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By

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FOREWORD

This report, 1492-1-F, "Duct Studies," was prepared by The University of Michigan Radiation Laboratory, Department of Electrical Engineering, under the direction of Professor Ralph E. Hiatt, Head of the Radiation Laboratory, and was written under Purchase Order 504-855029 to the Northrop Corporation, Norair Division.

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ABSTRACT

Problems in the high frequency regime of electromagnetic scattering by ducts and open waveguides whose walls obey an impedance boundary condition are investigated. In general, the impedance boundary condition means that the walls are absorbing electromagnetic energy. Three structures of particular interest are treated: the semi-infinite parallel plane waveguide, the semi-infinite circular cylinder, and the rectangular duct of finite length. When the surfaces are perfectly conducting, exact solutions for the first two structures are available, and these solutions are expanded asymptotically to yield term-by-term comparisons with the corresponding results of ray diffraction theory. When the surfaces are absorbing, exact solutions are not available, but the ray treatment is applied to advantage. In the case of the rectangular duct, the main problem is to determine the interior waveguide modes. This modal problem is non-separable due to the assumed impedance boundary conditions; nevertheless, information can be obtained by means of an asymptotic analysis applied to the transcendental eigenvalue equation of infinite order.

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I
INTRODUCTION

Problems of radiation and scattering by open waveguides with ideal boundary conditions have been studied extensively in the literature, and indeed, for some semi-infinite guides, exact solutions have been made available. In sharp distinction, however, comparatively few investigators have directed attention to such problems when the boundaries are not ideal -- for example, when the boundaries are nonconducting or absorbing in the electromagnetic case, or nonrigid in the acoustic case. Nevertheless, many practical applications arise for nonideal boundaries, and it would seem desirable to investigate the effects of these boundaries on the radiation and scattering properties of the structures involved. In this report, therefore, we shall investigate problems of electromagnetic scattering by ducts and open waveguides whose walls may be characterized by an impedance boundary condition, which in general means that the walls are absorbing electromagnetic energy. The wavelength of the incident field is assumed to be small compared to the obstacle dimensions, so that high frequency techniques may be utilized to advantage; however, in many cases the results remain useful even for wavelengths as large as the relevant dimensions of the scatterer.

Since we are assuming the validity of the high-frequency regime, it is natural to adopt ray-optical techniques to calculate the effects of diffraction due to the edges forming the mouth of an open waveguide or duct. In this case, the ray treatment is intimately dependent on known solutions for plane wave diffraction by a half plane, and in particular, we desire solutions for a half plane with arbitrary face impedances. For incidence normal to the edge of the half plane, the desired solutions are available from Maliuzhinets (1958, 1960); however, for oblique incidence on the edge, the solution is available only if the impedances on both sides of the half plane are identical (see, e.g., Bowman and Weston, 1968b). This means that when oblique incidence is involved in the ray treatment, both the interior and exterior surfaces of the open waveguide or duct must display closely similar electrical properties. The half plane solutions are summarized in Chapter II.

In the succeeding chapters, three structures of particular interest are treated: the semi-infinite parallel plane waveguide, the semi-infinite circular cylinder, and finally, the rectangular duct of finite length. When the surfaces are perfectly conducting, exact solutions for the first two structures are available. These configurations therefore represent important prototypes for testing the utility of the ray procedure, and also provide insight into the various diffraction mechanisms occurring at the mouth. For both of these canonical bodies we shall provide some new asymptotic representations of the scattered field which yield term-by-term comparisons with the corresponding ray-optical results. It is found that the ray method yields complete agreement for the primary diffraction, along with the first and second interaction contributions, but that for each successive interaction after the second, the ray optical result underestimates the asymptotic result obtained from the exact solution. The higher order interaction contributions are, however, small in magnitude, and the ray treatment is therefore expected to yield quite accurate results. When the surfaces are absorbing, exact solutions are not available, but the ray treatment is straightforward in view of the perfectly conducting results.

For the rectangular duct of finite length, the ray method is applied to obtain the primary diffraction fields generated at the mouth of the duct, although the simple ray expressions must be suitably modified to account for the finite lengths of the diffracting edges. From this information, the energy launched into the waveguide modes within the duct is to be determined. The modal problem, however, is nonseparable due to the assumed impedance boundary conditions; nevertheless, the modes may be determined by means of an asymptotic analysis applied to the transcendental eigenvalue equation of infinite order. These modes obey a general orthogonality condition that leads to the determination of the conversion coefficients which couple the induced aperture fields to the waveguide modes in the duct. The scattered field due to modes reflected from the closed termination of the duct are obtained by means of a Kirchhoff approximation over the aperture formed by the mouth.

II

THE HALF PLANE

2.1 Normal Incidence

Before we consider problems of diffraction by ducts and open-ended waveguides we must first examine the pertinent half plane results. To fix our notation, we shall employ natural units with free-space constants ϵ_o, μ_o set equal to unity and suppress the harmonic time dependence $\exp(-i\omega t)$ throughout. A plane electromagnetic wave of unit amplitude is assumed incident at an angle θ_o to the semi-infinite screen as shown in Fig. 2-1. The screen is assumed to be comprised of material of such a kind as to make the total tangential field components satisfy the following impedance boundary condition on the surface ($\hat{n}$ is the unit outward normal to the surface)

$$\underline{E} - (\hat{n} \cdot \underline{E})\,\hat{n} = \eta \hat{n} \wedge \underline{H} \quad , \tag{2.1}$$

where $\eta = \eta_1$ on the upper surface and $\eta = \eta_2$ on the lower surface. The face impedances η_1 and η_2 are complex constants whose real parts, because of energy considerations, must be non-negative. Further, the surface impedances are assumed to account for the presence of thin layers of highly refractive absorbing materials applied as a coating on a perfectly conducting half plane. Although the validity of the impedance boundary condition near an edge of the diffracting structure is then open to question (Weston, 1963), one nevertheless expects some general features of the scattering process to emerge when the overall effect of an absorber coating is treated in terms of a constant surface impedance (see, e.g., Bowman, 1967). It may be noted that if the surface is to be considered as perfectly conducting, then the corresponding impedance must be set identically equal to zero. In particular, a special case of interest would be that of a perfectly conducting

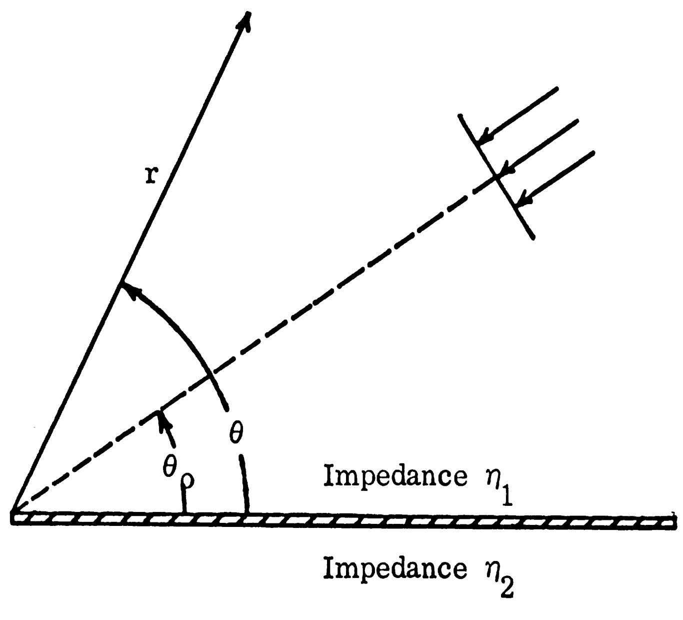

FIG. 2-1: PLANE WAVE INCIDENCE ON ABSORBING HALF PLANE WITH TWO FACE IMPEDANCES.

half plane coated on one side with radar absorbing material.

The exact solution for plane-wave scattering by an absorbing half plane with two face impedances is available from Maliuzhinets (1958, 1960), who treated the more general problem of diffraction by a wedge with arbitrary face impedances. For the application of ray-optics techniques, we shall require the asymptotic far-field form of the solution. This result, obtained by means of a steepest descent approximation to the exact contour integral solution, may be written in the form

$$u^d \sim \frac{1}{2i} e^{-i\pi/4} \frac{e^{ikp}}{(2\pi kp)^{1/2}} U(\theta, \theta_o) \tag{2.2}$$

representing a cylindrical wave emanating from the edge of the semi-infinite screen and produced by a plane wave incident at an angle θ_o to the screen (see Fig. 2-1, where the geometrical configuration is different from that used by Maliuzhinets). The total exact solution, u, which is a function of two complex quantities α_1 and α_2, is derived by imposing boundary conditions of the third kind,

$$\frac{1}{ikr} \frac{\partial u}{\partial \theta} \pm u \cos \alpha_{1,2} = 0 \qquad (\theta = 0, 2\pi) \; , \tag{2.3}$$

on the faces of the semi-infinite screen. The quantities α_1 and α_2 are constants whose real parts lie in the closed interval $[0, \pi/2]$. For observation angles θ bounded away from the geometric optics boundaries (i.e. for $\theta \neq \pi \pm \theta_o$), the amplitude factor $U(\theta, \theta_o)$ appearing in (2.2) is given by

$$U(\theta,\theta_o) = \frac{\sin(\theta_o/2)}{\psi(\pi-\theta_o)} \left\{ \frac{\psi(-\theta)}{\sin(\theta/2)+\cos(\theta_o/2)} + \frac{\psi(2\pi-\theta)}{\sin(\theta/2)-\cos(\theta_o/2)} \right\} \tag{2.4}$$

where the auxiliary function $\psi(\beta)$ is expressed in terms of a special meromorphic function $\psi_\pi(\beta)$ by the product

$$\psi(\beta) = \psi_\pi(\beta+\pi+\alpha_1)\,\psi_\pi(\beta+\pi-\alpha_1)\,\psi_\pi(\beta-\pi-\alpha_2)\,\psi_\pi(\beta-\pi+\alpha_2) \tag{2.5}$$

and $\psi_\pi(\beta)$ has the representation

$$\psi_\pi(\beta) = \exp\left\{-\frac{1}{8\pi}\int_0^\beta \frac{\pi\sin\nu - 2\sqrt{2}\,\pi\,\sin(\nu/2) + 2\nu}{\cos\nu}\,d\nu\right\} . \tag{2.6}$$

Maliuzhinets (1960) mentions that the special function $\psi_\pi(\beta)$, along with its generalization for the wedge problem, has been tabulated by M. P. Sacharowa, although no reference to the literature is given. The important analytical properties of the functions $\psi_\pi(\beta)$ and $\psi(\beta)$ are given in Appendix A.

An examination of the boundary conditions (2.1) and (2.3) indicates that the scalar function u may be employed to represent the half-plane solution for either of the two fundamental electromagnetic polarizations. Thus, in the case of H polarization, the function u represents the z component of the magnetic field H_z, provided we make the identifications

$$\eta_1 \equiv \cos\alpha_1^H , \qquad \eta_2 \equiv \cos\alpha_2^H , \tag{2.7}$$

while in the case of E polarization, it represents E_z under the identifications

$$\eta_1 \equiv 1/\cos\alpha_1^E , \qquad \eta_2 \equiv 1/\cos\alpha_2^E . \tag{2.8}$$

This property is a result of the natural duality of Maxwell's equations and the impedance boundary condition (2.1) under the transformation $E \to H$, $H \to -E$, $\eta \to 1/\eta$. The affix E or H is attached according as the boundary conditions describe E- or H- polarization states, and in the sequel, all functions will be written with such an affix whenever this polarization distinction is necessary.

An alternative representation for $U(\theta,\theta_o)$ may be obtained by successive use of the identity given in (A.10); in particular, after some trigonometric reduction we obtain the following expression, valid so long as $\theta \neq \pi \pm \theta_o$:

$$u(\theta,\theta_o) = \frac{P(\theta)\,P(\theta_o)}{\cos\theta+\cos\theta_o}\left\{\left(\frac{1}{2} - \cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2}\right) + \right.$$

$$\left. + \frac{1}{\sqrt{2}}\left(\cos\frac{\theta}{2} + \cos\frac{\theta_o}{2}\right)\left(\cos\frac{\alpha_2}{2} - \cos\frac{\alpha_1}{2}\right) + \cos\frac{\theta}{2}\cos\frac{\theta_o}{2}\right\} \tag{2.9}$$

with

$$P(\theta) = \frac{\sin(\theta/2)}{\psi(\pi-\theta)}\ \psi_\pi^4\,(\pi/2)\ . \tag{2.10}$$

The quantity $\psi_\pi(\pi/2)$ is given in (A.9). The expression in (2.9) is manifestly symmetrical in the two variables θ, θ_o, thereby confirming the principle of reciprocity: $U(\theta,\theta_o) = U(\theta_o, \theta)$. For $\alpha_1 = \alpha_2 = (\pi/2)$, corresponding to a perfect conductor with H polarization, we have upon employing (A.10) and (A.15)

$$\psi(\pi-\theta) = \frac{1}{2}\left[\psi_\pi(\frac{\pi}{2})\right]^4 \sin\frac{\theta}{2}\ ,$$

and therefore (2.9) reduces to

$$U(\theta,\theta_o) = \frac{4\cos(\theta/2)\cos(\theta_o/2)}{\cos\theta+\cos\theta_o}\ . \tag{2.11}$$

On the other hand, for $\alpha_1 = \alpha_2 \to \pm\, i\,\infty$, corresponding to a perfect conductor with E polarization, one finds asymptotically

$$\psi(\pi-\theta) \sim \frac{1}{4}\left[\psi_\pi(\frac{\pi}{2})\right]^4 e^{|\alpha_1|/2}\ ,$$

so that (2.9) now reduces to

$$U(\theta,\theta_o) = -\frac{4\sin(\theta/2)\sin(\theta_o/2)}{\cos\theta+\cos\theta_o} \quad . \tag{2.12}$$

These limiting expressions are in accord with known results for a perfectly conducting half plane.

In order to explore the effects of multiple interaction, such as the interaction existing between the two edges forming the mouth of a semi-infinite parallel plane waveguide, we shall also require an asymptotic expression for the half-plane solution along the ray-optics reflection boundary $\theta = \pi - \theta_o$. It may be shown from the Maliuzhinets (1958, 1960) contour integral solution that for $\theta = \pi - \theta_o$ the total field behaves as

$$u \sim e^{ikr\cos 2\theta_o} - \frac{1}{2}\,\frac{\cos\alpha_1 - \sin\theta_o}{\cos\alpha_1 + \sin\theta_o}\, e^{ikr} + O\left(\frac{1}{\sqrt{kr}}\right) \tag{2.13}$$

provided $0 < \theta_o < \pi$. For incidence from the lower half space $\pi < \theta_o < 2\pi$, the same expression obtains except that α_1 is replaced by α_2. The result in (2.13) is hardly surprising physically and shows that far along the reflection boundary the scattered field is given by the perfectly conducting result multiplied by the infinite flat plane reflection coefficient. However, to apply the ray-optical procedure to the semi-infinite parallel plane waveguide, we need to know the field generated along the reflection boundary by a line source located at a finite distance from the semi-infinite screen. Rather than solve the line source problem for an absorbing half plane, we shall draw upon an analogy (in view of Eq. 2.13) with the known result for an isotropic line source over a perfectly conducting half plane. Thus, for an incident field given by

$$u^i = H_o^{(1)}(k|\underline{r} - \underline{r_o}|) \sim \sqrt{\frac{2}{\pi k|\underline{r}-\underline{r_o}|}}\, e^{ik|\underline{r}-\underline{r_o}| - i\frac{\pi}{4}} \tag{2.14}$$

it is physically reasonable to take

$$u^s \sim -\frac{1}{2} \frac{\cos\alpha_1 - \sin\theta_o}{\cos\alpha_1 + \sin\theta_o} \sqrt{\frac{2}{\pi k(r+r_o)}}\, e^{ik(r+r_o) - i\frac{\pi}{4}} \tag{2.15}$$

as the scattered field far along the reflection boundary $\theta = \pi - \theta_o$, where $0 < \theta_o < \pi$. For $\pi < \theta_o < 2\pi$, replace α_1 by α_2 in (2.15).

2.2 Oblique Incidence

The exact solution for the problem of a plane electromagnetic wave incident at an oblique angle to the edge of an absorbing half plane was first presented by Senior (1959), who solved the coupled Wiener-Hopf integral equations which determine the currents induced on the surface of the sheet. As pointed out by Senior, the standard technique (Clemmow, 1951) used for the derivation of three-dimensional solutions from known two-dimensional solutions in the case of perfectly conducting cylindrical structures fails to yield correct results when the diffracting structures are imperfectly conducting. This is due to the coupling of TE and TM modes in the presence of dielectric or absorbing cylindrical bodies (see, e.g., Wait, 1955). On the basis of Senior's (1959) investigation, however, Williams (1960) subsequently developed a generalized technique by which the oblique incidence solution may be deduced directly from the known scalar solutions for a half plane with nonzero surface impedance. These treatments of diffraction by a half plane assume that the screen is characterized by a single constant surface impedance η, whereas for the purposes of our investigation it would be desirable to consider a semi-infinite sheet with two face impedances, say $\eta = \eta_1$ on the upper surface and $\eta = \eta_2$ on the lower surface. A particular case of interest would then be that of a perfectly conducting half plane coated on one side with radar absorbing material. Since the scalar solutions for plane-wave scattering by a half plane with arbitrary face impedances are available from Maliuzhinets (1958, 1960), we attempted (Bowman and Weston, 1968b) to carry out the Williams (1960) procedure to obtain the corresponding three-dimensional solutions for oblique incidence. Unfortunately, it came to light that the technique fails except in the case for which the impedances on both sides of the half plane are identical. The Williams technique appears to be intimately dependent upon the inherent symmetry of the problem with regard to the plane defined by the diffracting screen. In the following, therefore, it is necessary to assume $\eta_1 = \eta_2 = \eta$.

For later application to the rectangular duct, we are interested in the field diffracted into the far zone by the half plane. The diffraction screen will be taken to occupy the half plane $y=0$, $x>0$. In cylindrical coordinates (ρ, ϕ, z), as well as in spherical coordinates (r, θ, ϕ), the upper and lower surfaces of the half plane are prescribed by $\phi=0$ and $\phi=2\pi$, respectively. The primary excitation will be due to a plane wave of unit intensity propagating from the direction $\theta=(\pi/2)-\theta_o$, $\phi=\phi_o$:

$$\underline{E}^i = (-\hat{x}\cos\phi_o\sin\theta_o - \hat{y}\sin\phi_o\sin\theta_o + \hat{z}\cos\theta_o)\, e^{-ikS} \quad , \tag{2.16}$$

$$\underline{H}^i = (-\hat{x}\sin\phi_o + \hat{y}\cos\phi_o)\, e^{-ikS} \quad ,$$

with

$$S = x\cos\phi_o\cos\theta_o + y\sin\phi_o\cos\theta_o + z\sin\theta_o \quad . \tag{2.17}$$

The incident field is thus taken to be a pure TM mode (no H_z^i component); however, the other polarization case (TE incident mode) presents no difficulty in its treatment, and may indeed be obtained upon making use of the natural duality of Maxwell's equations and the impedance boundary conditions. When $\theta_o = 0$, the plane wave is incident normally to the edge of the semi-infinite screen.

