 using an extension of the analysis in I. For maximum generality, the dielectric is assumed to be inhomogeneous, and boundary conditions through the second order are constructed. Finally, in Section 7 we consider the case of a circular cylinder with a homogeneous coating of uniform thickness and use the modal solution to derive approximate boundary conditions for low and high contrast materials. These are in agreement with the preceding results.

2 Low Contrast Coating, Second Order

A metallic or other opaque body is covered with a layer of dielectric material whose properties may vary laterally as well as in depth, and is illuminated by an electromagnetic field. In terms of the orthogonal curvilinear coordinates α, β, γ with metric coefficients $h_\alpha, h_\beta, h_\gamma$, respectively, the region $\gamma_o - \tau < \gamma < \gamma_o$ consists of a dielectric material with permittivity ϵ and permeability μ. The lower surface $\gamma = \gamma_o - \tau$ is that of the body itself, and to preserve duality it is assumed that a standard impedance boundary condition is imposed here. Thus, at $\gamma = \gamma_o - \tau$

$$\hat{\gamma} \times \left(\hat{\gamma} \times \mathbf{E}^i\right) = -\eta_m \hat{\gamma} \times \mathbf{H}^i$$

implying

$$E_\alpha^i = -\eta_m H_\beta^i \,, \qquad E_\beta^i = \eta_m H_\alpha^i \tag{1}$$

where the affix i denotes the field in the dielectric and η_m is specified. The region $\gamma > \gamma_o$ is free space having propagation constant k_o and intrinsic impedance Z_o, and we seek boundary conditions which can be applied to the exterior field components at $\gamma = \gamma_o$ to simulate the scattering properties.

Following Weinstein [1969] the fields in the dielectric are expanded in Taylor series in γ. In particular

$$E_\alpha^i(\gamma_o - \tau) = E_\alpha^i - \tau \frac{\partial E_\alpha^i}{\partial \gamma} + O(\tau^2) \tag{2}$$

for small τ, where we show only the γ dependence and the fields on the right hand side are evaluated at $\gamma = \gamma_o - 0$. From Maxwell's equation with a time factor $e^{-i\omega t}$

$$\frac{1}{h_\alpha h_\gamma} \left\{ \frac{\partial}{\partial \gamma}(h_\alpha E_\alpha^i) - \frac{\partial}{\partial \alpha}(h_\gamma E_\gamma^i) \right\} = i k_o Z_o \frac{\mu}{\mu_o} H_\beta^i$$

giving

$$\frac{\partial E_\alpha^i}{\partial \gamma} = -E_\alpha^i \frac{\partial}{\partial \gamma}(\ln h_\alpha) + i k_o Z_o \frac{\mu}{\mu_o} h_\gamma H_\beta^i + \frac{1}{h_\alpha} \frac{\partial}{\partial \alpha}(h_\gamma E_\gamma^i) \,, \tag{3}$$

and therefore

$$E^i_\alpha(\gamma_o-\tau) = \left\{1+\tau\frac{\partial}{\partial\gamma}(\ln h_\alpha)\right\}E^i_\alpha - ik_oZ_o\tau\frac{\mu}{\mu_o}h_\gamma H^i_\beta - \frac{\tau}{h_\alpha}\frac{\partial}{\partial\alpha}(h_\gamma E^i_\gamma)+O(\tau^2). \tag{4}$$

Similarly

$$E^i_\beta(\gamma_o-\tau) = \left\{1+\tau\frac{\partial}{\partial\gamma}(\ln h_\beta)\right\}E^i_\beta + ik_oZ_o\tau\frac{\mu}{\mu_o}h_\gamma H^i_\alpha - \frac{\tau}{h_\beta}\frac{\partial}{\partial\beta}(h_\gamma E^i_\gamma)+O(\tau^2), \tag{5}$$

and by duality

$$H^i_\alpha(\gamma_o-\tau) = \left\{1+\tau\frac{\partial}{\partial\gamma}(\ln h_\alpha)\right\}H^i_\alpha + ik_oY_o\tau\frac{\epsilon}{\epsilon_o}h_\gamma E^i_\beta - \frac{\tau}{h_\alpha}\frac{\partial}{\partial\alpha}(h_\gamma H^i_\gamma)+O(\tau^2) \tag{6}$$

$$H^i_\beta(\gamma_o-\tau) = \left\{1+\tau\frac{\partial}{\partial\gamma}(\ln h_\beta)\right\}H^i_\beta - ik_oY_o\tau\frac{\epsilon}{\epsilon_o}h_\gamma E^i_\alpha - \frac{\tau}{h_\beta}\frac{\partial}{\partial\beta}(h_\gamma H^i_\gamma)+O(\tau^2). \tag{7}$$

On applying the boundary conditions (1) at the lower surface of the dielectric, we obtain

$$\left\{1+\tau\frac{\partial}{\partial\gamma}(\ln h_\alpha) - ik_o\tau\frac{\epsilon}{\epsilon_o}Y_o\eta_m h_\gamma\right\}E^i_\alpha - \frac{\tau}{h_\alpha}\frac{\partial}{\partial\alpha}(h_\gamma E^i_\gamma)$$
$$= -\eta_m\left[\left\{1+\tau\frac{\partial}{\partial\gamma}(\ln h_\beta) - ik_o\tau\frac{\mu}{\mu_o}\frac{Z_o}{\eta_m}h_\gamma\right\}H^i_\beta - \frac{\tau}{h_\beta}\frac{\partial}{\partial\beta}(h_\gamma H^i_\gamma)\right]+O(\tau^2) \tag{8}$$

$$\left\{1+\tau\frac{\partial}{\partial\gamma}(\ln h_\beta)-ik_o\tau\frac{\epsilon}{\epsilon_o}Y_o\eta_m h_\gamma\right\}E^i_\beta-\frac{\tau}{h_\beta}\frac{\partial}{\partial\beta}(h_\gamma E^i_\gamma)$$

$$=\eta_m\left[\left\{1+\tau\frac{\partial}{\partial\gamma}(\ln h_\alpha)-ik_o\tau\frac{\mu}{\mu_o}\frac{Z_o}{\eta_m}h_\gamma\right\}H^i_\alpha-\frac{\tau}{h_\alpha}\frac{\partial}{\partial\alpha}(h_\gamma H^i_\gamma)\right]+O(\tau^2)$$

and by imposing the boundary conditions at the dielectric-air interface $\gamma=\gamma_o$, (8) can be expressed in terms of the exterior fields as

$$E_\alpha-\frac{\tau}{h_\alpha}\frac{\partial}{\partial\alpha}\left(\frac{\epsilon_o}{\epsilon}h_\gamma E_\gamma\right)=-\Gamma\left\{H_\beta-\frac{\tau}{h_\beta}\frac{\partial}{\partial\beta}\left(\frac{\mu_o}{\mu}h_\gamma H_\gamma\right)\right\}+O(\tau^2)$$

$$E_\beta-\frac{\tau}{h_\beta}\frac{\partial}{\partial\beta}\left(\frac{\epsilon_o}{\epsilon}h_\gamma E_\gamma\right)=\Gamma_1\left\{H_\alpha-\frac{\tau}{h_\alpha}\frac{\partial}{\partial\alpha}\left(\frac{\mu_o}{\mu}h_\gamma H_\gamma\right)\right\}+O(\tau^2) \tag{9}$$

where

$$\Gamma=\eta_m\frac{1+\tau\frac{\partial}{\partial\gamma}(\ln h_\beta)-ik_o\tau\frac{\mu}{\mu_o}\frac{Z_o}{\eta_m}h_\gamma}{1+\tau\frac{\partial}{\partial\gamma}(\ln h_\alpha)-ik_o\tau\frac{\epsilon}{\epsilon_o}Y_o\eta_m h_\gamma},$$

$$\Gamma_1=\eta_m\frac{1+\tau\frac{\partial}{\partial\gamma}(\ln h_\alpha)-ik_o\tau\frac{\mu}{\mu_o}\frac{Z_o}{\eta_m}h_\gamma}{1+\tau\frac{\partial}{\partial\gamma}(\ln h_\beta)-ik_o\tau\frac{\epsilon}{\epsilon_o}Y_o\eta_m h_\gamma}. \tag{10}$$

We note that $\Gamma_1=1/\Gamma^*$ where the asterisk denotes the dual quantity.

In the context of GIBCs, (9) are second order boundary conditions, and correct to the first order in τ they can be written as

$$\hat\gamma\times\left(\hat\gamma\times\left\{\mathbf{E}-\tau\nabla\left(\frac{\epsilon_o}{\epsilon}h_\gamma E_\gamma\right)\right\}\right)=-\bar{\bar\eta}\cdot\hat\gamma\times\left\{\mathbf{H}-\tau\nabla\left(\frac{\mu_o}{\mu}h_\gamma H_\gamma\right)\right\} \tag{11}$$

with

$$\bar{\bar\eta}=\Gamma\hat\alpha\hat\alpha+\frac{1}{\Gamma^*}\hat\beta\hat\beta\,.$$

It can be verified that (11) satisfies the duality condition and its general form is consistent with (I.53).

In the particular case of a perfectly conducting substrate, $\eta_m = 0$ and (10) reduce to

$$\Gamma = \Gamma_1 = -ik_o\tau\frac{\mu}{\mu_o}Z_o h_\gamma \tag{12}$$

to the first order in τ. The impedance is now isotropic in spite of the curvature of the surface, and with this change (9) is unaffected, but because of the factor τ/η_m in (8), equations (9) and (11) are no longer correct in the limit as $\eta_m \to 0$. Indeed, from (8) we have

$$\hat{\gamma} \times \left(\hat{\gamma} \times \left\{\mathbf{E} - \tau\nabla\left(\frac{\epsilon_o}{\epsilon}h_\gamma E_\gamma\right)\right\}\right) = ik_o\tau\frac{\mu}{\mu_o}Z_o h_\gamma \hat{\gamma} \times \mathbf{H} \tag{13}$$

when $\eta_m = 0$.

3 Low Contrast Coating, Third Order

The third order boundary conditions can be found by retaining the terms $O(\tau^2)$ in the Taylor series expansions of the fields in the dielectric. Thus

$$E^i_\alpha(\gamma_o - \tau) = E^i_\alpha - \tau\frac{\partial E^i_\alpha}{\partial\gamma} + \frac{\tau^2}{2}\frac{\partial^2 E^i_\alpha}{\partial\gamma^2} + O(\tau^3) \tag{14}$$

and from (3)

$$\frac{\partial^2 E^i_\alpha}{\partial\gamma^2} = -E^i_\alpha\frac{\partial^2}{\partial\gamma^2}(\ln h_\alpha) - \frac{\partial}{\partial\gamma}(\ln h_\alpha)\left\{-E^i_\alpha\frac{\partial}{\partial\gamma}(\ln h_\alpha) + ik_o Z_o\frac{\mu}{\mu_o}h_\gamma H^i_\beta\right.$$
$$\left. + \frac{1}{h_\alpha}\frac{\partial}{\partial\alpha}(h_\gamma E^i_\gamma)\right\} + ik_o Z_o\frac{\partial}{\partial\gamma}\left(\frac{\mu}{\mu_o}h_\gamma H^i_\beta\right) + \frac{\partial}{\partial\gamma}\left\{\frac{1}{h_\alpha}\frac{\partial}{\partial\alpha}(h_\gamma E^i_\gamma)\right\}.$$

Also

$$\frac{\partial H^i_\beta}{\partial\gamma} = -H^i_\beta\frac{\partial}{\partial\gamma}(\ln h_\beta) + ik_o Y_o\frac{\epsilon}{\epsilon_o}h_\gamma E^i_\alpha + \frac{1}{h_\beta}\frac{\partial}{\partial\beta}(h_\gamma H^i_\gamma)$$

giving

$$\frac{\partial}{\partial\gamma}\left(\frac{\mu}{\mu_o}h_\gamma H^i_\beta\right) = -\frac{\mu}{\mu_o}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{h_\gamma}\frac{\mu_o}{\mu}\right)h_\gamma H^i_\beta$$
$$+ik_o Y_o N^2 h_\gamma^2 E^i_\alpha + \frac{\mu}{\mu_o}\frac{h_\gamma}{h_\beta}\frac{\partial}{\partial\beta}(h_\gamma H^i_\gamma),$$

and therefore

$$E^i_\alpha(\gamma_o-\tau)=\left\{1+\tau\frac{\partial}{\partial\gamma}(\ln h_\alpha)-\frac{\tau^2}{2}\left[\frac{\partial^2}{\partial\gamma^2}(\ln h_\alpha)-\left\{\frac{\partial}{\partial\gamma}(\ln h_\alpha)\right\}^2\right.\right.$$

$$\left.\left.+k_o^2N^2h_\gamma^2\right]\right\}E^i_\alpha-\tau\left\{1+\frac{\tau}{2}\frac{\partial}{\partial\gamma}(\ln h_\alpha)\right\}\frac{1}{h_\alpha}\frac{\partial}{\partial\alpha}(h_\gamma E^i_\gamma)$$

$$+\frac{\tau^2}{2}\frac{\partial}{\partial\gamma}\left\{\frac{1}{h_\alpha}\frac{\partial}{\partial\alpha}(h_\gamma E^i_\gamma)\right\}-ik_o\tau\frac{\mu}{\mu_o}Z_o\left\{1+\frac{\tau}{2}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha h_\beta}{h_\gamma}\frac{\mu_o}{\mu}\right)\right\}h_\gamma H^i_\beta$$

$$+ik_o\frac{\tau^2}{2}\frac{\mu}{\mu_o}Z_o\frac{h_\gamma}{h_\beta}\frac{\partial}{\partial\beta}(h_\gamma H^i_\gamma)+O(\tau^3)\,,$$

i.e.

$$E^i_\alpha(\gamma_o-\tau)=\left\{1+\tau\frac{\partial}{\partial\gamma}(\ln h_\alpha)-\frac{\tau^2}{2}\left[\frac{\partial^2}{\partial\gamma^2}(\ln h_\alpha)-\left\{\frac{\partial}{\partial\gamma}(\ln h_\alpha)\right\}^2\right.\right.$$

$$\left.\left.+k_o^2N^2h_\gamma^2\right]\right\}E^i_\alpha-\tau\left\{1+\tau\frac{\partial}{\partial\gamma}(\ln h_\alpha)\right\}\frac{1}{h_\alpha}\frac{\partial}{\partial\alpha}(h_\gamma E^i_\gamma)$$

$$+\frac{\tau^2}{2}\frac{1}{h_\alpha}\frac{\partial^2}{\partial\alpha\partial\gamma}(h_\gamma E^i_\gamma)-ik_o\tau\frac{\mu}{\mu_o}Z_o\left\{1+\frac{\tau}{2}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha h_\beta}{h_\gamma}\frac{\mu_o}{\mu}\right)\right\}h_\gamma H^i_\beta$$

$$+ik_o\frac{\tau^2}{2}\frac{\mu}{\mu_o}Z_o\frac{h_\gamma}{h_\beta}\frac{\partial}{\partial\beta}(h_\gamma H^i_\gamma)+O(\tau^3)\,. \qquad (15)$$

