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A DIGEST OF HERTZ'S ELECTROMAGNETISM

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Abstract

An attempt is being made to recast a masterpiece in the theory of electromagnetism by the great scientist in modern notations and to supply the derivations of several important equations in the work. By invoking the radiation condition in his formulation, an amendment has been made to clarify a criticism addressed to that theory. Also, upon removing the constraint placed on the current distribution in the original theory we have demonstrated a complete union of Hertz's theory of electromagnetism with Maxwell's theory. The distinct and independent features of Hertz's theory have been emphasized. The importance of this theoretical work by Hertz and its significance appear to have not been fully recognized.

Introduction

The work under discussion was published by Hertz in 1884 [1], twenty years after the publication of Maxwell's theory [2], and three years before the start of his renowned experimental verification of Maxwell's theory, a monumental work in the history of science. The paper has been examined by several authors [3, 4], but the importance of this theoretical work and its significance appear to have not been fully recognized. The purpose of the present study is to recast the entire work in modern notations, to fill in some detailed steps not found in Hertz's original paper, and most important of all, to deduce some new information which can be extracted from his theory. For the convenience of modern readers, all quantities are now defined in MKS system. Gibb's notations of vector analysis are used throughout.

The Formulation

According to Hertz, the following premises are adopted in his theory:

1. The principle of the unity of electric force and that of the unity of magnetic force. The two forces which Hertz referred to correspond to the electric and the magnetic fields in modern nomenclature.

2. The principle of the conservation of energy, that of the action and reaction as applied to systems of closed circuits; that of the superposition of electric and magnetic actions, and lastly, the well-known laws of the magnetic and electromagnetic actions of closed currents and of magnets.

There are two sets of equations in the original work. The first set consists of equations representing Ampère's law and Faraday's law for fields produced by electric current. For a steady current flow with electric current density $\mathbf{J}_e$ which is a function of time Ampère's law states

$$\nabla \times \mathbf{H} = \mathbf{J}_e \tag{1}$$

then

$$\nabla \cdot \mathbf{J}_e = 0 \tag{2}$$

In free space (air)

$$\nabla \cdot (\mu_o \mathbf{H}) = 0, \tag{3}$$

so we can introduce an electric vector potential function, denoted by $\mathbf{A}_e$ such that

$$\mu_o \mathbf{H} = \nabla \times \mathbf{A}_e \tag{4}$$

then, in view of (1), we obtain

$$\nabla \times \nabla \times \mathbf{A}_e = \mu_o \mathbf{J}_e.$$

By assuming

$$\nabla \cdot \mathbf{A}_e = 0, \tag{5}$$

the previous equation can be changed to

$$\nabla^2 \mathbf{A}_e = -\mu_o \mathbf{J}_e. \tag{6}$$

By defining

$$\mathbf{A}_e = \mu_o \mathbf{I}_e, \tag{7}$$

(4) becomes

$$\mathbf{H} = \frac{1}{\mu_o} \nabla \times \mathbf{A}_e = \nabla \times \mathbf{I}_e, \tag{8}$$

and (6) can be written in the form

$$\nabla^2 \mathbf{I}_e = -\mathbf{J}_e. \tag{9}$$

The quantity $\mathbf{I}_e$ will be designated as the electric current potential. It has the dimension of $[\mathbf{I}_e]$ = [Ampère].

Hertz invoked the induction law that the time rate of change of $\mathbf{I}_e$ would yield an electric field given by

$$\mathbf{E} = -\mu_o \frac{\partial \mathbf{I}_e}{\partial t}. \tag{10}$$

In an interpretation of the origin of (10), it appears that this induction law is a consequence of Faraday's law, namely,

$$\nabla \times \mathbf{E} = -\mu_o \frac{\partial \mathbf{H}}{\partial t}. \tag{11}$$

With **H** given by (8), (11) becomes

$$\nabla \times \left(\mathbf{E} + \mu_o \frac{\partial \mathbf{I}_e}{\partial t} \right) = 0. \tag{12}$$

A particular solution of the above equation is

$$\mathbf{E} = -\mu_o \frac{\partial \mathbf{I}_e}{\partial t}, \tag{13}$$

which is the same as (10). Equations (8) and (13) are two basic equations used by Hertz.