Away from the optical boundaries $\phi = \pi \pm \phi_o$, the diffracted field in the far zone is (Bowman and Weston, 1968b)

$$\underline{E}^d = -\frac{e^{-i\frac{\pi}{4}}}{4\cos\theta_o}\,\frac{e^{-ikz\sin\theta_o + ik\rho\cos\theta_o}}{(2\pi k\rho\cos\theta_o)^{1/2}}\,\frac{\underline{E}(\phi,\phi_o)}{\cos^2\phi\cos^2\theta_o + \sin^2\theta_o}\,,$$

$$\underline{H}^d = \frac{-e^{-i\frac{\pi}{4}}}{4\cos\theta_o}\,\frac{e^{-ikz\sin\theta_o + ik\rho\cos\theta_o}}{(2\pi k\rho\cos\theta_o)^{1/2}}\,\frac{\underline{H}(\phi,\phi_o)}{\cos^2\phi\cos^2\theta_o + \sin^2\theta_o}\,,$$

(2.18)

where

$$\underline{E}(\phi,\phi_o) = \hat{x}\left[\sin\theta_o\cos\phi\, G_1(\phi,\phi_o) + \sin\phi\, G_2(\phi,\phi_o)\right] +$$

$$+\hat{y}\left[\sin\theta_o\sin\phi\, G_1(\phi,\phi_o) - \cos\phi\, G_2(\phi,\phi_o)\right] + \hat{z}\cos\theta_o G_1(\phi,\phi_o)\,,$$

(2.19)

$$\underline{H}(\phi,\phi_o) = \hat{x}\left[\sin\phi\, G_1(\phi,\phi_o) - \sin\theta_o\cos\phi\, G_2(\phi,\phi_o)\right] -$$

$$-\hat{y}\left[\cos\phi\, G_1(\phi,\phi_o) + \sin\theta_o\sin\phi\, G_2(\phi,\phi_o)\right] - \hat{z}\cos\theta_o G_2(\phi,\phi_o)\,.$$

(2.20)

The functions $G_{1,2}(\phi,\phi_o)$ are given by

$$G_1(\phi,\phi_o) = ik\cos\theta_o\cos\phi\left[\frac{\cos\phi}{k}\,\frac{Q_e^E(\phi,\phi_o) + Q_o^E(\phi,\phi_o)}{\cos\phi + \cos\phi_o} + AQ_e^E(\phi,\phi_o) + \right.$$

$$\left. + DQ_o^E(\phi,\phi_o)\right] - ik\cos\theta_o\sin\theta_o\sin\phi\left[\frac{\sin\theta_o\sin\phi_o}{k}\,\frac{Q_o^H(\phi,\phi_o) + Q_e^H(\phi,\phi_o)}{\cos\phi + \cos\phi_o} + \right.$$

$$\left. + BQ_o^H(\phi,\phi_o) + CQ_e^H(\phi,\phi_o)\right],$$ (2.21)

$$G_2(\phi,\phi_o) = ik\cos\theta_o\sin\theta_o\sin\phi\left[\frac{\cos\phi}{k}\,\frac{Q_e^E(\phi,\phi_o)+Q_o^E(\phi,\phi_o)}{\cos\phi+\cos\phi_o} + AQ_e^E(\phi,\phi_o) + \right.$$

$$\left. + DQ_o^E(\phi,\phi_o)\right] + ik\cos\theta_o\cos\phi\left[\frac{\sin\theta_o\sin\phi_o}{k}\,\frac{Q_o^H(\phi,\phi_o)+Q_e^H(\phi,\phi_o)}{\cos\phi+\cos\phi_o} + \right.$$

$$\left. + BQ_o^H(\phi,\phi_o) + CQ_e^H(\phi,\phi_o)\right] , \tag{2.22}$$

where

$$Q_e(\phi,\phi_o) = -P(\phi)\,P(\phi_o)\cos\left[\alpha(\theta_o)\right] ,$$

$$Q_o(\phi,\phi_o) = P(\phi)\,P(\phi_o)\,2\cos(\phi/2)\cos(\phi_o/2) \tag{2.23}$$

with

$$P(\phi) = \frac{\sin(\phi/2)}{\psi(\pi-\phi)}\,\psi_\pi^4(\pi/2) , \tag{2.24}$$

and where the constants A and B are determined by

$$\left[\frac{Q_e^E(\frac{\pi}{2}+i\nu,\phi_o)}{Q_o^H(\frac{\pi}{2}+i\nu,\phi_o)}\,\frac{\tan\theta_o}{k(-i\tan\theta_o+\cos\phi_o)} - \frac{Q_e^E(\frac{\pi}{2}-i\nu,\phi_o)}{Q_o^H(\frac{\pi}{2}-i\nu,\phi_o)}\,\frac{\tan\theta_o}{k(i\tan\theta_o+\cos\phi_o)}\right]$$

$$= \frac{2\tan\theta_o\sin\theta_o\sin\phi_o}{ik(\tan^2\theta_o+\cos^2\phi_o)} + \frac{1}{i}\left[\frac{Q_e^E(\frac{\pi}{2}+i\nu,\phi_o)}{Q_o^H(\frac{\pi}{2}+i\nu,\phi_o)} + \frac{Q_e^E(\frac{\pi}{2}-i\nu,\phi_o)}{Q_o^H(\frac{\pi}{2}-i\nu,\phi_o)}\right]A , \tag{2.25}$$

$$\left[\frac{Q_o^H(\frac{\pi}{2}+i\nu,\phi_o)}{Q_e^E(\frac{\pi}{2}+i\nu,\phi_o)}\,\frac{\sin\theta_o\sin\phi_o}{k(-i\tan\theta_o+\cos\phi_o)} + \frac{Q_o^H(\frac{\pi}{2}-i\nu,\phi_o)}{Q_e^E(\frac{\pi}{2}-i\nu,\phi_o)}\,\frac{\sin\theta_o\sin\phi_o}{k(i\tan\theta_o+\cos\phi_o)}\right]$$

$$= \frac{-2\tan\theta_o\cos\phi_o}{k(\tan^2\theta_o+\cos^2\phi_o)} - \left[\frac{Q_o^H(\frac{\pi}{2}+i\nu,\phi_o)}{Q_e^E(\frac{\pi}{2}+i\nu,\phi_o)} + \frac{Q_o^H(\frac{\pi}{2}-i\nu,\phi_o)}{Q_e^E(\frac{\pi}{2}-i\nu,\phi_o)}\right]B . \tag{2.26}$$

In (2.25) and (2.26) the parameter ν is related to the angle of incidence θ_o through the relations

$$\sinh \nu = \tan \theta_o, \quad \cosh \nu = \sec \theta_o \ . \tag{2.27}$$

Finally, the constants C and D are obtained from (2.25) and (2.26) upon replacing Q_e^E by Q_o^E and Q_o^H by Q_e^H, in which instance A $\rightarrow$ D and B $\rightarrow$ C. In the above, the affix E or H is attached to indicate where the equations

$$\frac{1}{\eta} = \cos\theta_o \cos \alpha^E(\theta_o), \quad \eta = \cos\theta_o \cos \alpha^H(\theta_o) \tag{2.28}$$

should be used to determine the constants $\alpha^E(\theta_o)$ or $\alpha^H(\theta_o)$ from the impedance η. For normal incidence ($\theta_o = 0$), the equations in (2.28) are equivalent to those in (2.7) and (2.8) with $\alpha_1 = \alpha_2 = \alpha(0)$.

As a check, let us consider the perfectly conducting case which corresponds to $\alpha^H = (\pi/2)$ and $\alpha^E \rightarrow \pm i \infty$. From Appendix A we employ (A.26) and (A.27), along with (2.24) to find

$$\begin{aligned} Q_e^E(\mu, \phi_o) &= -8 \sin\frac{\mu}{2} \sin\frac{\phi_o}{2}, \quad & Q_o^E(\mu, \phi_o) &= 0, \\ Q_o^H(\mu, \phi_o) &= 8 \cos\frac{\mu}{2} \cos\frac{\phi_o}{2}, \quad & Q_e^H(\mu, \phi_o) &= 0 \ . \end{aligned} \tag{2.29}$$

Only the constants A and B need be evaluated, and with the help of

$$\frac{Q_e^E(\frac{\pi}{2} \pm i\nu, \phi_o)}{Q_o^H(\frac{\pi}{2} \pm i\nu, \phi_o)} = -e^{\pm i\theta_o} \tan\frac{\phi_o}{2}, \tag{2.30}$$

we obtain from (2.25) and (2.26), after some trigonometric reduction,

$$A = 0, \qquad B = \frac{-\sin\theta_o \sin\phi_o}{k\,(1+\cos\phi_o)} \quad . \tag{2.31}$$

With these values of the constants we finally have

$$G_1(\phi,\phi_o) = i\cos\theta_o(\cos^2\phi + \sin^2\theta_o \sin^2\phi)\,\frac{-8\sin(\phi/2)\sin(\phi_o/2)}{\cos\phi + \cos\phi_o}\,,$$

$$G_2(\phi,\phi_o) = 0, \tag{2.32}$$

which leads to agreement with the results for a perfectly conducting half plane.

Equation (2.18) is fundamentally important for the application of ray-optical techniques to problems involving planar structures that are coated with radar absorbing material, and in particular, the diffraction coefficient derived from (2.18) will play an important role in estimating the fields generated at the mouth of a fully-lined rectangular duct. For this purpose we specialize to the case of edge-on incidence, as illustrated in Fig. 2-2, since only principal-plane incidence will be considered in our treatment of the rectangular duct.

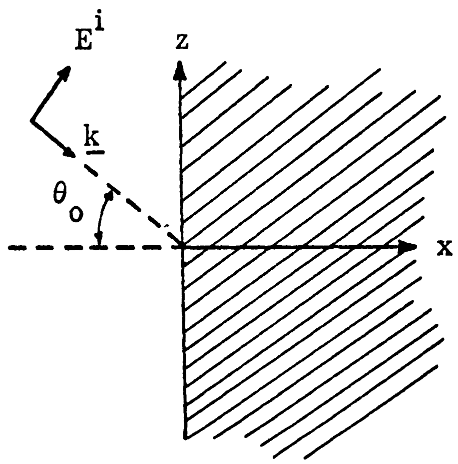

FIG. 2-2: EDGE-ON OBLIQUE INCIDENCE ON ABSORBING HALF PLANE.

For edge-on incidence ($\phi_o = \pi$) the incoming field becomes

$$\underline{E}^i = (\hat{x}\sin\theta_o + \hat{z}\cos\theta_o)\, e^{ik(x\cos\theta_o - z\sin\theta_o)}$$

$$\underline{H}^i = -\hat{y}\, e^{ik(x\cos\theta_o - z\sin\theta_o)} \quad . \tag{2.33}$$

Now, from (2.24) we see that $Q_o \to 0$ as $\phi_o \to \pi$, and from (2.25) and (2.26) it is evident that great care must be taken in deriving the correct expressions for $\underline{E}(\phi, \pi)$, $\underline{H}(\phi, \pi)$. Further, since we are ultimately interested only in the aperture fields for a rectangular duct, we need only consider the half plane diffraction fields along $\phi = \pi/2$, $3\pi/2$. There is much algebra involved, although simplification is achieved by noting the following relationships

$$P(\tfrac{\pi}{2} + i\nu) = \frac{4\cos\left(\frac{\pi}{4} - \frac{i\nu}{2}\right)\psi_\pi^2\left(\alpha_1 - i\nu + \frac{\pi}{2}\right)\psi_\pi^2\left(\alpha_1 + i\nu - \frac{\pi}{2}\right)}{\psi_\pi^4\left(\frac{\pi}{2}\right)\left(\cos\frac{\alpha}{2} + \cos\frac{i\nu}{2}\right)\left(\cos\frac{\alpha}{2} + \sin\frac{i\nu}{2}\right)}$$

$$P\left(\frac{\pi}{2} + i\nu\right) P\left(\frac{\pi}{2} - i\nu\right) = \frac{4}{1 + \cos\theta_o\cos\alpha}\,, \quad P\left(\frac{\pi}{2}\right) = \frac{2}{(1+\cos\alpha)^{1/2}} \quad . \tag{2.34}$$

It turns out that the diffraction fields of interest may be written as:

For $\phi = (\pi/2)$:

$$\underline{E}^d = \frac{i}{2} e^{-i\pi/4} \frac{e^{-ikz\sin\theta_o + ik\rho\cos\theta_o}}{(2\pi k\rho\cos\theta_o)^{1/2}} \left[\hat{x}\widetilde{M}(\theta_o) - (\hat{y}\sin\theta_o + \hat{z}\cos\theta_o) M(\theta_o)\right] ,$$

$$\underline{H}^d = \frac{i}{2} e^{-i\pi/4} \frac{e^{-ikz\sin\theta_o + ik\rho\cos\theta_o}}{(2\pi k\rho\cos\theta_o)^{1/2}} \left[-\hat{x} M(\theta_o) - (\hat{y}\sin\theta_o + \hat{z}\cos\theta_o)\widetilde{M}(\theta_o)\right] . \tag{2.35}$$

For $\phi = (3\pi/2)$:

$$\underline{E}^d = \frac{i}{2} e^{-i\pi/4} \frac{e^{-ikz\sin\theta_o + ik\rho\cos\theta_o}}{(2\pi k\rho\cos\theta_o)^{1/2}} \left[\hat{x}\widetilde{M}(\theta_o) + (\hat{y}\sin\theta_o - \hat{z}\cos\theta_o) M(\theta_o)\right],$$

$$\underline{H}^d = \frac{1}{2} e^{-i\pi/4} \frac{e^{-ikz\sin\theta_o + ik\rho\cos\theta_o}}{(2\pi k\rho\cos\theta_o)^{1/2}} \left[\hat{x} M(\theta_o) - (\hat{y}\sin\theta_o - \hat{z}\cos\theta_o) \widetilde{M}(\theta_o)\right], \tag{2.36}$$

where the M functions are

$$M(\theta_o) = \frac{\cos^2\theta_o\, \psi_\pi^8(\pi/2)\, P^E(\pi)}{(1+\eta)(\cos\theta_o + \eta)^{1/2} \left[e^{-i\frac{\theta_o}{2}} K(\theta_o) + e^{i\frac{\theta_o}{2}} K(-\theta_o)\right]},$$

$$\widetilde{M}(\theta_o) = \frac{i P^E(\pi) \left[e^{i\frac{\theta_o}{2}} K(\theta_o) - e^{-i\frac{\theta_o}{2}} K(-\theta_o)\right]}{\left[\eta\cos\theta_o(\eta\cos\theta_o + 1)\right]^{1/2} \left[e^{-i\frac{\theta_o}{2}} K(\theta_o) + e^{i\frac{\theta_o}{2}} K(-\theta_o)\right]}, \tag{2.37}$$

with

$$K(\theta_o) = \frac{\psi_\pi^2(\alpha^H - i\nu + \frac{\pi}{2})\, \psi_\pi^2(\alpha^H + i\nu - \frac{\pi}{2})\, \psi_\pi^2(\alpha^E + i\nu + \frac{\pi}{2})\, \psi_\pi^2(\alpha^E - i\nu - \frac{\pi}{2})}{\left(\cos\frac{\alpha^H}{2} + \cos\frac{i\nu}{2}\right)\left(\cos\frac{\alpha^H}{2} + \sin\frac{i\nu}{2}\right)\left(\cos\frac{\alpha^E}{2} + \cos\frac{i\nu}{2}\right)\left(\cos\frac{\alpha^E}{2} - \sin\frac{i\nu}{2}\right)}. \tag{2.38}$$

For a perfect conductor $\eta \to 0$, in which case

$$K(\theta_o) \to \frac{1}{2}\sqrt{\eta}\ \psi_\pi^8(\pi/2)\cos\theta_o\, e^{-i\frac{\theta_o}{2}} \quad ,$$

$$P^E(\theta_o) \to 2\,(2\eta\cos\theta_o)^{1/2} \quad ,$$

and $M(\theta_o) = 2\sqrt{2}$, $\widetilde{M}(\theta_o) = 0$. This checks with known results. On the other hand, for a perfect magnetic conductor, $M(\theta_o) = \widetilde{M}(\theta_o) = 0$; hence, there is no scattered field in the case of edge-on incidence, as expected. Finally, for normal incidence ($\theta_o = 0$) we have $M(0) = U^E(3\pi/2, \pi)$, $\widetilde{M}(0) = 0$, in agreement with Maliuzhinets' (1958, 1960) solution.

III

THE SEMI-INFINITE PARALLEL PLANE WAVEGUIDE

3.1 Perfectly Conducting Case

The problem of diffraction by two parallel half planes constitutes a valuable prototype for testing the utility of the ray optical method. On the one hand, exact solutions based on the Wiener-Hopf technique are widely known in the literature, while on the other hand, the ray method can be applied systematically and without difficulty. Ray-optical techniques and their relation to canonical problems with parallel plane geometries involving perfect conductors have been discussed extensively by Felsen and his collaborators (Yee and Felsen, 1967a, b; Felsen and Yee, 1968a, b; Yee, Felsen and Keller, 1968; Yee and Felsen, 1969). Their work has been centered on the study of reflection and radiation of waveguide modes incident on the open end of a waveguide, and they have reported that the ray method yields remarkably accurate results even at small ka values. In this section we shall discuss the plane-wave scattering case. Some new asymptotic developments of the exact solution are presented which yield a term-by-term comparison with the corresponding ray-optical results.