Similarly

$$E^i_\beta(\gamma_o-\tau)=\left\{1+\tau\frac{\partial}{\partial\gamma}(\ln h_\beta)-\frac{\tau^2}{2}\left[\frac{\partial^2}{\partial\gamma^2}(\ln h_\beta)-\left\{\frac{\partial}{\partial\gamma}(\ln h_\beta)\right\}^2\right.\right.$$

$$\left.\left.+k_o^2N^2h_\gamma^2\right]\right\}E^i_\beta-\tau\left\{1+\tau\frac{\partial}{\partial\gamma}(\ln h_\beta)\right\}\frac{1}{h_\beta}\frac{\partial}{\partial\beta}(h_\gamma E^i_\gamma)$$

$$+\frac{\tau^2}{2}\frac{1}{h_\beta}\frac{\partial^2}{\partial\beta\partial\gamma}(h_\gamma E^i_\gamma)+ik_o\tau\frac{\mu}{\mu_o}Z_o\left\{1+\frac{\tau}{2}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha h_\beta}{h_\gamma}\frac{\mu_o}{\mu}\right)\right\}h_\gamma H^i_\alpha$$

$$-ik_o\frac{\tau^2}{2}\frac{\mu}{\mu_o}Z_o\frac{h_\gamma}{h_\alpha}\frac{\partial}{\partial\alpha}(h_\gamma H^i_\gamma)+O(\tau^3), \qquad (16)$$

and for a perfectly conducting substrate the left hand sides of (15) and (16) are both zero. On applying the boundary conditions at the dielectric-air interface and using (A.1) we then obtain

$$\left\{1+\tau\frac{\partial}{\partial\gamma}(\ln h_\alpha)-\frac{\tau^2}{2}\left[\frac{\partial^2}{\partial\gamma^2}(\ln h_\alpha)-\left\{\frac{\partial}{\partial\gamma}(\ln h_\alpha)\right\}^2+k_o^2N^2h_\gamma^2\right]\right\}\left\{E_\alpha\right.$$
$$\left.-\tau\frac{1}{h_\alpha}\frac{\partial}{\partial\alpha}\left[\frac{\epsilon_o}{\epsilon}h_\gamma E_\gamma\left\{1+\frac{\tau}{2}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha h_\beta}{h_\gamma}\frac{\epsilon}{\epsilon_o}\right)\right\}+\frac{\tau}{2}\frac{\epsilon_o}{\epsilon}h_\gamma^2\nabla_s\cdot\left(\frac{\epsilon}{\epsilon_o}\mathbf{E}\right)\right]\right\}$$
$$=ik_o\tau\frac{\mu}{\mu_o}Z_oh_\gamma\left\{1+\frac{\tau}{2}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha h_\beta}{h_\gamma}\frac{\mu_o}{\mu}\right)\right\}\left\{H_\beta-\frac{\tau}{2}\frac{1}{h_\beta}\frac{\partial}{\partial\beta}\left(\frac{\mu_o}{\mu}h_\gamma H_\gamma\right)\right\}$$
$$+O(\tau^3)$$

$$\left\{1+\tau\frac{\partial}{\partial\gamma}(\ln h_\beta)-\frac{\tau^2}{2}\left[\frac{\partial^2}{\partial\gamma^2}(\ln h_\beta)-\left\{\frac{\partial}{\partial\gamma}(\ln h_\beta)\right\}^2+k_o^2N^2h_\gamma^2\right]\right\}\left\{E_\beta\right.$$
$$\left.-\tau\frac{1}{h_\beta}\frac{\partial}{\partial\beta}\left[\frac{\epsilon_o}{\epsilon}h_\gamma E_\gamma\left\{1+\frac{\tau}{2}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha h_\beta}{h_\gamma}\frac{\epsilon}{\epsilon_o}\right)\right\}+\frac{\tau}{2}\frac{\epsilon_o}{\epsilon}h_\gamma^2\nabla_s\cdot\left(\frac{\epsilon}{\epsilon_o}\mathbf{E}\right)\right]\right\}$$
$$=-ik_o\tau\frac{\mu}{\mu_o}Z_oh_\gamma\left\{1+\frac{\tau}{2}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha h_\beta}{h_\gamma}\frac{\mu_o}{\mu}\right)\right\}\left\{H_\alpha-\frac{\tau}{2}\frac{1}{h_\alpha}\frac{\partial}{\partial\alpha}\left(\frac{\mu_o}{\mu}h_\gamma H_\gamma\right)\right\}$$
$$+O(\tau^3)\,,$$

and correct to the second order in τ the boundary condition for the exterior field is

$$\hat{\gamma}\times\left(\hat{\gamma}\times\left\{\mathbf{E}-\tau\nabla\left[\frac{\epsilon_o}{\epsilon}h_\gamma E_\gamma\left\{1+\frac{\tau}{2}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha h_\beta}{h_\gamma}\frac{\epsilon_o}{\epsilon}\right)\right\}\right.\right.\right.$$
$$\left.\left.\left.+\frac{\tau}{2}\frac{\epsilon_o}{\epsilon}h_\gamma^2\nabla_s\cdot\left(\frac{\epsilon}{\epsilon_o}\mathbf{E}\right)\right]\right\}\right)=-\overline{\overline{\eta}}\cdot\hat{\gamma}\times\left\{\mathbf{H}-\frac{\tau}{2}\nabla\left(\frac{\mu_o}{\mu}h_\gamma H_\gamma\right)\right\}\quad(17)$$

where

$$\overline{\overline{\eta}}=\Gamma\hat{\alpha}\hat{\alpha}+\Gamma_1\hat{\beta}\hat{\beta}$$

with

$$
\begin{aligned}
\Gamma &= -ik_o\tau\frac{\mu}{\mu_o}Z_oh_\gamma\left\{1+\frac{\tau}{2}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{h_\alpha h_\gamma}\frac{\mu_o}{\mu}\right)\right\} \\
\Gamma_1 &= -ik_o\tau\frac{\mu}{\mu_o}Z_oh_\gamma\left\{1+\frac{\tau}{2}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta h_\gamma}\frac{\mu_o}{\mu}\right)\right\}\,.
\end{aligned}
\tag{18}
$$

To this order the effective surface impedance is anisotropic, but if terms $O(\tau^2)$ are negligible, the impedance becomes isotropic are (17) reduces to the second order condition (13).

For an impedance substrate we require also the expressions for H^i_α and H^i_β at $\gamma = \gamma_o - \tau$. From (15) and (16) using duality,

$$
\begin{aligned}
H^i_\alpha(\gamma_o-\tau) = \Bigg\{1+\tau\frac{\partial}{\partial\gamma}(\ln h_\alpha) - \frac{\tau^2}{2}\Bigg[\frac{\partial^2}{\partial\gamma^2}(\ln h_\alpha) - \left\{\frac{\partial}{\partial\gamma}(\ln h_\alpha)^2\right\} \\
+k_o^2N^2h_\gamma^2\Bigg]\Bigg\}H^i_\alpha - \tau\left\{1+\tau\frac{\partial}{\partial\gamma}(\ln h_\alpha)\right\}\frac{1}{h_\alpha}\frac{\partial}{\partial\alpha}(h_\gamma H^i_\gamma) \\
+\frac{\tau^2}{2}\frac{1}{h_\alpha}\frac{\partial^2}{\partial\alpha\partial\gamma}(h_\gamma H^i_\gamma) + ik_o\tau\frac{\epsilon}{\epsilon_o}Y_o\left\{1+\frac{\tau}{2}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha h_\beta}{h_\gamma}\frac{\epsilon_o}{\epsilon}\right)\right\}h_\gamma E^i_\beta \\
-ik_o\frac{\tau^2}{2}\frac{\epsilon}{\epsilon_o}Y_o\frac{h_\gamma}{h_\beta}\frac{\partial}{\partial\beta}(h_\gamma E^i_\gamma) + O(\tau^3)\,,
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
H^i_\beta(\gamma_o-\tau) = \Bigg\{1+\tau\frac{\partial}{\partial\gamma}(\ln h_\beta) - \frac{\tau^2}{2}\Bigg[\frac{\partial^2}{\partial\gamma^2}(\ln h_\beta) - \left\{\frac{\partial}{\partial\gamma}(\ln h_\beta)\right\}^2 \\
+k_o^2N^2h_\gamma^2\Bigg]\Bigg\}H^i_\beta - \tau\left\{1+\tau\frac{\partial}{\partial\gamma}(\ln h_\beta)\right\}\frac{1}{h_\beta}\frac{\partial}{\partial\beta}(h_\gamma H^i_\gamma) \\
+\frac{\tau^2}{2}\frac{1}{h_\beta}\frac{\partial^2}{\partial\beta\partial\gamma}(h_\gamma H^i_\gamma) - ik_o\tau\frac{\epsilon}{\epsilon_o}Y_o\left\{1+\frac{\tau}{2}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha h_\beta}{h_\gamma}\frac{\epsilon_o}{\epsilon}\right)\right\}h_\gamma E^i_\alpha \\
+ik_o\frac{\tau^2}{2}\frac{\epsilon}{\epsilon_o}Y_o\frac{h_\gamma}{h_\alpha}\frac{\partial}{\partial\alpha}(h_\gamma E^i_\gamma) + O(\tau^3)\,,
\end{aligned}
\tag{20}
$$

and from (15), (17) and the boundary condition (1),

$$\left\{1+\tau\frac{\partial}{\partial\gamma}(\ln h_\alpha)-\frac{\tau^2}{2}\left[\frac{\partial^2}{\partial\gamma^2}(\ln h_\alpha)-\left\{\frac{\partial}{\partial\gamma}(\ln h_\alpha)\right\}^2+k_o^2N^2h_\gamma^2\right]\right.$$

$$\left.-\,ik_o\tau\frac{\epsilon}{\epsilon_o}Y_o\eta_m\left[1+\frac{\tau}{2}\frac{\partial}{\partial\gamma}\left(\ln h_\alpha h_\beta\frac{\epsilon_o}{\epsilon}\right)\right]h_\gamma\right\}E_\alpha^i-\tau\left\{1+\tau\frac{\partial}{\partial\gamma}(\ln h_\alpha)\right.$$

$$\left.-ik_o\frac{\tau}{2}\frac{\epsilon}{\epsilon_o}Y_oh_\gamma\eta_m\right\}\frac{1}{h_\alpha}\frac{\partial}{\partial\alpha}(h_\gamma E_\gamma^i)+\frac{\tau^2}{2}\frac{1}{h_\alpha}\frac{\partial^2}{\partial\alpha\partial\gamma}(h_\gamma E_\gamma^i)$$

$$=-\eta_m\left(\left\{1+\tau\frac{\partial}{\partial\gamma}(\ln h_\beta)-\frac{\tau^2}{2}\left[\frac{\partial^2}{\partial\gamma^2}(\ln h_\beta)-\left\{\frac{\partial}{\partial\gamma}(\ln h_\beta)\right\}^2\right.\right.\right.$$

$$\left.\left.+\,k_o^2N^2h_\gamma^2\right]-ik_o\tau\frac{\mu}{\mu_o}\frac{Z_o}{\eta_m}\left[1+\frac{\tau}{2}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha h_\beta}{h_\gamma}\frac{\mu_o}{\mu}\right)\right]h_\gamma\right\}H_\beta^i$$

$$\left.-\,\tau\left\{1+\tau\frac{\partial}{\partial\gamma}(\ln h_\beta)-ik_o\frac{\tau}{2}\frac{\mu}{\mu_o}\frac{Z_o}{\eta_m}h_\gamma\right\}\frac{1}{h_\beta}\frac{\partial}{\partial\beta}(h_\gamma H_\gamma^i)+\frac{\tau^2}{2}\frac{1}{h_\beta}\frac{\partial^2}{\partial\beta\partial\gamma}(h_\gamma H_\gamma^i)\right)$$

$$+O(\tau^3)\,.$$

When the boundary condition at the dielectric interface is imposed we find, using (A.1) and (A.2),

$$\left\{1+\tau\frac{\partial}{\partial\gamma}(\ln h_\alpha)-\frac{\tau^2}{2}\left[\frac{\partial^2}{\partial\gamma^2}(\ln h_\alpha)-\left\{\frac{\partial}{\partial\gamma}(\ln h_\alpha)\right\}^2+k_o^2N^2h_\gamma^2\right]\right.$$

$$\left.-ik_o\tau\frac{\epsilon}{\epsilon_o}Y_o\eta_m\left[1+\frac{\tau}{2}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha h_\beta}{h_\gamma}\frac{\epsilon_o}{\epsilon}\right)\right]h_\gamma\right\}\left\{E_\alpha-\tau\frac{1}{h_\alpha}\frac{\partial}{\partial\alpha}\left[\frac{\epsilon_o}{\epsilon}h_\gamma E_\gamma\right.\right.$$

$$\left.\cdot\left\{1+\frac{\tau}{2}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha h_\beta}{h_\gamma}\frac{\epsilon}{\epsilon_o}\right)\right\}+\frac{\tau}{2}\frac{\epsilon_o}{\epsilon}h_\gamma^2\nabla_s\cdot\left(\frac{\epsilon}{\epsilon_o}\mathbf{E}\right)\right]$$

$$\left.-\,ik_o\frac{\tau^2}{2}\frac{\epsilon}{\epsilon_o}Y_o\eta_mh_\gamma\frac{1}{h_\alpha}\frac{\partial}{\partial\alpha}\left(\frac{\epsilon_o}{\epsilon}h_\gamma E_\gamma\right)\right\}=-\eta_m\left\{1+\tau\frac{\partial}{\partial\gamma}(\ln h_\beta)\right.$$

$$-\frac{\tau^2}{2}\left[\frac{\partial^2}{\partial\gamma^2}(\ln h_\beta)-\left\{\frac{\partial}{\partial\gamma}(\ln h_\beta)\right\}^2+k_o^2N^2h_\gamma^2\right]-ik_o\tau\frac{\mu}{\mu_o}\frac{Z_o}{\eta_m}$$

$$\cdot \left[1 + \frac{\tau}{2}\frac{\partial}{\partial\gamma}\left(\ln \frac{h_\alpha h_\beta}{h_\gamma}\frac{\mu_o}{\mu}\right)\right] h_\gamma\Big\} \Big\{H_\beta - \tau \frac{1}{h_\beta}\frac{\partial}{\partial\beta}\left[\frac{\mu_o}{\mu} h_\gamma H_\gamma \Big\{1 + \frac{\tau}{2}\frac{\partial}{\partial\gamma}\right.$$

$$\left. \cdot \left(\ln \frac{h_\alpha h_\beta}{h_\gamma}\frac{\mu}{\mu_o}\right) + \frac{\tau}{2}\frac{\mu_o}{\mu} h_\gamma^2 \nabla_s \cdot \left(\frac{\mu}{\mu_o}\mathbf{H}\right)\right] - ik_o \frac{\tau^2}{2}\frac{\mu}{\mu_o}\frac{Z_o}{\eta_m} h_\gamma \frac{1}{h_\beta}\frac{\partial}{\partial\beta}$$

$$\cdot \left(\frac{\mu_o}{\mu} h_\gamma H_\gamma\right)\Big\} + O(\tau^3)\,,$$

and correct to the second order in τ,

$$E_\alpha - \tau\left(1 + ik_o \frac{\tau}{2}\frac{\epsilon}{\epsilon_o} Y_o \eta_m h_\gamma\right)\frac{1}{h_\alpha}\frac{\partial}{\partial\alpha}\left[\frac{\epsilon_o}{\epsilon} h_\gamma E_\gamma \left\{1 + \frac{\tau}{2}\frac{\partial}{\partial\gamma}\left(\ln \frac{h_\alpha h_\beta}{h_\gamma}\frac{\epsilon}{\epsilon_o}\right)\right\}\right.$$

$$\left. + \frac{\tau}{2}\frac{\epsilon_o}{\epsilon} h_\gamma^2 \nabla_s \cdot \left(\frac{\epsilon}{\epsilon_o}\mathbf{E}\right)\right] = -\Gamma\Big\{H_\beta - \tau\left(1 + ik_o \frac{\tau}{2}\frac{\mu}{\mu_o}\frac{Z_o}{\eta_m} h_\gamma\right)\frac{1}{h_\beta}\frac{\partial}{\partial\beta}$$

$$\cdot \left[\frac{\mu_o}{\mu} h_\gamma H_\gamma \left\{1 + \frac{\tau}{2}\frac{\partial}{\partial\gamma}\left(\ln \frac{h_\alpha h_\beta}{h_\gamma}\frac{\mu}{\mu_o}\right)\right\} + \frac{\tau}{2}\frac{\mu_o}{\mu} h_\gamma^2 \nabla_s \cdot \left(\frac{\mu}{\mu_o}\mathbf{H}\right)\right]\Big\} \qquad (21)$$

with

$$\Gamma = \eta_m \Big\{1 + \tau\left[\frac{\partial}{\partial\gamma}(\ln h_\beta) - ik_o \frac{\mu}{\mu_o}\frac{Z_o}{\eta_m} h_\gamma\right] - \frac{\tau^2}{2}\left[\frac{\partial^2}{\partial\gamma^2}(\ln h_\beta) - \left\{\frac{\partial}{\partial\gamma}(\ln h_\beta)\right\}^2\right.$$

$$\left. + k_o^2 N^2 h_\gamma^2 + ik_o \frac{\mu}{\mu_o}\frac{Z_o}{\eta_m} h_\gamma \frac{\partial}{\partial\gamma}\left(\ln \frac{h_\alpha h_\beta}{h_\gamma}\frac{\mu_o}{\mu}\right)\right]\Big\}\Big\{1 + \tau\left[\frac{\partial}{\partial\gamma}(\ln h_\alpha)\right.$$

$$\left. - ik_o \frac{\epsilon_o}{\epsilon_o} Y_o \eta_m h_\gamma\right] - \frac{\tau^2}{2}\left[\frac{\partial^2}{\partial\gamma^2}(\ln h_\alpha) - \left\{\frac{\partial}{\partial\gamma}(\ln h_\alpha)\right\}^2\right.$$

$$\left. + k_o^2 N^2 h_\gamma^2 + ik_o \frac{\epsilon}{\epsilon_o} Y_o \eta_m h_\gamma \frac{\partial}{\partial\gamma}\left(\ln \frac{h_\alpha h_\beta}{h_\gamma}\frac{\epsilon_o}{\epsilon}\right)\right]\Big\}^{-1}. \qquad (22)$$