A second set of equations were introduced by Hertz based on the concept of magnetic current, which was interpreted by him as the rate of the change of magnetization, i.e.,

$$\mathbf{J}_m = -\mu_o \frac{\partial \mathbf{M}}{\partial t}. \tag{14}$$

The negative sign was introduced in (14) so that the magnetic Ampère's law would have the same form as the electric Ampère's law. Namely,

$$\nabla \times \mathbf{E} = \mathbf{J}_m. \tag{15}$$

We designate (15) as the 'magnetic Ampère's law' a term not used by Hertz. For the electric field produced by a magnetic current

$$\nabla \cdot (\epsilon_o \mathbf{E}) = 0. \tag{16}$$

A magnetic vector potential, denoted by $\mathbf{A}_m$, can thus be introduced such that

$$\epsilon_o \mathbf{E} = \nabla \times \mathbf{A}_m. \tag{17}$$

Substituting (17) into (15), we obtain

$$\nabla \times \nabla \times \mathbf{A}_m = \epsilon_o \mathbf{J}_m. \tag{18}$$

By assuming

$$\nabla \cdot \mathbf{A}_m = 0, \tag{19}$$

(18) reduces to

$$\nabla^2 \mathbf{A}_m = -\epsilon_o \mathbf{J}_m. \tag{20}$$

Let us define a magnetic current potential, denoted by $\mathbf{I}_m$, such that

$$\mathbf{I}_m = \frac{\mathbf{A}_m}{\epsilon_o}. \tag{21}$$

Then (17) can be written in the form

$$\mathbf{E} = \nabla \times \mathbf{I}_m. \tag{22}$$

The dimension of the magnetic current potential is $[\mathbf{I}_m] = [\text{volt}]$. The induction theorem used by Hertz for the magnetic field produced by a magnetic current is

$$\mathbf{H} = \epsilon_o \frac{\partial \mathbf{I}_m}{\partial t}. \tag{23}$$

We can give an interpretation of the origin of (23), by considering a 'magnetic Faraday's law' in the form

$$\nabla \times \mathbf{H} = \epsilon_o \frac{\partial \mathbf{E}}{\partial t}. \tag{24}$$

Substituting (22) into (24) we obtain

$$\nabla \times \left(\mathbf{H} - \epsilon_o \frac{\partial \mathbf{I}_m}{\partial t} \right) = 0. \tag{25}$$

A particular solution of (25) is

$$\mathbf{H} = \epsilon_o \frac{\partial \mathbf{I}_m}{\partial t},$$

which is the same as (23). We must call attention to the fact that the term $\epsilon_o \frac{\partial \mathbf{E}}{\partial t}$ in (24) should not be viewed as the displacement current in Maxwell's theory because the fields **E** and **H** described by (22) and (23) are produced by magnetic current. There is no electric current involved in this set. Equations (22) and (23) are the dual set of (8) and (10). Let us summarize the two basic sets of equations used by Hertz to develop this theory of electromagnetism to be disclosed shortly. They are:

$$\begin{cases} \mathbf{H} = \nabla \times \mathbf{I}_e & (26) \\ \mathbf{E} = -\mu_o \frac{\partial \mathbf{I}_e}{\partial t} & (27) \end{cases}$$

$$\begin{cases} \mathbf{E} = \nabla \times \mathbf{I}_m & (28) \\ \mathbf{H} = \epsilon_o \frac{\partial \mathbf{I}_m}{\partial t}, & (29) \end{cases}$$

where $\mathbf{I}_e$ and $\mathbf{I}_m$ denote, respectively, the electric current potential and the magnetic current potential which are solutions of the equations:

$$\nabla^2 \mathbf{I}_e = -\mathbf{J}_e \qquad (30)$$

$$\nabla^2 \mathbf{I}_m = -\mathbf{J}_m. \qquad (31)$$

The explicit expressions of $\mathbf{I}_e$ and $\mathbf{I}_m$ will be discussed later.