We consider an E-polarized plane wave incident from direction θ_o on a pair of perfectly conducting parallel half planes with geometry as illustrated in Fig. 3-1. By combining the results of Vajnshtejn (1954) and Clemmow (1951), we can write the far field as obtained from the exact solution in the following manner:

$$E^s \sim -\frac{1}{4\pi i}\sqrt{\frac{2\pi}{kr}}\, e^{i(kr-\frac{\pi}{4})}\, \frac{2\sin(\theta/2)\sin(\theta_o/2)}{\cos\theta+\cos\theta_o}\left\{\left(1+e^{2ika\sin\theta}\right)\left(1+e^{2ika\sin\theta_o}\right)e^{V+V_o}+\right.$$

$$\left.+\left(1-e^{2ika\sin\theta}\right)\left(1-e^{2ika\sin\theta_o}\right)e^{U+U_o}\right\}e^{-ika(\sin\theta+\sin\theta_o)}\,, \qquad (3.1)$$

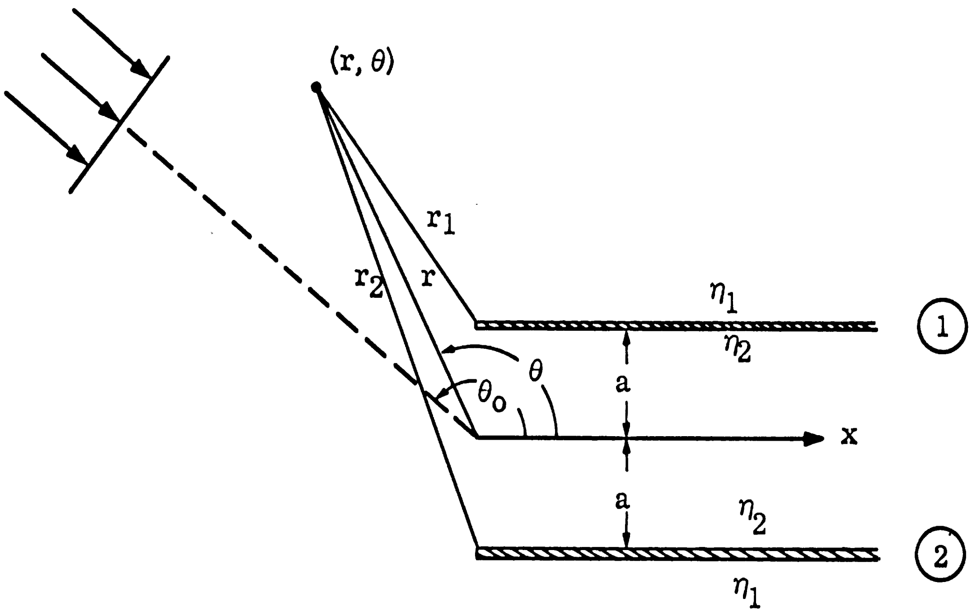

FIG. 3-1: PLANE WAVE INCIDENT ON A SEMI-INFINITE PARALLEL PLANE WAVEGUIDE WITH TWO SURFACE IMPEDANCES. IN THE PERFECTLY CONDUCTING CASE $\eta_1 = \eta_2 = 0$.

where we have assumed $\cos\theta < 0$, $\cos\theta_o < 0$, and where

$$V = \frac{1}{2\pi i}\int_{\frac{\pi}{2}-i\infty}^{-\frac{\pi}{2}+i\infty} \log\left(1+e^{2ika\cos\tau}\right)\frac{\cos\tau\,d\tau}{\sin\tau-\cos\theta},$$

$$U = \frac{1}{2\pi i}\int_{\frac{\pi}{2}-i\infty}^{-\frac{\pi}{2}+i\infty} \log\left(1-e^{2ika\cos\tau}\right)\frac{\cos\tau\,d\tau}{\sin\tau-\cos\theta}. \tag{3.2}$$

The quantities V_o, U_o are obtained from (3.2) upon replacing θ by θ_o. Now for $ka >> 1$, a steepest descent approximation yields (Vajnshtejn, 1954):

$$V \sim \frac{1}{2\pi i}\int_{-\infty}^{\infty} \log\left(1+e^{2ika-\frac{t^2}{2}}\right)\frac{dt}{t-\sqrt{2ka}\,e^{i\pi/4}\cos\theta}. \tag{3.3}$$

We expand the logarithm under the assumption that k has a vanishingly small positive imaginary part,

$$\log\left(1+e^{2ika-\frac{t^2}{2}}\right) = -\sum_{m=1}^{\infty}\frac{(-)^m e^{2imka}}{m}\,e^{-\frac{mt^2}{2}},$$

and integrate term by term to obtain

$$V \sim -\frac{1}{2}\,\mathrm{sgn}(\cos\theta)\sum_{m=1}^{\infty}\frac{(-)^m}{m}\,e^{2imka}\,G(\sqrt{m}\,w), \tag{3.4}$$

where $w = \sqrt{ka/2}\,|\cos\theta|$ and

$$G(\xi) = \frac{2}{\sqrt{\pi}}\, e^{-i2\xi^2} \int_{(1-i)\xi}^{\infty} e^{-\mu^2}\, d\mu . \tag{3.5}$$

For values of ka and θ such that $\sqrt{ka/2}\,|\cos\theta| >> 1$, we have

$$G(w) \sim \frac{e^{i\frac{\pi}{4}}}{\sqrt{2\pi}\, w} , \tag{3.6}$$

and consequently

$$e^{V} \sim 1 - \frac{e^{i\frac{\pi}{4}}}{\sqrt{4\pi ka}\cos\theta} \sum_{m=1}^{\infty} \frac{(-)^m}{m^{3/2}}\, e^{2imka} + O\left(\frac{1}{ka}\right) . \tag{3.7}$$

Similarly,

$$e^{U} \sim 1 - \frac{e^{i\frac{\pi}{4}}}{\sqrt{4\pi ka}\cos\theta} \sum_{m=1}^{\infty} \frac{1}{m^{3/2}}\, e^{2imka} + O\left(\frac{1}{ka}\right) . \tag{3.8}$$

When (3.7) and (3.8) are employed in (3.1), we obtain

$$\begin{aligned} E^{s} \sim \sqrt{\frac{2}{\pi kr}}\, e^{ikr+i\frac{\pi}{4}} \frac{2\sin(\theta/2)\sin(\theta_0/2)}{\cos\theta + \cos\theta_o} \Bigg\{ & \cos\left[ka(\sin\theta + \sin\theta_o)\right] + \\ & + \frac{e^{i\frac{\pi}{4}}}{\sqrt{4\pi ka}} \left(\frac{1}{\cos\theta} + \frac{1}{\cos\theta_o}\right) \sum_{m=0}^{\infty} \frac{e^{i(2m+1)2ka}}{(2m+1)^{3/2}} \cos\left[ka(\sin\theta - \sin\theta_o)\right] - \\ & - \frac{e^{i\frac{\pi}{4}}}{\sqrt{4\pi ka}} \left(\frac{1}{\cos\theta} + \frac{1}{\cos\theta_o}\right) \sum_{m=1}^{\infty} \frac{e^{i(2m)2ka}}{(2m)^{3/2}} \cos\left[ka(\sin\theta + \sin\theta_o)\right] + O\left(\frac{1}{ka}\right) \Bigg\} , \end{aligned} \tag{3.9}$$

which is the desired expansion of the scattered field. In order to interpret (3.9) physically, we now direct attention to the ray-optical calculation.

The primary diffraction due to the open-ended parallel plane waveguide is obtained in the ray method by neglecting the mutual interaction process that takes place between the two straight edges forming the waveguide mouth. In this first approximation, then, the two edges behave as independent semi-infinite screens, each in the absence of the other and each excited by the incident field alone. Noting that the distances r_1, r_2 are approximately $r \mp a \sin\theta$ as $r \to \infty$, and taking into account the phase of the incident plane wave at the two edges, we may at once write the scattered field from the parallel plane waveguide as a superposition of the two primary edge waves:

$$E^s \sim \sqrt{\frac{2}{\pi kr}}\, e^{ikr+i\frac{\pi}{4}} \frac{2\sin(\theta/2)\sin(\theta_o/2)}{\cos\theta+\cos\theta_o} \cos\left[ka(\sin\theta+\sin\theta_o)\right] \qquad (3.10)$$

provided $(\pi/2) < (\theta, \theta_o) \leq \pi$.

This approximation is valid so long as $ka >> 1$, so that the interaction between the edges is weak. However, for more closely spaced waveguides, it becomes necessary to include mutual interaction effects in order to improve the accuracy of the ray optical procedure. To calculate the secondary diffraction contribution, for example, each edge is considered to undergo an excitation due to a cylindrical wave emanating from the other edge in addition to the excitation provided by the incident plane-wave field. The cylindrical wave from the edge is assumed to emanate from an equivalent isotropic line source whose strength is chosen to provide the correct diffraction field in the direction toward the other edge. Knowing how the field of such a line source is diffracted by the half plane representing the other edge, one may write down the first interaction contribution to the far scattered field.

The physical situation becomes somewhat more complicated for the successive interactions past the first. This is due essentially to the fact that each edge lies precisely along the ray optical reflection boundary of the half plane corresponding to the other edge. Consider, for example, an isotropic line source of unit strength located at the position of edge ① in the presence of the half plane ②. The field scattered back toward the line source -- along the reflection boundary of plate ② -- is then given by

$$E^{BS} \sim -\frac{1}{2}\sqrt{\frac{2}{\pi k(r_2+2a)}}\; e^{ik(r_2+2a)-i\frac{\pi}{4}}, \tag{3.11}$$

and this cylindrical wave appears to emanate from an image line source located at a distance 2a behind the half plane ②. From this example, it may be seen that each successive interaction between the edges of the waveguide gives rise to a new image source and that an infinite number of image sources is required to account for the mulitple interactions. By adding the contributions due to the image sources, retaining in the process only terms through $O(1/\sqrt{ka})$, we obtain the following ray-optical result:

$$\begin{aligned} E^{s} \sim \sqrt{\frac{2}{\pi kr}}\; e^{ikr+i\frac{\pi}{4}} \cdot \frac{2\sin(\theta/2)\sin(\theta_o/2)}{\cos\theta+\cos\theta_o} \Bigg\{ \cos\left[ka(\sin\theta+\sin\theta_o)\right] + \\ + \frac{e^{i\frac{\pi}{4}}}{\sqrt{4\pi ka}}\left(\frac{1}{\cos\theta}+\frac{1}{\cos\theta_o}\right)\sum_{m=0}^{\infty}\frac{e^{i(2m+1)2ka}}{2^{2m}\sqrt{2m+1}}\cos\left[ka(\sin\theta-\sin\theta_o)\right] - \\ - \frac{e^{i\frac{\pi}{4}}}{\sqrt{4\pi ka}}\left(\frac{1}{\cos\theta}+\frac{1}{\cos\theta_o}\right)\sum_{m=1}^{\infty}\frac{e^{i(2m)2ka}}{2^{2m-1}\sqrt{2m}}\cos\left[ka(\sin\theta+\sin\theta_o)\right]+O\left(\frac{1}{ka}\right)\Bigg\}, \end{aligned} \tag{3.12}$$

which is to be compared with (3.9).

The two results in (3.9) and (3.12) differ because of the numerical factors in the denominators of the summands. It is seen, however, that the primary diffraction, along with the first and second interaction contributions, are in complete agreement. For each successive interaction after the second, the ray-optical result (3.12) underestimates the asymptotic result (3.9). In order to obtain a more accurate result from the ray method, which is based on diffraction of isotropic line sources, modifications are required in order to take into account the fact that the equivalent line sources are not isotropic. This is not necessary, however, because the exact edge interaction functions tell us immediately how to modify the ray-optical interaction functions, at least in the case of perfectly conducting surfaces.

3.2 Absorbing Case

Armed with the results of Section 3.1, we can now deal with the problem of scattering by an absorbing parallel plane waveguide, for which, unfortunately, an exact solution is not available. The geometry of the problem is illustrated in Fig. 3-1 where, as shown, the interior surfaces of the waveguide are governed by a constant surface impedance η_2, while on the exterior surfaces an impedance η_1 is prescribed. A plane wave is assumed incident upon the structure from the direction θ_o, where $(\pi/2) < \theta_o \leq \pi$.

We shall employ the ray-optical procedure to calculate the field scattered into the far zone by the edges forming the mouth of the waveguide. We make use of (2.2) to calculate the primary diffraction, which takes the form

$$u^s \sim \frac{1}{4\pi i}\sqrt{\frac{2\pi}{kr}}\, e^{i(kr-\frac{\pi}{4})}\left\{e^{-ika(\sin\theta+\sin\theta_o)}U(\theta,\theta_o)+e^{ika(\sin\theta+\sin\theta_o)}U(2\pi-\theta,2\pi-\theta_o)\right\} \tag{3.13}$$

where $(\pi/2) < (\theta,\theta_o) \leq \pi$. It may be noted that the function $U(2\pi-\theta, 2\pi-\theta_o)$ properly accounts for the fact that the lower plate ② is illuminated on the η_2 side rather than on the η_1 side as is the case for the upper plate ① . The edge interaction contributions are calculated just as in Section 3.1 except that in place of (3.11) the following equation (in view of Eq. 2.15) is employed:

$$u^{BS} \sim -\frac{1}{2}\,\frac{\cos\alpha_2 - 1}{\cos\alpha_2 + 1}\sqrt{\frac{2}{\pi k(r_2+2a)}}\; e^{ik(r_2+2a) - i\frac{\pi}{4}} \quad . \tag{3.14}$$

The final ray-optical expression for the far scattered field is

$$u^{s} \sim \frac{1}{4\pi i}\sqrt{\frac{2\pi}{kr}}\, e^{i(kr-\frac{\pi}{4})} \Bigg\{ e^{-ika(\sin\theta+\sin\theta_o)}\, U(\theta,\theta_o) + e^{ika(\sin\theta+\sin\theta_o)}\, U(2\pi-\theta, 2\pi-\theta_o) -$$

$$- \frac{e^{i\frac{\pi}{4}}}{\sqrt{4\pi ka}} \sum_{m=0}^{\infty} \frac{R^{2m} e^{i(2m+1)2ka}}{2^{2m+1}\sqrt{2m+1}} \Bigg[e^{-ika(\sin\theta-\sin\theta_o)}\, U(\theta, \tfrac{3\pi}{2})\, U(\tfrac{3\pi}{2}, 2\pi-\theta_o) +$$

$$+ e^{ika(\sin\theta-\sin\theta_o)}\, U(2\pi-\theta, \tfrac{3\pi}{2})\, U(\tfrac{3\pi}{2}, \theta_o) \Bigg] +$$

$$+ \frac{e^{i\frac{\pi}{4}}}{\sqrt{4\pi ka}} \sum_{m=1}^{\infty} \frac{R^{2m-1} e^{i(2m)2ka}}{2^{2m}\sqrt{2m}} \Bigg[e^{-ika(\sin\theta+\sin\theta_o)}\, U(\theta, \tfrac{3\pi}{2})\, U(\tfrac{3\pi}{2}, \theta_o) +$$

$$+ e^{ika(\sin\theta+\sin\theta_o)}\, U(2\pi-\theta, \tfrac{3\pi}{2})\, U(\tfrac{3\pi}{2}, 2\pi-\theta_o) \Bigg] + O\!\left(\frac{1}{ka}\right) \Bigg\} \tag{3.15}$$

where

$$R = \frac{\cos\alpha_2 - 1}{\cos\alpha_2 + 1} \quad . \tag{3.16}$$

It may be noted that each successive interaction is reduced in magnitude not only by the numerical factors appearing in the denominators of the summands, but also by the reflection coefficient R which for good absorbers is very close to zero. The higher interaction terms are thus negligible in the absorbing case, although in the perfectly conducting case they may play a more important role.

IV
THE SEMI-INFINITE CIRCULAR CYLINDER

4.1 Perfectly Conducting Case

The Wiener-Hopf technique has also been applied to yield exact solutions to the important problems of scattering and radiation from a hollow semi-infinite circular cylinder. This configuration therefore constitutes another important prototype for testing the application of the ray method. Levine and Schwinger (1948) (see also Jones, 1952, 1964; Morse and Feshbach, 1953; Noble, 1958) investigated the radiation and reflection of sound waves in an open-ended cylindrical tube, while Vajnshtejn (1954) treated the problem in more detail for both acoustic and electromagnetic radiation. The reflection and transmission properties of electromagnetic waves (H_{11} mode) in an open-ended circular pipe has also been studied intensively by Iijima (1952) although his report does not seem to be readily available. The problem of scattering of electromagnetic plane waves by a semi-infinite circular tube was treated by Pearson (1953) for axial incidence and by Bowman (1963) for general incidence; these results may be found in the exhaustive review of Einarsson et al (1966). Ray-optical analyses of modal reflection in an open-ended circular waveguide have been carried out for sound waves by Felsen and Yee (1968c) and for electromagnetic waves by Yee and Felsen (1969). These authors have also provided numerical comparisons of the ray-optical results with data computed from the exact solutions. Again, the ray method is shown to be capable of high accuracy even at the lower frequencies including the dominant mode regime. We shall direct attention here to the plane-wave scattering problem and present some new asymptotic developments. As in Section 3.1, these results provide a term-by-term comparison with the ray-optical expressions.

The physical configuration is illustrated in Fig. 4-1 where, in terms of cylindrical coordinates (ρ, ϕ, z), the semi-infinite tube is located at ρ=a,

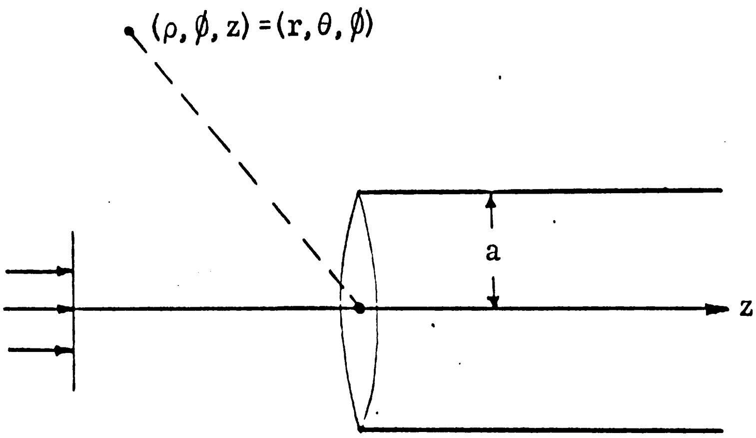

FIG. 4-1: PLANE WAVE INCIDENCE ON SEMI-INFINITE CIRCULAR CYLINDER.

$z > 0$. The incoming plane wave is propagating along the z axis with spatial variation prescribed by

$$\underline{E}^i = \hat{x}\, e^{ikz} \quad . \tag{4.1}$$

Part of this wave will propagate along the axis as a set of modes within the tube and part will be scattered into the space surrounding the tube. In terms of spherical coordinates (r, θ, ϕ), the field scattered into the far zone can be written in the form (Bowman, 1963)

$$\begin{aligned} E^s_\theta = H^s_\phi &= \frac{e^{ikr}}{r}\, f(\theta,\phi) \quad , \\ E^s_\phi = -H^s_\theta &= \frac{e^{ikr}}{r}\, g(\theta,\phi) \quad , \end{aligned} \tag{4.2}$$

where the angular distribution functions are given by

$$f(\theta,\phi) = \frac{\cos\phi\, J_1(ka\sin\theta)\, L(ka)}{2k\cos(\theta/2)\, M^2(ka)\, L(-ka\cos\theta)(1-\Delta^2)} \quad , \tag{4.3}$$

$$g(\theta,\phi) = \frac{a \sin\phi \, J_1'(ka\sin\theta) \, M(ka)}{\sin(\theta/2) \, M^2(ka) \, M(-ka\cos\theta)(1-\Delta^2)} \quad , \tag{4.4}$$

with

$$\Delta = \frac{1}{2ka} \frac{L(ka)}{M(ka)} \quad . \tag{4.5}$$

The functions $L(\omega)$, $M(\omega)$ are factorization functions analytic and non zero in the upper complex ω plane, and are defined by

$$\begin{aligned} L(\omega) \, L(-\omega) &= \pi \lambda J_1(\lambda) H_1^{(1)}(\lambda) \quad , \\ M(\omega) \, M(-\omega) &= \pi \lambda J_1'(\lambda) H_1^{(1)'}(\lambda) \quad , \end{aligned} \tag{4.6}$$

where

$$\lambda = \sqrt{(ka)^2 - \omega^2} \, , \qquad \mathrm{Im}\lambda > 0 \text{ for } \mathrm{Im}\,k > 0 \, . \tag{4.7}$$

The imaginary part of k is assumed to vanish in the final formulae.