Similarly

$$E_\beta - \tau\left(1 + ik_o\frac{\tau}{2}\frac{\epsilon}{\epsilon_o}Y_o\eta_m h_\gamma\right)\frac{1}{h_\beta}\frac{\partial}{\partial\beta}\left[\frac{\epsilon_o}{\epsilon}h_\gamma E_\gamma\left\{1 + \frac{\tau}{2}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha h_\beta}{h_\gamma}\frac{\epsilon}{\epsilon_o}\right)\right\}\right.$$
$$\left. + \frac{\tau}{2}\frac{\epsilon_o}{\epsilon}h_\gamma^2\nabla_s\cdot\left(\frac{\epsilon}{\epsilon_o}\mathbf{E}\right)\right] = \Gamma_1\left\{H_\alpha - \tau\left(1 + ik_o\frac{\tau}{2}\frac{\mu}{\mu_o}\frac{Z_o}{\eta_m}h_\gamma\right)\frac{1}{h_\alpha}\frac{\partial}{\partial\alpha}\right.$$
$$\left.\cdot\left[\frac{\mu_o}{\mu}h_\gamma H_\gamma\left\{1 + \frac{\tau}{2}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha h_\beta}{h_\gamma}\frac{\mu}{\mu_o}\right)\right\} + \frac{\tau}{2}\frac{\mu_o}{\mu}h_\gamma^2\nabla_s\cdot\left(\frac{\mu}{\mu_o}\mathbf{H}\right)\right]\right\} \quad (23)$$

with

$$\Gamma_1 = \eta_m\left\{1 + \tau\left[\frac{\partial}{\partial\gamma}(\ln h_\alpha) - ik_o\frac{\mu}{\mu_o}\frac{Z_o}{\eta_m}h_\alpha\right] - \frac{\tau^2}{2}\left[\frac{\partial^2}{\partial\gamma^2}(\ln h_\alpha) - \left\{\frac{\partial}{\partial\gamma}(\ln h_\alpha)\right\}^2\right.\right.$$
$$\left.\left. + k_o^2N^2h_\gamma^2 + ik_o\frac{\mu}{\mu_o}\frac{Z_o}{\eta_m}h_\gamma\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha h_\beta}{h_\gamma}\frac{\mu_o}{\mu}\right)\right]\right\}\left\{1 + \tau\left[\frac{\partial}{\partial\gamma}(\ln h_\beta)\right.\right.$$
$$\left. - ik_o\frac{\epsilon}{\epsilon_o}Y_o\eta_m h_\gamma\right] - \frac{\tau^2}{2}\left[\frac{\partial^2}{\partial\gamma^2}(\ln h_\beta) - \left\{\frac{\partial}{\partial\gamma}(\ln h_\beta)\right\}^2\right.$$
$$\left.\left. + k_o^2N^2h_\gamma^2 + ik_o\frac{\epsilon}{\epsilon_o}Y_o\eta_m h_\gamma\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha h_\beta}{h_\gamma}\frac{\epsilon_o}{\epsilon}\right)\right]\right\}^{-1} \quad (24)$$

and we observe that Γ and Γ_1 differ only in having α and β interchanged.

More significantly, $\Gamma_1 = 1/\Gamma^*$ where the asterisk denotes the dual quantity, and (21) and (23) can be expressed in vector form as

$$\hat{\gamma}\times\left(\hat{\gamma}\times\left\{\mathbf{E} - \tau\left(1 + ik_o\frac{\tau}{2}\frac{\epsilon}{\epsilon_o}Y_o\eta_m h_\gamma\right)\nabla\left[\frac{\epsilon_o}{\epsilon}h_\gamma E_\gamma\left\{1 + \frac{\tau}{2}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha h_\beta}{h_\gamma}\frac{\epsilon}{\epsilon_o}\right)\right\}\right.\right.\right.$$
$$\left.\left.\left. + \frac{\tau}{2}\frac{\epsilon_o}{\epsilon}h_\gamma^2\nabla_s\cdot\left(\frac{\epsilon}{\epsilon_o}\mathbf{E}\right)\right]\right\}\right) = -\bar{\bar{\eta}}\cdot\hat{\gamma}\times\left\{\mathbf{H} - \tau\left(1 + ik_o\frac{\tau}{2}\frac{\mu}{\mu_o}\frac{Z_o}{\eta_m}h_\gamma\right)\right.$$
$$\left.\cdot\nabla\left[\frac{\mu_o}{\mu}h_\gamma H_\gamma\left\{1 + \frac{\tau}{2}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha h_\beta}{h_\gamma}\frac{\mu}{\mu_o}\right)\right\} + \frac{\tau}{2}\frac{\mu_o}{\mu}h_\gamma^2\nabla_s\cdot\left(\frac{\mu}{\mu_o}\mathbf{H}\right)\right]\right\} \quad (25)$$

with

$$\bar{\bar{\eta}} = \Gamma\hat{\alpha}\hat{\alpha} + \frac{1}{\Gamma^*}\hat{\beta}\hat{\beta}\,. \quad (26)$$

The boundary condition (25) is a third order one which satisfies duality and reduces to the second order condition (11) when terms $O(\tau^2)$ are neglected. However, as was true with (11), we cannot recover the result for a perfectly conducting substrate (see (17)) by simply putting $\eta_m = 0$.

A case of particular interest is a circular cylinder with a homogeneous dielectric coating. Putting $\alpha = \phi, \beta = z, \gamma = \rho$ where ρ, ϕ, z are cylindrical polar coordinates so that $h_\alpha = \rho, h_\beta = h_\gamma = 1$, (25) becomes

$$\hat{\rho} \times \left(\hat{\rho} \times \left\{\mathbf{E} - \tau\nabla\left[\left\{1 + \frac{\tau}{2}\left(\frac{1}{\rho} + ik_o\frac{\epsilon}{\epsilon_o}Y_o\eta_m\right)\right\}\frac{\epsilon_o}{\epsilon}E_\rho + \frac{\tau}{2}\nabla_s \cdot \mathbf{E}\right]\right\}\right)$$

$$= -\overline{\overline{\eta}} \cdot \hat{\rho} \times \left\{\mathbf{H} - \tau\nabla\left[\left\{1 + \frac{\tau}{2}\left(\frac{1}{\rho} + ik_o\frac{\mu}{\mu_o}\frac{Z_o}{\eta_m}\right)\right\}\frac{\mu_o}{\mu}H_\rho + \frac{\tau}{2}\nabla_s \cdot \mathbf{H}\right]\right\} \quad (27)$$

with

$$\Gamma = \eta_m\left\{1 - \tau\left(1 + \frac{\tau}{2\rho}\right)ik_o\frac{\mu}{\mu_o}\frac{Z_o}{\eta_m} - \frac{1}{2}(k_o\tau N)^2\right\}$$

$$\cdot\left\{1 - \tau\left(1 + \frac{\tau}{2\rho}\right)\left(ik_o\frac{\epsilon}{\epsilon_o}Y_o\eta_m - \frac{1}{\rho}\right) - \frac{\tau^2}{2}\left(k_o^2N^2 - \frac{1}{\rho^2}\right)\right\}^{-1} \quad (28)$$

Finally, we note that (25) has the general form

$$\hat{\gamma} \times \left(\hat{\gamma} \times \left\{\mathbf{E} - \nabla\left[A\frac{\epsilon_o}{\epsilon}h_\gamma E_\gamma + B\nabla_s \cdot \mathbf{E}\right]\right\}\right) = -\left(\Gamma\hat{\alpha}\hat{\alpha} + \frac{1}{\Gamma^*}\hat{\beta}\hat{\beta}\right)$$

$$\cdot\hat{\gamma} \times \left\{\mathbf{H} - \nabla\left[A^*\frac{\mu_o}{\mu}h_\gamma H_\gamma + B^*\nabla_s \cdot \mathbf{H}\right]\right\} \quad (29)$$

where Γ, A and B are geometry and material dependent, and as befits a third order boundary condition, there are three quantities at our disposal to simulate the scattering properties of the surface.

4 High Contrast Coating, Zeroth Order

We now consider the problem of a high contrast coating consisting of a dielectric having $|N| \gg 1$. The treatment is based on the method of Rytov [1940] and makes use of the analysis in I, but in addition to the inward propagating field considered there, we must also include an outward propagating field produced by reflection at the dielectric-substrate boundary $\gamma = \gamma_o - \tau$. The properties of such a field are given in Appendix B. For generality, we again assume that the standard impedance boundary condition (1) is imposed at $\gamma = \gamma_o - \tau$, and we start by considering the zeroth order approximation.

For the inward propagating field

$$\overline{\mathcal{E}} = \mathbf{A}e^{ik_o\psi/q}, \quad \overline{\mathcal{H}} = \mathbf{B}e^{ik_o\psi/q} \tag{30}$$

with

$$A_\alpha = -K_1 B_\beta, \quad A_\beta = K_2 B_\beta \tag{31}$$

at any point in the dielectric, and to the zeroth order

$$K_1 = K_2 = 1. \tag{32}$$

Since $\psi = 0$ at $\gamma = \gamma_o$,

$$\psi(\gamma) = -\int_{\gamma_o}^{\gamma} v\,d\gamma = -q\int_{\gamma_o}^{\gamma} N\,d\gamma$$

where the integral is simply the optical path length, and for brevity we write

$$\int_{\gamma_o}^{\gamma_o-\tau} N\,d\gamma = -L$$

so that

$$\psi(\gamma_o - \tau) = qL. \tag{33}$$

In the particular case of a homogeneous dielectric, $L = N\tau$.

Similarly, for the outward propagating field,

$$\overline{\mathcal{E}}' = \mathbf{A}e^{ik_o\psi'/q}, \quad \overline{\mathcal{H}}' = \mathbf{B}e^{ik_o\psi'/q} \tag{34}$$

with

$$A'_\alpha = K'_1 B'_\beta, \quad A'_\beta = -K'_2 B'_\alpha \tag{35}$$

and to the zeroth order

$$K_1' = K_2' = 1\,. \tag{36}$$

Also

$$\begin{aligned}\psi'(\gamma) &= q\left(\int_{\gamma_o-\tau}^{\gamma_o} + \int_{\gamma_o-\tau}^{\gamma}\right) N d\gamma \\ &= qL + q\int_{\gamma_o-\tau}^{\gamma} N d\gamma\end{aligned}$$

so that

$$\psi'(\gamma_o - \tau) = qL \tag{37}$$

and

$$\psi'(\gamma_o) = 2qL\,. \tag{38}$$

The total field in the dielectric is

$$\mathbf{E}^{\text{in}} = \frac{1}{\sqrt{\epsilon}}(\overline{\mathcal{E}} + \overline{\mathcal{E}}')\,, \quad \mathbf{H}^{\text{in}} = \frac{1}{\sqrt{\mu}}(\overline{\mathcal{H}} + \overline{\mathcal{H}}')\,. \tag{39}$$

At $\gamma = \gamma_o - \tau$ the boundary condition (1) requires that

$$\begin{aligned}\mathcal{E}_\alpha + \mathcal{E}_\alpha' &= -\eta_m Y(\mathcal{H}_\beta + \mathcal{H}_\beta') \\ \mathcal{E}_\beta + \mathcal{E}_\beta' &= \eta_m Y(\mathcal{H}_\alpha + \mathcal{H}_\alpha')\end{aligned} \tag{40}$$

and therefore

$$\begin{aligned}A_\alpha e^{ik_o\psi/q} + A_\alpha' e^{ik_o\psi'/q} &= -\eta_m Y\left(B_\beta e^{ik_o\psi/q} + B_\beta' e^{ik_o\psi'/q}\right) \\ A_\beta e^{ik_o\psi/q} + A_\beta' e^{ik_o\psi'/q} &= \eta_m Y\left(B_\alpha e^{ik_o\psi/q} + B_\alpha' e^{ik_o\psi'/q}\right)\end{aligned}$$

where the phases are evaluated at $\gamma = \gamma_o - \tau$. Hence, from (31)-(33), (35), (36) and (38),

$$\begin{aligned}B_\alpha' &= \frac{1-\eta_m Y}{1+\eta_m Y} B_\alpha\,, \\ B_\beta' &= \frac{1-\eta_m Y}{1+\eta_m Y} B_\beta\,.\end{aligned} \tag{41}$$

At the upper surface $\gamma = \gamma_o$

$$E_\alpha^{\text{in}} = \frac{1}{\sqrt{\epsilon}}\left(A_\alpha + A'_\alpha e^{2ik_oL}\right) ,$$

and using (31), (32), (34), (35) and (41),

$$E_\alpha^{\text{in}} = \frac{1}{\sqrt{\epsilon}}\left\{-1 + \frac{1-\eta_m Y}{1+\eta_m Y}e^{2ik_oL}\right\} B_\beta .$$

Similarly

$$E_\beta^{\text{in}} = \frac{1}{\sqrt{\epsilon}}\left\{1 - \frac{1-\eta_m Y}{1+\eta_m Y}e^{2ik_oL}\right\} B_\alpha$$

and

$$\begin{aligned} H_\alpha^{\text{in}} &= \frac{1}{\sqrt{\mu}}\left\{1 + \frac{1-\eta_m Y}{1+\eta_m Y}e^{2ik_oL}\right\} B_\alpha , \\ H_\beta^{\text{in}} &= \frac{1}{\sqrt{\mu}}\left\{1 + \frac{1-\eta_m Y}{1+\eta_m Y}e^{2ik_oL}\right\} B_\beta . \end{aligned}$$

On matching to the exterior fields we then have

$$\frac{E_\alpha}{H_\beta} = \frac{E_\alpha^{\text{in}}}{H_\beta^{\text{in}}} = iZ\frac{\tan k_oL + i\eta_m Y}{1 - i\eta_m Y \tan k_oL} \tag{42}$$

and

$$\frac{E_\beta}{H_\alpha} = \frac{E_\beta^{\text{in}}}{H_\alpha^{\text{in}}} = -iZ\frac{\tan k_oL + i\eta_m Y}{1 - i\eta_m Y \tan k_oL} , \tag{43}$$

and the resulting boundary condition can be written as

$$\hat{\gamma} \times (\gamma \times \mathbf{E}) = -\eta\hat{\gamma} \times \mathbf{H} \tag{44}$$

with

$$\eta = -iZ\frac{\tan k_oL + i\eta_m Y}{1 - i\eta_m Y \tan k_oL} . \tag{45}$$

We observe that the boundary condition is entirely local in character and independent of any transverse variation in the properties of the dielectric. Any normal variation enters in only through the optical path length L, and (44) is identical to the boundary condition obtained by considering a plane wave at normal incidence on a locally flat structure.

5 High contrast Coating, First Order

The procedure is very similar to that for the zeroth order case. To the first order

$$
\begin{aligned}
K_1 &= 1 + \frac{1}{2ik_oNh_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right) \\
K_2 &= 1 + \frac{1}{2ik_oNh_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{h_\alpha}Z\right)
\end{aligned} \tag{46}
$$

$$
\begin{aligned}
K_1' &= 1 - \frac{1}{2ik_oNh_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right) \\
K_2' &= 1 - \frac{1}{2ik_oNh_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{h_\alpha}Z\right)
\end{aligned} \tag{47}
$$

and from the boundary condition at $\gamma = \gamma_o - \tau$

$$
\begin{aligned}
-K_1B_\beta e^{ik_o\psi/q} + K_1'B_\beta' e^{ik_o\psi'/q} &= -\eta_m Y\left(B_\beta e^{ik_o\psi/q} + B_\beta' e^{ik_o\psi'/q}\right) \\
K_2B_\alpha e^{ik_o\psi/q} - K_2'B_\alpha' e^{ik_o\psi'/q} &= \eta_m Y\left(B_\alpha e^{ik_o\psi/q} + B_\alpha' e^{ik_o\psi'/q}\right)
\end{aligned}
$$

giving

$$
\begin{aligned}
B_\alpha' &= \frac{K_2 - \eta_m Y}{K_2' + \eta_m Y}B_\alpha\,, \\
B_\beta' &= \frac{K_1 - \eta_m Y}{K_1' + \eta_m Y}B_\beta\,.
\end{aligned} \tag{48}
$$

At the upper surface $\gamma = \gamma_o$

$$
E_\alpha^{\text{in}} = \frac{1}{\sqrt{\epsilon}}\left\{-K_1B_\beta + K_1'B_\beta' e^{2ik_oL}\right\}
$$

and when the expression (48) for B_β' is inserted, we obtain

$$
E_\alpha^{\text{in}} = \frac{1}{\sqrt{\epsilon}}\left\{-K_1 + K_1'\frac{K_1 - \eta_m Y}{K_1' + \eta_m Y}e^{2ik_oL}\right\}B_\beta\,.
$$

