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FOREWORD

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ABSTRACT

As part of an investigation concerning radar scattering by a rectangular duct lined with absorbing materials, some problems associated with the diffraction of plane electromagnetic waves incident obliquely (with respect to the edge) on an absorbing half plane are considered. The half plane is initially assumed to be governed by an impedance boundary condition wherein the impedances on the upper and lower surfaces of the diffracting screen may differ from one another; however, the investigation then reveals that a rigorous solution appears possible only in the case for which both impedances are identical. This means that in order to apply ray optical techniques to estimate the field generated at the mouth of the duct, it is necessary to consider a duct which is coated on both the interior and exterior surfaces with absorbing materials manifesting closely similar electrical properties. In connection with the modes sustained by a rectangular waveguide whose four walls obey an impedance boundary condition, the boundary conditions are found to be inseparable and no explicit results for the modes are available at present. On the other hand, closed form solutions can be obtained for the modes that exist in a rectangular guide in which two parallel walls are absorbing while the remaining two walls are perfectly conducting. These modes are studied as a preliminary to the more difficult fully-lined duct problem. Also briefly discussed are the modes sustained in a circular waveguide whose wall obeys an impedance boundary condition.

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INTRODUCTION

The purpose of this contract is to investigate the radar cross section of an air intake duct similar to that found on jet aircraft. In order to reduce the radar cross section, the interior of the duct is supposed to be lined with a layer of radar absorbing material, thereby providing an absorption of the energy launched into waveguide modes which may be reflected by the termination of the duct and reradiated back toward the radar transmitter. The opening of the duct is taken to be rectangular, although the cross sectional shape gradually changes to circular at the rear of the duct. The tasks of the contract are, therefore, divided into two parts; the first being to calculate the energy launched into the duct and its consequent reradiation based upon an unknown reflection coefficient at the termination of the duct, and the second being to calculate the effect of the gradual cross sectional change of shape of the duct on the waveguide modes carrying the energy. From this knowledge, it is hoped to produce a good estimate for the radar cross section due to the interior of the duct.

The theoretical approach we adopt is to use ray optical techniques to calculate the field generated at the mouth of the duct and from this information to determine the energy launched into the waveguide modes. As a first approximation to this very complicated problem we have previously studied (Bowman and Weston, 1968) the two-dimensional problem of a plane wave incident upon a semi-infinite parallel plane waveguide wherein the outside surfaces of the guide are perfectly conducting and the interior surfaces of the guide obey an impedance boundary condition. Ray optical techniques were employed to obtain both the field generated at the mouth of the guide and the field scattered into the far zone by the edges forming the waveguide opening. The ray treatment is intimately dependent on the known solution for plane wave diffraction by a half plane with arbitrary face impedances (Maliuzhinets, 1958, 1960). The ray optical methods were also applied to the case of a perfectly conducting parallel plane waveguide for which an exact solution is is available, and the results were compared with the asymptotic expansion of the exact solution. It was found that the two methods yield complete agreement for the primary diffraction, along with the first and second interaction contributions, but that for each successive interaction after the second, the ray optical result underestimates the asymptotic result obtained from the exact solution. The higher order interaction contributions are, however, small in magnitude, and the ray treatment is therefore expected to yield accurate results.

For application to a rectangular duct, the ray optical procedure must be extended to include the case of oblique incidence on the edges forming the mouth of the duct, and this aspect of the problem will be treated in the present report. We begin by seeking the three-dimensional generalization of the Maliuzhinets (1958, 1960) scalar solution for diffraction by a half plane with arbitrary face impedances. Unfortunately, it developes that such a generalization does not seem possible (by the method employed here) except in the case where the impedances on both sides of the plane are identical (Senior, 1959; Williams, 1960). This means that in order to apply the ray optical technique it is necessary to consider a duct which is coated on both the interior and exterior surfaces with the same absorbing material, or at least with absorbers manifesting closely similar electrical properties. Expressions for the diffracted fields due to a plane electromagnetic wave incident at an oblique angle on a half plane governed by a single constant impedance are presented in the following chapters. The investigation of the induced aperture fields for a fully-lined rectangular duct is near completion, and a detailed derivation will be presented in the next interim report.

In connection with the modes sustained by a fully-lined rectangular guide, the boundary conditions are inseparable and no explicit results for the modes are available at present. On the other hand, closed form solutions can be obtained when two parallel walls of the guide are perfectly conducting. These modes are studied as a preliminary to the more difficult fully-lined duct problem. Also discussed are the modes supported by circular waveguides with absorbing walls.

II

OBLIQUE INCIDENCE ON AN ABSORBING HALF PLANE

The exact solution for the problem of a plane electromagnetic wave incident at an oblique angle on an absorbing half plane was first presented by Senior (1959), who solved the coupled Wiener-Hopf integral equations which determine the currents induced on the surface of the sheet. As pointed out by Senior, the standard technique (Clemmow, 1951) used for the derivation of three-dimensional solutions from known two-dimensional solutions in the case of perfectly conducting cylindrical structures fails to yield correct results when the diffracting structures are imperfectly conducting. This is due to the coupling of TE and TM modes in the presence of dielectric or absorbing cylindrical bodies (see, e.g. Wait, 1955). On the basis of Senior's (1959) investigation, however, Williams (1960) subsequently developed a generalized technique by which the oblique incidence solution may be deduced directly from the known scalar solutions for a half plane with nonzero surface impedance. These treatments of diffraction by a half plane assume that the screen is characterized by a single constant surface impedance η, whereas for the purposes of our investigation it would be desirable to consider a semi-infinite sheet with two face impedances, say $\eta = \eta_1$ on the upper surface and $\eta = \eta_2$ on the lower surface. A particular case of interest would then be that of a perfectly conducting half plane coated on one side with radar absorbing material. Since the scalar solutions for plane-wave scattering by a half plane with arbitrary face impedances are available from Maliuzhinets (1958,1960), we have attempted to carry out the Williams (1960) procedure to obtain the corresponding three-dimensional solutions for oblique incidence. Unfortunately, it has come to light that the technique fails except in the case for which the impedances on both sides of the half plane are identical. The Williams technique appears to be intimately dependent upon the inherent symmetry of the problem with regard to the plane defined by the diffracting screen. Nevertheless, in order to highlight where the procedure breaks down, we shall assume $\eta_1 \neq \eta_2$ until it is no longer feasible to do so, at which point is is necessary to impose the condition $\eta_1 = \eta_2$ in order to gain a rigorous solution. Future efforts may yield a more general method than the Williams technique.