A most remarkable feature of Hertz's theory is the casting of the electric field given by (27) into a form of (28) by an *equivalent* magnetic current potential, denoted herein by $\mathbf{I}'_{m1}$. Thus let

$$-\mu_o \frac{\partial \mathbf{I}_e}{\partial t} = \nabla \times \mathbf{I}'_{m1}. \qquad (32)$$

The prime on $\mathbf{I}'_{m1}$ is merely a notation, not a differential sign. This is the most important step in his theory. It invokes the concept of the interaction or coupling of two otherwise separate systems, the consequence of which shows that Ampère's law, both electrical and magnetic individually, is not sufficient to describe electromagnetic phenomena. The end result of his theory is the natural appearance of the electric displacement current in the electric system and the magnetic displacement current in the magnetic system. This logic and its development are quite different from that of Maxwell. In certain aspects, it is richer in its physical content. Hertz did not use the word 'equivalent' to introduce $\mathbf{I}'_{m1}$. This terminology is our suggestion. The meaning of the subscript '$m1$' will be evident later. By taking the curl of (32) we obtain

$$-\mu_o \frac{\partial}{\partial t} \nabla \times \mathbf{I}_e = \nabla \times \nabla \times \mathbf{I}'_{m1}. \tag{33}$$

By assuming

$$\nabla \cdot \mathbf{I}'_{m1} = 0, \tag{34}$$

we can reduce (33) to the form

$$\nabla^2 \mathbf{I}'_{m1} = \mu_o \frac{\partial}{\partial t} \nabla \times \mathbf{I}_e. \tag{35}$$

Now we introduce a new function $\mathbf{I}_{e1}$ such that

$$\mathbf{I}'_{m1} = -\mu_o \frac{\partial}{\partial t} \nabla \times \mathbf{I}_{e1}, \tag{36}$$

and substitute it into (35). We obtain

$$\nabla^2 \mathbf{I}_{e1} = -\mathbf{I}_e. \tag{37}$$

In (37), we treat $\mathbf{I}_e$ as a known function which is a solution of (30) and $\mathbf{I}_{e1}$ as an unknown function to be determined from (37). Once $\mathbf{I}_{e1}$ is so determined we can find $\mathbf{I}'_{m1}$ based on (36), i.e.,

$$\mathbf{I}'_{m1} = -\mu_o \frac{\partial}{\partial t} \nabla \times \mathbf{I}_{e1}. \tag{38}$$

The magnetic field due to this equivalent magnetic current potential can be cast in the form of (29). Denoting this magnetic field by $\mathbf{H}_1$, we obtain

$$\mathbf{H}_1 = \epsilon_o \frac{\partial \mathbf{I}'_{m1}}{\partial t} = -\mu_o \epsilon_o \frac{\partial^2}{\partial t^2} \nabla \times \mathbf{I}_{e1}. \tag{39}$$

In the words of Hertz this is the *corrected* part of the magnetic field which has to be added to the magnetic field due to $\mathbf{I}_e$ alone. The total magnetic field is now given by

$$\begin{aligned} \mathbf{H}_2 &= \mathbf{H} + \mathbf{H}_1 \\ &= \nabla \times \left[\mathbf{I}_e - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{I}_{e1} \right], \end{aligned} \tag{40}$$

where $c = 1/(\mu_o \epsilon_o)^{\frac{1}{2}}$ denotes the velocity of light in empty space. It is interesting to observe that in his original work this constant appears very early in his formulation because his quantities are defined in the absolute electric and magnetic units. The change of the electric current potential as seen in (40) also changes the corresponding electric field. Thus,

$$\mathbf{E}_2 = \mathbf{E} + \mathbf{E}_1 = -\mu_o \frac{\partial}{\partial t} \left[\mathbf{I}_e - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{I}_{e1} \right]. \tag{41}$$

The iteration process can be continued on indefinitely to yield a series for the total electric current potential, denoted by $\mathbf{P}_e$, given by

$$\mathbf{P}_e = \left[\mathbf{I}_e - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{I}_{e1} \cdots (-1)^n \frac{1}{c^{2n}} \frac{\partial^{2n}}{\partial t^{2n}} \mathbf{I}_{en} + \cdots \right]. \tag{42}$$