Similarly

$$
\begin{aligned}
E_\beta^{\text{in}} &= \frac{1}{\sqrt{\epsilon}}\left\{K_2 - K_2' \frac{K_2 - \eta_m Y}{K_2' + \eta_m Y} e^{2ik_oL}\right\} B_\alpha \\
H_\alpha^{\text{in}} &= \frac{1}{\sqrt{\mu}}\left\{1 + \frac{K_2 - \eta_m Y}{K_2' + \eta_m Y} e^{2ik_oL}\right\} B_\alpha \\
H_\beta^{\text{in}} &= \frac{1}{\sqrt{\mu}}\left\{1 + \frac{K_1 - \eta_m Y}{K_1' + \eta_m Y} e^{2ik_oL}\right\} B_\beta \,,
\end{aligned}
$$

and on matching to the exterior fields we find

$$
\begin{aligned}
\frac{E_\alpha}{H_\beta} &= \frac{E_\alpha^{\text{in}}}{H_\beta^{\text{in}}} = -Z\frac{K_1(K_1' + \eta_m Y) - K_1'(K_1 - \eta_m Y)e^{2ik_oL}}{K_1' + \eta_m Y + (K_1 - \eta_m Y)e^{2ik_oL}} \\
\frac{E_\beta}{H_\alpha} &= \frac{E_\beta^{\text{in}}}{H_\alpha^{\text{in}}} = Z\frac{K_2(K_2' + \eta_m Y) - K_2'(K_2 - \eta_m Y)e^{2ik_oL}}{K_2' + \eta_m Y + (K_2 - \eta_m Y)e^{2ik_oL}} \,,
\end{aligned}
$$

which can be written as

$$
\begin{aligned}
E_\alpha &= iZ\frac{\{2K_1K_1' + \eta_m Y(K_1 - K_1')\}\tan k_oL + i\eta_m Y(K_1 + K_1')}{K_1 + K_1' + i(K_1 - K_1' - 2\eta_m Y)\tan k_oL} H_\beta \\
E_\beta &= -iZ\frac{\{2K_2K_2' + \eta_m Y(K_2 - K_2')\}\tan k_oL + i\eta_m Y(K_2 + K_2')}{K_2 + K_2' + i(K_2 - K_2' - 2\eta_m Y)\tan k_oL} H_\alpha
\end{aligned}
\qquad (49)
$$

But

$$
K_1 + K_1' = K_2 + K_2' = 2 \,,
$$

and to the first order in $|N|^{-1}$,

$$
K_1 K_1' = K_2 K_2' = 1
$$

The boundary conditions are therefore

$$
E_\alpha = -\Gamma H_\beta \,, \qquad E_\beta = \Gamma_1 H_\alpha \qquad (50)
$$

where

$$\begin{aligned}
\Gamma &= -iZ\frac{\left\{1+\dfrac{\eta_m Y}{2ik_oNh_\gamma}\dfrac{\partial}{\partial\gamma}\left(\ln\dfrac{h_\alpha}{h_\beta}Z\right)\right\}\tan k_oL + i\eta_m Y}{1-i\eta_m Y\left\{1-\dfrac{Z/\eta_m}{2ik_oNh_\gamma}\dfrac{\partial}{\partial\gamma}\left(\ln\dfrac{h_\alpha}{h_\beta}Z\right)\right\}\tan k_oL} \\
\Gamma_1 &= -iZ\frac{\left\{1+\dfrac{\eta_m Y}{2ik_oNh_\gamma}\dfrac{\partial}{\partial\gamma}\left(\ln\dfrac{h_\beta}{h_\alpha}Z\right)\right\}\tan k_oL + i\eta_m Y}{1-i\eta_m Y\left\{1-\dfrac{Z/\eta_m}{2ik_oNh_\gamma}\dfrac{\partial}{\partial\gamma}\left(\ln\dfrac{h_\beta}{h_\alpha}Z\right)\right\}\tan k_oL},
\end{aligned} \tag{51}$$

and (50) can be expressed as

$$\hat{\gamma}\times(\hat{\gamma}\times\mathbf{E}) = -\overline{\overline{\eta}}\cdot\hat{\gamma}\times\mathbf{H} \tag{52}$$

with

$$\overline{\overline{\eta}} = \Gamma\hat{\alpha}\hat{\alpha} + \Gamma_1\hat{\beta}\hat{\beta}\,. \tag{53}$$

Since $\Gamma_1 = 1/\Gamma^*$ where the asterisk denotes the dual quantity, the boundary condition (52) satisfies duality, and reduces to (44) when terms $O(|N|^{-1})$ are neglected. Here again, any lateral variation in the dielectric properties has no effect.

In the particular case of a planar surface z = constant we choose $\alpha = x, \beta = y, \gamma = z$ and then

$$\Gamma = \Gamma_1 = -iZ\frac{\left\{1+\dfrac{\eta_m Y}{2ik_oN}\dfrac{\partial}{\partial z}(\ln Z)\right\}\tan k_oL + i\eta_m Y}{1-i\eta_m Y\left\{1-\dfrac{Z/\eta_m}{2ik_oN}\dfrac{\partial}{\partial z}(\ln Z)\right\}\tan k_oL}, \tag{54}$$

but for the cylindrical surface ρ = constant ($\alpha = \phi, \beta = z, \gamma = \rho$ implying $h_\alpha = \rho, h_\beta = h_\gamma = 1$)

$$\Gamma = -iZ\frac{\left\{1+\frac{\eta_m Y}{2ik_oN}\left(\frac{1}{\rho}+\frac{\partial}{\partial\rho}(\ln Z)\right)\right\}\tan k_oL + i\eta_m Y}{1-i\eta_m Y\left\{1-\frac{Z/\eta_m}{2ik_oN}\left(\frac{1}{\rho}+\frac{\partial}{\partial\rho}(\ln Z)\right)\right\}\tan k_oL}$$

$$\Gamma_1 = -iZ\frac{\left\{1+\frac{\eta_m Y}{2ik_oN}\left(-\frac{1}{\rho}+\frac{\partial}{\partial\rho}(\ln Z)\right)\right\}\tan k_oL + i\eta_m Y}{1-i\eta_m Y\left\{1-\frac{Z/\eta_m}{2ik_oN}\left(-\frac{1}{\rho}+\frac{\partial}{\partial\rho}(\ln Z)\right)\right\}\tan k_oL}. \tag{55}$$

Finally, for an arbitrary surface with a perfectly conducting ($\eta_m = 0$) substrate,

$$\Gamma = -iZ\left\{\cot k_oL + \frac{1}{2k_oNh_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right)\right\}^{-1}$$

$$\Gamma_1 = -iZ\left\{\cot k_oL + \frac{1}{2k_oNh_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{h_\alpha}Z\right)\right\}^{-1}. \tag{56}$$

6 High Contrast Coating, Second Order

This is important in order to demonstrate the practical advantages of a higher boundary condition, and it is unfortunate that the analysis is significantly harder than in the previous two cases.

For the inward propagating field the expressions analogous to (31) but accurate to the second order are

$$A_\alpha = -K_1B_\beta - L\frac{1}{h_\alpha}\frac{\partial B}{\partial\alpha} + M\frac{1}{h_\beta}\frac{\partial A}{\partial\beta}$$

$$A_\beta = K_2B_\alpha - L\frac{1}{h_\beta}\frac{\partial B}{\partial\beta} + M\frac{1}{h_\alpha}\frac{\partial A}{\partial\alpha} \tag{57}$$

(see (I.43) and (I.47)) where

$$K_1 = 1 + \frac{1}{2ik_oNh_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right) - \frac{1}{(2k_oNh_\gamma)^2}\left[\frac{1}{2}\left\{\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right)\right\}^2 + \frac{\partial^2}{\partial\gamma^2}\left(\ln\frac{h_\alpha}{h_\beta}Z\right) - \frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right)\frac{\partial}{\partial\gamma}(\ln h_\gamma N)\right]$$

$$K_2 = 1 + \frac{1}{2ik_oNh_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{h_\alpha}Z\right) - \frac{1}{(2k_oNh_\gamma)^2}\left[\frac{1}{2}\left\{\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{h_\alpha}Z\right)\right\}^2 + \frac{\partial^2}{\partial\gamma^2}\left(\ln\frac{h_\beta}{h_\alpha}Z\right) - \frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{h_\alpha}Z\right)\frac{\partial}{\partial\gamma}(\ln h_\gamma N)\right] \tag{58}$$

$$\begin{aligned} L\frac{\partial B}{\partial t} &= \frac{\sqrt{\epsilon}}{2k_o^2Nh_\gamma}\frac{\partial}{\partial t}\left[\frac{Z}{N}h_\gamma\hat{\gamma}\cdot\nabla\times\frac{\mathbf{B}}{\sqrt{\mu}}\right] \\ M\frac{\partial A}{\partial t} &= \frac{\sqrt{\mu}}{2k_o^2Nh_\gamma}\frac{\partial}{\partial t}\left[\frac{1}{ZN}h_\gamma\hat{\gamma}\cdot\nabla\times\frac{\mathbf{A}}{\sqrt{\epsilon}}\right] . \end{aligned} \tag{59}$$

Similarly, for the outward propagating field

$$\begin{aligned} A'_\alpha &= K'_1B'_\beta + L\frac{1}{h_\alpha}\frac{\partial B'}{\partial\alpha} + M\frac{1}{h_\beta}\frac{\partial A'}{\partial\beta} \\ A'_\beta &= -K'_2B'_\alpha + L\frac{1}{h_\beta}\frac{\partial B'}{\partial\beta} - M\frac{1}{h_\alpha}\frac{\partial A'}{\partial\alpha} \end{aligned} \tag{60}$$

(see (B.14) and (B.16)) where

$$\begin{aligned} K'_1 &= K_1 - \frac{1}{ik_oNh_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right) \\ K'_2 &= K_2 - \frac{1}{ik_oNh_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{h_\alpha}Z\right) , \end{aligned} \tag{61}$$

and to the required order it is sufficient to insert the zeroth order approximations to the fields in the derivative terms in (57) and (60).

At any point in the dielectric

$$
\begin{aligned}
E^{\text{in}}_{\alpha,\beta} &= \frac{1}{\sqrt{\epsilon}}\left\{A_{\alpha,\beta}e^{ik_o\psi/q} + A'_{\alpha,\beta}e^{ik_o\psi'/q}\right\} \\
H^{\text{in}}_{\alpha,\beta} &= \frac{1}{\sqrt{\mu}}\left\{B_{\alpha,\beta}e^{ik_o\psi/q} + B'_{\alpha,\beta}e^{ik_o\psi'/q}\right\}
\end{aligned}
$$

and when the boundary condition (1) is imposed, we obtain

$$
\left(-K_1B_\beta - L\frac{1}{h_\alpha}\frac{\partial B}{\partial\alpha} + M\frac{1}{h_\beta}\frac{\partial A}{\partial\beta}\right)e^{ik_o\psi/q} + \left(K'_1B'_\beta + L\frac{1}{h_\alpha}\frac{\partial B'}{\partial\alpha}\right.
$$

$$
\left. + M\frac{1}{h_\beta}\frac{\partial A'}{\partial\beta}\right)e^{ik_o\psi'/q} = -\eta_mY\left(B_\beta e^{ik_o\psi/q} + B'_\beta e^{ik_o\psi/q}\right)
$$

$$
\left(K_2B_\alpha - L\frac{1}{h_\beta}\frac{\partial B}{\partial\beta} - M\frac{1}{h_\alpha}\frac{\partial A}{\partial\alpha}\right)e^{ik_o\psi/q} + \left(-K'_2B'_\alpha + L\frac{1}{h_\beta}\frac{\partial B'}{\partial\beta}\right.
$$

$$
\left. - M\frac{1}{h_\alpha}\frac{\partial A'}{\partial\alpha}\right)e^{ik_o\psi'/q} = \eta_mY\left(B_\alpha e^{ik_o\psi/q} + B'_\alpha e^{ik_o\psi/q}\right)
$$

giving

$$
(K'_1 + \eta_mY)B'_\beta = (K_1 - \eta_mY)B_\beta + L\frac{1}{h_\alpha}\frac{\partial}{\partial\alpha}(B - B') - M\frac{1}{h_\beta}\frac{\partial}{\partial\beta}(A + A') \tag{62}
$$

$$
(K'_2 + \eta_mY)B'_\alpha = (K_2 - \eta_mY)B_\alpha + L\frac{1}{h_\beta}\frac{\partial}{\partial\beta}(B - B') - M\frac{1}{h_\alpha}\frac{\partial}{\partial\alpha}(A + A') \tag{63}
$$

At the upper surface $\gamma = \gamma_o$

$$
\begin{aligned}
E_{\alpha,\beta} &= E^{\text{in}}_{\alpha,\beta} = \frac{1}{\sqrt{\epsilon}}\left\{A_{\alpha,\beta} + A'_{\alpha,\beta}e^{2ik_oL}\right\} \\
H_{\alpha,\beta} &= H^{\text{in}}_{\alpha,\beta} = \frac{1}{\sqrt{\mu}}\left\{B_{\alpha,\beta} + B'_{\alpha,\beta}e^{2ik_oL}\right\}\ .
\end{aligned} \tag{64}
$$

Hence, from (57), (60) and (62),

$$\begin{aligned}
\sqrt{\epsilon}E_\alpha &= -K_1 B_\beta - L\frac{1}{h_\alpha}\frac{\partial B}{\partial \alpha} + M\frac{1}{h_\beta}\frac{\partial A}{\partial \beta} + \left\{ K_1' B_\beta' + L\frac{1}{h_\alpha}\frac{\partial B'}{\partial \alpha} \right. \\
&\quad \left. + M\frac{1}{h_\beta}\frac{\partial A'}{\partial \beta} \right\} e^{2ik_oL} \\
&= -K_1 B_\beta - L\frac{1}{h_\alpha}\frac{\partial B}{\partial \alpha} + M\frac{1}{h_\beta}\frac{\partial A}{\partial \beta} + \left\{ \frac{K_1'}{K_1' + \eta_m Y}\left[(K_1 - \eta_m Y)B_\beta \right. \right. \\
&\quad \left. + L\frac{1}{h_\alpha}\frac{\partial}{\partial \alpha}(B - B') - M\frac{1}{h_\beta}\frac{\partial}{\partial \beta}(A - A') \right] \\
&\quad \left. + L\frac{1}{h_\alpha}\frac{\partial B'}{\partial \alpha} + M\frac{1}{h_\beta}\frac{\partial A'}{\partial \beta} \right\} e^{2ik_oL} \\
&= \frac{1}{K_1' + \eta_m Y}\left\{ \left[K_1'(K_1 - \eta_m Y)e^{2ik_oL} - K_1(K_1' + \eta_m Y) \right] B_\beta \right. \\
&\quad + \left[K_1' e^{2ik_oL} - (K_1' + \eta_m Y) \right] \left(L\frac{1}{h_\alpha}\frac{\partial B}{\partial \alpha} - M\frac{1}{h_\beta}\frac{\partial A'}{\partial \beta} \right) \\
&\quad \left. + \eta_m Y e^{2ik_oL} \left(L\frac{1}{h_\alpha}\frac{\partial B'}{\partial \alpha} + M\frac{1}{h_\beta}\frac{\partial A'}{\partial \beta} \right) \right\}
\end{aligned}$$

giving

$$\begin{aligned}
B_\beta &= \frac{1}{K_1'(K_1 - \eta_m Y)e^{2ik_oL} - K_1(K_1' + \eta_m Y)}\left\{ (K_1' + \eta_m Y)\sqrt{\epsilon}E_\alpha \right. \\
&\quad - \left[K_1' e^{2ik_oL} - (K_1' + \eta_m Y) \right] \left(L\frac{1}{h_\alpha}\frac{\partial B}{\partial \alpha} - M\frac{1}{h_\beta}\frac{\partial A}{\partial \beta} \right) \\
&\quad \left. - \eta_m Y e^{2ik_oL} \left(L\frac{1}{h_\alpha}\frac{\partial B'}{\partial \alpha} + M\frac{1}{h_\beta}\frac{\partial A'}{\partial \beta} \right) \right\} \qquad (65)
\end{aligned}$$

and similarly

$$B_\alpha = \frac{1}{K_2'(K_2 - \eta_m Y)e^{2ik_oL} - K_2(K_2' + \eta_m Y)}\left\{ -(K_2' + \eta_m Y)\sqrt{\epsilon}E_\beta \right.$$

$$+\left[K_2'e^{2ik_oL}-(K_2'+\eta_mY)\right]\left(L\frac{1}{h_\beta}\frac{\partial B}{\partial\beta}+M\frac{1}{h_\alpha}\frac{\partial A}{\partial\alpha}\right)$$

$$+\eta_mYe^{2ik_oL}\left(L\frac{1}{h_\beta}\frac{\partial B'}{\partial\beta}-M\frac{1}{h_\alpha}\frac{\partial A'}{\partial\alpha}\right)\Bigg\}. \qquad (66)$$

Also

$$\sqrt{\mu}H_\alpha=B_\alpha+\frac{e^{2ik_oL}}{K_2'+\eta_mY}\left\{(K_2-\eta_mY)B_\alpha-L\frac{1}{h_\beta}\frac{\partial}{\partial\beta}(B-B')\right.$$

$$\left.-M\frac{1}{h_\alpha}\frac{\partial}{\partial\alpha}(A+A')\right\}$$

giving

$$B_\alpha=\frac{1}{(K_2-\eta_mY)e^{2ik_oL}+(K_2'+\eta_mY)}\left\{(K_2'+\eta_mY)\sqrt{\mu}H_\alpha\right.$$

$$\left.+e^{2ik_oL}\left[L\frac{1}{h_\beta}\frac{\partial}{\partial\beta}(B-B')+M\frac{1}{h_\alpha}\frac{\partial}{\partial\alpha}(A+A')\right]\right\} \qquad (67)$$

and

$$B_\beta=\frac{1}{(K_1-\eta_mY)e^{2ik_oL}+(K_1'+\eta_mY)}\left\{(K_1'+\eta_mY)\sqrt{\mu}H_\beta\right.$$

$$\left.-e^{2ik_oL}\left[L\frac{1}{h_\alpha}\frac{\partial}{\partial\alpha}(B-B')-M\frac{1}{h_\beta}\frac{\partial}{\partial\beta}(A+A')\right]\right\}. \qquad (68)$$