Explicit expressions for the factorization functions have been given by Vajnshtejn (1954), who also derived their asymptotic approximations. In particular, for $ka \gg 1$ and $\cos\theta < 0$, we have

$$L(-ka\cos\theta) = e^U \, , \tag{4.8}$$

where U is given asymptotically by (Vajnshtejn, 1954)

$$U \sim \frac{1}{2\pi i} \int_{-\infty}^{\infty} \log\left(1 - e^{2iq - \frac{t^2}{2}}\right) \frac{dt}{t + \sqrt{2ka} \, e^{i\pi/4} \cos\theta} \tag{4.9}$$

with

$$q = ka - \frac{\pi}{4} + \frac{3}{8ka} \quad . \tag{4.10}$$

The Method of Section 3.1 may now be employed to obtain the following alternative asymptotic representation:

$$L(-ka\cos\theta) \sim 1 + \frac{e^{i\pi/4}}{\sqrt{4\pi ka}\cos\theta} \sum_{m=1}^{\infty} (-i)^m m^{-3/2} e^{2imka} + O\left(\frac{1}{ka}\right) \qquad (4.11)$$

provided $\sqrt{ka/2}\,|\cos\theta| \gg 1$. In similar fashion we obtain

$$M(-ka\cos\theta) \sim 1 + \frac{e^{i\pi/4}}{\sqrt{4\pi ka}\cos\theta} \sum_{m=1}^{\infty} i^m m^{-3/2} e^{2imka} + O\left(\frac{1}{ka}\right) . \qquad (4.12)$$

The assumption $\cos\theta \leq 0$ means that we are confining our attention to observation points in the backward half-space $z < 0$.

When the immediately preceding results are employed in (4.2) through (4.4), the field scattered into the backward half-space is found to take the form

$$E_\theta^s \sim \frac{e^{ikr}}{2kr} \frac{J_1(ka\sin\theta)\cos\phi}{\cos(\theta/2)} \left\{ 1 + \frac{e^{i\pi/4}}{\sqrt{\pi ka}} \sum_{m=1}^{\infty} i^m m^{-3/2} e^{2imka} \times \left[1 - (-)^m \sec\theta\cos^2(\theta/2)\right] \right\} , \qquad (4.13)$$

$$E_\phi^s \sim \frac{a\,e^{ikr}}{r} \frac{J_1'(ka\sin\theta)\sin\phi}{\sin(\theta/2)} \left\{ 1 - \frac{e^{i\pi/4}}{\sqrt{\pi ka}} \frac{\sin^2(\theta/2)}{\cos\theta} \sum_{m=1}^{\infty} i^m m^{-3/2} e^{2imka} \right\} . \qquad (4.14)$$

For $ka\sin\theta \gg 1$ the scattered field is thus

$$\underline{E}^s \sim \hat{\phi}\, \frac{e^{ikr}}{r}\left(\frac{2a}{\pi k \sin\theta}\right)^{1/2} \frac{\cos(ka\sin\theta - \frac{\pi}{4})\sin\phi}{\sin(\theta/2)} \left\{1 - \frac{e^{i\pi/4}}{\sqrt{\pi ka}} \times \right.$$

$$\left. \times\, \frac{\sin^2(\theta/2)}{\cos\theta} \sum_{m=1}^{\infty} i^m\, m^{-3/2}\, e^{2imka} \right\} , \tag{4.15}$$

whereas, for backscattering ($\theta=\pi$)

$$\underline{E}^{BS} \sim -\hat{x}\, \frac{a\, e^{ikr}}{2r} \left\{1 + \frac{e^{i\pi/4}}{\sqrt{\pi ka}} \sum_{m=1}^{\infty} i^m\, m^{-3/2}\, e^{2imka}\right\} . \tag{4.16}$$

It is interesting to note that the high-frequency backscattering cross section is

$$\sigma^{BS} \sim \pi a^2 + O\left(\frac{1}{\sqrt{ka}}\right) , \tag{4.17}$$

which is just the area of the mouth. The result obtained by Ross (1967) for backscattering along the axis is apparently in error.

It is a relatively straightforward task to provide a ray-optics description of the scattering process for axial incidence. For observation at (θ, ϕ) bounded away from the backscattering direction, the points of diffraction (or scattering centers) on the rim of the open tube are located at ϕ and $\phi + \pi$. This means that we can begin with the parallel plane result in (3.12) specialized to the case $\theta_o = \pi$,

$$E^s \sim -\sqrt{\frac{2}{\pi kr}}\, e^{ikr + i\pi/4}\, \frac{e^{ika\sin\theta} + e^{-ika\sin\theta}}{2\sin(\theta/2)} \left\{1 - \right.$$

$$\left. - \frac{e^{i\pi/4}}{\sqrt{\pi ka}} \frac{\sin^2(\theta/2)}{\cos\theta}\left[\sum_{m=1}^{\infty} \frac{e^{i(2m)2ka}}{2^{2m-1}(2m)^{1/2}} - \sum_{m=0}^{\infty} \frac{e^{i(2m+1)2ka}}{2^{2m}(2m+1)^{1/2}}\right]\right\} , \tag{4.18}$$

and suitably modify this result to account for the axial caustic of the circular tube and for the divergence of rays due to the curvature of the rim. Each time a ray traverses the axial caustic, its phase must be retarded by the amount $\pi/2$; therefore, the summand in the first summation in (4.18) requires a factor $(-i)^{2m}$, while that in the second summation takes a factor $(-i)^{2m+1}$. Furthermore, the term $e^{ika\sin\phi}$ which is due to the diffraction point at $\phi + \pi$ must be multiplied by $(-i)$ since rays diffracted at $\phi + \pi$ pass through the caustic to arrive at (θ, ϕ). The curvature of the rim is taken into account by incorporating an overall divergence factor $(a/r \sin\theta)^{1/2}$, which is the same for both diffraction points under consideration (see e.g. Felsen and Yee, 1968c). Finally, the incident polarization is

$$\hat{x} = -\hat{\phi}\sin\phi + \hat{\rho}\cos\phi$$

and only the $\hat{\phi}$ component contributes since it is the component tangential to the edge at ϕ and $\phi + \pi$. (The field scattered by the normal component is identically zero for edge-on incidence.) When these modifications are applied to (4.18), the ray-optical result for the semi-infinite circular cylinder becomes (for $ka\sin\theta \gg 1$)

$$\underline{E}^s \sim \hat{\phi}\, \frac{e^{ikr}}{r} \left(\frac{2a}{\pi k \sin\theta}\right)^{1/2} \frac{\cos(ka\sin\theta - \frac{\pi}{4})\sin\phi}{\sin(\theta/2)} \left\{ 1 - \frac{e^{i\pi/4}}{\sqrt{\pi ka}} \frac{\sin^2(\theta/2)}{\cos\theta} \times \right. \tag{4.19}$$

$$\left. \times \sum_{m=1}^{\infty} \frac{i^m e^{2imka}}{2^{m-1}\sqrt{m}} \right\} ,$$

in agreement with (4.15) up to and including the m=2 term. Complete agreement would have been obtained if we had started with (3.9) rather than (3.12).

In the backscattering case ($\theta=\pi$) a slightly modified ray method is required because all points on the rim contribute to the backscattered field. It is simplest to integrate the contributions around the rim à la Siegel (1959).

The final ray-optical result is then

$$\underline{E}^{BS} \sim \hat{x}\,\frac{a\,e^{ikr}}{2r}\left\{1+\frac{e^{i\pi/4}}{\pi ka}\sum_{m=1}^{\infty}\frac{i^m\,e^{2imka}}{2^{m-1}\sqrt{m}}\right\} \tag{4.20}$$

in agreement with (4.16) up to and including the m=2 term. Again, complete agreement would have been obtained by starting with the exact parallel plane result.

In the case of off-axis plane wave excitation of the semi-infinite hollow cylinder, both the exact solution and the ray-optics description are considerably more complicated. Let the incident plane wave be

$$\underline{E}^{i} = (\hat{x}\cos\alpha\cos\beta + \hat{y}\sin\alpha + \hat{z}\cos\alpha\sin\beta)\, e^{ik(x\sin\beta - z\cos\beta)}\,, \tag{4.21}$$

so that the incident wave is propagating parallel to the (x, z) plane from the direction $\theta = \beta$, $\phi = \pi$. The angle α determines the state of polarization of the wave. If the observation angle is outside the domain of reflected cylindrical waves, that is

$$\theta > \pi - \beta\,,$$

then the far-zone scattered field is comprised solely of the diffraction contribution and is given by (4.2) with (Bowman, 1963)

$$f(\theta,\phi) = \frac{a\cos\alpha}{\cos(\theta/2)\cos(\beta/2)}\sum_{n=-\infty}^{\infty}\frac{e^{in\phi}\,J_n(ka\sin\theta)\,J_n(ka\sin\beta)}{L_n(-ka\cos\theta)L_n(-ka\cos\beta)}\times$$

$$\times\left[\frac{2\cos^2(\theta/2)\cos^2(\beta/2)}{\cos\theta+\cos\beta}+\frac{\Delta_n^2}{1-\Delta_n^2}\right] - \tag{4.22}$$

$$-\frac{ia\sin\alpha}{\cos(\theta/2)\sin(\beta/2)}\sum_{n=-\infty}^{\infty}\frac{e^{in\phi}\,J_n(ka\sin\theta)J_n'(ka\sin\beta)}{L_n(-ka\cos\theta)M_n(-ka\cos\beta)}\;\frac{\Delta_n}{1-\Delta_n^2}\,,$$

$$g(\theta,\phi) = \frac{a\sin\alpha}{\sin(\theta/2)\sin(\beta/2)} \sum_{n=-\infty}^{\infty} \frac{e^{in\phi} J_n'(ka\sin\theta) J_n'(ka\sin\beta)}{M_n(-ka\cos\theta) M_n(-ka\cos\beta)} \times$$

$$\times \left[\frac{2\sin^2(\theta/2)\sin^2(\beta/2)}{\cos\theta + \cos\beta} - \frac{\Delta_n^2}{1-\Delta_n^2} \right] + \tag{4.23}$$

$$+ \frac{i a\cos\alpha}{\sin(\theta/2)\cos(\beta/2)} \sum_{n=-\infty}^{\infty} \frac{e^{in\phi} J_n'(ka\sin\theta) J_n(ka\sin\beta)}{M_n(-ka\cos\theta) L_n(-ka\cos\beta)} \frac{\Delta_n}{1-\Delta_n^2} ,$$

where

$$\Delta_n = \frac{n}{2ka} \frac{L_n(ka)}{M_n(ka)} . \tag{4.24}$$

The factorization functions $L_n(\omega)$, $M_n(\omega)$ are analytic in the upper complex ω plane and are defined by

$$\begin{aligned} L_n(\omega) L_n(-\omega) &= \pi\lambda\, J_n(\lambda) H_n^{(1)}(\lambda) , \\ M_n(\omega) M_n(-\omega) &= \pi\lambda\, J_n'(\lambda) H_n^{(1)'}(\lambda) , \end{aligned} \tag{4.25}$$

with λ as in (4.7)

For simplification, we shall first restrict our consideration to observation points in the plane of symmetry — the (x, z) plane — and further, we shall consider only source and observation angles such that $\cos\beta < 0$ and $\cos\theta < 0$. In this case the factorization functions approach unity as $ka \to \infty$, and the leading terms for the angular distribution functions are

$$\begin{Bmatrix} f(\theta,\pi) \\ f(\theta,0) \end{Bmatrix} \sim a\cos\alpha \frac{2\cos(\theta/2)\cos(\beta/2)}{\cos\theta + \cos\beta} \sum_{n=-\infty}^{\infty} \begin{Bmatrix} (-1)^n \\ 1 \end{Bmatrix} J_n(ka\sin\theta) J_n(ka\sin\beta) , \tag{4.26}$$

$$\begin{Bmatrix} g(\theta,\pi) \\ g(\theta,0) \end{Bmatrix} \sim a\sin\alpha \, \frac{2\sin(\theta/2)\sin(\beta/2)}{\cos\theta+\cos\beta} \sum_{n=-\infty}^{\infty} \begin{Bmatrix} (-1)^n \\ 1 \end{Bmatrix} J_n'(ka\sin\theta) J_n'(ka\sin\beta) \; . \tag{4.27}$$

The summations immediately above can be carried out explicitly by means of the addition theorems for Bessel functions. We obtain

$$\begin{Bmatrix} f(\theta,\pi) \\ f(\theta,0) \end{Bmatrix} \sim a\cos\alpha \, \frac{2\cos(\theta/2)\cos(\beta/2)}{\cos\theta+\cos\beta} \, J_o\left[ka(\sin\theta \pm \sin\beta)\right] \; , \tag{4.28}$$

$$\begin{Bmatrix} g(\theta,\pi) \\ g(\theta,0) \end{Bmatrix} \sim \pm a\sin\alpha \, \frac{2\sin(\theta/2)\sin(\beta/2)}{\cos\theta+\cos\beta} \, J_1'\left[ka(\sin\theta \pm \sin\beta)\right] \; . \tag{4.29}$$

In the case of backscattering $(\theta = \beta, \; \phi = \pi)$ the outgoing field is

$$\underline{E}^{BS} \sim \frac{a\, e^{ikr}}{r\cos\beta} \Big[\hat{y}\sin\alpha\sin^2(\beta/2) \, J_1'(2ka\sin\beta) -$$

$$- (\hat{x}\cos\beta + \hat{z}\sin\beta)\cos\alpha\cos^2(\beta/2) \, J_o(2ka\sin\beta)\Big] \; . \tag{4.30}$$

For axial incidence $(\beta = \pi)$ agreement with the leading term in (4.16) is obtained (now the incident field is polarized in the $\hat{y}$ direction). Moreover, for $ka\sin\beta \gg 1$

$$\underline{E}^{BS} \sim \frac{e^{ikr}}{r\cos\beta} \left(\frac{a}{\pi k\sin\beta}\right)^{1/2} \cos\left(2ka\sin\beta - \frac{\pi}{4}\right) \Big[\hat{y}\sin\alpha\sin^2(\beta/2) -$$

$$- (\hat{x}\cos\beta + \hat{z}\sin\beta)\cos\alpha\cos^2(\beta/2)\Big] \; , \tag{4.31}$$

which is in agreement with the ray-optical calculation (see e.g. Ross, 1967).

Further insight is gained by examining the ray structure for general observation locations. If $\cos\beta < 0$, $\cos\theta < 0$ then at least two diffracted rays pass through each observation point in the far zone. These rays correspond to the existence of at least two points of diffraction (or scattering centers) on the rim forming the mouth of the open waveguide. Let us consider the set of diffracted rays arising from an induced source element at $\phi = \phi_s$ on the curved rim, which may be regarded as locally straight because of the large waveguide assumption $ka \gg 1$. In general the incident ray strikes the locally straight edge at an oblique angle so that the family of rays emanating from the point of diffraction ϕ_s forms a right circular cone. Now, the equation

$$\frac{r\sin\theta\,\sin(\phi-\phi_s)}{\left[r^2+a^2-2ra\sin\theta\cos(\phi-\phi_s)\right]^{1/2}} = \pm\cos\psi\,,\quad 0<\psi<\frac{\pi}{2} \tag{4.32}$$

describes a right circular semi-cone of central angle 2ψ with apex on the rim at ϕ_s and with interior axis in the $\hat{\phi}_s$ or $-\hat{\phi}_s$ direction depending, respectively, on whether the upper or lower sign is chosen. The angle ψ is determined by ray-optics considerations; in particular, ψ is the angle formed by the incident ray and the tangent to the edge at ϕ_s. Since

$$\hat{k} = \hat{x}\sin\beta - \hat{z}\sin\beta,\quad \hat{\phi}_s = -\hat{x}\sin\phi_s + \hat{y}\cos\phi_s\,, \tag{4.33}$$

and since we want $0<\psi<(\pi/2)$, we can take

$$\cos\psi = \left|\hat{k}\cdot\hat{\phi}_s\right| = \left|\sin\beta\sin\phi_s\right| \quad . \tag{4.34}$$

From our geometry it is seen that the diffracted rays must lie on semi-cones with axes in the directions $\mp\phi_s$ for $\sin\phi_s \gtrless 0$. In both cases, the "ray cone" emanating from the point of diffraction at ϕ_s is described by the equation

$$\frac{r\sin\theta\sin(\phi-\phi_s)}{\left[r^2+a^2-2ra\sin\theta\cos(\phi-\phi_s)\right]^{1/2}} = -\sin\beta\sin\phi_s \quad . \tag{4.35}$$

In the far zone ($r\to\infty$) the cone equation (4.35) becomes

$$\sin\theta\,\sin(\phi-\phi_s) = -\sin\beta\sin\phi_s \quad . \tag{4.36}$$

For a fixed observation location (θ, ϕ), Eq.(4.36) can be solved for ϕ_s, and since both ϕ_s and $\pi + \phi_s$ will appear as solutions, it is clear that two diametrically opposite points of diffraction will yield a diffracted ray passing through the point (θ, ϕ). Thus, without loss of generality, we may assume that $0 \leq \phi_s < \pi$ provided the two rays from ϕ_s and $\pi + \phi_s$ are taken into account. It may be noted that (4.36) is identically satisfied for all ϕ_s if $\theta = \beta = \pi$ or if $\theta = \beta$, $\phi = 0$. The former case corresponds to backscattering with axial incidence; the latter case is described by (4.28) and (4.29) in which the angular functions $f(\beta, 0)$, $g(\beta, 0)$ are governed by Bessel functions whose arguments vanish. In both cases all points on the rim contribute to the diffracted field.