We shall employ natural units with free-space constants ϵ_o, μ_o set equal to unity and suppress the harmonic time dependence $\exp(-i\omega t)$ throughout. The diffraction screen will be taken to occupy the half plane $y = 0$, $x > 0$. In cylindrical coordinates (ρ, ϕ, z), as well as in spherical coordinates (r, θ, ϕ), the upper and lower surfaces of the half plane are prescribed by $\phi = 0$ and $\phi = 2\pi$, respectively. The primary excitation will be due to a plane wave of unit intensity propagating from the direction (θ_o, ϕ_o):

$$\underline{E}^i = (-\hat{x}\cos\phi_o\sin\theta_o - \hat{y}\sin\phi_o\sin\theta_o + \hat{z}\cos\theta_o)e^{-ikS} ,$$
$$\underline{H}^i = (-\hat{x}\sin\phi_o + \hat{y}\cos\phi_o)e^{-ikS} , \tag{2.1}$$

with

$$S = x\cos\phi_o\cos\theta_o + y\sin\phi_o\cos\theta_o + z\sin\theta_o = \rho\cos(\phi - \phi_o)\cos\theta_o + z\sin\theta_o . \tag{2.2}$$

The incident field is thus taken to be a pure TM mode (no H_z^i component); however, the other polarization case (TE incident mode) presents no difficulty in its treatment, and may indeed be obtained upon making use of the natural duality of Maxwell's equations and the impedance boundary conditions. When $\theta_o = 0$, the plane wave is incident normally to the edge of the semi-infinite screen.

Since, as we have mentioned earlier, there is a coupling between outgoing TE and TM modes when the half plane is imperfectly conducting, the total electromagnetic field must be expressed as a linear combination of the two basic types of modes, and these may be derived from the electric and magnetic Hertz potentials Π and $\tilde{\Pi}$ as follows: for the TM (or E) modes:

$$\underline{E} = \left\{ \hat{x} \frac{\partial^2}{\partial x \partial z} + \hat{y} \frac{\partial}{\partial y \partial z} + \hat{z} \left(\frac{\partial^2}{\partial z} + k^2 \right) \right\} \Pi \quad ,$$

$$\underline{H} = \left\{ \hat{x} \frac{\partial}{\partial y} - \hat{y} \frac{\partial}{\partial x} \right\} (-ik) \Pi \, , \tag{2.3}$$

while for the TE (or H) modes:

$$\underline{E} = \left\{ \hat{x} \frac{\partial}{\partial y} - \hat{y} \frac{\partial}{\partial x} \right\} (ik) \tilde{\Pi} \, ,$$

$$\underline{H} = \left\{ \hat{x} \frac{\partial^2}{\partial x \partial z} + \hat{y} \frac{\partial^2}{\partial y \partial z} + \hat{z} \left(\frac{\partial^2}{\partial z^2} + k^2 \right) \right\} \tilde{\Pi} \, . \tag{2.4}$$

The incident plane wave specified by (2.1) and (2.2) is derivable from the Hertz potentials

$$\Pi^i = \frac{e^{-ikS}}{k^2 \cos\theta_o} \, , \quad \tilde{\Pi}^i = 0 \, , \tag{2.5}$$

and in view of the z dependence here and the fact that the diffracting structure itself is two-dimensional, it is reasonable to take the total Hertz potentials in the form

$$\Pi = e^{-ikz \sin\theta_o} U(x,y), \quad \tilde{\Pi} = e^{-ikz \sin\theta_o} \tilde{U}(x,y,) \, , \tag{2.6}$$

where U and $\tilde{U}$ are both solutions of the equation

$$\left(\nabla^2_{xy} + k^2 \cos^2\theta_o \right) U = 0 \, . \tag{2.7}$$

The screen is assumed to be comprised of material of such a kind as to make the total tangential field components satisfy the following impedance boundary condition on the surface:

$$\underline{E} = (\hat{n} \cdot \underline{E}) \hat{n} = \eta \, \hat{n} \wedge \underline{H} \tag{2.8}$$

where $\eta = \eta_1$ on the upper surface, $\eta = \eta_2$ on the lower surface, and $\hat{n}$ denotes the unit outward normal to the surface. The face impedances η_1 and η_2 are complex constants whose real parts, because of energy considerations, must be non-negative. For the rectangular components of the total field the boundary condition (2. 8) is such that

$$E_z = -\eta_1 H_x, \qquad E_x = \eta_1 H_z \qquad \text{on } \phi = 0,$$
$$E_z = \eta_2 H_x, \qquad E_x = -\eta_2 H_z \qquad \text{on } \phi = 2\pi. \tag{2.9}$$

On $\phi = 0$ these boundary conditions lead to the coupled equations

$$k^2 \cos^2\theta_o U = ik\eta_1 \frac{\partial U}{\partial y} + ik\eta_1 \sin\theta_o \frac{\partial \tilde{U}}{\partial x},$$
$$\eta_1 k^2 \cos^2\theta_o \tilde{U} = -ik \sin\theta_o \frac{\partial U}{\partial x} + ik \frac{\partial \tilde{U}}{\partial y}, \tag{2.10}$$

while on $\phi = 2\pi$ the corresponding equations are obtained from (2. 10) upon replacing η_1 by $-\eta_2$.

Following Williams (1960) we can uncouple the boundary conditions on U and $\tilde{U}$ by considering certain linear combinations of the derivatives of these potentials; in particular, one may show by invoking (2.7) and (2.10) that the scalar functions f_1 and f_2 defined by

$$f_1 = i \frac{\partial \tilde{U}}{\partial x} + i \sin\theta_o \frac{\partial U}{\partial y}, \qquad f_2 = -i \frac{\partial U}{\partial x} + i \sin\theta_o \frac{\partial \tilde{U}}{\partial y} \tag{2.11}$$

satisfy the following boundary conditions:

$$\frac{\partial f_1}{\partial y} = -ik\eta_1 f_1, \qquad \frac{\partial f_2}{\partial y} = -\frac{ik}{\eta_1} f_2 \qquad \text{on } \phi = 0,$$
$$\frac{\partial f_1}{\partial y} = ik\eta_2 f_1, \qquad \frac{\partial f_2}{\partial y} = \frac{ik}{\eta_2} f_2 \qquad \text{on } \phi = 2\pi. \tag{2.12}$$

The scalar functions f_1 and f_2 are thus directly related to the solutions of the corresponding two-dimensional diffraction problems for H and E polarization, respectively. The original potentials U and $\tilde{U}$ are related to f_1 and f_2 by Poisson-type differential equations.

$$-i\left(\frac{\partial^2 U}{\partial x^2}+\sin^2\theta_o\,\frac{\partial^2 U}{\partial y^2}\right)=\frac{\partial f_2}{\partial x}-\sin\theta_o\,\frac{\partial f_1}{\partial y}\,,$$

$$i\left(\frac{\partial^2 \tilde{U}}{\partial x^2}+\sin^2\theta_o\,\frac{\partial^2 \tilde{U}}{\partial y^2}\right)=\frac{\partial f_1}{\partial x}+\sin\theta_o\,\frac{\partial f_2}{\partial y} \qquad (2.13)$$

which follow from (2.11). The original boundary-value problem has now been reduced to two independent boundary value problems for f_1 and f_2, and because of (2.13) four arbitrary constants will be involved in the final expressions for U and $\tilde{U}$. It turns out (Senior, 1959) that these constants are determined by the condition that the final solution should contain no plane-wave terms other than the incident and reflected waves of geometrical optics.