The relation between the successive electric current potentials is governed by the equation

$$\nabla^2 \mathbf{I}_{en} = -\mathbf{I}_{e(n-1)}. \tag{43}$$

It is understood that

$$\mathbf{I}_{e(-1)} = \mathbf{J}_e \tag{44}$$

$$\mathbf{I}_{e0} = \mathbf{I}_e. \tag{45}$$

The total electromagnetic field due to $\mathbf{P}_e$ is then given by

$$\mathbf{H}_e = \nabla \times \mathbf{P}_e \tag{46}$$

$$\mathbf{E}_e = -\mu_o \frac{\partial \mathbf{P}_e}{\partial t}. \tag{47}$$

By means of the same technique, it is obvious that the fields produced by the magnetic current can be developed in a similar manner. In fact, by considering (28) and (29) as the dual of (26) and (27), we can deduce the following key equations:

$$\mathbf{P}_m = \left[\mathbf{I}_m - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{I}_{m1} \cdots (-1)^n \frac{1}{c^{2n}} \frac{\partial^{2n}}{\partial t^{2n}} \mathbf{I}_{mn} + \ldots \right] \tag{48}$$

$$\mathbf{E}_m = \nabla \times \mathbf{P}_m \tag{49}$$

$$\mathbf{H}_m = \epsilon_o \frac{\partial \mathbf{P}_m}{\partial t} \tag{50}$$

The relation between the successive magnetic current potentials is given by

$$\nabla^2 \mathbf{I}_{mn} = -\mathbf{I}_{m(n-1)} \tag{51}$$

with

$$\mathbf{I}_{m(-1)} = \mathbf{J}_m \tag{52}$$

and

$$\mathbf{I}_{mo} = \mathbf{I}_m. \tag{53}$$

Based on (42) and (43), Hertz showed that $\mathbf{P}_e$ satisfies the Helmholtz wave equation. The proof is as follows. By taking the Laplacian of (42) we obtain

$$\nabla^2 \mathbf{P}_e = \nabla^2 \sum_{n=0}^{\infty} (-1)^n \frac{1}{c^{2n}} \frac{\partial^{2n}}{\partial t^{2n}} \mathbf{I}_{en}. \tag{54}$$

Since

$$\nabla^2 \mathbf{I}_{eo} = -\mathbf{J}_e, \tag{55}$$

and

$$\nabla^2 \mathbf{I}_{en} = -\mathbf{I}_{e(n-1)}, \quad n \geq 1. \tag{56}$$

(54) can be written in the form

$$\begin{aligned}
\nabla^2 \mathbf{P}_e &= -\mathbf{J}_e - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \sum_{n=1}^{\infty} (-1)^n \frac{1}{c^{2(n-1)}} \frac{\partial^{2(n-1)}}{\partial t^{2(n-1)}} I_{e(n-1)} \\
&= -\mathbf{J}_e + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \sum_{n=0}^{\infty} (-1)^n \frac{1}{c^{2n}} \frac{\partial^{2n}}{\partial t^{2n}} \mathbf{I}_{en} \\
&= -\mathbf{J}_e + \frac{1}{c^2} \frac{\partial^2 \mathbf{P}_e}{\partial t^2},
\end{aligned}$$

or

$$\nabla^2 \mathbf{P}_e - \frac{1}{c^2}\frac{\partial^2 \mathbf{P}_e}{\partial t^2} = -\mathbf{J}_e, \tag{57}$$

which is the vector Helmholtz equation.