Provided the two cases cited immediately above are excluded, the field scattered into the far zone is of the form

$$\underline{E}^s \sim \frac{e^{ikr}}{r} \underline{D} \left\{ \sqrt{\frac{a \sin^2 \psi}{(\hat{r}-\hat{k}) \cdot \hat{\rho}(\phi_s)}} \, e^{ika \left[(\hat{k}-\hat{r}) \cdot \hat{\rho}(\phi_s)\right] + i\pi/4} + \right. \tag{4.37}$$

$$\left. + \sqrt{\frac{a \sin^2 \psi}{(\hat{r}-\hat{k}) \cdot \hat{\rho}(\pi + \phi_s)}} \, e^{ika \left[(\hat{k}-\hat{r}) \cdot \hat{\rho}(\pi + \phi_s)\right] + i\pi/4} \right\}$$

where $\underline{D}$ is an appropriate vector diffraction coefficient (with a factor $e^{i\pi/4}$ excluded) obtained from the half plane solutions and where $\hat{k}$ is given in (4.33) while

$$\hat{r} = \hat{x} \sin\theta \cos\phi + \hat{y} \sin\theta \sin\phi + \hat{z} \cos\theta,$$
$$\hat{p}(\phi_s) = \hat{x} \cos\phi_s + \hat{y} \sin\phi_s \; . \tag{4.38}$$

Combining (4.33), (4.36) and (4.38) we obtain

$$(\hat{r}-\hat{k}) \cdot \hat{\rho}(\phi_s) = \sin\theta \cos(\phi - \phi_s) - \sin\beta \cos\phi_s$$

$$= \frac{\sin\theta \sin\phi}{\sin\phi_s} = \operatorname{sgn}(\sin\phi)(\sin^2\theta + \sin^2\beta - 2\sin\theta\sin\beta\cos\phi)^{1/2} \; ; \qquad (4.39)$$

thus the scattered field (4.37) behaves as

$$\underline{E}^s \sim \frac{e^{ikr}}{r} \underline{D} \left(\frac{2a\sin^2\psi}{\Omega}\right)^{1/2} \cos\left(2ka\Omega - \frac{\pi}{4}\right) \qquad (4.40)$$

where

$$2\Omega = (\sin^2\theta + \sin^2\beta - 2\sin\theta\sin\beta\cos\phi)^{1/2}, \quad 0 < \Omega < 1 . \qquad (4.41)$$

It may be noted that Ω vanishes only if $\theta = \beta = \pi$ or if $\theta = \beta, \ \phi = 0$, which are the cases we have excluded above. In the case of backscattering ($\theta = \beta, \ \phi = \pi$) the field in (4.40) will reduce to that in (4.31).

4.2 Absorbing Case

In view of the results of Sections 3.2 and 4.1, the problem of axial incidence upon an absorbing semi-infinite cylinder may now be treated without difficulty. The geometry of the configuration remains the same as in Fig. 4-1, except that the interior surface of the cylinder is assumed to be governed by a surface impedance η_2, while the exterior surface is governed by an impedance η_1.

As pointed out in Section 4.1, the parallel plane result may be suitably modified by ray-optical considerations to yield the circular cylinder result, and indeed, in the case of perfectly conducting surfaces the exact parallel plane result can be used to yield the exact cylinder result. When the surfaces are absorbing, we begin with the parallel plane result in (3.15) with $\theta_0 = \pi$, and then incorporate the appropriate ray-optical modifications that are necessary to account for the axial caustic of the circular tube and for the curvature of the rim forming the mouth of the tube. For axial incidence given by (4.1) and for $ka\sin\theta \gg 1$, the far field scattered by the absorbing semi-infinite cylinder is found to take the form

$$\underline{E}^s \sim -\hat{\phi}\, u^E \sin\phi + \hat{\theta}\, u^H \cos\phi\,, \tag{4.42}$$

where

$$u \sim \frac{-e^{ikr}}{4r}\left(\frac{2a}{\pi k \sin\theta}\right)^{1/2} \left\{ e^{-ika\sin\theta + i\pi/4} \left[U(\theta,\pi) + \right.\right.$$

$$\left. + U(\theta, 3\pi/2)\, U(3\pi/2, \pi) \frac{e^{i\pi/4}}{\sqrt{\pi ka}} \sum_{m=1}^{\infty} \frac{i^m R^{m-1} e^{2imka}}{2^{m+1}\sqrt{m}} \right] +$$

$$+ e^{ika\sin\theta - i\pi/4} \left[U(2\pi-\theta, \pi) + U(2\pi-\theta, 3\pi/2)\, \times \right.$$

$$\left.\left. \times\; U(3\pi/2, \pi) \frac{e^{i\pi/4}}{\sqrt{\pi ka}} \sum_{m=1}^{\infty} \frac{i^m \cdot R^{m-1} e^{2imka}}{2^{m+1}\sqrt{m}} \right] \right\} \quad . \tag{4.43}$$

In (4.42) the affix H or E is attached to indicate where (2.7) or (2.8) should be used to determine the constants $\alpha^H_{1,2}$, $\alpha^E_{1,2}$ that are implicit in the functions above. For a perfect conductor,

$$U^H(\theta,\pi) = 0, \;\; U(\theta,\pi) = \frac{2}{\sin(\theta/2)} \quad ,$$

$$U^E(\theta, 3\pi/2)\, U^E(3\pi/2, \pi) = -\frac{8\sin(\theta/2)}{\cos\theta} \quad ,$$

and agreement with (4.19) is obtained.

To obtain the backscattered field, we begin with (3.15) specialized to the case $\theta_o = \theta = \pi$, and incorporate only the ray-optical caustic corrections. The outgoing electric field from a diffraction point on the rim at ϕ then has the form

$$-\frac{e^{i(kr+\pi/4)}}{\sqrt{2\pi kr}}\, 2\left(\underline{E}^i_{\parallel} S^E - \underline{E}^i_{\perp} S^H\right) \quad , \tag{4.44}$$

where $\underline{E}^i_{\parallel}$ and $\underline{E}^i_{\perp}$ are the components of the incident field parallel and perpendicular, respectively, to the rim at the point ϕ:

$$\underline{E}^i_{\parallel} = \hat{x}\sin^2\phi - \hat{y}\sin\phi\cos\phi\,, \qquad (4.45)$$
$$\underline{E}^i_{\perp} = \hat{x}\cos^2\phi + \hat{y}\sin\phi\cos\phi\ .$$

The quantity S is given by

$$2S = U(\pi,\pi) + U^2(\pi,\frac{3\pi}{2})\frac{e^{i\pi/4}}{\sqrt{\pi ka}}\sum_{m=1}^{\infty}\frac{i^m R^{m-1} e^{2imka}}{2^{m+1}\sqrt{m}}\ . \qquad (4.46)$$

To convert the field (4.44) to a three dimensional field, we multiply (4.44) by Siegel's (1959) factor $e^{-i\pi/4}(k/2\pi r)^{1/2}\, a\, d\phi$ and to add up the contributions from all points on the rim, we integrate over ϕ from 0 to π (rather than 2π since we began with the parallel plane result). The backscattered field is then

$$\underline{E}^{BS} \sim -\hat{x}\ \frac{ae^{ikr}}{2r}\,(S^E - S^H)\ . \qquad (4.47)$$

For a perfect conductor $S^H=0$, and S^E becomes

$$S^E = 1 + \frac{e^{i\pi/4}}{\sqrt{\pi ka}}\sum_{m=1}^{\infty}\frac{i^m e^{2imka}}{2^m\sqrt{m}}\ ,$$

so that agreement with (4.20) is established. On the other hand, for a perfect absorber ($\eta_1=\eta_2=1$) we have $S^E=S^H$ in agreement with Weston's (1963) general absorber theorem. Weston's theorem states that any body whose shape is invariant under a rotation of 90^o about an axis will yield a vanishing backscattered field if a plane wave is incident along the symmetry axis and if the surface impedance of the body is unity. The circular cylinder under consideration here is a special case governed by Weston's theorem.

In the case of off-axis incidence with the incoming field given by (4.21), the backscattered field (for which $\theta = \beta$, $\phi=\pi$) may be derived in the form, provided $ka\sin\beta \gg 1$ and $\cos\beta < 0$,

$$\underline{E}^{BS} = \hat{y}\, u^E \sin\alpha + (\hat{x}\cos\beta + \hat{y}\sin\beta) u^H \cos\alpha \tag{4.48}$$

where

$$u \sim \frac{-e^{ikr}}{4r}\left(\frac{a}{\pi k\sin\beta}\right)^{1/2} \left\{ e^{-2ika\sin\theta + i\frac{\pi}{4}}\, U(\beta,\beta) + \right.$$
$$\left. + e^{2ika\sin\theta - i\frac{\pi}{4}}\, U(2\pi-\beta, 2\pi-\beta) \right\} . \tag{4.49}$$

Equation (4.48) represents the appropriate generalization of (4.31), and indeed, if the surfaces are perfectly conducting, then

$$U^E(\beta,\beta) = -\frac{2\sin^2(\beta/2)}{\cos\beta} \quad , \quad U^H(\beta,\beta) = \frac{2\cos^2(\beta/2)}{\cos\beta} \quad ,$$

which yields agreement with the result in (4.31).

V

THE RECTANGULAR DUCT

We now direct attention to the problem of scattering by a rectangular duct of uniform cross section and finite length. The walls of the duct are assumed to be absorbing and such that an impedance boundary condition may be applied. Rather than determine the field scattered directly by the edges forming the mouth of the duct -- a task which is easily accomplished in view of the preceding sections -- we are interested in determining the contribution to the radar cross section due to energy launched into waveguide modes which may be reflected by the closed termination of the duct and reradiated back toward the transmitter. The first step is to calculate, by means of ray optics, the diffraction fields generated at the mouth of the duct by the incident plane wave. This information must then be used to calculate the conversion coefficients which couple the induced aperture fields to the waveguide modes in the duct. In the process, the modal structure for a rectangular waveguide whose four walls obey an impedance boundary condition must be examined. Because of the boundary conditions, the modal problem is not separable; however, the modes may be determined approximately by means of an asymptotic analysis applied to the transcendental eigenvalue equation of infinite order. Orthogonality relations for the modes may be derived by using the Lorentz reciprocity theorem in conjunction with the boundary conditions and symmetry conditions. Finally, the scattered field due to modes reflected from the duct termination are obtained by means of a Kirchhoff approximation over the aperture formed by the mouth.

5.1 Aperture Fields

The geometrical configuration for the rectangular duct is displayed in Fig. 5-1, where the origin of the Cartesian coordinates is taken at the center of the guide cross section and in the plane of the open mouth. The duct is assumed to be terminated at a distance L from the mouth by a perfectly conducting plate transverse to the z axis; the waveguide is thus short circuited at the termination and the modes suffer no decrease in amplitude at the point of reflection. The incident plane wave is taken as

$$\underline{E}^i = (\hat{y}\cos\theta_o - \hat{z}\sin\theta_o)\, e^{-ik(y\sin\theta_o + z\cos\theta_o)}$$

$$\underline{H}^i = \hat{x}\, e^{-ik(y\sin\theta_o + z\cos\theta_o)} \tag{5.1}$$

and is therefore propagating in a principal plane of the waveguide structure. Finally, an impedance boundary condition of the type (2.1) will be imposed on the walls of the duct. For simplicity, it will be assumed that both the interior and exterior surfaces of the duct are governed by a single constant impedance η. This assumption is made because the plane wave in (5.1) is incident obliquely to edges ③ and ④ (see Fig. 5-1), and as pointed out in Section 2.2, the diffraction coefficient for oblique incidence on an absorbing half plane is available only if the impedances on both sides of the half plane are identical.

The primary diffraction fields generated at the mouth of the duct by the field (5.1) will be determined from ray diffraction theory with suitable modifications to account for the finite lengths of the diffracting edges. We begin with the fields due to edges ③, ④, and employ the half plane results (2.35) and (2.36). When the change in coordinate systems is accounted for, the half plane diffraction fields take the following form. For edge ③:

$$\underline{E}^d = \frac{i}{2} e^{-i\pi/4} \frac{e^{-iky\sin\theta_o + ik\rho_3\cos\theta_o}}{(2\pi k\rho_3\cos\theta_o)^{1/2}} \left[(\hat{x}\sin\theta_o - \hat{y}\cos\theta_o)\, M(\theta_o) - \hat{z}\tilde{M}(\theta_o)\right],$$

$$\underline{H}^d = \frac{i}{2} e^{-i\pi/4} \frac{e^{-iky\sin\theta_o + ik\rho_3\cos\theta_o}}{(2\pi k\rho_3\cos\theta_o)^{1/2}} \left[(\hat{x}\sin\theta_o - \hat{y}\cos\theta_o)\, \tilde{M}(\theta_o) + \hat{z}M(\theta_o)\right]. \tag{5.2}$$

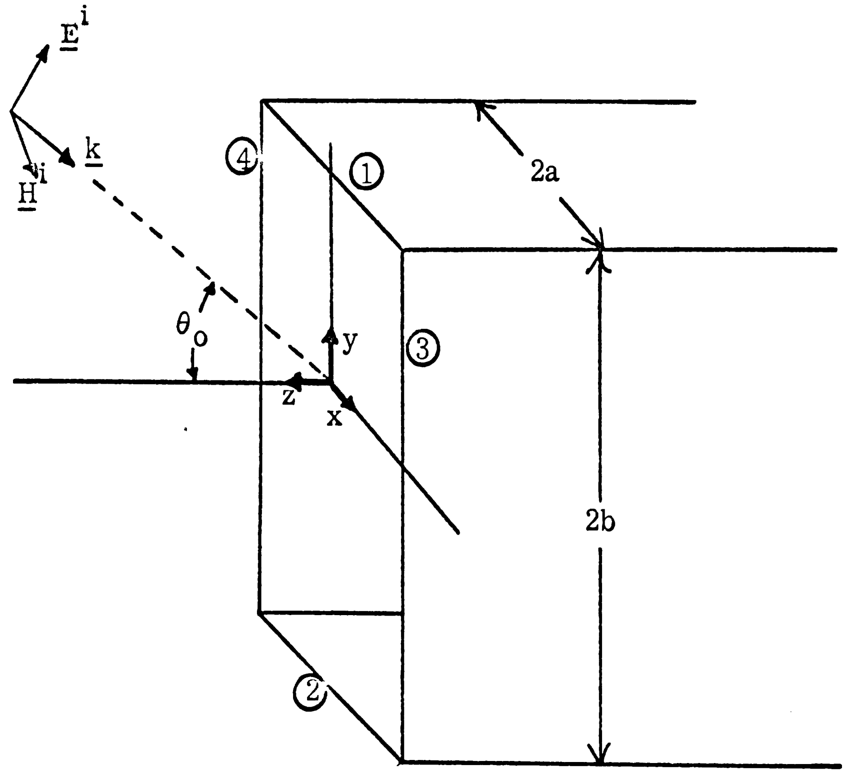

FIG. 5-1: PLANE WAVE INCIDENCE ON RECTANGULAR DUCT.

For edge ④:

$$\underline{E}^d = \frac{i}{2} e^{-i\pi/4} \frac{e^{-iky\sin\theta_o + ik\rho_4\cos\theta_o}}{(2\pi k\rho_4\cos\theta_o)^{1/2}} \left[-(\hat{x}\sin\theta_o + \hat{y}\cos\theta_o)M(\theta_o) - \hat{z}\tilde{M}(\theta_o)\right] ,$$

$$\underline{H}^d = \frac{i}{2} e^{-i\pi/4} \frac{e^{-iky\sin\theta_o + ik\rho_4\cos\theta_o}}{(2\pi k\rho_4\cos\theta_o)^{1/2}} \left[(\hat{x}\sin\theta_o + \hat{y}\cos\theta_o)\tilde{M}(\theta_o) - \hat{z}M(\theta_o)\right] , \tag{5.3}$$

where $\rho_3 = (a-x)$, $\rho_4 = (a+x)$, and the M functions are given in (2.37). In these functions the quantities $\alpha^E(\theta_o)$, $\alpha^H(\theta_o)$ appear implicitly and are determined by (2.28).

To account for the finite lengths of the edges, we consider the integral

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{-iky'\sin\theta_o} \frac{e^{ik\left[\rho^2+(y-y')^2\right]^{1/2}}}{\left[\rho^2+(y-y')^2\right]^{1/2}}\, dy'$$

$$= \frac{1}{2} e^{-iky\sin\theta_o} H_o^{(1)}(k\rho\cos\theta_o) \sim e^{-i\pi/4} \frac{e^{-iky\sin\theta_o + ik\rho\cos\theta_o}}{(2\pi k\rho\cos\theta_o)^{1/2}} . \tag{5.4}$$

In view of (5.4), better results for the aperture fields are expected if the quantity

$$\frac{i}{2} e^{-i\pi/4} (2\pi k\rho\cos\theta_o)^{-1/2} e^{-iky\sin\theta_o - ik\rho\cos\theta_o} \tag{5.5}$$

appearing in (5.2) and (5.3) is replaced by the <u>finite</u> integral

$$\frac{1}{4\pi}\int_{-b}^{b} e^{-iky'\sin\theta_o} \frac{e^{ik\left[\rho^2+(y-y')^2\right]^{1/2}}}{\left[\rho^2+(y-y')^2\right]^{1/2}}\, dy' . \tag{5.6}$$

The integral (5.6) physically corresponds to a finite line source with traveling wave current. Let us denote this integral by $I_b(\rho, y, \theta_o)$, then the aperture fields due to edges ③ and ④ may be taken as

$$\underline{E} = I_b(a-x, y; \theta_o) \left[(\hat{x}\sin\theta_o - \hat{y}\cos\theta_o)\, M(\theta_o) - \hat{z}\,\tilde{M}(\theta_o)\right] -$$

$$- I_b(a+x, y; \theta_o) \left[(\hat{x}\sin\theta_o + \hat{y}\cos\theta_o)\, M(\theta_o) + \hat{z}\,\tilde{M}(\theta_o)\right], \qquad (5.7)$$

$$\underline{H} = I_b(a-x, y; \theta_o) \left[(\hat{x}\sin\theta_o - \hat{y}\cos\theta_o)\, \tilde{M}(\theta_o) + \hat{z}\,M(\theta_o)\right] +$$

$$+ I_b(a+x, y; \theta_o) \left[(\hat{x}\sin\theta_o + \hat{y}\cos\theta_o)\, \tilde{M}(\theta_o) - \hat{z}\,M(\theta_o)\right]. \qquad (5.8)$$

The quantity $M(\theta_o)$ is due to an electric line source, while the quantity $\tilde{M}(\theta_o)$ is due to a magnetic line source.