In order to obtain expressions for f_1 and f_2 we now turn to a modification of the Maliuzhinets (1958, 1960) two-dimensional solution and consider the function

$$\chi = e^{-ik\cos\theta_o(x\cos\phi_o+y\sin\phi_o)}+\frac{1}{8\pi i}\int_C d\mu\, e^{ik\cos\theta_o(x\cos\mu+|y|\sin\mu)}\,\frac{Q(\mu,\phi_o;y)}{\cos\mu+\cos\phi_o}\,, \qquad (2.14)$$

where

$$Q(\mu,\phi_o;y)=\left[V(\mu,\phi_o)+V(2\pi-\mu,\phi_o)\right]+\mathrm{sgn}(y)\left[V(\mu,\phi_o)-V(2\pi-\mu,\phi_o)\right] \qquad (2.15)$$

with $V(\mu,\phi_o)$ defined as

$$V(\mu,\phi_o)=\frac{\psi_\pi^8(\frac{\pi}{2})\sin\frac{\mu}{2}\sin\frac{\phi_o}{2}}{\psi(\pi-\mu)\,\psi(\pi-\phi_o)}\left\{\left(\frac{1}{2}-\cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2}\right)+\right.$$

$$\left.+\frac{1}{\sqrt{2}}\left(\cos\frac{\mu}{2}+\cos\frac{\phi_o}{2}\right)\left(\cos\frac{\alpha_2}{2}-\cos\frac{\alpha_1}{2}\right)+\cos\frac{\mu}{2}\cos\frac{\phi_o}{2}\right\}. \qquad (2.16)$$

The auxiliary function $\psi(\beta)$ in (2.16) was introduced by Maliuzhinets (1958) and expressed in terms of a special meromorphic function $\psi_\pi(\beta)$ by the product

$$\psi(\beta) = \psi_\pi(\beta+\pi+\alpha_1)\psi_\pi(\beta+\pi-\alpha_1)\psi_\pi(\beta-\pi-\alpha_2)\psi_\pi(\beta-\pi+\alpha_2) \ , \tag{2.17}$$

where $\psi_\pi(\beta)$ has the representation

$$\psi_\pi(\beta) = \exp\left\{ -\frac{1}{8\pi} \int_0^\beta \frac{\pi \sin\nu - 2\pi\sqrt{2}\sin(\nu/2) + 2\nu}{\cos\nu} \, d\nu \right\} \ . \tag{2.18}$$

The important analytical properties of the functions $\psi_\pi(\beta)$ and $\psi(\beta)$ are given in Appendix A. The quantities α_1 and α_2 are complex constants whose real parts lie in the interval $[0, \pi/2]$; these constants are related to the impedances η_1 and η_2, as we shall see. The contour C in (2.14) is shown in Fig. 2-1 and consists of a path in the complex μ plane along which $\cos\mu$ ranges through real values from ∞ to $-\infty$, although it passes above the pole $\mu = \pi - \phi_0$. Now the important feature about the function χ defined above is that it obeys the boundary conditions

$$\frac{\partial\chi}{\partial y} \pm ik\chi\cos\theta_0\cos\alpha_{1,2} = 0 \qquad \text{on } \phi = 0,\ 2\pi \ . \tag{2.19}$$

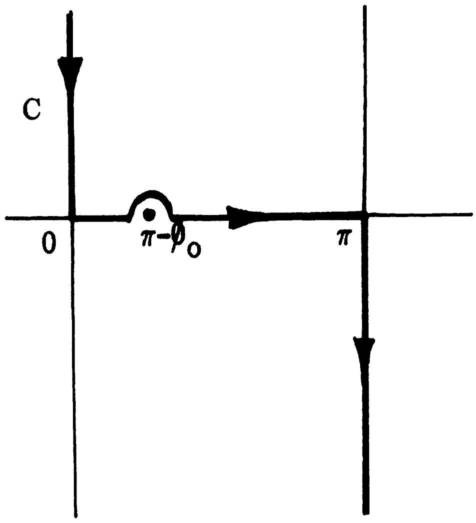

FIG. 2-1: PATH C IN COMPLEX μ PLANE .

By comparison with (2.12) we see that χ satisfies boundary conditions appropriate to f_1 provided we define $\alpha_{1,2}$ by the identification

$$\eta_{1,2} = \cos\theta_0 \cos\alpha^H_{1,2} \ , \tag{2.20}$$

whereas conditions appropriate to f_2 obtain if we make the identification

$$\eta_{1,2} = \frac{1}{\cos\theta_0 \cos\alpha^E_{1,2}} \ . \tag{2.21}$$

The affix E or H is attached according as the boundary conditions describe E- or H-polarization states, and in the sequel, the function χ as well as other functions will be written with such an affix whenever this polarization distinction is necessary. In all instances the notation is employed simply to indicate where (2.20) or (2.21) should be used to determine the constants $\alpha_{1,2}$ from the impedances $\eta_{1,2}$.

The function χ taken by itself is not sufficient to satisfy all the requirements of our problem, such as the requirements at the edge, although it is sufficient to yield the proper incident and reflected fields of geometrical optics. Thus we must seek other solutions to (2.7), subject to the boundary conditions (2.19), but which do not contain plane-wave components. Such a solution would be, for example, the function

$$\frac{\partial\chi}{\partial x} + ik\,\chi \cos\theta_0 \cos\phi_0 \ ; \tag{2.22}$$

however, it turns out that yet another independent solution is required. Williams (1960) obtains two independent solutions by taking separately the even and odd parts with respect to y of the function (2.22). Thus consider the functions (even and odd in y)

$$\chi_e = \frac{1}{8\pi i}\int_C d\mu\, e^{ik\cos\theta_0 (x\cos\mu + |y|\sin\mu)} \left[V(\mu,\phi_0) + V(2\pi-\mu,\phi_0)\right] \ ,$$

$$\chi_o = \frac{\mathrm{sgn}(y)}{8\pi i}\int_C d\mu\, e^{ik\cos\theta_0 (x\cos\mu + |y|\sin\mu)} \left[V(\mu,\phi_0) - V(2\pi-\mu,\phi_0)\right] \ . \tag{2.23}$$

These, however, satisfy the boundary conditions (2.19) only in the case $\alpha_1=\alpha_2$, and it is at this point that the Williams procedure breaks down for the general case of arbitrary face impedances.