On equating the expressions for B_β and rearranging the terms, we obtain

$$E_\alpha-\frac{(K_1'+\eta_mY)(e^{2ik_oL}-1)}{(K_1-\eta_mY)e^{2ik_oL}+(K_1'+\eta_mY)}\frac{1}{\sqrt{\epsilon}}L\frac{1}{h_\alpha}\frac{\partial B}{\partial\alpha}$$

$$-\frac{(K_1-\eta_mY)(e^{2ik_oL}-1)e^{2ik_oL}}{(K_1-\eta_mY)e^{2ik_oL}+(K_1'+\eta_mY)}\frac{1}{\sqrt{\epsilon}}L\frac{1}{h_\alpha}\frac{\partial B'}{\partial\alpha}$$

$$=\frac{K_1'(K_1-\eta_mY)e^{2ik_oL}-K_1(K_1'+\eta_mY)}{(K_1-\eta_mY)e^{2ik_oL}+(K_1'+\eta_mY)}Z\Big\{H_\beta$$

$$- \quad \frac{(K_1' + \eta_m Y)(e^{2ik_oL} - 1)}{K_1'(K_1 - \eta_m Y)e^{2ik_oL} - K_1(K_1' + \eta_m Y)} \frac{1}{\sqrt{\mu}} M \frac{1}{h_\beta} \frac{\partial A}{\partial B}$$

$$+ \quad \frac{(K_1 - \eta_m Y)(e^{2ik_oL} - 1)e^{2ik_oL}}{K_1'(K_1 - \eta_m Y)e^{2ik_oL} - K_1(K_1' + \eta_m Y)} \frac{1}{\sqrt{\mu}} M \frac{1}{h_\beta} \frac{\partial A'}{\partial \beta} \Bigg\}, \quad (69)$$

and from the analogous expressions for B_α

$$E_\beta - \quad \frac{(K_2' + \eta_m Y)(e^{2ik_oL} - 1)}{(K_2 - \eta_m Y)e^{2ik_oL} + (K_2' + \eta_m Y)} \frac{1}{\sqrt{\epsilon}} L \frac{1}{h_\beta} \frac{\partial B}{\partial \beta}$$

$$- \quad \frac{(K_2 - \eta_m Y)(e^{2ik_oL} - 1)e^{2ik_oL}}{(K_2 - \eta_m Y)e^{2ik_oL} + (K_2' + \eta_m Y)} \frac{1}{\sqrt{\epsilon}} L \frac{1}{h_\beta} \frac{\partial B'}{\partial \beta}$$

$$= \quad - \quad \frac{K_2'(K_2 - \eta_m Y)e^{2ik_oL} - K_2(K_2' + \eta_m Y)}{(K_2 - \eta_m Y)e^{2ik_oL} + (K_2' + \eta_m Y)} Z \Bigg\{ H_\alpha$$

$$- \quad \frac{(K_2' + \eta_m Y)(e^{2ik_oL} - 1)}{K_2'(K_2 - \eta_m Y)e^{2ik_oL} - K_2(K_2' + \eta_m Y)} \frac{1}{\sqrt{\mu}} M \frac{1}{h_\alpha} \frac{\partial A}{\partial \alpha}$$

$$+ \quad \frac{(K_2 - \eta_m Y)(e^{2ik_oL} - 1)e^{2ik_oL}}{K_2'(K_2 - \eta_m Y)e^{2ik_oL} - K_2(K_2' + \eta_m Y)} \frac{1}{\sqrt{\mu}} M \frac{1}{h_\alpha} \frac{\partial A'}{\partial \alpha} \Bigg\}. \quad (70)$$

In the terms involving L and M is is sufficient to replace all quantities by their zeroth order approximation, and since $K_1 = K_2 = K_1' = K_2' = 1$ to this order, (69) and (70) reduce to

$$E_\alpha - a(e^{2ik_oL} - 1)\frac{1}{\sqrt{\epsilon}} L \frac{1}{h_\alpha} \frac{\partial B}{\partial \alpha} - abe^{2ik_oL}(e^{2ik_oL} - 1)\frac{1}{\sqrt{\epsilon}} L \frac{1}{h_\alpha} \frac{\partial B'}{\partial \alpha}$$

$$= -\Gamma \Bigg\{ H_\beta + a^*(e^{2ik_oL} - 1)\frac{1}{\sqrt{\mu}} M \frac{1}{h_\beta} \frac{\partial A}{\partial \beta}$$

$$+ a^* b^* e^{2ik_oL}(e^{2ik_oL} - 1)\frac{1}{\sqrt{\mu}} M \frac{1}{h_\beta} \frac{\partial A'}{\partial \beta} \Bigg\} \quad (71)$$

$$E_\beta - a(e^{2ik_oL} - 1)\frac{1}{\sqrt{\epsilon}} L \frac{1}{h_\beta} \frac{\partial B}{\partial \beta} - abe^{2ik_oL}(e^{2ik_oL} - 1)\frac{1}{\sqrt{\epsilon}} L \frac{1}{h_\beta} \frac{\partial B'}{\partial \beta}$$

$$= \Gamma_1 \left\{ H_\alpha + a^*(e^{2ik_oL} - 1)\frac{1}{\sqrt{\mu}} M \frac{1}{h_\alpha}\frac{\partial A}{\partial \alpha} \right.$$

$$\left. + a^* b^* e^{2ik_oL}(e^{2ik_oL} - 1)\frac{1}{\sqrt{\mu}} M \frac{1}{h_\alpha}\frac{\partial A'}{\partial \alpha} \right\} \qquad (72)$$

where

$$\begin{aligned} \Gamma &= -Z\frac{K_1'(K_1 - \eta_m Y)e^{2ik_oL} - K_1(K_1' + \eta_m Y)}{(K_1 - \eta_m Y)e^{2ik_oL} + (K_1' + \eta_m Y)} \\ \Gamma_1 &= -Z\frac{K_2'(K_2 - \eta_m Y)e^{2ik_oL} - K_2(K_2' + \eta_m Y)}{(K_2 - \eta_m Y)e^{2ik_oL} + (K_2' + \eta_m Y)} \end{aligned} \qquad (73)$$

$$\begin{aligned} a &= \left\{ 1 + \frac{1 - \eta_m Y}{1 + \eta_m Y} e^{2ik_oL} \right\}^{-1} \\ b &= \frac{1 - \eta_m Y}{1 + \eta_m Y} \end{aligned} \qquad (74)$$

and the asterisk denotes the dual quantity. Thus

$$a^* = \left\{ 1 - \frac{1 - \eta_m Y}{1 + \eta_m Y} e^{2ik_oL} \right\}^{-1} \qquad (75)$$

and

$$b^* = -b\,.$$

We remark that the admittance Y appearing in these quntities is evaluated at the surface $\gamma = \gamma_o - \tau$ of the substrate. To this same order (67) and (68) give

$$\frac{B_{\alpha,\beta}}{\sqrt{\mu}} = aH_{\alpha,\beta}\,. \qquad (76)$$

Also, from (31) and (32)

$$A_\alpha = -B_\beta\,, \qquad A_\beta = B_\alpha$$

and since (see (65) and (66))

$$B_\beta = -a^* \sqrt{\epsilon}\, E_\alpha\,, \qquad B_\alpha = a^* \sqrt{\epsilon}\, E_\beta\,,$$

we have

$$\frac{A_{\alpha,\beta}}{\sqrt{\epsilon}} = a^* E_{\alpha,\beta}\,. \tag{77}$$

Moreover, from (41)

$$B'_{\alpha,\beta} = b B_{\alpha,\beta}$$

implying

$$\frac{B'_{\alpha,\beta}}{\sqrt{\mu}} = abH_{\alpha,\beta} \tag{78}$$

and (see (35))

$$A'_\alpha = B'_\alpha\,, \qquad A'_\beta = -B'_\alpha$$

so that

$$\frac{A'_{\alpha,\beta}}{\sqrt{\epsilon}} = a^* b^* E_{\alpha,\beta}\,. \tag{79}$$

Thus

$$\begin{aligned}
\frac{1}{\sqrt{\epsilon}} L \frac{1}{h_t}\frac{\partial B}{\partial t} &= \frac{1}{2k_o^2 N h_\gamma}\frac{1}{h_t}\frac{\partial}{\partial t}\left[\frac{Z}{N} h_\gamma \cdot \nabla \times (a\mathbf{H})\right] \\
&= \frac{1}{2k_o^2 N h_\gamma}\frac{1}{h_t}\frac{\partial}{\partial t}\left[\frac{Z}{N} h_\gamma \hat{\gamma} \cdot \{-ik_o Y_o a\mathbf{E} - \mathbf{H} \times \nabla a\}\right] \\
&= \frac{1}{2ik_o N h_\gamma}\frac{1}{h_t}\frac{\partial}{\partial t}\left[a\frac{\epsilon_o}{\epsilon} h_\gamma \left\{E_\gamma - \frac{1}{ik_o} Z_o \mathbf{H} \cdot \hat{\gamma} \times \nabla(\ln a)\right\}\right]
\end{aligned}$$

and similarly

$$\begin{aligned}
\frac{1}{\sqrt{\epsilon}} L \frac{1}{h_t}\frac{\partial B'}{\partial t} &= \frac{1}{2ik_o N h_\gamma}\frac{1}{h_t}\frac{\partial}{\partial t}\left[ab\frac{\epsilon_o}{\epsilon} h_\gamma \left\{E_\gamma - \frac{1}{ik_o} Z_o \mathbf{H} \cdot \hat{\gamma} \times \nabla(\ln ab)\right\}\right] \\
\frac{1}{\sqrt{\mu}} M \frac{1}{h_t}\frac{\partial A}{\partial t} &= -\frac{1}{2ik_o N h_\gamma}\frac{1}{h_t}\frac{\partial}{\partial t}\left[a^*\frac{\mu_o}{\mu} h_\gamma \left\{H_\gamma + \frac{1}{ik_o} Y_o \mathbf{E} \cdot \hat{\gamma} \times \nabla(\ln a^*)\right\}\right] \\
\frac{1}{\sqrt{\mu}} M \frac{1}{h_t}\frac{\partial A'}{\partial t} &= -\frac{1}{2ik_o N h_\gamma}\frac{1}{h_t}\frac{\partial}{\partial t}\left[a^* b^*\frac{\mu_o}{\mu} h_\gamma \left\{H_\gamma + \frac{1}{ik_o} Y_o \mathbf{E} \cdot \hat{\gamma} \times \nabla(\ln a^* b^*)\right\}\right]
\end{aligned}$$

Equation (71) then becomes

$$E_\alpha - \frac{a}{2ik_oNh_\gamma}(e^{2ik_oL}-1)\frac{1}{h_\alpha}\frac{\partial}{\partial\alpha}\left[a\frac{\epsilon_o}{\epsilon}h_\gamma\left\{E_\gamma - \frac{1}{ik_o}Z_o\mathbf{H}\cdot\hat{\gamma}\times\nabla(\ln a)\right\}\right]$$

$$-\frac{ab}{2ik_oNh_\gamma}(e^{2ik_oL}-1)e^{2ik_oL}\frac{1}{h_\alpha}\frac{\partial}{\partial\alpha}\left[ab\frac{\epsilon_o}{\epsilon}h_\gamma\left\{E_\gamma - \frac{1}{ik_o}Z_o\mathbf{H}\cdot\hat{\gamma}\times\nabla(\ln ab)\right\}\right]$$

$$= -\Gamma\left\{H_\beta - \frac{a^*}{2ik_oNh_\gamma}(e^{2ik_oL}-1)\frac{1}{h_\beta}\frac{\partial}{\partial\beta}\left[a^*\frac{\mu_o}{\mu}h_\gamma\{H_\gamma\right.\right.$$

$$\left.\left. + \frac{1}{ik_o}Y_o\mathbf{E}\cdot\hat{\gamma}\times\nabla(\ln a^*)\right\}\right] - \frac{a^*b^*}{2ik_oNh_\gamma}(e^{2ik_oL}-1)e^{2ik_oL}\frac{1}{h_\beta}\frac{\partial}{\partial\beta}$$

$$\left.\cdot\left[a^*b^*\frac{\mu_o}{\mu}h_\gamma\left\{H_\gamma + \frac{1}{ik_o}Y_o\mathbf{E}\cdot\hat{\gamma}\times\nabla(\ln a^*b^*)\right\}\right]\right\}$$

with an analogous result for (72) , and these can be written as

$$\hat{\gamma}\times\left(\hat{\gamma}\times\left\{\mathbf{E} - \frac{a}{2ik_oNh_\gamma}(e^{2ik_oL}-1)\nabla\left[a\frac{\epsilon_o}{\epsilon}h_\gamma\left\{E_\gamma - \frac{1}{ik_o}Z_o\mathbf{H}\cdot\hat{\gamma}\times\nabla(\ln a)\right\}\right]\right.\right.$$

$$\left.\left. - \frac{ab}{2ik_oNh_\gamma}(e^{2ik_oL}-1)e^{2ik_oL}\nabla\left[ab\frac{\epsilon_o}{\epsilon}h_\gamma\left\{E_\gamma - \frac{1}{ik_o}Z_o\mathbf{H}\cdot\hat{\gamma}\times\nabla(\ln ab)\right\}\right]\right\}\right)$$

$$= -\overline{\overline{\eta}}\cdot\hat{\gamma}\times\left\{\mathbf{H} - \frac{a^*}{2ik_oNh_\gamma}(e^{2ik_oL}-1)\nabla\left[a^*\frac{\mu_o}{\mu}h_\gamma\{H_\gamma\right.\right.$$

$$\left. + \frac{1}{ik_o}Y_o\mathbf{E}\cdot\hat{\gamma}\times\nabla(\ln a^*)\right\}\right] - \frac{a^*b^*}{2ik_oNh_\gamma}(e^{2ik_oL}-1)e^{2ik_oL}\nabla\left[a^*b^*\frac{\mu_o}{\mu}h_\gamma\{H_\gamma\right.$$

$$\left.\left. + \frac{1}{ik_o}Y_o\mathbf{E}\cdot\hat{\gamma}\times\nabla(\ln a^*b^*)\right\}\right]\right\} \tag{80}$$

with

$$\overline{\overline{\eta}} = \Gamma\hat{\alpha}\hat{\alpha} + \Gamma_1\hat{\beta}\hat{\beta}\,. \tag{81}$$

It can be verified that (80) satisfies duality and $\Gamma_1 = 1/\Gamma^*$.

For a homogeneous dielectric coating of constant thickness, a, b, a^* and b^* are independent of α and β, and the boundary condition reduces to

$$\hat{\gamma} \times \left(\hat{\gamma} \times \left\{ \mathbf{E} - \frac{w}{2ik_oNh_\gamma} \nabla \left(\frac{\epsilon_o}{\epsilon} h_\gamma E_\gamma \right) \right\} \right) = -\overline{\overline{\eta}} \cdot \hat{\gamma} \times \left\{ \mathbf{H} - \frac{w^*}{2ik_oNH_\gamma} \nabla \left(\frac{\mu_o}{\mu} h_\gamma H_\gamma \right) \right\} \tag{82}$$

where

$$w = a^2 \left(1 + b^2 e^{2ik_oL} \right) \left(e^{2ik_oL} - 1 \right) . \tag{83}$$

We observe that if

$$k_oL = k_oN\tau = m\pi$$

where m is an integer, then

$$\Gamma = \Gamma_1 = \eta_m$$

(see (73)), and the boundary condition becomes

$$\hat{\gamma} \times (\hat{\gamma} \times \mathbf{E}) = -\eta_m \hat{\gamma} \times \mathbf{H} . \tag{84}$$

The coating is now invisible, and as we shall see later, this is a consequence of ignoring the thickness τ in all amplitude factors.