It should be emphasized that the wave equation for $\mathbf{P}_e$ as stated by (57) was derived under the condition that $\nabla \cdot \mathbf{J}_e = 0$ and its derivation does not depend on the explicit expressions for $\mathbf{I}_{en}$. Only (43) is needed. By eliminating $\mathbf{P}_e$ between (46) and (47), with the aid of (57), one finds

$$\nabla \times \mathbf{E}_e = -\mu_o \frac{\partial \mathbf{H}_e}{\partial t} \tag{58}$$

$$\nabla \times \mathbf{H}_e = \mathbf{J}_e + \epsilon_o \frac{\partial \mathbf{E}_e}{\partial t}. \tag{59}$$

These are two of the differential equations found in Maxwell's theory now derived from Hertz's theory under the constraint $\nabla \cdot \mathbf{J}_e = 0$, hence $\nabla \cdot (\epsilon_o \mathbf{E}) = 0$. One important feature of Hertz's theory is the natural appearance of the displacement current $\epsilon_o \frac{\partial \mathbf{E}_e}{\partial t}$ in (59).

The constraint on $\mathbf{J}_e$ can be removed. But before we do that, let us complete the discussion of Hertz's work, particularly in regard to his solutions for $\mathbf{I}_{en}$ for $n \geq 0$.

Solutions for $\mathbf{P}_e$

The series expansion for $\mathbf{P}_e$ derived by Hertz as given by (42) contain the potential functions $\mathbf{I}_{en}$. An expression for $\mathbf{I}_{en}$ was found in the original paper without showing its derivation. We attempt to give a derivation of that expression using a procedure that is presumably similar to Hertz's original scheme. We start with $\mathbf{I}_{eo}$, which is a solution of the equation

$$\nabla^2 \mathbf{I}_{eo} = -\mathbf{I}_{e(-1)} = -\mathbf{J}_e. \tag{60}$$

One particular solution of $\mathbf{I}_{eo}$ in free space is given by

$$\mathbf{I}_{eo}(\mathbf{R}) = \frac{1}{4\pi}\int\int\int \frac{\mathbf{J}_e(\mathbf{R}')}{|\mathbf{R} - \mathbf{R}'|} dV', \tag{61}$$

or simply

$$\mathbf{I}_{eo} = \frac{1}{4\pi} \int \frac{\mathbf{J}_e}{r} d\tau, \tag{62}$$

which will be written in the form

$$\mathbf{I}_{eo} = \frac{1}{4\pi} \int \mathbf{J}_e K_o(r) d\tau, \tag{63}$$

with $K_o(r) = 1/r$. To find $\mathbf{I}_{e1}$ which is a solution of

$$\nabla^2 \mathbf{I}_{e1} = -\mathbf{I}_{eo}, \tag{64}$$

we let

$$\mathbf{I}_{e1} = \frac{1}{4\pi} \int \mathbf{J}_e K_1(r) d\tau. \tag{65}$$

Substituting (63) and (65) into (64), we see that the function $K_1(r)$ must satisfy the equation

$$\nabla^2 K_1(r) = -K_o(r) = -1/r,$$

or

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dK_1}{dr} \right) = -\frac{1}{r}. \tag{66}$$

The particular solution for K_1 is

$$K_1(r) = -r/2. \tag{67}$$

Following the same procedure progressively, one finds

$$\mathbf{I}_{en} = \frac{1}{4\pi} \int \mathbf{J}_e K_n(r) d\tau, \tag{68}$$

with

$$K_n(r) = (-1)^n r^{2n-1}/(2n)!. \tag{69}$$

hence

$$\begin{aligned}\mathbf{P}_e &= \sum_{n=0}(-1)^n \frac{\partial^{2n}}{\partial(ct)^{2n}}\mathbf{I}_{en} \\ &= \frac{1}{4\pi}\int \sum_{n=0} \frac{r^{2n-1}}{(2n)!}\frac{\partial^{2n}\mathbf{J}_e}{\partial(ct)^{2n}}d\tau. \end{aligned} \tag{70}$$

This is the expression for $\mathbf{P}_e$ found in Hertz's original paper, although he used the component form of $\mathbf{P}_e$ in stating his result. It can be proved by means of (68) that $\nabla \cdot \mathbf{I}_{en} = 0$ when $\nabla \cdot \mathbf{J}_e = 0$. This is a verification of the postulate stated by (34) and passed on to $\mathbf{I}_{e1}$ and subsequently to $\mathbf{I}_{en}$.