Before continuing with the case $\alpha_1=\alpha_2$, let us seek a solution to (2.7) and (2.19) in the form

$$\int_C d\mu\, e^{ik\cos\theta_o(x\cos\mu+|y|\sin\mu)}\, S(\mu) \tag{2.24}$$

where $S(\mu)$ is independent of y. The boundary conditions (2.19) demand that

$$\int_C d\mu\, e^{ikx\cos\theta_o\cos\mu}(\sin\mu+\cos\alpha_{1,2})S(\mu) = 0\,, \tag{2.25}$$

and in order to satisfy (2.25) one may show, with certain analyticity assumptions concerning $S(\mu)$, that the integrands in (2.25) must be odd functions of μ. Therefore, we must have

$$\frac{S(\mu)}{S(-\mu)} = \frac{\sin\mu-\cos\alpha_1}{\sin\mu+\cos\alpha_1} = \frac{\sin\mu-\cos\alpha_2}{\sin\mu+\cos\alpha_2}\,, \tag{2.26}$$

which is possibly only if $\alpha_1=\alpha_2$.

From this point on we assume that $\alpha_1=\alpha_2$, in which case χ_e and χ_o in (2.23) take the form

$$\chi_e = \frac{1}{8\pi i}\int_C d\mu\, e^{ik\cos\theta_o(x\cos\mu+|y|\sin\mu)}\, Q_e(\mu,\phi_o)\,,$$

$$\chi_o = \frac{\mathrm{sgn}(y)}{8\pi i}\int_C d\mu\, e^{ik\cos\theta_o(x\cos\mu+|y|\sin\mu)}\, Q_o(\mu,\phi_o) \tag{2.27}$$

where

$$Q_e(\mu,\phi_o) = -\frac{\sin(\mu/2)\sin(\phi_o/2)}{\psi(\pi-\mu)\,\psi(\pi-\phi_o)}\;\psi_\pi^8\left(\frac{\pi}{2}\right)\cos\alpha_1\,,$$

$$Q_o(\mu,\phi_o) = \frac{\sin(\mu/2)\sin(\phi_o/2)}{\psi(\pi-\mu)\psi(\pi-\phi_o)}\;\psi_\pi^8\left(\frac{\pi}{2}\right)2\cos\frac{\mu}{2}\cos\frac{\phi_o}{2}. \tag{2.28}$$

The function $Q(\mu,\phi_o;y)$ in (2.15) may now be written as

$$Q(\mu,\phi_o;y)=Q_e(\mu,\phi_o)+\mathrm{sgn}(y)Q_o(\mu,\phi_o) \quad . \tag{2.29}$$

The relationships between the Q functions here defined and the "split" functions $K_+(k\cos\mu)$ and $L_+(k\cos\mu)$ of Senior (1952) may be determined from Appendix A.

The scalar functions f_1 and f_2 may be expressed as

$$f_1=\frac{\sin\theta_o\sin\phi_o}{k}\chi^H+C\chi_e^H+B\chi_o^H ,$$

$$f_2=\frac{-i}{k^2\cos\theta_o}\frac{\partial\chi^E}{\partial x}+A\chi_e^E+D\chi_o^E , \tag{2.30}$$

where the form of f_1 and f_2 have been chosen to ensure the appropriate incident field behavior

$$f_1^i=\frac{\sin\theta_o\sin\phi_o}{k}e^{-ik\cos\theta_o(x\cos\phi_o+y\sin\phi_o)}$$

$$f_2^i=-\frac{\cos\phi_o}{k}e^{-ik\cos\theta_o(x\cos\phi_o+y\sin\phi_o)} , \tag{2.31}$$

and where the constants A, . . . , D are closely related to those of Senior (1959). In integral form f_1 and f_2 are given by

$$f_1=f_1^i+\frac{1}{8\pi i}\int_C d\mu\, e^{ik\cos\theta_o(x\cos\mu+|y|\sin\mu)}S^H(\mu,\phi_o;y) ,$$

$$f_2=f_2^i+\frac{1}{8\pi i}\int_C d\mu\, e^{ik\cos\theta_o(x\cos\mu+|y|\sin\mu)}S^E(\mu,\phi_o;y) \tag{2.32}$$

with

$$S^H(\mu,\phi_o;y)=\frac{\sin\theta_o\sin\phi_o}{k}\frac{Q^H(\mu,\phi_o;y)}{\cos\mu+\cos\phi_o}+CQ_e^H(\mu,\phi_o)+BQ_o^H(\mu,\phi_o)\mathrm{sgn}(y),$$

$$S^E(\mu,\phi_o;y)=\frac{\cos\mu}{k}\frac{Q^E(\mu,\phi_o;y)}{\cos\mu+\cos\phi_o}+AQ_e^E(\mu,\phi_o)+DQ_o^E(\mu,\phi_o)\mathrm{sgn}(y) . \tag{2.33}$$

It follows from (2.13) and (2.32) that U and $\tilde{U}$ may be written as

$$U=U^i-\frac{\sec^2\theta_o}{8\pi k^2}\int_C d\mu e^{ik\cos\theta_o(x\cos\mu+|y|\sin\mu)}\frac{G_1(\mu,\phi_o;y)}{\cos^2\mu+\sin^2\theta_o\sin^2\mu},$$

$$\tilde{U}=\frac{\sec^2\theta_o}{8\pi k^2}\int_C d\mu e^{ik\cos\theta_o(x\cos\mu+|y|\sin\mu)}\frac{G_2(\mu,\phi_o;y)}{\cos^2\mu+\sin^2\theta_o\sin^2\mu}, \tag{2.34}$$

where

$$G_1(\mu,\phi_o,y)=ik\cos\theta_o\cos\mu S^E(\mu,\phi_o;y)-ik\cos\theta_o\sin\theta_o\sin\mu S^H(\mu,\phi_o;y)\mathrm{sgn}(y),$$

$$G_2(\mu,\phi_o;y)=ik\cos\theta_o\cos\mu S^H(\mu,\phi_o;y)+ik\cos\theta_o\sin\theta_o\sin\mu S^E(\mu,\phi_o;y)\mathrm{sgn}(y). \tag{2.35}$$