To illustrate the boundary condition (82), consider the special case of a circular cylinder of outer radius ρ_o. Putting $\alpha = \phi, \beta = z, \gamma = \rho$ so that $h_\alpha = \rho, h_\beta = h_\gamma = 1$, the expressions for K_1 and K_2 computed from (58) are

$$K_1 = 1 + \frac{1}{2ik_oN\rho_o} + \frac{1}{2(2k_oN\rho_o)^2}$$

$$K_2 = 1 - \frac{1}{2ik_oN\rho_o} - \frac{3}{2(2k_oN\rho_o)^2} .$$

Hence, from (61),

$$K_1' = 1 - \frac{1}{2ik_oN\rho_o} + \frac{1}{2(2k_oN\rho_o)^2}$$

$$K_2' = 1 + \frac{1}{2ik_oN\rho_o} - \frac{3}{2(2k_oN\rho_o)^2}$$

and (73) now give

$$\Gamma = -iZ\frac{\left\{1-\frac{i\eta_m Y}{2k_oN\rho_o}+\frac{3}{2(2k_oN\rho_o)^2}\right\}\tan k_oN\tau + i\eta_m Y}{1-i\eta_m Y\left\{1+\frac{i}{\eta_m Y}\frac{1}{2k_oN\rho_o}-\frac{1}{2(2k_oN\rho_o)^2}\right\}\tan k_oN\tau}$$

$$\Gamma_1 = -iZ\frac{\left\{1+\frac{i\eta_m Y}{2k_oN\rho_o}-\frac{1}{2(2k_oN\rho_o)^2}\right\}\tan k_oN\tau + i\eta_m Y}{1-i\eta_m Y\left\{1-\frac{i}{\eta_m Y}\frac{1}{2k_oN\rho_o}+\frac{3}{2(2k_oN\rho_o)^2}\right\}\tan k_oN\tau}\,. \qquad (85)$$

As expected, $\Gamma_1 = 1/\Gamma^*$. Finally, from (74),

$$w = i\tan k_oN\tau\,\frac{1+(\eta_m Y)^2 - 2i\eta_m Y\tan k_oN\tau}{(1-i\eta_m Y\tan k_oN\tau)^2} \qquad (86)$$

and therefore

$$w^* = -i\tan k_oN\tau\,\frac{1+(\eta_m Y)^2 - 2i\eta_m Y\tan k_oN\tau}{(\tan k_oN\tau + i\eta_m Y)^2}\,. \qquad (87)$$

In the particular case of a perfectly conducting substrate ($\eta_m = 0$)

$$\Gamma = -iZ\left\{1+\frac{3}{2(2k_oN\rho_o)^2}\right\}\left\{\cot k_oN\tau + \frac{1}{2k_oN\rho_o}\right\}^{-1}$$

$$\Gamma_1 = -iZ\left\{1-\frac{1}{2(2k_oN\rho_o)^2}\right\}\left\{\cot k_oN\tau - \frac{1}{2k_oN\rho_o}\right\}^{-1} \qquad (88)$$

and

$$w = \frac{1}{w^*} = i\tan k_oN\tau\,.$$

For a first order boundary condition, terms $O(|N|^{-2})$ are neglected, and the expressions (88) for Γ and Γ_1 reduce to (56).

7 Coated Circular Cylinder

From the modal series solution for a circular cylinder covered with a homogeneous dielectric layer of constant thickness, it is relatively easy to derive approximate boundary conditions of almost any order corresponding to low and high contrast coatings, and these can serve to check the preceding results.

A circular cylinder of outer radius ρ_o consists of a core (or substrate) of radius ρ_1 where an impedance boundary condition is imposed and a homogeneous dielectric coating of thickness $\tau = \rho_o - \rho_1$. If the cylinder is illuminated with an H polarized plane wave incident in a plane perpendicular to its axis, the non-zero components of the field are H_z, E_ρ, E_ϕ, and these can be expressed as:

$\rho \geq \rho_o$:

$$
\begin{aligned}
H_z &= \sum_{n=0}^{\infty} \epsilon_n(-i)^n \left\{J_n(k_o\rho) + R_n H_n^{(1)}(k_o\rho)\right\} \cos n\phi \\
E_\rho &= -i\frac{Z_o}{k_o\rho} \sum_{n=0}^{\infty} \epsilon_n(-i)^n n \left\{J_n(k_o\rho) + R_n H_n^{(1)}(k_o\rho)\right\} \sin n\phi \\
E_\phi &= -iZ_o \sum_{n=0}^{\infty} \epsilon_n(-i)^n \left\{J_n'(k_o\rho) + R_n H_n^{(1)\prime}(k_o\rho)\right\} \cos n\phi
\end{aligned}
$$

$\rho_o \geq \rho \geq \rho_1$:

$$
\begin{aligned}
H_z &= \sum_{n=0}^{\infty} \epsilon_n(-i)^n \left\{a_n J_n(Nk_o\rho) + b_n H_n^{(1)}(Nk_o\rho)\right\} \cos n\phi \\
E_\rho &= -i\frac{Z}{Nk_o\rho} \sum_{n=0}^{\infty} \epsilon_n(-i)^n n \left\{a_n J_n(Nk_o\rho) + b_n H_n^{(1)}(nk_o\rho)\right\} \sin n\phi \\
E_\phi &= -iZ \sum_{n=0}^{\infty} \epsilon_n(-i)^n \left\{a_n J_n'(Nk_o\rho) + b_n H_n^{(1)\prime}(Nk_o\rho)\right\} \cos n\phi
\end{aligned}
$$

where the prime denotes the derivative. At $\rho = \rho_1$ the boundary condition is

$$E_\phi = -\eta_m H_z$$

and this gives

$$b_n = Q a_n$$

with

$$Q = -\frac{J_n'(s) + iY\eta_m J_n(s)}{H_n^{(1)\prime}(s) + iY\eta_m H_n^{(1)}(s)} \tag{89}$$

and $s = Nk_o\rho_1$. By enforcing the continuity of H_z and E_ϕ at $\rho = \rho_o$ we then obtain

$$R_n = -\frac{J_n'(k_o\rho_o) + iY_oPJ_n(k_o\rho_o)}{H_n^{(1)\prime}(k_o\rho_o) + iY_oPH_n^{(1)}(k_o\rho_o)} \tag{90}$$

where

$$P = iZ\,\frac{J_n'(t) + QH_n^{(1)\prime}(t)}{J_n(t) + QH_n^{(1)}(t)} \tag{91}$$

and $t = Nk_o\rho_o$.

The quantity P can be used to specify an approximate boundary condition which is imposed at $\rho = \rho_o$ and reproduces R_n to some desired accuracy. Consider, for example, the third order boundary condition (29) which can be written as

$$\hat{\rho}\times\left(\hat{\rho}\times\left\{\mathbf{E} - \nabla\left[A\frac{\epsilon_o}{\epsilon}E_\rho + B\nabla_s\cdot\mathbf{E}\right]\right\}\right)$$

$$= -\left(\Gamma\hat{\phi}\hat{\phi} + \frac{1}{\Gamma^*}\hat{z}\hat{z}\right)\cdot\hat{\rho}\times\left\{\mathbf{H} - \nabla\left[A^*\frac{\mu_o}{\mu}H_\rho + B^*\nabla_s\cdot\mathbf{H}\right]\right\}. \tag{92}$$

For H-poalrization this reduces to

$$\left(1 - \frac{B}{\rho_o^2}\frac{\partial^2}{\partial\phi^2}\right)E_\phi - \frac{A}{\rho_o}\frac{\epsilon_o}{\epsilon}\frac{\partial E_\rho}{\partial\phi} = -\Gamma H_z \tag{93}$$

and when the expressions for the field components in $\rho \geq \rho_o$ are inserted, we find that R_n is as shown in (90) with

$$P = \frac{\Gamma + \dfrac{iZ_o}{k_o\rho_o^2}\dfrac{\epsilon_o}{\epsilon}An^2}{1 + B\dfrac{n^2}{\rho_o^2}}. \tag{94}$$

A third order boundary condition suffices to the extent that (91) can be written in this form. For a second order condition $B = 0$ and for a first (or zeroth) order, $A = 0$ as well. In the latter case P is independent of n.

We consider first a low contrast coating for which $|\delta|$ is small where

$$\delta = t - s = Nk_o\tau .$$

Then

$$J_n(s) = J_n(t) - \delta J_n'(t) + \frac{\delta^2}{2} J_n''(t) + O(\delta^3)$$

and since

$$J_n''(t) = -\left(1 - \frac{n^2}{t^2}\right) J_n(t) - \frac{1}{t} J_n'(t)$$

from Bessel's equation, we have

$$J_n(s) = \left\{1 - \frac{\delta^2}{2}\left(1 - \frac{n^2}{t^2}\right)\right\} J_n(t) - \delta\left(1 + \frac{\delta}{2t}\right) J_n'(t) + O(\delta^3) .$$

Also

$$J_n'(s) = J_n'(t) - \delta J_n''(t) + \frac{\delta^2}{2} J_n'''(t) + O(\delta^3)$$

and because

$$J_n'''(t) = \frac{1}{t}\left(1 - \frac{3n^2}{t^2}\right) J_n(t) - \left(1 - \frac{n^2+2}{t^2}\right) J_n'(t) ,$$

we have

$$J_n'(s) = \delta\left\{1 - \frac{n^2}{t^2} + \frac{\delta}{2t}\left(1 - \frac{3n^2}{t^2}\right)\right\} J_n(t) + \left\{1 + \frac{\delta}{t} - \frac{\delta^2}{2}\left(1 - \frac{n^2+2}{t^2}\right)\right\} J_n'(t) + O(\delta^3) .$$

There are similar expressions for $H_n^{(1)}(s)$ and $H_n^{(1)\prime}(s)$, and when these are inserted into (89) and then into (91) we obtain

$$P = -iZ\frac{iY\eta_m\left\{1-\frac{\delta^2}{2}\left(1-\frac{n^2}{t^2}\right)\right\}+\delta\left\{1-\frac{n^2}{t^2}+\frac{\delta}{2t}\left(1-\frac{3n^2}{t^2}\right)\right\}}{1+\frac{\delta}{t}-\frac{\delta^2}{2}\left(1-\frac{n^2+2}{t^2}\right)-iY\eta_m\delta\left(1+\frac{\delta}{2t}\right)}+O(\delta^3)\,. \tag{95}$$

To identify the coefficients in (94) we rewrite (95) as

$$\begin{aligned} P &= \eta_m\frac{1-\frac{i\delta}{Y\eta_m}\left(1+\frac{\delta}{2t}\right)-\frac{\delta^2}{2}+\frac{n^2\delta}{t^2}\left\{\frac{i}{Y\eta_m}\left(1+\frac{3\delta}{2t}\right)+\frac{\delta}{2}\right\}}{1-\delta\left(iY\eta_m-\frac{1}{t}\right)-\frac{\delta^2}{2}\left(1+\frac{iY\eta_m}{t}-\frac{2}{t^2}\right)+\frac{n^2\delta^2}{2t^2}}+O(\delta^3) \\ &= \left[\eta_m\frac{1-\frac{i\delta}{Y\eta_m}\left(1+\frac{\delta}{2t}\right)-\frac{\delta^2}{2}}{1-\delta\left(iY\eta_m-\frac{1}{t}\right)-\frac{\delta^2}{2}\left(1+\frac{iY\eta_m}{t}-\frac{2}{t^2}\right)}+iZ\frac{n^2\delta}{t^2}\left\{1\right.\right. \\ &\quad \left.\left.+\frac{\delta}{2}\left(iY\eta_m+\frac{1}{t}\right)\right\}\right]\left[1+\frac{n^2\delta^2}{2t^2}\right]^{-1}+O(\delta^3) \end{aligned}$$

and to the second order in δ

$$\Gamma = \eta_m\frac{1-\tau\left(1+\frac{\tau}{2\rho_o}\right)ik_o\frac{\mu}{\mu_o}\frac{Z_o}{\eta_m}-\frac{1}{2}(k_oN\tau)^2}{1-\tau\left(1+\frac{\tau}{2\rho_o}\right)\left(ik_o\frac{\epsilon}{\epsilon_o}Y_o\eta_m-\frac{1}{\rho_o}\right)-\frac{\tau^2}{2}\left(k_o^2N^2-\frac{1}{\rho_o^2}\right)} \tag{96}$$

$$A = \tau\left\{1+\frac{\tau}{2}\left(\frac{1}{\rho_o}+ik_o\frac{\epsilon}{\epsilon_o}Y_o\eta_m\right)\right\} \tag{97}$$

$$B = \frac{1}{2}\tau^2\,. \tag{98}$$

These agree with the results in Section 3 (see (27) and (28)) and also reduce to the solution for a perfectly conducting substrate (see (17) and (18)) on putting $\eta_m = 0$.

To include terms in δ^3 leads to an expression for P involving n^4 and this requires a fourth (or higher) order boundary condition, but it is worth noting that we can actually reproduce the terms in n^2 using only a second order boundary condition. This is evident since B is proportional to δ^2 and if we put $B = 0$ correpsonding to a second order condition, the expression (96) for Γ is unaffected, but (97) is replaced by

$$A = \tau \left\{ 1 + \tau \left(\frac{1}{2\rho_o} + ik_o \frac{\epsilon}{\epsilon_o} Y_o \eta_m \right) \right\} . \tag{99}$$

We now turn to the more difficult problem of a high contrast coating for which $|s|, |t| \gg 1$. For large arguments

$$H_n^{(1)}(t) = \sqrt{\frac{2}{\pi t}} e^{i\alpha_t} f_1$$

with

$$\alpha_t = t - n\frac{\pi}{2} - \frac{\pi}{4}$$

where f_1 is a series in powers of $1/t$. Similarly

$$\begin{aligned} H_n^{(1)'}(t) &= \sqrt{\frac{2}{\pi t}} e^{i\alpha_t} g_1 \\ H_n^{(1)}(s) &= \sqrt{\frac{2}{\pi s}} e^{i\alpha_s} f_2 \\ H_n^{(1)'}(s) &= \sqrt{\frac{2}{\pi s}} e^{i\alpha_s} g_2 \end{aligned}$$

with

$$\alpha_s = s - n\frac{\pi}{2} - \frac{\pi}{4}$$

where f_1 and g_2 differ from f_1 and g_1 in having t replaced by s. Since

$$J_n(t) = \frac{1}{2} \left\{ H_n^{(1)}(t) + H_n^{(2)}(t) \right\}$$

it follows that

$$J_n(t) = \sqrt{\frac{2}{\pi t}}\,\frac{1}{2}\left(f_1 e^{i\alpha_t} + \overline{f}_1 e^{i\alpha_t}\right)$$

where the bar denotes the complex conjugate. There are analogous expressions for $J_n'(t)$, $J_n(s)$ and $J_n'(s)$, and when these are substituted into (89) we find

$$Q = -\frac{1}{2}\left\{1 + \frac{\overline{g}_2 + iY\eta_m\overline{f}_2}{g_2 + iY\eta_m f_2}e^{-2i\alpha_s}\right\}$$

and hence

$$P = iZ\,\frac{g_1(\overline{g}_2 + iY\eta_m\overline{f}_2)e^{i\delta} - \overline{g}_1(g_2 + iY\eta_m f_2)e^{-i\delta}}{f_1(\overline{g}_2 + iY\eta_m\overline{f}_2)e^{i\delta} - \overline{f}_1(g_2 + iY\eta_m f_2)e^{-i\delta}} \tag{100}$$

where, as before, $\delta = t - s = \alpha_t - \alpha_s$.

From Bowman et al [1987, p. 53]

$$\begin{aligned} f_1 &= 1 - \frac{4n^2-1}{8it} - \frac{(4n^2-1)(4n^2-9)}{128t^2} \\ &+ \frac{(4n^2-1)(4n^2-9)(4n^2-25)}{3072it^3} + O(t^{-4}) \end{aligned}$$

and

$$\begin{aligned} g_1 &= i\left\{1 - \frac{4n^2+3}{8it} - \frac{(4n^2-1)(4n^2+15)}{128t^2}\right. \\ &+ \left.\frac{(4n^2-1)(4n^2-9)(4n^2+35)}{3072it^3} + O(t^{-4})\right\}. \end{aligned}$$

If we ignore the difference between s and t except in the phase,

$$\begin{aligned} g_1\overline{g}_2e^{i\delta} - \overline{g}_1 g_2 e^{-i\delta} &= 2i\left\{1 - \frac{1}{8t^2}(4n^2-3) + O(t^{-4})\right\}\sin\delta \\ g_1\overline{f}_2e^{i\delta} - \overline{g}_1 f_2 e^{-i\delta} &= 2i\left\{\cos\delta - \frac{1}{2t}\left[1 + \frac{3}{8t^2}(4n^2-1)\right]\sin\delta + O(t^{-4})\right\} \\ f_1\overline{g}_2e^{i\delta} - \overline{f}_1 g_2 e^{-i\delta} &= -2i\left\{\cos\delta + \frac{1}{2t}\left[1 + \frac{3}{8t^2}(4n^2-1)\right]\sin\delta + O(t^{-4})\right\} \end{aligned}$$

and

$$f_1\bar{f}_2e^{i\delta} - \bar{f}_1f_2e^{-i\delta} = 2i\left\{1 + \frac{1}{8t^2}(4n^2-1) + O(t^{-4})\right\}\sin\delta\,,$$

giving

$$P = -iZ\,\frac{\tan\delta\left\{1 - \frac{1}{8t^2}(4n^2-3) - \frac{iY\eta_m}{2t}\left[1 + \frac{3}{8t^2}(4n^2-1)\right]\right\} + iY\eta_m}{1 - iY\eta_m\tan\delta\left\{1 + \frac{1}{8t^2}(4n^2-1) + \frac{i}{2Y\eta_m t}\left[1 + \frac{3}{8t^2}(4n^2-1)\right]\right\}}$$

$$+O(t^{-4})\,. \quad (101)$$

Comparison with (94) now shows that to the third order in $1/t$,

$$\Gamma = -iZ\,\frac{\tan\delta\left\{1 + \frac{3}{2(2Nk_o\rho_o)^2} - \frac{iY\eta_m}{2Nk_o\rho_o}\left[1 - \frac{3}{2(2Nk_o\rho_o)^2}\right]\right\} + iY\eta_m}{1 - iY\eta_m\tan\delta\left\{1 - \frac{1}{2(2Nk_o\rho_o)^2} + \frac{i}{Y\eta_m}\frac{1}{2Nk_o\rho_o}\left[1 - \frac{3}{2(2Nk_o\rho_o)^2}\right]\right\}}$$

$$A = \frac{1}{2Nk_o}\,\frac{\tan\delta}{1 - iY\eta_m\tan\delta}\left\{1 + \frac{1}{2Nk_o\rho_o}\left(3iY\eta_m - \frac{\tan\delta}{1 - iY\eta_m\tan\delta}\right)\right\} \quad (102)$$

$$B = -\frac{1}{2(Nk_o)^2}\,\frac{iY\eta_m\tan\delta}{1 - iY\eta_m\tan\delta}\left\{1 + \frac{1}{2Nk_o\rho_o}\left(\frac{3i}{Y\eta_m} - \frac{\tan\delta}{1 - iY\eta_m\tan\delta}\right)\right\}$$

with $\delta = Nk_o\tau$, and these complete the specification of the third order boundary condition (92).