According to the wave equation for $\mathbf{P}_e$ stated by (57), there are, mathematically, two independent solutions given by

$$[\mathbf{P}_e]_r = \frac{1}{4\pi}\int \frac{\mathbf{J}_e(t-r/c)}{r}d\tau, \tag{71}$$

$$[\mathbf{P}_e]_a = \frac{1}{4\pi}\int \frac{\mathbf{J}_e(t+r/c)}{r}d\tau, \tag{72}$$

in free space. The function $[\mathbf{P}_e]_r$ is designated as the retarded potential and $[\mathbf{P}_e]_a$ as the advanced potential. The Taylor series expansions of $\mathbf{J}_e(t-r/c)$ and $\mathbf{J}_e(t+r/c)$ are given by

$$\mathbf{J}(t-r/c) = \left[\mathbf{J}_e + r\frac{\partial \mathbf{J}_e}{\partial(ct)} + \frac{r^2}{2}\frac{\partial^2 \mathbf{J}_e}{\partial(ct)^2} + \cdots\right] \tag{73}$$

$$\mathbf{J}(t+r/c) = \left[\mathbf{J}_e - r\frac{\partial \mathbf{J}_e}{\partial(ct)} + \frac{r^2}{2}\frac{\partial^2 \mathbf{J}_e}{\partial(ct)^2} - \cdots\right]. \tag{74}$$

Thus, Hertz's expression for $\mathbf{P}_e$ as stated by (70) is the arithematic mean of $[\mathbf{P}_e]_r$ and $[\mathbf{P}_e]_a$, i.e.,

$$\mathbf{P}_e = \frac{1}{2}\left\{[\mathbf{P}_e]_r + [\mathbf{P}_e]_a\right\} \tag{75}$$

This relation was first pointed out by Havas [3], after commenting on a paper by Zatkis [4] who misinterpreted Hertz's expression for $\mathbf{P}_e$. According to Havas, Hertz's solution for $\mathbf{P}_e$ is not acceptable from the physical point of view, which is certainly true. However, the 'shortcoming' can be readily remedied. In fact, the concept of retarded potential was introduced by Hertz several years later [5]. If he had pursued the matter further that shortcoming could have been easily removed.

In any event, there are two approaches by which Hertz's result can be properly modified to yield the correct answer. The missing criterion is the radiation condition which Hertz never mentioned in his paper. If we add to Hertz's expression for $\mathbf{P}_e$ a term denoted by $\mathbf{P}_o$, which is a solution of the homogeneous wave equation

$$\nabla^2 \mathbf{P}_o - \frac{1}{c^2}\frac{\partial^2 \mathbf{P}_o}{\partial t^2} = 0, \tag{76}$$

then the radiation condition can be satisfied by a proper choice of $\mathbf{P}_o$. The desired function is obviously given by

$$\mathbf{P}_o = \frac{1}{2}\left\{[\mathbf{P}_e]_r - [\mathbf{P}_e]_a\right\}. \tag{77}$$

Since both $[\mathbf{P}_e]_r$ and $[\mathbf{P}_e]_a$ satisfy (57), their difference certainly satisfies (76). Let the resultant potential function be denoted by $\mathbf{P}_{eT}$, then

$$\begin{aligned} \mathbf{P}_{eT} &= \mathbf{P}_e + \mathbf{P}_o \\ &= \frac{1}{2}\left\{[\mathbf{P}_e]_r + [\mathbf{P}_e]_a\right\} + \frac{1}{2}\left\{[\mathbf{P}_e]_r - [\mathbf{P}_e]_a\right\} \\ &= [\mathbf{P}_e]_r, \end{aligned} \tag{78}$$

which is the desired answer.