But the integrands in (2.34) have spurious poles in the μ plane at the zeros of

$$\cos^2\mu+\sin^2\theta_o\sin^2\mu=\cos^2\mu\cos^2\theta_o+\sin^2\theta_o=0\ , \tag{2.36}$$

that is, at

$$\mu=\frac{\pi}{2}\mp i\nu\ , \tag{2.37}$$

where ν satisfies the equations

$$\sinh\nu=\tan\theta_o,\qquad \cosh\nu=\sec\theta_o\ . \tag{2.38}$$

In order that U and $\tilde{U}$ be free of plane-wave terms other than the incident and refelcted waves of geometrical optics, we must have

$$G_1(\frac{\pi}{2}\mp i\nu,\phi_o;y)=0,\qquad G_2(\frac{\pi}{2}\mp i\nu,\phi_o;y)=0 \tag{2.39}$$

for all y. Thus (2.39) must hold separately for both the even and the odd parts in y. If we write

$$G_{1,2}(\mu,\phi_o;y)=G^e_{1,2}(\mu,\phi_o)+G^o_{1,2}(\mu,\phi_o)\mathrm{sgn}(y) \tag{2.40}$$

where, for example

$$G^e_1(\mu,\phi_o)=ik\cos\theta_o\cos\mu Q^E_e(\mu,\phi_o)\left[\frac{\cos\mu}{k(\cos\mu+\cos\phi_o)}+A\right]-$$

$$-\ ik\cos\theta_o\sin\theta_o\sin\mu\, Q^H_o(\mu,\phi_o)\left[\frac{\sin\theta_o\sin\phi_o}{k(\cos\mu+\cos\phi_o)}+B\right], \tag{2.41}$$

$$G_1^o(\mu,\phi_o)=ik\cos\theta_o\cos\mu\, Q_o^E(\mu,\phi_o)\left[\frac{\cos\mu}{k(\cos\mu+\cos\phi_o)}+D\right]-$$

$$-\,ik\cos\theta_o\sin\theta_o\sin\mu\, Q_e^H(\mu,\phi_o)\left[\frac{\sin\theta_o\sin\phi_o}{k(\cos\mu+\cos\phi_o)}+C\right], \qquad (2.42)$$

then A and B are determined by

$$G_1^e\left(\frac{\pi}{2}\mp i\nu,\phi_o\right)=0\,, \qquad (2.43)$$

or explicitly,

$$i\,\frac{Q_o^E(\frac{\pi}{2}-i\nu,\phi_o)}{Q_e^H(\frac{\pi}{2}-i\nu,\phi_o)}\left[\frac{i\tan\theta_o}{k(i\tan\theta_o+\cos\phi_o)}+A\right]-\left[\frac{\sin\theta_o\sin\phi_o}{k(i\tan\theta_o+\cos\phi_o)}+B\right]=0\,,$$

$$-i\,\frac{Q_o^E(\frac{\pi}{2}+i\nu,\phi_o)}{Q_e^H(\frac{\pi}{2}+i\nu,\phi_o)}\left[\frac{-i\tan\theta_o}{k(-i\tan\theta_o+\cos\phi_o)}+A\right]-\left[\frac{\sin\theta_o\sin\phi_o}{k(-i\tan\theta_o+\cos\phi_o)}+B\right]=0\,. \qquad (2.44)$$

Solving for A and B, we obtain

$$\left[\frac{Q_e^E(\frac{\pi}{2}+i\nu,\phi_o)}{Q_o^H(\frac{\pi}{2}+i\nu,\phi_o)}\,\frac{\tan\theta_o}{k(-i\tan\theta_o+\cos\phi_o)}-\frac{Q_e^E(\frac{\pi}{2}-i\nu,\phi_o)}{Q_o^H(\frac{\pi}{2}-i\nu,\phi_o)}\,\frac{\tan\theta_o}{k(i\tan\theta_o+\cos\phi_o)}\right]$$

$$=\frac{2\tan\theta_o\sin\theta_o\sin\phi_o}{ik(\tan^2\theta_o+\cos^2\phi_o)}+\frac{1}{i}\left[\frac{Q_e^E(\frac{\pi}{2}+i\nu,\phi_o)}{Q_o^H(\frac{\pi}{2}+i\nu,\phi_o)}+\frac{Q_e^E(\frac{\pi}{2}-i\nu,\phi_o)}{Q_o^H(\frac{\pi}{2}-i\nu,\phi_o)}\right]A\,, \qquad (2.45)$$

$$\left[\frac{Q_o^H(\frac{\pi}{2}+i\nu,\phi_o)}{Q_e^E(\frac{\pi}{2}+i\nu,\phi_o)}\,\frac{\sin\theta_o\sin\phi_o}{k(-i\tan\theta_o+\cos\phi_o)}+\frac{Q_o^H(\frac{\pi}{2}-i\nu,\phi_o)}{Q_e^E(\frac{\pi}{2}-i\nu,\phi_o)}\,\frac{\sin\theta_o\sin\phi_o}{k(i\tan\theta_o+\cos\phi_o)}\right]$$

$$=\frac{-2\tan\theta_o\cos\phi_o}{k(\tan^2\theta_o+\cos^2\phi_o)}-\left[\frac{Q_o^H(\frac{\pi}{2}+i\nu,\phi_o)}{Q_e^E(\frac{\pi}{2}+i\nu,\phi_o)}+\frac{Q_o^H(\frac{\pi}{2}-i\nu,\phi_o)}{Q_e^E(\frac{\pi}{2}-i\nu,\phi_o)}\right]B\,. \qquad (2.46)$$

The constants C and D are similarly determined by

$$G_1^o\left(\frac{\pi}{2}\mp i\nu,\ \phi_o\right)=0\,; \qquad (2.47)$$

however, they may be obtained from (2.45) and (2.46) upon replacing Q_e^E by Q_o^E and Q_o^H by Q_e^H , in which instance A → D and B → C . Finally, it is easy to verify that the vanishing of $G_2(\frac{\pi}{2} \mp i\nu, \phi_o; y)$ yields the same constants A, . . . , D .