To include terms in $1/t^4$ would require a boundary condition of fourth (or higher) order, but even a second order condition (having $B = 0$) is sufficient to reproduce terms through the third order in $1/t$. Putting $B = 0$ in (94) the expression for A becomes

$$A = \frac{1}{2Nk_o}\frac{\tan\delta}{(1 - iY\eta_m\tan\delta)^2}\left\{1 + (Y\eta_m)^2 - 2iY\eta_m\tan\delta\right.$$

$$\left. + \frac{1}{2Nk_o\rho_o}\left[iY\eta_m(2 - iY\eta_m\tan\delta) + \frac{\tan\delta + iY\eta_m}{1 - iY\eta_m\tan\delta}\right]\right\} \quad (103)$$

with Γ as before, and obviously the same boundary condition is sufficient to match terms through the second order in $1/t$. When the third order terms are omitted from (102) and (103) we obtain

$$
\begin{aligned}
\Gamma &= -iZ\frac{\tan\delta\left\{1-\frac{iY\eta_m}{2Nk_o\rho_o}+\frac{3}{2(2nk_o\rho_o)^2}\right\}+iY\eta_m}{1-iY\eta_m\tan\delta\left\{1+\frac{i}{Y\eta_m}\frac{1}{2Nk_o\rho_o}-\frac{1}{2(2Nk_o\rho_o)^2}\right\}} \\
A &= \frac{1}{2Nk_o}\tan\delta\frac{1+(Y\eta_m)^2-2iY\eta_m\tan\delta}{(1-iY\eta_m\tan\delta)^2}\,,
\end{aligned}
$$

and the boundary condition is then identical (see (82), (85) and (86)) to the one derived in Section 6.

For all of the above contrast conditions, if $\delta = m\pi$ where m is an integer, then $\tan\delta = 0$ and $\Gamma = \eta_m$ with $A = B = 0$. In other words, the incident field sees only the substrate and the coating is invisible. As we shall now show, this is a consequence of ignoring the difference between s and t in the amplitudes.

If the coating thickness is too large to be ignored,

$$
\frac{1}{s^m} \simeq \frac{1}{t^m}(1+m\Delta)
$$

to the first power of $\Delta = \tau/t$. Since Δ is of the zeroth order in $1/N$, the retention of terms through the third order in $1/t$ now leads to an expression for P involving n^4 that demands a fourth (or higher) order boundary condition. On the other hand, through the second order in $1/t$,

$$
\begin{aligned}
g_1\overline{g}_2e^{i\delta}-\overline{g}_1g_2e^{-i\delta} &= 2i\left\{\left[1-\frac{1+\Delta}{8t^2}(4n^2-3)\right]\sin\delta-\frac{\Delta}{8t}(4n^2+3)\cos\delta\right\} \\
g_1\overline{f}_2e^{i\delta}-\overline{g}_1f_2e^{-i\delta} &= 2i\left\{\left[1+\frac{3\Delta}{16t^2}(4n^2-1)\right]\cos\delta\right. \\
&\qquad \left.-\frac{1}{2t}\left[1-\frac{\Delta}{4}(4n^2-1)\right]\sin\delta\right\} \\
f_1\overline{g}_2e^{i\delta}-\overline{f}_1g_2e^{-i\delta} &= -2i\left\{\left[1-\frac{3\Delta}{16t^2}(4n^2-1)\right]\cos\delta\right.
\end{aligned}
$$

$$+\frac{1}{2t}\left[1+\frac{\Delta}{4}(4n^2+3)\right]\sin\delta\Big\}$$

$$f_1\overline{f}_2e^{i\delta}-\overline{f}_1f_2e^{-i\delta} = 2i\left\{\left[1+\frac{1+\Delta}{8t^2}(4n^2-1)\right]\sin\delta-\frac{\Delta}{8t}(4n^2-1)\cos\delta\right\}$$

and when these are substituted into (100) we find

$$\begin{aligned} P &= -iZ\left(\tan\delta\left\{1-\frac{iY\eta_m}{2t}\left[1-\frac{\Delta}{4}(4n^2-1)\right]-\frac{1+\Delta}{8t^2}(4n^2-3)\right\}\right. \\ &\quad \left.+\,iY\eta_m\left\{1+\frac{i}{Y\eta_m}\frac{\Delta}{8t}(4n^2+3)+\frac{3\Delta}{16t^2}(4n^2-1)\right\}\right) \\ &\quad \cdot\left(1+iY\eta_m\frac{\Delta}{8t}(4n^2-1)-\frac{3\Delta}{16t^2}(4n^2-1)\right. \\ &\quad \left.-\,iY\eta_m\tan\delta\left\{1+\frac{i}{Y\eta_m}\frac{1}{2t}\left[1+\frac{\Delta}{4}(4n^2+3)\right]+\frac{1+\Delta}{8t^2}(4n^2-1)\right\}\right)^{-1}. \end{aligned} \tag{104}$$

If $\Delta = 0$ this reduces to (101) with the terms in $\frac{1}{t^3}$ omitted, but if $\delta = m\pi$ (so that $\tan\delta = 0$)

$$P = \eta_m\frac{1+\frac{i}{Y\eta_m}\frac{\Delta}{8t}(4n^2+3)+\frac{3\Delta}{16t^2}(4n^2-1)}{1+iY\eta_m\frac{\Delta}{8t}(4n^2-1)-\frac{3\Delta}{16t^2}(4n^2-1)} \tag{105}$$

and the coating is no longer invisible. Equation (104) can be used to construct second and third order boundary conditions analogous to those given by Senior and Volakis [1989] for a planar layer.

8 Conclusions

A problem of considerable practical interest is the scattering from a body covered with a layer of dielectric, and one way to simplify it is to simulate the surface using an approximate boundary condition of appropriate order. For an inhomogeneous dielectric whose inner and outer boundaries are coordinate surfaces in an orthogonal curvilinear coordinate system, two different situations have been considered, and in each instance, boundary conditions of three different orders have been derived. At least for a homogeneous dielectric the results are sufficiently simple to be usable, and in the special case of a circular cylinder with a coating of uniform thickness, the boundary conditions have been confirmed by starting from the known modal expansion for the scattered field.

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Appendix A: Matching $\frac{\partial}{\partial\gamma}(h_\gamma E^i_\gamma)$

At the boundary of the layer the boundary conditions give

$$E^i_\alpha = E_\alpha\,, \quad E^i_\beta = E_\beta\,, \quad E^i_\gamma = \frac{\epsilon_o}{\epsilon}E_\gamma$$

and we must use these to match $\frac{\partial}{\partial\gamma}(h_\gamma H^i_\gamma)$ to the exterior field.

Since

$$\nabla\cdot\mathbf{E}^i = \frac{1}{h_\alpha h_\beta h_\gamma}\left\{\frac{\partial}{\partial\alpha}(h_\beta h_\gamma E^i_\alpha) + \frac{\partial}{\partial\beta}(h_\alpha h_\gamma E^i_\beta) + \frac{\partial}{\partial\gamma}(h_\alpha h_\beta E^i_\gamma)\right\}$$

and

$$\begin{aligned}\frac{\partial}{\partial\gamma}(h_\alpha h_\beta E^i_\gamma) &= \frac{\partial}{\partial\gamma}\left(\frac{h_\alpha h_\beta}{h_\gamma}h_\gamma E^i_\gamma\right)\\ &= \frac{h_\alpha h_\beta}{h_\gamma}\frac{\partial}{\partial\gamma}(h_\gamma E^i_\gamma) + h_\gamma E^i_\gamma\frac{\partial}{\partial\gamma}\left(\frac{h_\alpha h_\beta}{h_\gamma}\right)\,,\end{aligned}$$

we have

$$\nabla\cdot\mathbf{E}^i = \frac{1}{h_\gamma^2}\frac{\partial}{\partial\gamma}(h_\gamma E^i_\gamma) + \frac{E^i_\gamma}{h_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha h_\beta}{h_\gamma}\right)$$
$$+\frac{1}{h_\alpha h_\beta h_\gamma}\left\{\frac{\partial}{\partial\alpha}(h_\beta h_\gamma E^i_\alpha) + \frac{\partial}{\partial\beta}(h_\alpha h_\gamma E^i_\beta)\right\}\,.$$

But

$$\nabla\cdot(\epsilon\mathbf{E}^i) = 0$$

and therefore

$$\nabla\cdot\mathbf{E}^i = -\frac{\epsilon_o}{\epsilon}\mathbf{E}^i\cdot\nabla\frac{\epsilon}{\epsilon_o}\,.$$

Hence

$$\frac{\partial}{\partial\gamma}(h_\gamma E^i_\gamma) = -\frac{\epsilon_o}{\epsilon}h_\gamma^2\left\{\frac{E^i_\alpha}{h_\alpha}\frac{\partial}{\partial\alpha}\left(\frac{\epsilon}{\epsilon_o}\right) + \frac{E^i_\beta}{h_\beta}\frac{\partial}{\partial\beta}\left(\frac{\epsilon}{\epsilon_o}\right) + \frac{E^i_\gamma}{h_\gamma}\frac{\partial}{\partial\gamma}\left(\frac{\epsilon}{\epsilon_o}\right)\right\}$$

$$-h_\gamma E^i_\gamma\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha h_\beta}{h_\gamma}\right) - \frac{h_\gamma}{h_\alpha h_\beta}\left\{\frac{\partial}{\partial\alpha}(h_\beta h_\gamma E^i_\gamma) + \frac{\partial}{\partial\beta}(h_\alpha h_\gamma E^i_\beta)\right\}$$

$$= -\frac{\epsilon_o}{\epsilon}h_\gamma^2\left\{\frac{E_\alpha}{h_\alpha}\frac{\partial}{\partial\alpha}\left(\frac{\epsilon}{\epsilon_o}\right) + \frac{E_\beta}{h_\beta}\frac{\partial}{\partial\beta}\left(\frac{\epsilon}{\epsilon_o}\right)\right\} - \left(\frac{\epsilon_o}{\epsilon}\right)^2 h_\gamma E_\gamma\frac{\partial}{\partial\gamma}\left(\frac{\epsilon}{\epsilon_o}\right)$$

$$-\frac{\epsilon_o}{\epsilon}h_\gamma E_\gamma\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha h_\beta}{h_\gamma}\right) - \frac{h_\gamma}{h_\alpha h_\beta}\left\{\frac{\partial}{\partial\alpha}(h_\beta h_\gamma E^i_\alpha) + \frac{\partial}{\partial\beta}(h_\alpha h_\gamma E^i_\beta)\right\}$$

$$= -\frac{\epsilon_o}{\epsilon}h_\gamma^2\frac{1}{h_\alpha h_\beta h_\gamma}\left\{\frac{\partial}{\partial\alpha}\left(h_\beta h_\gamma\frac{\epsilon}{\epsilon_o}E_\alpha\right) + \frac{\partial}{\partial\beta}\left(h_\alpha h_\gamma\frac{\epsilon}{\epsilon_o}E_\beta\right)\right\}$$

$$-\frac{\epsilon_o}{\epsilon}h_\gamma E_\gamma\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha h_\beta}{h_\gamma}\frac{\epsilon}{\epsilon_o}\right) ,$$

and if the surface divergence is defined as

$$\nabla_s\cdot\overline{P} = \frac{1}{h_\alpha h_\beta h_\gamma}\left\{\frac{\partial}{\partial\alpha}(h_\beta h_\gamma P_\alpha) + \frac{\partial}{\partial\beta}(h_\alpha h_\gamma P_\beta)\right\} ,$$

then

$$\frac{\partial}{\partial\gamma}(h_\gamma E^i_\gamma) = -\frac{\epsilon_o}{\epsilon}h_\gamma^2\left\{\nabla_s\cdot\left(\frac{\epsilon}{\epsilon_o}\mathbf{E}\right) + \frac{E_\gamma}{h_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha h_\beta}{h_\gamma}\frac{\epsilon}{\epsilon_o}\right)\right\} . \quad (A.1)$$

Similarly

$$\frac{\partial}{\partial\gamma}(h_\gamma H^i_\gamma) = -\frac{\mu_o}{\mu}h_\gamma^2\left\{\nabla_s\cdot\left(\frac{\mu}{\mu_o}\mathbf{H}\right) + \frac{H_\gamma}{h_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha h_\beta}{h_\gamma}\frac{\mu}{\mu_o}\right)\right\} . \quad (A.2)$$

Appendix B: Outward Propagating Fields

To apply the analysis in I to the problem of a high contrast coating, it is necessary to determine the properties of an outward propagating field in the dielectric, accurate to the second order in $|N|^{-1}$.

For such a field $\nabla\psi = v\hat{\gamma}$ and therefore

$$\mathbf{A}_n + \hat{\gamma} \times \mathbf{B}_n = -\frac{\sqrt{\mu}}{ik_o v}\nabla \times \frac{\mathbf{B}_{n-1}}{\sqrt{\mu}}$$

$$\mathbf{B}_n - \hat{\gamma} \times \mathbf{A}_n = \frac{\sqrt{\epsilon}}{ik_o v}\nabla \times \frac{\mathbf{A}_{n-1}}{\sqrt{\epsilon}} \tag{B.1}$$

for $n = 0, 1, 2, \ldots$ with $\mathbf{A}_{-1} = \mathbf{B}_{-1} = 0$. Hence, to the zeroth order,

$$A_{o\alpha} = B_{o\beta}\,, \quad A_{o\beta} = -B_{o\alpha}\,, \quad A_{o\gamma} = B_{o\gamma} = 0 \tag{B.2}$$

For the first order fields

$$A_{1\alpha} - B_{1\beta} = \frac{\sqrt{\mu}}{ik_o v h_\beta h_\gamma}\frac{\partial}{\partial\gamma}\left(\frac{h_\beta B_{o\beta}}{\sqrt{\mu}}\right)$$

$$A_{1\beta} + B_{1\alpha} = -\frac{\sqrt{\mu}}{ik_o v h_\alpha h_\gamma}\frac{\partial}{\partial\gamma}\left(\frac{h_\alpha B_{o\alpha}}{\sqrt{\mu}}\right)$$

$$B_{1\alpha} + A_{1\beta} = \frac{\sqrt{\epsilon}}{ik_o v h_\beta h_\gamma}\frac{\partial}{\partial\gamma}\left(\frac{h_\beta B_{o\alpha}}{\sqrt{\epsilon}}\right) \tag{B.3}$$

$$B_{1\beta} - A_{1\alpha} = \frac{\sqrt{\epsilon}}{ik_o v h_\alpha h_\gamma}\frac{\partial}{\partial\gamma}\left(\frac{h_\alpha B_{o\beta}}{\sqrt{\epsilon}}\right)$$

on using (B.2). Also

$$A_{1\gamma} = -\frac{\sqrt{\mu}}{ik_o v h_\alpha h_\beta}\left\{\frac{\partial}{\partial\alpha}\left(\frac{h_\beta B_{o\beta}}{\sqrt{\mu}}\right) - \frac{\partial}{\partial\beta}\left(\frac{h_\alpha B_{o\alpha}}{\sqrt{\mu}}\right)\right\}$$

$$B_{1\gamma} = \frac{\sqrt{\epsilon}}{ik_o v h_\alpha h_\beta}\left\{\frac{\partial}{\partial\alpha}\left(\frac{h_\beta A_{o\beta}}{\sqrt{\mu}}\right) - \frac{\partial}{\partial\beta}\left(\frac{h_\alpha A_{o\alpha}}{\sqrt{\mu}}\right)\right\}\,. \tag{B.4}$$