Another approach to find $\mathbf{P}_{eT}$ is more complicated but it follows closely Hertz's original analysis with a modification. Returning to the differential equation for $\mathbf{I}_{eo}$, namely,

$$\nabla^2 \mathbf{I}_{eo} = -\mathbf{I}_{e(-1)} = -\mathbf{J}_e, \tag{79}$$

we observe that the general solution for $\mathbf{I}_{eo}$ is

$$\mathbf{I}_{eo} = \frac{1}{4\pi}\int \mathbf{J}_e \left(\frac{1}{r} + c_o\right) d\tau, \tag{80}$$

where c_o is an arbitrary constant (function of time). In Hertz's treatment he let $c_o = 0$. Now if we choose c_o such that

$$c_o \mathbf{J}_e = \frac{\partial \mathbf{J}_e}{\partial(ct)}, \tag{81}$$

then

$$\mathbf{I}_{eo} = \frac{1}{4\pi} \int \left[\frac{\mathbf{J}_e}{r} + \frac{\partial \mathbf{J}_e}{\partial(ct)} \right] d\tau. \tag{82}$$

For $\mathbf{I}_{e1}$ we let

$$\mathbf{I}_{e1} = \frac{1}{4\pi} \int \left[\mathbf{J}_e K_1(r) + \frac{\partial \mathbf{J}_e}{\partial(ct)} K_1^*(r) \right]. \tag{83}$$

The functions $K_1(r)$ and K_1^* are solution of the equations

$$\nabla^2 K_1(r) \;=\; -1/r \tag{84}$$

$$\nabla^2 K_1^*(r) \;=\; -1. \tag{85}$$

The particular solutions are

$$K_1(r) \;=\; -r/2!$$

$$K_1^*(r) \;=\; -r^2/3!\,,$$

hence

$$\mathbf{I}_{e1} = \frac{-1}{4\pi} \int \left(\frac{r}{2!}\,\mathbf{J}_e + \frac{r^2}{3!}\,\frac{\partial \mathbf{J}_e}{\partial(ct)} \right) d\tau. \tag{86}$$

Similarly,

$$\mathbf{I}_{e2} = \frac{1}{4\pi} \int \left(\frac{r^3}{4!}\,\mathbf{J}_e + \frac{r^4}{5!}\,\frac{\partial \mathbf{J}_e}{\partial(ct)} \right) d\tau. \tag{87}$$

Using these modified expressions for $\mathbf{I}_{en}$ in (42), now representing $\mathbf{P}_{eT}$, we find

$$\mathbf{P}_{eT} = \frac{1}{4\pi} \int \sum_{n=0} \frac{r^{2n-1}}{n!}\,\frac{\partial^n \mathbf{J}_e}{\partial(ct)^n} d\tau, \tag{88}$$

which is the same as $[\mathbf{P}_e]_r$. It is seen that the new expression contains terms with odd derivatives of the current function, these terms represent precisely the mean value of the difference of the retarded and the advanced potentials. Of course, this approach is guided by our anticipation that the resulting potential function should represent a retarded potential. The method, however, is essientially based on Hertz's formulation with a proper modification.

Non-solenoidal Distribution of $\mathbf{J}_e$

So far we have been dealing with Hertz's theory under the constraint $\nabla \cdot \mathbf{J}_e = 0$. For non-solenoidal current distribution the equation of continuity reads:

$$\nabla \cdot \mathbf{J}_e = -\frac{\partial \rho}{\partial t}. \tag{89}$$

In view of (59) we must have

$$\nabla \cdot (\epsilon_o \mathbf{E}) = \rho, \tag{90}$$

which corresponds to Gauss law for time varying charge. With this removal of the original constraint we have obtained the complete system of Maxwell's equations from Hertz's theory of electromagnetism based on an independent method quite distinct from Maxwell's path. As far as the function for $\mathbf{P}_{eT}$ or $[\mathbf{P}_e]_r$ is concerned, the expression represented by (71), or its equivalent (88), is still valid, but $\nabla \cdot \mathbf{P}_{eT}$ is no longer vanishing. It can be shown that

$$\nabla \cdot \mathbf{P}_{eT} = \frac{1}{4\pi} \int \frac{1}{r} \frac{\partial \rho(t - r/c)}{\partial t} d\tau. \tag{91}$$

This part of the theory was investigated thoroughly by Hertz later in 1889 [5]. The consolidation of Maxwell's theory could have been established by Hertz theoretically in 1884 if he had pursued a little further based on his own model.