The solution for oblique incidence is now completely determined. Upon combining results, we can write the total electromagnetic fields in the form

$$\underline{E} = \underline{E}^i - \frac{\sec\theta_o}{8\pi} e^{-ikz\cos\theta_o} \int_C d\mu\, e^{ik\cos\theta_o (x\cos\mu + |y| \sin\mu)} \frac{\underline{E}(\mu, \phi_o; y)}{\cos^2\mu\cos^2\theta_o + \sin^2\theta_o} ,$$

$$\underline{H} = \underline{H}^i - \frac{\sec\theta_o}{8\pi} e^{-ikz\cos\theta_o} \int_C d\mu\, e^{ik\cos\theta_o (x\cos\mu + |y| \sin\mu)} \frac{\underline{H}(\mu, \phi_o; y)}{\cos^2\mu\cos^2\theta_o + \sin^2\theta_o} , \tag{2.48}$$

where

$$\underline{E}(\mu, \phi_o; y) = \hat{x}\left[\sin\theta_o\cos\mu G_1(\mu, \phi_o; y) + \sin\mu G_2(\mu, \phi_o; y)\mathrm{sgn}(y)\right] + $$
$$+ \hat{y}\left[\sin\theta_o\sin\mu G_1(\mu, \phi_o; y)\mathrm{sgn}(y) - \cos\mu G_2(\mu, \phi_o; y)\right] + \hat{z}\cos\theta_o G_1(\mu, \phi_o; y) , \tag{2.49}$$

$$\underline{H}(\mu, \phi_o; y) = \hat{x}\left[\sin\mu G_1(\mu, \phi_o; y)\mathrm{sgn}(y) - \sin\theta_o\cos\mu G_2(\mu, \phi_o; y)\right] - $$
$$- \hat{y}\left[\cos\mu G_1(\mu, \phi_o; y) + \sin\theta_o\sin\mu G_2(\mu, \phi_o; y)\mathrm{sgn}(y)\right] - \hat{z}\cos\theta_o G_2(\mu, \phi_o; y) . \tag{2.50}$$

Away from the optical boundaries $\phi = \pi \pm \phi_o$, the diffracted field in the far zone is

$$\underline{E}^d = -\frac{e^{-i\pi/4}}{4\cos\theta_o} \frac{e^{-ikz\sin\theta_o + ik\rho\cos\theta_o}}{(2\pi k\rho\cos\theta_o)^{1/2}} \frac{\underline{E}(\phi, \phi_o)}{\cos^2\phi\cos^2\theta_o + \sin^2\theta_o} ,$$

$$\underline{H}^d = -\frac{e^{-i\pi/4}}{4\cos\theta_o} \frac{e^{-ikz\sin\theta_o + ik\rho\cos\theta_o}}{(2\pi k\rho\cos\theta_o)^{1/2}} \frac{\underline{H}(\phi, \phi_o)}{\cos^2\phi\cos^2\theta_o + \sin^2\theta_o} , \tag{2.51}$$

where

$$\underline{E}(\phi, \phi_o) = \hat{x}\left[\sin\theta_o\cos\phi G_1(\phi, \phi_o) + \sin\phi G_2(\phi, \phi_o)\right] + $$
$$+ \hat{y}\left[\sin\theta_o\sin\phi G_1(\phi, \phi_o) - \cos\phi G_2(\phi, \phi_o)\right] + \hat{z}\cos\theta_o G_1(\phi, \phi_o) , \tag{2.52}$$

$$\underline{H}(\phi,\phi_o)=\hat{x}\left[\sin\phi G_1(\phi,\phi_o)-\sin\theta_o\cos\phi G_2(\phi,\phi_o)\right]-$$

$$-\hat{y}\left[\cos\phi G_1(\phi,\phi_o)+\sin\theta_o\sin\phi G_2(\phi,\phi_o)\right]-\hat{z}\cos\phi_o G_2(\phi,\phi_o)\ , \tag{2.53}$$

with

$$G_{1,2}(\phi,\phi_o)=G^e_{1,2}(\phi,\phi_o)+G^o_{1,2}(\phi,\phi_o)\ . \tag{2.54}$$

Equations (2.51) through (2.54) are valid for all ϕ and ϕ_o except $\phi \cong \pi \pm \phi_o$.

As a check, let us consider the perfectly conducting case which corresponds to $\alpha_1^H = (\pi/2)$ and $\alpha_1^E \rightarrow \pm i\infty$. Employing from Appendix A (A.26) and (A.27), along with (2.28), we find

$$Q_e^E(\mu,\phi_o) = -8\sin\frac{\mu}{2}\sin\frac{\phi_o}{2}\ ,\qquad Q_o^E(\mu,\phi_o) = 0,$$

$$Q_o^H(\mu,\phi_o) = 8\cos\frac{\mu}{2}\cos\frac{\phi_o}{2}\ ,\qquad Q_e^H(\mu,\phi_o) = 0\ . \tag{2.55}$$

Only the constants A and B need be evaluated, and with the help of

$$\frac{Q_e^E(\frac{\pi}{2}\pm i\nu,\phi_o)}{Q_o^H(\frac{\pi}{2}\pm i\nu,\phi_o)} = -e^{\pm i\theta_o}\tan\frac{\phi_o}{2}\ , \tag{2.56}$$

we obtain from (2.45) and (2.46), after some trigonometric reduction,

$$A = 0,\qquad B = \frac{-\sin\theta_o\sin\phi_o}{k(1+\cos\phi_o)}\ . \tag{2.57}$$

With these values of the constants we finally have

$$G_1(\mu,\phi_o;y)=ik\cos\theta_o(\cos^2\mu+\sin^2\theta_o\sin^2\mu)\frac{-8\sin(\mu/2)\sin(\phi_o/2)}{\cos\mu+\cos\phi_o}, \tag{2.58}$$

$$G_2(\mu,\phi_o;y) = 0\ ,$$

which leads to agreement with the results for a perfectly conducting half plane.

Equation (2.51) is fundamentally important for the application of ray-optical techniques to problems involving planar structures that are coated with radar absorbing material, and in particular, the diffraction coefficient derived from (2.51) will play an important role in estimating the fields generated at the mouth of a fully-lined rectangular duct. As pointed out in the Introduction, however, the condition $\eta_1 = \eta_2$ restricts our consideration to ducts whose walls are coated on both the interior and exterior surfaces with absorbers of closely similar electrical properties. A detailed derivation of the aperture fields — some aspects of which are not complete at present — will be given in the next interim report.

III

MODES IN RECTANGULAR WAVEGUIDES WITH IMPEDANCE BOUNDARY CONDITIONS

The problem of determining the modal structure of a rectangular waveguide whose four walls are absorbing is complicated by the fact that the boundary conditions are inseparable, and no simple representation for the eigenfunctions is presently available. Contrarywise, closed form solutions can be obtained for the modes that exist in a rectangular waveguide in which two parallel walls obey an impedance boundary condition while the remaining two walls are perfectly conducting.