The expressions for $A_{1\alpha} - B_{1\beta}$ are identical if

$$\frac{\partial B_{o\beta}}{\partial \gamma} = -\frac{1}{2} B_{o\beta} \frac{\partial}{\partial \gamma} \left(\ln \frac{h_\alpha h_\beta}{\sqrt{\epsilon\mu}} \right) \tag{B.5}$$

and then

$$A_{1\alpha} - B_{1\beta} = -\frac{1}{2ik_o v h_\gamma} B_{o\beta} \frac{\partial}{\partial \gamma} \left(\ln \frac{h_\alpha}{h_\beta} Z \right) . \tag{B.6}$$

Similarly, the expressions for $A_{1\beta} + B_{1\alpha}$ are identical if

$$\frac{\partial B_{o\alpha}}{\partial \gamma} = -\frac{1}{2} B_{o\alpha} \frac{\partial}{\partial \gamma} \left(\ln \frac{h_\alpha h_\beta}{\sqrt{\epsilon\mu}} \right) \tag{B.7}$$

and then

$$A_{1\beta} + B_{1\alpha} = \frac{1}{2ik_o v h_\gamma} B_{o\alpha} \frac{\partial}{\partial \gamma} \left(\ln \frac{h_\beta}{h_\alpha} Z \right) . \tag{B.8}$$

Thus, to the first order in q

$$A_\alpha = A_{o\alpha} + qA_{1\alpha} = B_{o\beta} + qB_{1\beta} - \frac{q}{2ik_o v h_\gamma} B_{o\beta} \frac{\partial}{\partial \gamma} \left(\ln \frac{h_\alpha}{h_\beta} Z \right)$$

i.e.

$$A_\alpha = \left\{ 1 - \frac{1}{2ik_o N h_\gamma} \frac{\partial}{\partial \gamma} \left(\ln \frac{h_\alpha}{h_\beta} Z \right) \right\} B_\beta , \tag{B.9}$$

and

$$A_\beta = A_{o\beta} + qA_{1\beta} = -B_{o\alpha} - qB_{1\alpha} + \frac{q}{2ik_o v h_\gamma} B_{o\alpha} \frac{\partial}{\partial \gamma} \left(\ln \frac{h_\beta}{h_\alpha} Z \right)$$

i.e.

$$A_\beta = -\left\{ 1 - \frac{1}{2ik_o N h_\gamma} \frac{\partial}{\partial \gamma} \left(\ln \frac{h_\beta}{h_\alpha} Z \right) \right\} B_\alpha . \tag{B.10}$$

For the second order fields

$$\begin{aligned}
A_{2\alpha} - B_{2\beta} &= -\frac{\sqrt{\mu}}{ik_o v h_\beta h_\gamma}\left\{\frac{\partial}{\partial\beta}\left(\frac{h_\gamma B_{1\gamma}}{\sqrt{\mu}}\right) - \frac{\partial}{\partial\gamma}\left(\frac{h_\beta B_{1\beta}}{\sqrt{\mu}}\right)\right\} \\
&= \frac{1}{ik_o v h_\gamma}\left\{\frac{\partial B_{1\beta}}{\partial\gamma} + B_{1\beta}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{\sqrt{\mu}}\right) - \frac{\sqrt{\mu}}{h_\beta}\frac{\partial}{\partial\beta}\left(\frac{h_\gamma B_{1\gamma}}{\sqrt{\mu}}\right)\right\} \\
A_{2\beta} + B_{2\alpha} &= -\frac{\sqrt{\mu}}{ik_o v h_\alpha h_\gamma}\left\{\frac{\partial}{\partial\gamma}\left(\frac{h_\alpha B_{1\alpha}}{\sqrt{\mu}}\right) - \frac{\partial}{\partial\alpha}\left(\frac{h_\gamma B_{1\gamma}}{\sqrt{\mu}}\right)\right\} \\
&= -\frac{1}{ik_o v h_\gamma}\left\{\frac{\partial B_{1\alpha}}{\partial\gamma} + B_{1\alpha}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{\sqrt{\mu}}\right) - \frac{\sqrt{\mu}}{h_\alpha}\frac{\partial}{\partial\alpha}\left(\frac{h_\gamma B_{1\gamma}}{\sqrt{\mu}}\right)\right\} \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(B.11)} \\
B_{2\alpha} + A_{2\beta} &= \frac{\sqrt{\epsilon}}{ik_o v h_\beta h_\gamma}\left\{\frac{\partial}{\partial\beta}\left(\frac{h_\gamma A_{1\gamma}}{\sqrt{\epsilon}}\right) - \frac{\partial}{\partial\gamma}\left(\frac{h_\beta A_{1\beta}}{\sqrt{\epsilon}}\right)\right\} \\
&= -\frac{1}{ik_o v h_\gamma}\left\{\frac{\partial A_{1\beta}}{\partial\gamma} + A_{1\beta}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{\sqrt{\epsilon}}\right) - \frac{\sqrt{\epsilon}}{h_\beta}\frac{\partial}{\partial\beta}\left(\frac{h_\gamma A_{1\gamma}}{\sqrt{\epsilon}}\right)\right\} \\
B_{2\beta} - A_{2\alpha} &= \frac{\sqrt{\epsilon}}{ik_o v h_\alpha h_\gamma}\left\{\frac{\partial}{\partial\gamma}\left(\frac{h_\alpha A_{1\alpha}}{\sqrt{\epsilon}}\right) - \frac{\partial}{\partial\alpha}\left(\frac{h_\gamma A_{1\gamma}}{\sqrt{\epsilon}}\right)\right\} \\
&= \frac{1}{ik_o v h_\gamma}\left\{\frac{\partial A_{1\alpha}}{\partial\gamma} + A_{1\alpha}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{\sqrt{\epsilon}}\right) - \frac{\sqrt{\epsilon}}{h_\alpha}\frac{\partial}{\partial\alpha}\left(\frac{h_\gamma A_{1\gamma}}{\sqrt{\epsilon}}\right)\right\} .
\end{aligned}$$

Hence

$$\begin{aligned}
A_{2\alpha} - B_{2\beta} = \frac{1}{2ik_o v h_\gamma}&\left\{\frac{\partial}{\partial\gamma}(B_{1\beta} - A_{1\alpha}) + B_{1\beta}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{\sqrt{\mu}}\right)\right. \\
&\left. - A_{1\alpha}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{\sqrt{\epsilon}}\right) - \frac{\sqrt{\mu}}{h_\beta}\frac{\partial}{\partial\beta}\left(\frac{h_\gamma B_{1\gamma}}{\sqrt{\mu}}\right) + \frac{\sqrt{\epsilon}}{h_\alpha}\frac{\partial}{\partial\alpha}\left(\frac{h_\gamma A_{1\gamma}}{\sqrt{\epsilon}}\right)\right\} \\
= \frac{1}{2ik_o v h_\gamma}&\left\{\frac{1}{2ik_o}\frac{\partial}{\partial\gamma}\left[\frac{1}{v h_\gamma}B_{o\beta}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right)\right] - \frac{B_{o\beta}}{2ik_o v h_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right)\right. \\
&\left.\cdot\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{\sqrt{\epsilon}}\right) - B_{1\beta}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right) - \frac{\sqrt{\mu}}{h_\beta}\frac{\partial}{\partial\beta}\left(\frac{h_\gamma B_{1\gamma}}{\sqrt{\mu}}\right) + \frac{\sqrt{\epsilon}}{h_\alpha}\frac{\partial}{\partial\alpha}\left(\frac{h_\gamma A_{1\gamma}}{\sqrt{\epsilon}}\right)\right\}
\end{aligned}$$

giving

$$A_{2\alpha} - B_{2\beta} = -\frac{B_{1\beta}}{2ik_o v h_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right) - \frac{B_{o\beta}}{(2k_o v h_\gamma)^2}\left[\frac{1}{2}\left\{\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right)\right\}^2\right.$$

$$\left. + \frac{\partial^2}{\partial\gamma^2}\left(\ln\frac{h_\alpha}{h_\beta}Z\right) - \frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right)\frac{\partial}{\partial\gamma}(\ln v h_\gamma)\right]$$

$$+\frac{\sqrt{\mu}}{2k_o^2 v h_\gamma h_\beta}\frac{\partial}{\partial\beta}\left\{\frac{h_\gamma}{vZ}\hat{\gamma}\cdot\nabla\times\frac{\mathbf{A}_o}{\sqrt{\epsilon}}\right\} + \frac{\sqrt{\epsilon}}{2k_o^2 v h_\gamma h_\alpha}\frac{\partial}{\partial\alpha}\left\{\frac{h_\gamma Z}{v}\hat{\gamma}\cdot\nabla\times\frac{\mathbf{B}_o}{\sqrt{\mu}}\right\}, \tag{B.12}$$

and

$$A_{2\beta} + B_{2\alpha} = -\frac{1}{2ik_o v h_\gamma}\left\{\frac{\partial}{\partial\gamma}(B_{1\alpha} + A_{1\beta}) + B_{1\alpha}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{\sqrt{\mu}}\right)\right.$$

$$\left. + A_{1\beta}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{\sqrt{\epsilon}}\right) - \frac{\sqrt{\mu}}{h_\alpha}\frac{\partial}{\partial\alpha}\left(\frac{h_\gamma B_{1\gamma}}{\sqrt{\mu}}\right) - \frac{\sqrt{\epsilon}}{h_\beta}\frac{\partial}{\partial\beta}\left(\frac{h_\gamma A_{1\gamma}}{\sqrt{\epsilon}}\right)\right\}$$

$$= -\frac{1}{2ik_o v h_\gamma}\left\{\frac{1}{2ik_o}\frac{\partial}{\partial\gamma}\left[\frac{1}{vh_\gamma}B_{o\alpha}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{h_\alpha}Z\right)\right] + \frac{B_{o\alpha}}{2ik_o v h_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{h_\alpha}Z\right)\right.$$

$$\left. \cdot\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{\sqrt{\epsilon}}\right) - B_{1\alpha}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{h_\alpha}Z\right) - \frac{\sqrt{\mu}}{h_\alpha}\frac{\partial}{\partial\alpha}\left(\frac{h_\gamma B_{1\gamma}}{\sqrt{\mu}}\right) - \frac{\sqrt{\epsilon}}{h_\beta}\frac{\partial}{\partial\beta}\left(\frac{h_\gamma A_{1\gamma}}{\sqrt{\epsilon}}\right)\right\}$$

giving

$$A_{2\beta} + B_{2\alpha} = \frac{B_{1\alpha}}{2ik_o v h_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{h_\alpha}Z\right) + \frac{B_{o\alpha}}{(2k_o v h_\gamma)^2}\left[\frac{1}{2}\left\{\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{h_\alpha}Z\right)\right\}^2\right.$$

$$\left. + \frac{\partial^2}{\partial\gamma^2}\left(\ln\frac{h_\beta}{h_\alpha}Z\right) - \frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{h_\alpha}Z\right)\frac{\partial}{\partial\gamma}(\ln v h_\gamma)\right]$$

$$-\frac{\sqrt{\mu}}{2k_o^2 v h_\gamma h_\alpha}\frac{\partial}{\partial\alpha}\left\{\frac{h_\gamma}{vZ}\hat{\gamma}\cdot\nabla\times\frac{\mathbf{A}_o}{\sqrt{\epsilon}}\right\} + \frac{\sqrt{\epsilon}}{2k_o^2 v h_\gamma h_\beta}\frac{\partial}{\partial\beta}\left\{\frac{h_\gamma Z}{v}\hat{\gamma}\cdot\nabla\times\frac{\mathbf{B}_o}{\sqrt{\mu}}\right\}. \tag{B.13}$$

It follows that to the second order in q

$$A_\alpha = A_{o\alpha} + qA_{1\alpha} + q^2 A_{2\alpha}$$

$$= B_{o\beta} + qB_{1\beta} + q^2 B_{2\beta} - \frac{q}{2ik_o v h_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right)(B_{o\beta} + qB_{1\beta})$$

$$-\frac{q^2}{(2k_o v h_\gamma)^2}\left[\frac{1}{2}\left\{\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right)\right\}^2 + \frac{\partial^2}{\partial\gamma^2}\left(\ln\frac{h_\alpha}{h_\beta}Z\right)\right.$$

$$\left.-\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right)\frac{\partial}{\partial\gamma}(\ln v h_\gamma)\right] + \frac{q^2\sqrt{\mu}}{2k_o^2 v h_\gamma h_\beta}\frac{\partial}{\partial\beta}\left\{\frac{h_\gamma}{vZ}\hat{\gamma}\cdot\nabla\times\frac{\mathbf{A}_o}{\sqrt{\epsilon}}\right\}$$

$$+\frac{q^2\sqrt{\epsilon}}{2k_o^2 v h_\gamma h_\alpha}\frac{\partial}{\partial\alpha}\left\{\frac{h_\gamma Z}{v}\hat{\gamma}\cdot\nabla\times\frac{\mathbf{B}_o}{\sqrt{\mu}}\right\}$$

so that

$$A_\alpha = K_1' B_\beta + \frac{\sqrt{\mu}}{2k_o^2 N h_\gamma h_\beta}\frac{\partial}{\partial\beta}\left\{\frac{1}{NZ}h_\gamma\hat{\gamma}\cdot\nabla\times\frac{\mathbf{A}_o}{\sqrt{\epsilon}}\right\}$$

$$+\frac{\sqrt{\epsilon}}{2k_o^2 N h_\gamma h_\alpha}\frac{\partial}{\partial\alpha}\left\{\frac{Z}{N}h_\gamma\hat{\gamma}\cdot\nabla\times\frac{\mathbf{B}_o}{\sqrt{\mu}}\right\} \qquad (B.14)$$

where

$$K_1' = 1 - \frac{1}{2ik_o N h_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right) - \frac{1}{(2k_o N h_\gamma)^2}\left[\frac{1}{2}\left\{\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right)\right\}^2\right.$$

$$\left.+\frac{\partial^2}{\partial\gamma^2}\left(\ln\frac{h_\alpha}{h_\beta}Z\right) - \frac{\partial}{\partial\gamma}\left(\ln\frac{h_\alpha}{h_\beta}Z\right)\frac{\partial}{\partial\gamma}(\ln h_\gamma N)\right], \qquad (B.15)$$

and

$$A_\beta = A_{o\beta} + qA_{1\beta} + q^2 A_{2\beta}$$

$$= -B_{o\alpha} - qB_{1\alpha} - q^2 B_{2\alpha} + \frac{q}{2ik_o v h_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{h_\alpha}Z\right)(B_{o\alpha} + qB_{1\alpha})$$

$$+\frac{q^2}{(2k_o v h_\gamma)^2}\left[\frac{1}{2}\left\{\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{h_\alpha}Z\right)\right\}^2 + \frac{\partial^2}{\partial\gamma^2}\left(\ln\frac{h_\beta}{h_\alpha}Z\right)\right.$$

$$\left.-\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{h_\alpha}Z\right)\frac{\partial}{\partial\gamma}(\ln v h_\gamma)\right] - \frac{q^2\sqrt{\mu}}{2k_o^2 v h_\gamma h_\alpha}\frac{\partial}{\partial\alpha}\left\{\frac{h_\gamma}{vZ}\hat{\gamma}\cdot\nabla\times\frac{\mathbf{A}_o}{\sqrt{\epsilon}}\right\}$$

$$+\frac{q^2\sqrt{\epsilon}}{2k_o^2 v h_\gamma h_\beta}\frac{\partial}{\partial\beta}\left\{\frac{h_\gamma Z}{v}\hat{\gamma}\cdot\nabla\times\frac{\mathbf{B}_o}{\sqrt{\mu}}\right\}$$

so that

$$A_\beta = -K_2' B_\alpha - \frac{\sqrt{\mu}}{2k_o^2 N h_\gamma h_\alpha}\frac{\partial}{\partial\alpha}\left\{\frac{1}{NZ}h_\gamma\hat{\gamma}\cdot\nabla\times\frac{\mathbf{A}_o}{\sqrt{\epsilon}}\right\}$$

$$+\frac{\sqrt{\epsilon}}{2k_o^2 N h_\gamma h_\beta}\frac{\partial}{\partial\beta}\left\{\frac{Z}{N}h_\gamma\hat{\gamma}\cdot\nabla\times\frac{\mathbf{B}_o}{\sqrt{\mu}}\right\} \qquad (B.16)$$

where

$$K_2' = 1 - \frac{1}{2ik_o N h_\gamma}\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{h_\alpha}Z\right) - \frac{1}{(2k_o N h_\gamma)^2}\left[\frac{1}{2}\left\{\frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{h_\alpha}Z\right)\right\}^2\right.$$

$$\left.+\frac{\partial^2}{\partial\gamma^2}\left(\ln\frac{h_\beta}{h_\alpha}Z\right) - \frac{\partial}{\partial\gamma}\left(\ln\frac{h_\beta}{h_\alpha}Z\right)\frac{\partial}{\partial\gamma}(\ln h_\gamma N)\right]. \qquad (B.17)$$