The modification of Hertz's theory for the magnetic current model can be executed in the same manner. The most convenient derivation is to apply the duality principle by replacing $(\mathbf{J}_e, \rho_e, \mathbf{P}_{eT}, \mathbf{E}_e, \mathbf{H}_e, \mu_o, \epsilon_o)$ in the electric model by $(\mathbf{J}_m, 0, \mathbf{P}_{mT}, -\mathbf{H}_m, \mathbf{E}_m, \epsilon_o, \mu_o)$ in the magnetic model.

In Retrospect

Several authors in the past have reviewed the paper under discussion, but it is our opinion that the value of this masterpiece was not fully recognized. It is remarkable that an alternative method was available to derive Maxwell's equations based on a quite different approach. Even though a constraint was originally imposed on the method, the fact that a retarded potential formula can be extracted from the formulation demonstrates convincingly the power of his method. As has been shown, by removing that constraint and invoking the equation of continuity and the Gauss law, the complete system of Maxwell's equations evolve from Hertz's theory. For the magnetic current model, since there is no magnetic charge, the condition $\nabla \cdot \mathbf{J}_m = 0$ is not a constraint. It is a mathematical statement of a physical law. The theory for the magnetic current model is, therefore, complete by itself provided that a homogeneous solution is added to the original magnetic current potential function. This portion of Hertz's theory appears to be not covered in Maxwell's theory. The physical insight of Hertz's work seems to be not well appreciated in the past. In a comment by the renowned physicist Max Planck [6], Hertz's derivation of Maxwell's equations is considered to be peculiar. We feel that the derivation was very brilliant, logical, and not peculiar at all. Hertz's modest conclusion of his own theory "... if the choice rests only between the usual systems of electromagnetics and Maxwell's, the latter is certainly to be preferred ..." might have casted a negative image in the eyes of later scholars.

It may be of interest to remark that forty years ago one of the present authors (C.T.T.) considered a method [7] to extend Rayleigh's theory of diffraction of electromagnetic waves by small bodies [8]. In that method, we expand $\mathbf{E}$ and $\mathbf{H}$ for a monochromatic oscillating field into two series in the form

$$\mathbf{E} = \sum_{n=0} (ik)^n \mathbf{E}_n \tag{92}$$

$$\mathbf{H} = \sum_{n=0} (ik)^n \mathbf{H}_n, \tag{93}$$

where $k = \omega(\mu_o \epsilon_o)^{\frac{1}{2}}$. According to Maxwell's theory, in free space,

$$\nabla \times \mathbf{E} = ikZ_o\mathbf{H}, \tag{94}$$

$$\nabla \times \mathbf{H} = -ik\frac{\mathbf{E}}{Z_o}, \quad Z_o = \left(\frac{\mu_o}{\epsilon_o}\right)^{\frac{1}{2}}. \tag{95}$$

By substituting (92) and (93) into (94) and (95) and equating the terms of the same power of $(ik)^n$, we obtain, for $n \geq 1$,

$$\nabla \times (Z_o\mathbf{H}_n) = -\mathbf{E}_{n-1} \tag{96}$$

$$\nabla \times \mathbf{E}_n = Z_o\mathbf{H}_{n-1}. \tag{97}$$

In particular, for $n = 1$

$$\nabla \times (Z_o\mathbf{H}_1) = -\mathbf{E}_o \tag{98}$$

$$\nabla \times \mathbf{E}_1 = Z_o\mathbf{H}_o. \tag{99}$$

In a later paper by Stevenson [9] the solution for $\mathbf{E}_n$, $\mathbf{H}_n$ for $n \geq 1$ are found systematically by an interative method based on a known pair of solution for $\mathbf{E}_o$ and $\mathbf{H}_o$. The similarity of the approaches between the works of Stephenson/Tai and that of Hertz is quite evident. However, Hertz's theory is more profound and authorative. Both of these authors were not aware of Hertz's iterative method. In fact, even Lord Rayleigh did not quote Hertz's theory relating a quasi-static field with a dynamic field or a wave theory. The scientific community had done a great injustice in not fully recognizing the value of this work which probably had prompted Hertz to search vigorously for the experimental evidence in verifying Maxwell's theory, now confirmed theoretically by an independent approach.

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