The geometry for the rectangular waveguide is displayed in Fig. 3-1, where the origin of the Cartesian coordinates is taken at the center of the guide cross section. An impedance boundary condition of the type in Eq. (2.8) will be imposed on the walls of the waveguide, with the top and bottom face characterized by an impedance η_1 and the side faces by an impedance η_2 . In terms of the field

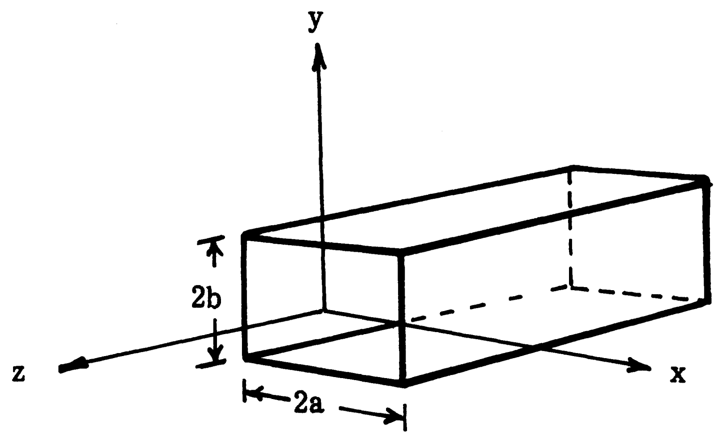

FIG. 3-1: COORDINATES AND DIMENSIONS OF RECTANGULAR WAVEGUIDE.

components, the boundary conditions are as follows:

(i) Top face y=b:
$$E_x = -\eta_1 H_z, \qquad E_z = \eta_1 H_x, \tag{3.1}$$

(ii) Bottom face y=-b:
$$E_x = \eta_1 H_z, \qquad E_z = -\eta_1 H_x, \tag{3.2}$$

(iii) Side face x=a:
$$E_y = \eta_2 H_z, \qquad E_z = -\eta_2 H_y, \tag{3.3}$$

(iv) Side face x=-a:
$$E_y = -\eta_2 H_z, \qquad E_z = \eta_2 H_y . \tag{3.4}$$

The fields will be expressed in terms of the electric and magnetic Hertz potentials Π, $\tilde{\Pi}$ as in Eqs. (2.3) and (2.4), and in addition, these potentials will be taken in the form

$$\Pi = e^{ihz}\chi(x,y), \quad \tilde{\Pi} = e^{ihz}\,\Psi(x,y), \tag{3.5}$$

where h is an unknown parameter and where both χ and Ψ satisfy

$$\left[\nabla^2_{xy} + (k^2 - h^2)\right]\chi = 0 . \tag{3.6}$$

Consider first the boundary conditions on the top and bottom faces. In terms of the potentials χ and Ψ, these boundary conditions become, for $y = \pm b$;

$$\mp i\left(h\frac{\partial\chi}{\partial x} + k\frac{\partial\Psi}{\partial y}\right) = \eta_1 \alpha^2 \Psi ,$$
$$\pm i\,\eta_1\left(h\frac{\partial\Psi}{\partial x} - k\frac{\partial\chi}{\partial y}\right) = \alpha^2 \chi , \tag{3.7}$$

with
$$\alpha^2 = k^2 - h^2 . \tag{3.8}$$

We define two classes of modes:

$$\text{(I).}\quad \Psi = \sin\beta y\, F_1(x), \qquad \chi = \cos\beta y\, F_2(x)$$

$$\text{(II).}\quad \Psi = \cos\beta y\, G_1(x), \qquad \chi = \sin\beta y\, G_2(x)\ , \tag{3.9}$$

where $F_{1,2}$ and $G_{1,2}$ satisfy the differential equation

$$\frac{d^2F}{dx^2} + (\alpha^2-\beta^2)F = 0\ . \tag{3.10}$$

For these modes it is immediately seen that the boundary conditions at y=b are identical to those at y=-b, and in particular, the modes of class (I) satisfy

$$-i\left(h\frac{dF_2}{dx} + k\beta\, F_1\right) = \eta_1\alpha^2F_1\tan\beta b\ ,$$

$$i\eta_1\left(h\,\frac{dF_1}{dx} + k\beta F_2\right) = \alpha^2F_2\cot\beta\, b\ , \tag{3.11}$$

whereas the modes of class (II) satisfy

$$i\eta_1\left(h\frac{dG_1}{dx} - k\beta G_2\right) = \alpha^2G_2\tan\beta b\ ,$$

$$-i\left(h\,\frac{dG_2}{dx} - k\beta\, G_1\right) = \eta_1\,\alpha^2G_1\cot\beta b \tag{3.12}$$

It should be noted that (3.12) may be obtained from (3.11) upon replacing $F_1(x)$ by $-G_2(x)$, $F_2(x)$ by $G_1(x)$, and then replacing η_1 by $(1/\eta_1)$. The modes of class (I) may be further distinguished into two classes depending on the parameter β; in particular Eqs. (3.10) and (3.11) are satisfied in the following two cases:

$$\text{(Ia)}\quad F_1 = -\frac{k}{\beta h}\frac{dF_2}{dx} \quad \text{for} \quad \tan\beta b \equiv -\frac{i\beta}{k\eta_1}\ ,$$

$$\text{(Ib)}\quad F_2 = -\frac{k}{\beta h}\frac{dF_1}{dx} \quad \text{for} \quad \tan\beta b \equiv -\frac{ik}{\beta\eta_1}\ . \tag{3.13}$$

similar results follow for the modes of class (II):

$$\text{(IIa)} \qquad G_2 = \frac{k}{\beta h}\frac{dG_1}{dx} \quad \text{for} \quad \tan\beta b \equiv -\frac{i\beta}{k}\eta_1$$

$$\text{(IIb)} \qquad G_1 = \frac{k}{\beta h}\frac{dG_2}{dx} \quad \text{for} \quad \tan\beta b \equiv -\frac{ik}{\beta}\eta_1 \ . \tag{3.14}$$

Now we consider the boundary conditions on the side walls. These become, for $x = \pm a$:

$$\mp i\left(k\frac{\partial\Psi}{\partial x} - h\frac{\partial\chi}{\partial y}\right) = \eta_2\alpha^2\Psi ,$$

$$\mp i\,\eta_2\left(k\frac{\partial\chi}{\partial x} + h\frac{\partial\Psi}{\partial y}\right) = \alpha^2\chi \ . \tag{3.15}$$