In the case of the remaining two edges, ① and ②, the half plane diffraction fields are determined from (2.2) with due account given to the phase of the incident field at the points of diffraction. For edge ①:

$$\underline{E}^d = \hat{z}\,\frac{i}{2}\, e^{-i\pi/4}\, \frac{e^{-ikb\sin\theta_o + ik\rho_1}}{(2\pi k\rho_1)^{1/2}}\, U^H\!\left(\frac{3\pi}{2}, \pi-\theta_o\right),$$

$$\underline{H}^d = -\hat{x}\,\frac{i}{2}\, e^{-i\pi/4}\, \frac{e^{-ikb\sin\theta_o + ik\rho_1}}{(2\pi k\rho_1)^{1/2}}\, U^H\!\left(\frac{3\pi}{2}, \pi-\theta_o\right). \qquad (5.9)$$

For edge ②:

$$\underline{E}^d = -\hat{z}\,\frac{i}{2}\, e^{-i\pi/4}\, \frac{e^{ikb\sin\theta_o + ik\rho_2}}{(2\pi k\rho_2)^{1/2}}\, U^H\!\left(\frac{3\pi}{2}, \pi+\theta_o\right),$$

$$\underline{H}^d = -\hat{x}\,\frac{i}{2}\, e^{-i\pi/4}\, \frac{e^{ikb\sin\theta_o + ik\rho_2}}{(2\pi k\rho_2)^{1/2}}\, U^H\!\left(\frac{3\pi}{2}, \pi+\theta_o\right), \qquad (5.10)$$

where $\rho_1 = (b-y)$ and $\rho_2 = (b+y)$. Here the impedances on the two sides need not be equal; however, we shall still adhere to our assumption that the impedances are identical. We also note that multiple interaction between edges ① and ② can be taken into account by incorporating the results of Section 3.2. Finally, since

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{ik\left[\rho^2+(x-x')^2\right]^{1/2}}}{\left[\rho^2+(x-x')^2\right]^{1/2}}\,dx' = \frac{1}{2}\,H_0^{(1)}(k\rho) \sim e^{-i\pi/4}\,\frac{e^{ik\rho}}{(2\pi k\rho)^{1/2}}\,, \tag{5.11}$$

it is clear that better results for the aperture fields can be obtained upon replacing

$$e^{-i\pi/4}\,(2\pi k\rho)^{-1/2}\,e^{ik\rho} \tag{5.12}$$

by the finite integral

$$\frac{1}{2\pi i}\int_{-a}^{a} \frac{e^{ik\left[\rho^2+(x-x')^2\right]^{1/2}}}{\left[\rho^2+(x-x')^2\right]^{1/2}}\,dx'. \tag{5.13}$$

The total aperture fields are obtained by combining the above results for all the edges ①, ..., ④. We define

$$I_b(\rho, y; \theta_o) = \frac{1}{4\pi}\int_{-b}^{b} e^{-iky'\sin\theta_o}\,\frac{e^{ik\left[\rho^2+(y-y')^2\right]^{1/2}}}{\left[\rho^2+(y-y')^2\right]^{1/2}}\,dy'\,, \tag{5.14}$$

$$I_a(\rho, x) = \frac{1}{4\pi}\int_{-a}^{a} \frac{e^{ik\left[\rho^2+(x-x')^2\right]^{1/2}}}{\left[\rho^2+(x-x')^2\right]^{1/2}}\,dx'\,, \tag{5.15}$$

then the aperture fields may be written in the form

$$\underline{E} = I_b(a-x, y; \theta_o)\left[(\hat{x}\sin\theta_o - \hat{y}\cos\theta_o)\, M(\theta_o) - \hat{z}\tilde{M}(\theta_o)\right] -$$

$$- I_b(a+x, y; \theta_o)\left[(\hat{x}\sin\theta_o + \hat{y}\cos\theta_o)\, M(\theta_o) + \hat{z}\tilde{M}(\theta_o)\right] +$$

$$+ \hat{z}\, I_a(b-y, x)\, U^H(\theta_o)\, e^{-ikb\sin\theta_o} - \hat{z}\, I_a(b+y, x)\, U^H(-\theta_o)\, e^{ikb\sin\theta_o} , \tag{5.16}$$

$$\underline{H} = I_b(a-x, y; \theta_o)\left[(\hat{x}\sin\theta_o - \hat{y}\cos\theta_o)\, \tilde{M}(\theta_o) + \hat{z}\, M(\theta_o)\right] +$$

$$+ I_b(a+x, y; \theta_o)\left[(\hat{x}\sin\theta_o + \hat{y}\cos\theta_o)\, \tilde{M}(\theta_o) - \hat{z}\, M(\theta_o)\right] -$$

$$- \hat{x}\, I_a(b-y, x)\, U^H(\theta_o)\, e^{-ikb\sin\theta_o} - \hat{x}\, I_a(b+y, x)\, U^H(-\theta_o)\, e^{ikb\sin\theta_o} , \tag{5.17}$$

where the M functions are given by (2.37) with (2.28) and (2.38), and where

$$U^H(\pm\theta_o) = \frac{\psi_\pi^4(\pi/2)\cos(\theta_o/2)}{(1+\eta)^{1/2}\,\psi^H(\theta_o)\cos\theta_o}\left[\eta \pm \sqrt{2}\,\sin(\theta_o/2)\right] \tag{5.18}$$

with $\eta = \cos\alpha^H$. For $\theta_o = 0$: $M(0) = U^E(0)$, $\tilde{M}(0) = 0$.

5.2 Boundary Conditions in Rectangular Waveguide

The problem of determining the modal structure in a rectangular waveguide whose four walls are absorbing is greatly complicated by the fact that the boundary conditions are inseparable. Investigations of alternative techniques for obtaining the fields inside the duct have revealed that the boundary conditions are most convenient when expressed in terms of the electric fields tangential to the normal cross section of the duct. In this section we shall display a modified form of the standard impedance boundary conditions.

The geometry for the rectangular waveguide is as illustrated in Fig. 5-1, although now we are considering a waveguide of infinite length. An impedance boundary condition (2.1) will be imposed on the four interior walls of the waveguide, with the walls characterized by a single constant impedance η. In terms of the field components the boundary conditions are as follows:

$$\text{On } y = \pm b: \qquad E_x = \mp \eta H_z, \quad E_z = \pm \eta H_x, \tag{5.19}$$

$$\text{On } x = \pm a: \qquad E_y = \pm \eta H_z, \quad E_z = \mp \eta H_y. \tag{5.20}$$

We select the transverse components E_x, E_y as the two fundamental field quantities. In terms of these components, Maxwell's equations allow us to rewrite the boundary conditions in the form

$$\text{On } y = \pm b: \qquad \frac{\partial E_y}{\partial y} = \pm ik\eta E_y, \quad \frac{\partial E_x}{\partial y} = \pm \frac{ik}{\eta} E_x + \frac{\partial E_y}{\partial x}, \tag{5.21}$$

$$\text{On } x = \pm a: \qquad \frac{\partial E_x}{\partial x} = \pm ik\eta E_x, \quad \frac{\partial E_y}{\partial x} = \pm \frac{ik}{\eta} E_y + \frac{\partial E_x}{\partial y}. \tag{5.22}$$

Two of these boundary conditions are uncoupled, whereas the remaining coupled boundary conditions appear to have the simplest form -- indeed, other choices of basic field quantities, aside from H_x and H_y, lead to much more complicated expressions involving second derivatives. In terms of the transverse magnetic field, the boundary conditions may be expressed as

$$\text{On } y = \pm b: \qquad \frac{\partial H_y}{\partial y} = \pm \frac{ik}{\eta} H_y, \quad \frac{\partial H_x}{\partial y} = \pm ik\eta H_x + \frac{\partial H_y}{\partial x}, \tag{5.23}$$

$$\text{On } x = \pm a: \qquad \frac{\partial H_x}{\partial x} = \pm \frac{ik}{\eta} H_x, \quad \frac{\partial H_y}{\partial x} = \pm ik\eta H_y + \frac{\partial H_x}{\partial y}, \tag{5.24}$$

and are, of course, derivable from (5.21), (5.22) by invoking the natural duality of Maxwell's equations and the impedance boundary condition.

5.3 Scalar Eigenfunctions

The boundary conditions in (5.21) and (5.22) strongly suggest that we examine first the eigenfunctions satisfying the uncoupled relations. Such eigenfunctions would be appropriate for a rectangular acoustic waveguide with non-rigid walls. The electromagnetic eigenfunctions will be represented as expansions with respect to the acoustic eigenfunctions.

The scalar functions of interest are determined by the differential equations

$$\Psi_m''(x) + \beta_m^2 \Psi_m(x) = 0 \quad , \qquad \chi_n''(y) + \gamma_n^2 \chi_n(y) = 0 \tag{5.25}$$

and the boundary conditions

$$\Psi_m'(a) = ik\eta \Psi_m(a) \quad , \qquad \chi_n'(b) = ik\eta\chi_n(b) \quad . \tag{5.26}$$

For convenience, let us consider explicitly only the set of functions $\left\{\Psi_m(x)\right\}$; all relationships for the set $\left\{\chi_n(y)\right\}$ follow by simple notational changes. The functions $\Psi_m(x)$ will be defined as follows:

$$\begin{aligned} \frac{\Psi_m(x)}{\Psi_m(a)} &= \frac{\cos\beta_m x}{\cos\beta_m a} && \text{for } m = 0, 2, 4, \ldots \\ \frac{\Psi_m(x)}{\Psi_m(a)} &= \frac{\sin\beta_m x}{\sin\beta_m a} && \text{for } m = 1, 3, 5, \ldots \end{aligned} \tag{5.27}$$

where the boundary conditions are met by adjusting β_m so that

$$\begin{aligned} \beta_m \tan\beta_m a &= -ik\eta && \text{(m even)} \\ \beta_m \cot\beta_m a &= ik\eta && \text{(m odd)} \end{aligned} \tag{5.28}$$

and where $\Psi_m(a)$ is given by

$$\Psi_m^2(a) = \frac{\beta_m^2}{(\beta_m^2 - k^2\eta^2)\,a - ik\eta} \, . \tag{5.29}$$

With these definitions, $\{\Psi_m(x)\}$ forms a normal orthogonal set of eigen-functions such that

$$\int_{-a}^{a} \Psi_m(x)\,\Psi_p(x)\,dx = \delta_m^p \tag{5.30}$$

where $\delta_m^p = 0\ (m \neq p)$ and $\delta_m^m = 1$. The set of derivatives $\{\Psi'_m(x)\}$ forms a set of functions with the interesting property

$$\int_{-a}^{a} \Psi'_m(x)\Psi'_p(x)\,dx - 2ik\eta\,\Psi_m(a)\Psi_p(a) = \beta_m^2\,\delta_m^p \, , \tag{5.31}$$

provided m, p are either both even or both odd. This follows easily from integration by parts:

$$\int_{-a}^{a} \Psi'_m(x)\,\Psi'_p(x)\,dx = \Big[\Psi'_m(x)\,\Psi_p(x)\Big]_{-a}^{a} - \int_{-a}^{a} \Psi''_m(x)\,\Psi_p(x)\,dx$$

$$= 2ik\eta\Psi_m(a)\,\Psi_p(a) + \beta_m^2 \int_{-a}^{a} \Psi_m(x)\Psi_p(x)\,dx$$

$$= 2ik\,\eta\,\Psi_m(a)\,\Psi_p(a) + \beta_m^2\,\delta_m^p \, .$$

Because of the orthogonality properties (5.30) and (5.31), expansions of a function f(x) with respect to the systems $\{\Psi_m(x)\}$ and $\{\Psi'_m(x)\}$ take the form

$$f(x) = \sum_m \Psi_m(x) \int_{-a}^{a} f(x)\Psi_m(x)\, dx \ , \tag{5.32}$$

$$f(x) = \sum_m \frac{\Psi'_m(x)}{\beta_m^2} \left\{ \int_{-a}^{a} f(x)\Psi'_m(x)\, dx - f(a)\Psi_m(a) + f(-a)\Psi_m(-a) \right\}. \tag{5.33}$$

For our later application, some particular expansions will be most useful; for example,

$$\frac{\cos\mu x}{\mu \sin\mu a + ik\eta\cos\mu a} = 2 \sum_m^{e} \frac{\Psi_m(a)\Psi_m(x)}{\mu^2 - \beta_m^2} \ ,$$

$$\frac{\sin\mu x}{\mu\cos\mu a - ik\eta\sin\mu a} = -2 \sum_m^{o} \frac{\Psi_m(a)\Psi_m(x)}{\mu^2 - \beta_m^2} \ , \tag{5.34}$$

where the affix e or o indicates whether the sum is over even m or odd m. The summations in (5.34) are unformly convergent in $|x| \le a$; however, the differentiated (term-by-term) series are uniformly convergent only in the interval $|x| \le a - \epsilon$, $0 < \epsilon < a$. For this reason, it may be important to have expansions whose derivatives are also uniformly convergent at the boundary $x = a$; in particular, the series

$$\frac{\cos\mu x}{\mu(\mu\cos\mu a - ik\eta\sin\mu a)} = -2 \sum_m^{o} \frac{\Psi_m(a)\Psi'_m(x)}{\beta_m^2(\mu^2 - \beta_m^2)} \ ,$$

$$\frac{\sin\mu x}{\mu(\mu\sin\mu a + ik\eta\cos\mu a} = -2 \sum_m^{e} \frac{\Psi_m(a)\Psi'_m(x)}{\beta_m^2(\mu^2 - \beta_m^2)} \tag{5.35}$$

and their derivatives are uniformly convergent in $|x| \le a$.

Equations (5.34) and (5.35) lead to the following relations:

$$\frac{\Psi'_m(x)}{\Psi_m(a)} = 2(\beta_m^2 - k^2\eta^2) \sum_p^o \frac{\Psi_p(a)\Psi_p(x)}{\beta_m^2 - \beta_p^2} \qquad \text{(m even)}$$

$$= 2(\beta_m^2 - k^2\eta^2) \sum_p^e \frac{\Psi_p(a)\Psi_p(x)}{\beta_m^2 - \beta_p^2} \qquad \text{(m odd)} \tag{5.36}$$

$$\frac{\Psi_m(x)}{\Psi_m(a)} = -2(\beta_m^2 - k^2\eta^2) \sum_p^o \frac{\Psi_p(a)\Psi'_p(x)}{\beta_p^2(\beta_m^2 - \beta_p^2)} \qquad \text{(m even)}$$

$$= -2(\beta_m^2 - k^2\eta^2) \sum_p^e \frac{\Psi_p(a)\Psi'_p(x)}{\beta_p^2(\beta_m^2 - \beta_p^2)} \qquad \text{(m odd)} \tag{5.37}$$

all valid for $|x| \leq a$. We also obtain

$$\frac{1}{\mu^2 - \mu'^2} \frac{\mu \sin\mu a \cos\mu' a - \mu' \sin\mu' a \cos\mu a}{(\mu \sin\mu a + ik\eta \cos\mu a)(\mu' \sin\mu' a + ik\eta \cos\mu' a)} =$$

$$= -2 \sum_p^e \frac{\Psi_p^2(a)}{(\mu^2 - \beta_p^2)(\mu'^2 - \beta_p^2)} ,$$

$$\frac{1}{\mu^2 - \mu'^2} \frac{\mu \cos\mu a \sin\mu' a - \mu' \cos\mu' a \sin\mu a}{(\mu \cos\mu a - ik\eta \sin\mu a)(\mu' \cos\mu' a - ik\eta \sin\mu' a)} = \tag{5.38}$$

$$= -2 \sum_p^o \frac{\Psi_p^2(a)}{(\mu^2 - \beta_p^2)(\mu'^2 - \beta_p^2)} .$$

Other sets of scalar eigenfunctions are obtained upon replacing η by $1/\eta$. These will be denoted as $\tilde{\Psi}_m(x)$ and $\tilde{\chi}_n(y)$, where

$$\tilde{\Psi}''_m(x) + \tilde{\beta}^2_m \, \tilde{\Psi}_m(x) = 0 \quad , \quad \tilde{\chi}''_n(y) + \tilde{\gamma}^2_n \tilde{\chi}_n(y) = 0 \tag{5.39}$$

with boundary conditions

$$\tilde{\Psi}'_m(a) = \frac{ik}{\eta} \, \tilde{\Psi}_m(a) \quad , \quad \tilde{\chi}'_n(b) = \frac{ik}{\eta} \, \tilde{\chi}_n(b) \ . \tag{5.40}$$

The set $\left\{\tilde{\Psi}_m(x)\right\}$ is related to $\left\{\Psi_m(x)\right\}$ through the expansions, valid for $|x| \leq a$,

$$\begin{aligned} \frac{\tilde{\Psi}_m(x)}{\tilde{\Psi}_m(a)} &= 2ik\left(\eta - \frac{1}{\eta}\right) \sum_p^{e} \frac{\Psi_p(a)\Psi_p(x)}{\tilde{\beta}^2_m - \beta^2_p} \qquad \text{(m even)} \\ &= 2ik\left(\eta - \frac{1}{\eta}\right) \sum_p^{o} \frac{\Psi_p(a)\Psi_p(x)}{\tilde{\beta}^2_m - \beta^2_p} \qquad \text{(m odd)} \ . \end{aligned} \tag{5.41}$$

Also, we have for $|x| \leq a$,

$$\begin{aligned} \frac{\tilde{\Psi}'_m(x)}{\tilde{\Psi}_m(a)} &= 2(\tilde{\beta}^2_m - k^2) \sum_p^{o} \frac{\Psi_p(a)\Psi_p(x)}{\tilde{\beta}^2_m - \beta^2_p} \qquad \text{(m even)} \\ &= 2(\tilde{\beta}^2_m - k^2) \sum_p^{e} \frac{\Psi_p(a)\Psi_p(x)}{\tilde{\beta}^2_m - \beta^2_p} \qquad \text{(m odd)} \ . \end{aligned} \tag{5.42}$$

Finally we note

$$4k^2\left(\eta-\frac{1}{\eta}\right)^2 \tilde{\psi}_m^2(a) \sum_p^e \frac{\tilde{\psi}_p^2(a)}{(\tilde{\beta}^2_{m'}-\beta^2_p)(\beta^2_p-\tilde{\beta}^2_{m'})} = \delta_m^{m'} \quad (m, m' \text{ both even}) ,$$

$$4k^2\left(\eta-\frac{1}{\eta}\right)^2 \tilde{\psi}_m^2(a) \sum_p^o \frac{\tilde{\psi}_p^2(a)}{(\tilde{\beta}^2_{m'}-\beta^2_p)(\beta^2_p-\tilde{\beta}^2_{m'})} = \delta_m^{m'} \quad (m, m' \text{ both odd}) . \tag{5.43}$$

Many other expansions can be derived. The eigenfunctions $\tilde{\psi}_m(x)$, $\tilde{\chi}_n(y)$ are particularly appropriate for representations of the magnetic field [compare the boundary conditions (5.40) with those in (5.23), (5.24)].

5.4 Approximate Eigenvalues for Scalar Functions

In general the tangent-cotangent equations (see, for example, eqs. 5.28) that determine the (complex) eigenvalues β_m, $\tilde{\beta}_m$, γ_n, $\tilde{\gamma}_n$ must be solved by numerical or graphical techniques. In some cases, however, they may be solved appro ̇mately. Let us consider the equations (5.28)

$$\beta_m \tan\beta_m a = -K \quad (m \text{ even})$$
$$\beta_m \cot\beta_m a = K \quad (m \text{ odd}) \tag{5.44}$$

where $K = ik\eta$. For $|Ka| >> 1$ (or for $m\pi << |Ka|$) an approximate solution is given by

$$\beta_m a \sim \frac{\pi}{2} \, \frac{(m+1)\,Ka}{Ka-1} \; . \tag{5.45}$$

On the other hand, for $|Ka| << 1$ (or for $m\pi >> |Ka|$) we have

$$\beta_m a \sim \frac{m\pi}{2}\left[1-\frac{4Ka}{m^2\pi^2}\right] \quad (m \neq 0)$$
$$\beta_m a \sim \sqrt{-Ka} \quad (m = 0) \tag{5.46}$$

from which it is clear that the eigenvalues and eigenfunctions of high order (m large) approach those of ordinary Fourier analysis. This very important property is characteristic of Sturm-Liouville problems, and strongly suggests that an asymptotic approach may be applied in the investigation of the electromagnetic modes in absorbing rectangular waveguides.

5.5 Modes in Rectangular Waveguides with Absorbing Walls

We are now in a position to investigate the electromagnetic modes, which will be obtained by representing the transverse electric field as Fourier series with respect to the systems $\{\psi_m(x)\}$ and $\{\chi_n(y)\}$. The first step is to decompose the solutions into four independent parts

$$\begin{aligned} E_y &= E_y^{ee} + E_y^{eo} + E_y^{oe} + E_y^{oo} \ , \\ E_x &= E_x^{oo} + E_x^{oe} + E_x^{eo} + E_x^{ee} \ , \end{aligned} \tag{5.47}$$

where the superscripts o, e refer to odd and even parity, and the first refers to this property with respect to the variable x, and the second with regard to the variable y. The independent solutions are formed by the pairs

$$(E_x^{oo}, E_y^{ee}), (E_x^{oe}. E_y^{eo}), (E_x^{eo}, E_y^{oe}), (E_x^{ee}, E_y^{oo}) \ . \tag{5.48}$$

Considering the first pair, for example, we can write the fields as

$$\begin{aligned} E_y^{ee} &= e^{-ihz} \sum_n^e A_n^e \frac{\chi_n(b)\chi_n(y)\cos\mu_n x}{\mu_n \sin\mu_n a + \frac{ik}{\eta}\cos\mu_n a} \ , \\ E_x^{oo} &= e^{-ihz} \sum_m^o B_m^o \frac{\Psi_m(a)\Psi_m(x)\sin\nu_m y}{\nu_m\cos\nu_m b - \frac{ik}{\eta}\sin\nu_m b} \ , \end{aligned} \tag{5.49}$$

where $h^2 = k^2 - \alpha^2$, $\operatorname{Im} h < 0$, $\mu_n^2 = \alpha^2 - \gamma_n^2$, and $\nu_m^2 = \alpha^2 - \beta_m^2$. The uncoupled boundary conditions in (5.21), (5.22) are thereby automatically satisfied. An alternative representation is

$$E_y^{ee} = -e^{-ihz} \sum_n^e \sum_m^o \frac{A_n^e a_{nm}^e \chi_n(y) \Psi_m(a) \Psi'_m(x)}{\beta_m^2 \chi_n(b)} ,$$

$$E_x^{oo} = e^{-ihz} \sum_m^o \sum_n^e \frac{B_m^o b_{mn}^o \Psi_m(x) \chi_n(b) \chi'_n(y)}{\gamma_n^2 \Psi_m(a)} , \tag{5.50}$$

where

$$a_{nm}^e = \frac{2\chi_n^2(b)\mu_n(\mu_n \cos\mu_n a - ik\eta \sin\mu_n a)}{(\mu_n^2 - \beta_m^2)(\mu_n \sin\mu_n a + \frac{ik}{\eta} \cos\mu_n a)} ,$$

$$b_{mn}^o = -\frac{2\Psi_m^2(a)\nu_m(\nu_m \sin\nu_m b + ik\eta \cos\nu_m b)}{(\nu_m^2 - \gamma_n^2)(\nu_m \cos\nu_m b - \frac{ik}{\eta} \sin\nu_m b)} . \tag{5.51}$$

Yet another representation is

$$E_y^{ee} = 2e^{-ihz} \sum_n^e \sum_m^o A_n^e \frac{\chi_n(b)\chi_n(y)\tilde{\Psi}_m(a)\tilde{\Psi}_m(x)}{\mu_n^2 - \tilde{\beta}_m^2} ,$$

$$E_x^{oo} = -2e^{-ihz} \sum_m^o \sum_n^e B_m^o \frac{\Psi_m(a)\Psi_m(x)\tilde{\chi}_n(b)\tilde{\chi}_n(y)}{\nu_m^2 - \tilde{\gamma}_n^2} . \tag{5.52}$$

The eigenvalues h are determined by applying the coupled boundary conditions in (5.21), (5.22). To this end, we employ the first two representations. On y = b the relation

$$\left(\frac{\partial}{\partial y} - \frac{ik}{\eta}\right) E_x = \frac{\partial E_y}{\partial x} \tag{5.53}$$

becomes

$$\sum_m^o B_m^o \Psi_m(a) \Psi_m(x) = \sum_m^o \sum_n^e A_n^e a_{nm}^e \Psi_m(a) \Psi_m(x) \, , \tag{5.54}$$

so that

$$B_m^o = \sum_n^e A_n^e a_{nm}^e \quad \text{(m odd)} \; . \tag{5.55}$$

Similarly, the coupled boundary condition on $x = a$ yields

$$A_n^e = \sum_m^o B_m^o b_{mn}^o \quad \text{(n even)} \; . \tag{5.56}$$

We thus obtain two sets of homogeneous equations:

$$\begin{aligned} &\sum_s^e A_s^e \sum_m^o \left\{ a_{sm}^e b_{mn}^o - \delta_s^n \right\} = 0 \quad \text{(n even)} \\ &\sum_s^o B_s^o \sum_n^e \left\{ b_{sn}^o a_{nm}^e - \delta_s^m \right\} = 0 \quad \text{(m odd)} \; . \end{aligned} \tag{5.57}$$

The eigenvalues h (or α) are determined by the requirement that the determinant vanishes, that is

$$\left\| \sum_m^o \left(a_{sm}^e b_{mn}^o - \delta_s^n \right) \right\| = 0 \qquad \text{(n, s even)} \; . \tag{5.58}$$

Thus the eigenvalues are specified by zeros of a transcendental determinant of infinite order.

Approximate expressions for the high order modes can be derived by recognizing that only a few terms contribute significantly to the determinant. To illustrate, we assume M, N to be large integers and set

$$\alpha^2 = \beta_M^2 + \gamma_N^2 + \delta^2 \qquad \text{(M odd, N even)} \tag{5.59}$$

where

$$\left|\beta_m^2\right| >> \left|\delta^2\right| , \quad \left|\gamma_N^2\right| >> \left|\delta^2\right| . \tag{5.60}$$

Then the only significant contribution to the determinant arises for $s = n = N$, $m = M$, and the eigenvalue equation (5.58) reduces to

$$a_{NM}^{e}\, b_{MN}^{o} = 1 . \tag{5.61}$$

Now, to a first approximation in δ^2 we have, upon expanding (5.51),

$$a_{NM}^{e} \simeq - \frac{\chi_N^2(b)\left[1 + \frac{\delta^2}{4\beta_M^2}\left\{2a\Psi_M^2(a) + 1\right\}\right]}{\Psi_M^2(a)\left[\left(1 - \frac{k^2}{\beta_M^2}\right) + \frac{\delta^2}{2\beta_M^2}\left\{1 + ika\left(\eta - \frac{1}{\eta}\right)\right\}\right]} ,$$

$$b_{NM}^{o} \simeq - \frac{\Psi_M^2(a)\left[1 + \frac{\delta^2}{4\gamma_N^2}\left\{2b\chi_N^2(b) + 1\right\}\right]}{\chi_n^2(b)\left[\left(1 - \frac{k^2}{\gamma_N^2}\right) + \frac{\delta^2}{2\gamma_N^2}\left\{1 + ikb\left(\eta - \frac{1}{\eta}\right)\right\}\right]} ; \tag{5.62}$$

thus, in view of (5.61) and (5.62) we see that δ^2 is specified approximately by a simple algebraic equation. For $\left|\beta_M^2\right| >> k^2$, $\left|\gamma_N^2\right| >> k^2$ and $ka > 1$, $kb > 1$ which implies $\left|\beta_M^2\right| >> (k/a)$, $\left|\gamma_N^2\right| >> (k/b)$ we have

$$\delta^2 \sim \frac{-4k^2(\gamma_N^2+\beta_M^2)}{(\gamma_N^2+\beta_M^2) - 2i\,(\eta - \frac{1}{\eta})(ka\,\gamma_N^2 + kb\,\beta_M^2)} \quad . \tag{5.63}$$

When $\eta = 1$ note that $\delta^2 \sim -4k^2$. Also, when $ka = kb = 1$

$$\delta^2 \sim \frac{-4k^2}{1-2i(\eta - \frac{1}{\eta})} \quad , \tag{5.64}$$

and one may expect the technique to be valid for the low order modes in the case of resonance frequencies $ka \sim O(1)$, $kb \sim O(1)$. Increased accuracy can be achieved by allowing more terms in the determinant to enter.

The remaining pairs of modes are treated similarly:

$$E_y^{eo} = e^{-ihz} \sum_n^{o} A_n^e \frac{\chi_n(b)\,\chi_n(y)\cos\mu_n x}{\mu_n \sin\mu_n a + \frac{ik}{\eta}\cos\mu_n a} = -e^{-ihz} \sum_n^{o} \sum_m^{o} \frac{A_n^e a_{nm}^e \chi_n(y) \Psi_m(a) \Psi'_m(x)}{\beta_m^2 \chi_n(b)} ,$$

$$E_x^{oe} = e^{-ihz} \sum_m^{o} B_m^e \frac{\Psi_m(a)\Psi_m(x)\cos\nu_m y}{\nu_m \sin\nu_m b + \frac{ik}{\eta}\cos\nu_m b} = -e^{-ihz} \sum_m^{o} \sum_n^{o} \frac{B_m^e b_{mn}^e \Psi_m(x) \chi_n(b) \chi'_n(y)}{\gamma_n^2 \Psi_m(a)} , \tag{5.65}$$

$$E_y^{oe} = e^{-ihz} \sum_n^{e} A_n^o \frac{\chi_n(b)\,\chi_n(y)\sin\mu_n x}{\mu_n \cos\mu_n a - \frac{ik}{\eta}\sin\mu_n a} = e^{-ihz} \sum_n^{e} \sum_m^{e} \frac{A_n^o a_{nm}^o \chi_n(y) \Psi_m(a) \Psi'_m(x)}{\beta_m^2 \chi_n(b)} ,$$

$$E_x^{eo} = e^{-ihz} \sum_m^{e} B_m^o \frac{\Psi_m(a)\Psi_m(x)\sin\nu_m y}{\nu_m \cos\nu_m b - \frac{ik}{\eta}\sin\nu_m b} = e^{-ihz} \sum_m^{e} \sum_m^{e} \frac{B_m^o b_{mn}^o \Psi_m(x) \chi_n(b) \chi'_n(y)}{\gamma_n^2 \Psi_m(a)} , \tag{5.66}$$

$$E_y^{oo} = e^{-ihz} \sum_n^o A_n^o \frac{\chi_n(b)\chi_n(y)\sin\mu_m x}{\mu_n \cos\mu_n a - \frac{ik}{\eta}\sin\mu_n a} = e^{-ihz} \sum_n^o \sum_m^e \frac{A_n^o a_{nm}^o \chi_n(y)\Psi_m(a)\Psi_m'(x)}{\beta_m^2 \chi_n(b)},$$

$$E_x^{ee} = e^{-ihz} \sum_m^e B_m^e \frac{\Psi_m(a)\Psi_m(x)\cos\nu_m y}{\nu_m \sin\nu_m b + \frac{ik}{\eta}\cos\nu_m b} = -e^{-ihz} \sum_m^e \sum_n^o \frac{B_m^e b_{mn}^e \Psi_m(x)\chi_n(b)\chi_n'(y)}{\gamma_n^2 \Psi_m(a)},$$

(5.67)

where a_{nm}^e, b_{mn}^o are as defined in (5.51) and

$$a_{nm}^o = -\frac{2\chi_n^2(b)\mu_n(\mu_n \sin\mu_n a + ik\eta\cos\mu_n a)}{(\mu_n^2 - \beta_m^2)(\mu_n \cos\mu_n a - \frac{ik}{\eta}\sin\mu_n a)},$$

(5.68)

$$b_{mn}^e = \frac{2\Psi_m^2(a)\nu_m(\nu_m \cos\nu_m b - ik\eta\sin\nu_m b)}{(\nu_m^2 - \gamma_n^2)(\nu_m \sin\nu_m b + \frac{ik}{\eta}\cos\nu_m b)}.$$

The uncoupled boundary conditions are again automatically satisfied, whereas the coupled relations demand

$$B_m^e = -\sum_n^o A_n^e a_{nm}^e \qquad \text{(m odd)}$$

$$A_n^e = -\sum_m^o B_m^e b_{mn}^e \qquad \text{(n odd)}$$

(5.69)

$$B_m^o = -\sum_n^e A_n^o a_{nm}^o \qquad \text{(m even)}$$

$$A_n^o = -\sum_m^e B_m^o b_{mn}^o \qquad \text{(n even)}$$

(5.70)

$$B^e_m = \sum_n^o A^o_n a^o_{nm} \qquad (m \text{ even})$$

$$A^o_n = \sum_m^e B^e_m b^e_{mn} \qquad (n \text{ odd}) \tag{5.71}$$

so the eigenvalue equations become

$$\left\| \sum_m^o \left(a^e_{sm} b^e_{mn} - \delta^n_s \right) \right\| = 0 \qquad (n, s \text{ odd})$$

$$\left\| \sum_m^e \left(a^o_{sm} b^o_{mn} - \delta^n_s \right) \right\| = 0 \qquad (n, s \text{ even}) \tag{5.72}$$

$$\left\| \sum_m^e \left(a^o_{sm} b^e_{mn} - \delta^n_s \right) \right\| = 0 \qquad (n, s \text{ odd})$$

In the asymptotic approximation $a^o_{nm} \sim a^e_{nm}$, $b^o_{mn} \sim b^e_{mn}$, so that the approximate expressions for δ^2 have identical forms for all mode pairs.

The result of the above analysis is that for each high order mode characterized by a pair of large integers M, N, the unknown eigenvalue h_{MN} can be determined approximately and is given by

$$h^2_{MN} \sim k^2 - \beta^2_M - \gamma^2_N - \delta^2 \, , \quad \operatorname{Im} h_{MN} < 0 \tag{5.73}$$

where δ^2 appears in (5.63). In the case of resonance frequencies such that $ka \sim O(1)$, $kb \sim O(1)$, the asymptotic expressions are expected to yield good results even for the low order modes. For other frequencies, increased accuracy can be achieved by retaining higher orders in δ^2 and by including more terms in the determinant close to $s = n = N$, $m = M$. The technique is basically a truncation procedure used in conjunction with asymptotic analysis.

As such, iteration methods can be used to advantage in order to gain more accurate evaluations of the eigenvalues. Once the eigenvalues are known, then the field expressions for the modes are determined to within an arbitrary multiplicative constant.

5.6 Orthogonality of Vector Modes

In order to determine the fields excited in the duct by the incident plane wave, we need to know the orthogonality properties of the electromagnetic modes sustained by the waveguide structure. We begin with the reciprocity statement

$$\underline{\nabla} \cdot (\underline{E} \wedge \underline{H}' - \underline{E}' \wedge \underline{H}) = 0 \tag{5.74}$$

which we write in the form

$$\underline{\nabla}_t \cdot (\underline{E} \wedge \underline{H}' - \underline{E}' \wedge \underline{H}) + \frac{\partial}{\partial z} \hat{z} \cdot (\underline{E} \wedge \underline{H}' - \underline{E}' \wedge \underline{H}) = 0 \tag{5.75}$$

where

$$\underline{\nabla}_t = \underline{\nabla} - \hat{z}(\hat{z} \cdot \underline{\nabla}) \ .$$

Now assume $(\underline{E}, \underline{H})$, $(\underline{E}', \underline{H}')$ have z-dependences e^{ihz}, $e^{ih'z}$, respectively, then

$$\underline{\nabla}_t \cdot (\underline{E} \wedge \underline{H}' - \underline{E}' \wedge \underline{H}) + i(h+h')\, \hat{z} \cdot (\underline{E} \wedge \underline{H}' - \underline{E}' \wedge \underline{H}) = 0 \ . \tag{5.76}$$

Integrate over the guide cross section and use the two-dimensional divergence theorem

$$\int_A \underline{\nabla}_t \cdot \underline{A} \, ds = \int_C (\hat{z} \wedge \underline{A}) \cdot (\hat{n} \wedge \hat{z}) \, dc = - \int_C \hat{n} \cdot \underline{A} \, dc \ , \tag{5.77}$$

to find

$$\int_A \underline{\nabla}_t \cdot (\underline{E} \wedge \underline{H}' - \underline{E}' \wedge \underline{H})\, dS = \int_C \left[\hat{z} \wedge (\underline{E} \wedge \underline{H}' - \underline{E}' \wedge \underline{H})\right] \cdot (\hat{n} \wedge \hat{z})\, dc$$

$$= \int_C \left[(\hat{z} \cdot \underline{H}')\, \underline{E} - (\hat{z} \cdot \underline{H})\, \underline{E}' + (\hat{z} \cdot \underline{E}')\, \underline{H} - (\hat{z} \cdot \underline{E})\, \underline{H}'\right] \cdot (\hat{n} \wedge \hat{z})\, dc \; . \tag{5.78}$$

However, the boundary conditions on C are

$$(\hat{z} \cdot \underline{E}) = -\eta\; \underline{H} \cdot (\hat{n} \wedge \hat{z})$$
$$\underline{E} \cdot (\hat{n} \wedge \hat{z}) = \eta\, (\hat{z} \cdot \underline{H}) \;, \tag{5.79}$$

and thus

$$(\hat{z} \cdot \underline{H}') \left[\underline{E} \cdot (\hat{n} \wedge \hat{z})\right] = (\hat{z} \cdot \underline{H}) \left[\underline{E}' \cdot (\hat{n} \wedge \hat{z})\right] \;,$$
$$(\hat{z} \cdot E') \left[\underline{H} \cdot (\hat{n} \wedge \hat{z})\right] = (\hat{z} \cdot E) \left[\underline{H}' \cdot (\hat{n} \wedge \hat{z})\right] \; . \tag{5.80}$$

Therefore

$$\int_A \underline{\nabla}_t \cdot (\underline{E} \wedge \underline{H}' - \underline{E}' \wedge \underline{H})\, dS \equiv 0 \;, \tag{5.81}$$

and when (5.81) is used in conjunction with (5.76), we find

$$\int_A \hat{z} \cdot (\underline{E} \wedge \underline{H}' - \underline{E}' \wedge \underline{H})\, dS = 0 \qquad \text{if } -h \neq h' \; . \tag{5.82}$$

Associated with the mode $(\underline{E}'_t\,,\ E'_z\,;\ \underline{H}'_t\,,\ H'_z)\, e^{ih'z}$ there is also a mode $(\underline{E}'_t,\ -E'_z\,;\ -\underline{H}'_t,\ H'_z)\, e^{-ih'z}$; therefore, we have

$$\int_A \hat{z} \cdot (\underline{E} \wedge \underline{H}' + \underline{E}' \wedge \underline{H})\, dS = 0 \quad \text{if } h \neq h'. \tag{5.83}$$

Adding the two orthogonality relations (5.82), (5.83) gives

$$\int_A \hat{z}\cdot(\underline{E}\wedge\underline{H}')\,dS = 0 \quad \text{if } h^2 \neq h'^2 \ . \tag{5.84}$$

This relation can be used to determine the modes generated in the duct. Note also that the orghogonality discussed here is valid for uniform waveguides of arbitrary cross sectional shape. For other treatments of orthogonality see Marcuvitz (1954), Kino (1955), Bresler and Marcuvitz (1956, 1957), and Bresler, Joshi and Marcuvitz (1958).

5.7 Determination of the Scattered Field

To obtain the scattered field due to modes reflected from the termination of the duct, we shall employ a Kirchhoff approximation. We begin with

$$\underline{E}(\underline{x}) = \int_S \left[(\hat{n}\wedge\underline{E})\wedge\nabla' G + (\hat{n}\cdot\underline{E})\,\nabla' G + ik(\hat{n}\wedge\underline{H})\,G\right] dS' \ ,$$

$$\underline{H}(\underline{x}) = \int_S \left[(\hat{n}\wedge\underline{H})\wedge\nabla' G + (\hat{n}\cdot H)\,\nabla' G - ik(\hat{n}\wedge\underline{E})\,G\right] dS' \ , \tag{5.85}$$

where the normal vector $\hat{n}$ points into the region of interest and

$$G(\underline{x}, \underline{x}') = \frac{e^{ikR}}{4\pi R} \ , \ R = \left|\underline{x} - \underline{x}'\right| \ . \tag{5.86}$$

Take $S = A + S_1$ where A is the aperture formed by the mouth and S_1 is the outside surface of the duct. We now assume that the currents on S_1 due to modes reflected from the rear of the duct are zero to a first approximation. The surface integrals thereby reduce to integrals over the aperture A. In the far zone we then have

$$\underline{E}^s(\underline{x}) \sim -\frac{e^{ikr}}{4\pi ikr}\,\underline{k}\wedge\int_A \left[(\hat{n}\wedge\underline{H})\wedge\underline{k} + k(\hat{n}\wedge\underline{E})\right] e^{-i\underline{k}\cdot\underline{x}'}\,dS' \ ,$$

$$\underline{H}^s(\underline{x}) \sim \frac{e^{ikr}}{4\pi ikr}\,\underline{k}\wedge\int_A \left[(\hat{n}\wedge\underline{E})\wedge\underline{k} - k(\hat{n}\wedge\underline{H})\right] e^{-i\underline{k}\cdot\underline{x}'}\,dS' \ , \tag{5.87}$$

where $\underline{k} = k(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$ is the wave vector of the radiated wave and $\hat{n} = \hat{z}$. To complete the Kirchhoff approximation, we take the field on A to be comprised of the modes reflected from the termination of the duct.

Physically, the incident field excites modes within the duct which progress toward the termination of the duct. These modes are then reflected from the closed end, return to be reflected from the open end, are again reflected from the closed end, and so on. As a result, oscillations are set up inside the duct, although for each reflection at the open end, energy is radiated into the surrounding medium. In general the modes are damped exponentially as they progress along the axis of the duct, and since we are considering the length L to be large, it will be sufficient as a first approximation to take into account only the first reflection from the termination. The amount of energy launched into each mode can be obtained by matching the normal mode expansion at the mouth to the aperture fields derived in Section 5.1. To accomplish this we use the orthogonality conditions derived in Section 5.6.

For simplification of notation, we shall attach a subscript σ to denote a normal mode, where in general σ may be a multiple index that accounts for the parity of the mode in the x, y variations and also tabulates the doubly infinite set of integers M, N which order the eigenvalues h. In addition, the z dependence is separated out by representing the transverse electric and magnetic fields for a mode as

$$\underline{E}_t = \underline{E}_\sigma(x,y)\, e^{-ih_\sigma z}$$
$$\underline{H}_t = \underline{H}_\sigma(x,y)\, e^{-ih_\sigma z} \qquad (5.88)$$

where $\underline{H}_\sigma(x,y)$ may be obtained from $\underline{E}_\sigma(x,y)$ through Maxwell's equations:

$$H_x = \frac{-1}{hk}\left[\left(\frac{\partial^2}{\partial y^2} - h^2\right) E_y + \frac{\partial^2 E_x}{\partial x \partial y}\right],$$
$$H_y = \frac{1}{hk}\left[\left(\frac{\partial^2}{\partial x^2} - h^2\right) E_x + \frac{\partial^2 E_y}{\partial x \partial y}\right]. \qquad (5.89)$$

Alternative representations for $\underline{E}_\sigma(x, y)$ may be obtained from Section 5.5.

It may be noted from (5.87) where $\hat{n} = \hat{z}$ that the z components of the electromagnetic fields in the integrals are of no consequence in calculating the scattered fields. The transverse fields $(\hat{z} \wedge \underline{E})$ and $(\hat{z} \wedge \underline{H})$ are evaluated at the aperture formed by the open end of the duct and are represented by normal mode expansions:

$$\hat{z} \wedge \underline{E} = - \sum_\sigma \frac{e^{-2ih_\sigma L} (\hat{z} \wedge \underline{E}_\sigma) \int_A \hat{z} \cdot (\underline{E}_A \wedge \underline{H}_\sigma) \, dS}{\int_A \hat{z} \cdot (\underline{E}_\sigma \wedge \underline{H}_\sigma) \, dS},$$

$$\hat{z} \wedge \underline{H} = \sum_\sigma \frac{e^{-2ih_\sigma L} (\hat{z} \wedge \underline{H}_\sigma) \int_A \hat{z} \cdot (\underline{E}_\sigma \wedge \underline{H}_A) \, dS}{\int_A \hat{z} \cdot (\underline{E}_\sigma \wedge \underline{H}_\sigma) \, dS} \qquad (5.90)$$

where $\underline{E}_A$, $\underline{H}_A$ are the aperture fields (5.16), (5.17) generated by the incident plane wave. The factor $\exp(-2ih_\sigma L)$ in (5.90) accounts for the fact that the modes have traversed the length of the duct and returned to the mouth. Since $\mathrm{Im}\, h_\sigma < 0$, this factor means that the summations in (5.90) may be effectively truncated, for example, at some σ for which $\mathrm{Re}\, h_\sigma = 0$. The minus sign in the expression for $(\hat{z} \wedge \underline{E})$ accounts for the fact that the electric field is reflected by the terminating plate with reflection coefficient (-1). The scattered field due to the reflected interior modes is obtained upon employing (5.90) for the integrands in (5.87).

APPENDIX A

PROPERTIES OF THE FUNCTIONS $\psi_\pi(\beta)$ AND $\psi(\beta)$

The meromorphic function $\psi_\pi(\beta)$ is defined as

$$\psi_\pi(\beta) = \exp\left\{-\frac{1}{8\pi}\int_0^\beta \frac{\pi \sin v - 2\sqrt{2}\,\pi \sin(v/2) + 2v}{\cos v}\,dv\right\}, \tag{A.1}$$

from which it will be observed that $\psi_\pi(\beta)$ is an even function of β whose logarithmic derivative is given by

$$\frac{\psi_\pi'(\beta)}{\psi_\pi(\beta)} = -\frac{1}{8}\frac{\sin\beta}{\cos\beta} + \frac{\sqrt{2}}{4}\frac{\sin(\beta/2)}{\cos\beta} - \frac{1}{4\pi}\frac{\beta}{\cos\beta}. \tag{A.2}$$

By means of the elementary integrals

$$\int_0^\beta \frac{\sin v}{\cos v}\,dv = -\ln(\cos\beta),$$

$$\sqrt{2}\int_0^\beta \frac{\sin(v/2)}{\cos v}\,dv = -\ln\left[\frac{\sqrt{2}\cos(\beta/2) - 1}{\sqrt{2}\cos(\beta/2) + 1}\cdot\frac{\sqrt{2}+1}{\sqrt{2}-1}\right],$$

we obtain the following alternative representations for $\psi_\pi(\beta)$:

$$\psi_\pi(\beta) = \left[\frac{\sqrt{2}\cos(\beta/2)+1}{\sqrt{2}+1}\right]^{\frac{1}{2}}\frac{1}{(\cos\beta)^{1/8}}\exp\left\{-\frac{1}{4\pi}\int_0^\beta \frac{v\,dv}{\cos v}\right\}, \tag{A.3}$$

$$\psi_\pi(\beta) = \left[\frac{\sqrt{2}\cos(\beta/2)+1}{\sqrt{2}+1}\right]^{\frac{1}{2}}\exp\left\{\frac{1}{8\pi}\int_0^\beta \frac{\pi\sin v - 2v}{\cos v}\,dv\right\}. \tag{A.4}$$

When $|\beta| < (\pi/2)$, the integral in (A.3) can be expanded as

$$\int_0^\beta \frac{v\,dv}{\cos v} = \frac{\beta^2}{2} + \frac{1}{2}\frac{\beta^4}{4} + \frac{5}{24}\frac{\beta^6}{6} + \frac{61}{720}\frac{\beta^8}{8} + \ldots + \frac{(-)^n E_{2n}}{(2n)!}\frac{\beta^{2n+2}}{2n+2} + \ldots, \tag{A.5}$$

where E_{2n} are the Euler numbers. When $\beta = i\infty$, we have (Gröbner and Hofreiter 1949)

$$\frac{1}{\pi}\int_0^{i\infty}\frac{v\,dv}{\cos v} = -\frac{1}{\pi}\int_0^\infty \frac{x\,dx}{\cosh x} = -\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} = -b, \tag{A.6}$$

where b = $(2/\pi)K$ with K = 0.9159656... (Catalan's constant). On the other hand, when $\beta = (\pi/2)$, we employ (A.4) with a change of integration variable $v = (\pi/2) - u$ to find

$$\psi_\pi(\pi/2) = \left[\frac{2}{\sqrt{2}+1}\right]^{\frac{1}{2}} \exp\left\{\frac{1}{8}\int_0^{\pi/2} \frac{\cos u - 1}{\sin u}\, du + \frac{1}{4\pi}\int_0^{\pi/2} \frac{u\, du}{\sin u}\right\}. \tag{A.7}$$

The first integral is elementary and the second integral is (Gröbner and Hofreiter 1949)

$$\frac{1}{\pi}\int_0^{\pi/2} \frac{u\, du}{\sin u} = \frac{2}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = b. \tag{A.8}$$

We obtain then

$$\psi_\pi(\pi/2) = \left[\frac{2^{\frac{1}{2}}}{\sqrt{2}+1}\, e^{b/2}\right]^{\frac{1}{4}}. \tag{A.9}$$

It is easy to verify the following fundamental identity (Maliuzhinets 1958)

$$\psi_\pi(\beta + \tfrac{1}{2}\pi)\psi_\pi(\beta - \tfrac{1}{2}\pi) = [\psi_\pi(\pi/2)]^2 \cos(\beta/4), \tag{A.10}$$

and by successive application of (A.10) one obtains

$$\psi_\pi(\beta + \pi)\psi_\pi(\beta - \pi) = \frac{1}{2}\frac{[\psi_\pi(\pi/2)]^4}{[\psi_\pi(\beta)]^2}[\cos(\beta/2) + \cos(\pi/4)], \tag{A.11}$$

$$\psi_\pi\left(\beta + \frac{3\pi}{2}\right)\psi_\pi\left(\beta - \frac{3\pi}{2}\right) = \tfrac{1}{2}[\psi_\pi(\pi/2)]^2 \frac{\cos(\beta/2)}{\cos(\beta/4)}. \tag{A.12}$$

From this last equation we observe that the zeros of $\psi_\pi(\beta)$ which are closest to the point $\beta = 0$ and the corresponding poles are the points $\beta = \pm(5\pi/2)$ and $\beta = \pm(7\pi/2)$, respectively. From Eq. (A.10) one also derives

$$\frac{\psi_\pi(\beta + \pi)}{\psi_\pi(\beta - \pi)} = \frac{\cos(\frac{1}{4}\beta + \frac{1}{8}\pi)}{\cos(\frac{1}{4}\beta - \frac{1}{8}\pi)}, \tag{A.13}$$

$$\frac{\psi_\pi(\beta + 2\pi)}{\psi_\pi(\beta - 2\pi)} = \cot(\tfrac{1}{2}\beta + \tfrac{1}{4}\pi). \tag{A.14}$$

The function $\psi(\beta)$ is expressed in terms of the function $\psi_\pi(\beta)$ by the product :

$$\psi(\beta) = \psi_\pi(\beta + \pi + \alpha_1)\psi_\pi(\beta + \pi - \alpha_1)\psi_\pi(\beta - \pi - \alpha_2)\psi_\pi(\beta - \pi + \alpha_2), \tag{A.15}$$

from which, by means of (A.14), we derive

$$\frac{\psi(\pi+\beta)}{\psi(\pi-\beta)}=\frac{\psi_\pi(\beta+\alpha_1+2\pi)\psi_\pi(\beta-\alpha_1+2\pi)}{\psi_\pi(\beta+\alpha_1+2\pi)\psi_\pi(\beta-\alpha_1-2\pi)}$$

$$=\cot\left(\tfrac{1}{2}\beta+\tfrac{1}{2}\alpha_1+\tfrac{1}{4}\pi\right)\cot\left(\tfrac{1}{2}\beta-\tfrac{1}{2}\alpha_1+\tfrac{1}{4}\pi\right)=\frac{\cos\alpha_1-\sin\beta}{\cos\alpha_1+\sin\beta}\,, \tag{A.16}$$

and similarly

$$\frac{\psi(-\pi-\beta)}{\psi(-\pi+\beta)}=\frac{\cos\alpha_2-\sin\beta}{\cos\alpha_2+\sin\beta}\;. \tag{A.17}$$

Another identity of interest may be derived through the application of (A.13); in particular,

$$\frac{\psi(\pi-\beta)}{\psi(-\pi+\beta)}=\frac{\left[\cos(\frac{\beta}{2}+\frac{\pi}{4})-\cos(\frac{\alpha_1}{2})\right]\left[\cos(\frac{\beta}{2}-\frac{\pi}{4})+\cos(\frac{\alpha_2}{2})\right]}{\left[\cos(\frac{\beta}{2}-\frac{\pi}{4})+\cos(\frac{\alpha_1}{2})\right]\left[\cos(\frac{\beta}{2}+\frac{\pi}{4})-\cos(\frac{\alpha_2}{2})\right]}\,, \tag{A.18}$$

from which it follows that

$$\frac{\psi(\pi-\beta)}{\psi(-\pi+\beta)}\cdot\frac{\psi(-\pi-\beta)}{\psi(\pi+\beta)}=\frac{\sin\beta+\cos\alpha_1}{\sin\beta-\cos\alpha_1}\cdot\frac{\sin\beta-\cos\alpha_2}{\sin\beta+\cos\alpha_2} \tag{A.19}$$

in agreement with the results in (A.16) and (A.17).

If $\alpha_1=\alpha_2$ (in which case the two surface impedances are equal), Eq. (A.18) reduces to

$$\psi(\pi-\beta)=\psi(-\pi+\beta)\,. \tag{A.20}$$

In this instance $\psi(\beta)$ is related to the "split" functions $K_+(k\cos\beta)$ and $L_+(k\cos\beta)$ of Senior 1952) as follows:

$$K_+(k\cos\beta)=\frac{\cos(\beta/2)}{2\psi(\beta)}\left[\psi_\pi(\tfrac{\pi}{2})\right]^4\sqrt{\cos\alpha_1}\qquad\text{with }\cos\alpha_1=\frac{1}{\eta}\,, \tag{A.21}$$

$$L_+(k\cos\beta)=\frac{\cos(\beta/2)}{2\psi(\beta)}\left[\psi_\pi(\tfrac{\pi}{2})\right]^4\sqrt{\cos\alpha_1}\qquad\text{with }\cos\alpha_1=\eta\,,$$

where the constant η represents the complex impedance of the half-plane. Senior's functions may be defined by the equations

$$K_+(k\cos\beta)K_+(-k\cos\beta)=\frac{\sin\beta}{1+\eta\sin\beta}\,, \qquad (A.23)$$

$$L_+(k\cos\beta)L_+(-k\cos\beta)=\frac{\eta\sin\beta}{\eta+\sin\beta}\,, \qquad (A.24)$$

whereas $\psi(\beta)$ with $\alpha_1=\alpha_2$ satisfies the relation [employ (A.10) and (A.12)]

$$\psi\left(\frac{\pi}{2}-\beta\right)\psi\left(\frac{\pi}{2}+\beta\right)=\frac{1}{8}\left[\psi_\pi\left(\frac{\pi}{2}\right)\right]^8(\cos\beta+\cos\alpha_1)\,. \qquad (A.25)$$

It is easily verified that (A.25) is consistent with (A.21) through (A.24).

For $\alpha_1=\alpha_2=(\pi/2)$, corresponding to a perfect conductor with H polarization, we have

$$\psi(\beta)=\frac{1}{2}\left[\psi_\pi\left(\frac{\pi}{2}\right)\right]\cos(\beta/2)\,; \qquad (A.26)$$

on the other hand, for $\alpha_1=\alpha_2\to\pm i\infty$, corresponding to a perfect conductor with E polarization, one finds asymptotically

$$\psi(\beta)\sim\frac{1}{4}\left[\psi_\pi\left(\frac{\pi}{2}\right)\right]^4 e^{|\alpha_1|/2}\,. \qquad (A.27)$$

These limiting expressions are valuable when checking the results of calculations against known perfectly conducting results.

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13. ABSTRACT

Problems in the high frequency regime of electromagnetic scattering by ducts and open waveguides whose walls obey an impedance boundary condition are investigated. In general, the impedance boundary condition means that the walls are absorbing electromagnetic energy. Three structures of particular interest are treated: the semi-infinite parallel plane waveguide, the semi-infinite circular cylinder, and the rectangular duct of finite length. When the surfaces are perfectly conducting, exact solutions for the first two structures are available, and these solutions are expanded asymptotically to yield term-by-term comparisons with the corresponding results of ray diffraction theory. When the surfaces are absorbing, exact solutions are not available, but the ray treatment is applied to advantage. In the case of the rectangular duct, the main problem is to determine the interior waveguide modes. This modal problem is non-separable due to the assumed impedance boundary conditions; nevertheless, information can be obtained by means of an asymptotic analysis applied to the transcendental eigenvalue equation of infinite order.

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