First consider modes of class (Ia); the boundary conditions (3.15) are, for $x = \pm a$:

$$\eta_2\alpha^2 F_1(x) = 0$$

$$k\frac{dF_1(x)}{dx} \mp i\,\eta_2(k^2-\beta^2)F_1(x) = 0. \tag{3.16}$$

Since, by virtue of (3.10), $F_1(x)$ has the general form

$$F_1(x) = A\sin\left(\sqrt{\alpha^2-\beta^2}\,x\right) + B\cos\left(\sqrt{\alpha^2-\beta^2}\,x\right) , \tag{3.17}$$

it follows that a non-trivial solution exists only if $\eta_2=0$, and this is also true for the other modes. When $\eta_2 = 0$, in which case the side walls are perfectly conducting, the modes of class (Ia) or (IIa) may be represented in the form (n=0, 1, 2, ...)

$$\left\{F^n_{1,o} \text{ or } G^n_{1,o}\right\} = \sin\left[\left(n+\frac{1}{2}\right)\frac{\pi x}{a}\right] \quad \text{with} \sqrt{\alpha^2-\beta^2} = \left(n+\frac{1}{2}\right)\frac{\pi}{a} ,$$

$$\left\{F^n_{1,e} \text{ or } G^n_{1,e}\right\} = \cos\left(\frac{n\pi x}{a}\right) \qquad \text{with} \sqrt{\alpha^2-\beta^2} = \frac{n\pi}{a} , \tag{3.18}$$

where F_1 or G_1 is chosen according to the specification of β. Similarly, the modes of class (Ib) or (IIb) may be written as

$$\left\{F^n_{2,o} \text{ or } G^n_{2,o}\right\} = \sin\left(\frac{n\pi x}{a}\right) \text{ with } \sqrt{\alpha^2-\beta^2} = \frac{n\pi}{a} \ ,$$

$$\left\{F^n_{2,e} \text{ or } G^n_{2,e}\right\} = \cos\left[(n+\frac{1}{2})\frac{\pi x}{a}\right] \text{with } \sqrt{\alpha^2-\beta^2} = (n+\frac{1}{2})\frac{\pi}{a} \ . \qquad (3.19$$

The modes are now completely determined, and the eigenvalues h are given by

$$h = \left[k^2-\beta^2-\frac{n^2\pi^2}{a^2}\right]^{1/2} \text{ or } h = \left[k^2-\beta^2-(n+\frac{1}{2})^2\frac{\pi^2}{a^2}\right]^{1/2} \qquad (3.20)$$

depending on the mode classification.

IV

MODES IN A CIRCULAR WAVEGUIDE WITH IMPEDANCE BOUNDARY CONDITIONS

In this chapter we devote attention to the modal structure sustained by a circular waveguide whose wall is lined with absorbing material. The electromagnetic field may again be represented in terms of electric and magnetic Hertz potentials Π and $\tilde{\Pi}$, respectively, and in cylindrical coordinates (ρ, ϕ, z) the relationships of the fields to the potentials are:

$$\underline{E} = \left\{ \hat{\rho}\, \frac{\partial^2}{\partial\rho\,\partial z} + \frac{\hat{\phi}}{\rho}\, \frac{\partial^2}{\partial\phi\,\partial z} + \hat{z} \left(\frac{\partial^2}{\partial z^2} + k^2 \right) \right\} \Pi ,$$

$$\underline{H} = \left\{ \frac{\hat{\rho}}{\rho}\, \frac{\partial}{\partial\phi} - \hat{\phi}\, \frac{\partial}{\partial\rho} \right\} (-ik)\, \Pi \tag{4.1}$$

for TM (or E) modes, and

$$\underline{E} = \left\{ \frac{\hat{\rho}}{\rho}\, \frac{\partial}{\partial\phi} - \hat{\phi}\, \frac{\partial}{\partial\rho} \right\} (ik)\, \tilde{\Pi} ,$$

$$\underline{H} = \left\{ \hat{\rho}\, \frac{\partial^2}{\partial\rho\,\partial z} + \frac{\hat{\phi}}{\rho}\, \frac{\partial^2}{\partial\phi\,\partial z} + \hat{z} \left(\frac{\partial^2}{\partial z^2} + k^2 \right) \right\} \tilde{\Pi} \tag{4.2}$$

for TE (or H) modes. Boundary conditions of the impedance type (2.8) will be imposed; in particular, for $\rho = a$ we demand that

$$E_z = -\eta\, H_\phi , \qquad E_\phi = \eta H_z . \tag{4.3}$$

Because of these conditions, only hybrid EH or HE modes will exist in general.

The Hertz potentials will be represented in the form

$$\Pi(\rho,\phi,z) = e^{ihz}\, \chi(\rho,\phi), \quad \tilde{\Pi}(\rho,\phi,z) = e^{ihz}\, \psi(\rho,\phi,z) , \tag{4.4}$$

where h is an unknown parameter. With this assumption, the boundary conditions become

$$-\frac{i\eta}{\rho}\left\{k\frac{\partial\chi}{\partial\rho}+h\frac{\partial\Psi}{\partial\phi}\right\}=\alpha^2\chi\,,$$

$$-\frac{i}{\rho}\left\{k\frac{\partial\Psi}{\partial\rho}-h\frac{\partial\chi}{\partial\phi}\right\}=\eta\,\alpha^2\Psi\,, \tag{4.5}$$

with $\alpha^2 = k^2-h^2$ as in (3.8). Now it is sufficient to consider modes of the form

$$\chi = A\,e^{in\phi}J_n(\alpha\rho), \quad \Psi = Be^{in\phi}J_n(\alpha\rho)\,, \tag{4.6}$$

in which case the conditions (4.5) yield

$$A\left\{\frac{ka}{u}J_n'(u)+\frac{i}{\eta}J_n(u)\right\}+iB\left\{\frac{nha}{u^2}J_n(u)\right\}=0\,,$$

$$-iA\left\{\frac{nha}{u^2}J_n(u)\right\}+B\left\{\frac{ka}{u}J_n'(u)+i\eta J_n(u)\right\}=0\,, \tag{4.7}$$

where $u = (k^2-h^2)^{1/2}a$. A non-trivial solution of the homogeneous equations in (4.7) exists provided the determinate vanishes; thus we obtain the eigenvalue equation

$$\left[\frac{ka}{u}\frac{J_n'(u)}{J_n(u)}-\frac{i}{\eta}\right]\left[\frac{ka}{u}\frac{J_n'(u)}{J_n(u)}-i\eta\right]=\frac{n^2}{u^2}\left[\left(\frac{ka}{u}\right)^2-1\right] \tag{4.8}$$

which determines the eigenvalues u. In the symmetric case (n=0) the eigenvalue equation uncouples into two equations, one for the E_o mode and one for the H_o mode:

$$\frac{ka}{u}\frac{J_1(u)}{J_o(u)}=\frac{1}{i\eta} \qquad \text{for } E_o \text{ mode}\,,$$

$$\frac{ka}{u}\frac{J_1(u)}{J_o(u)}=\frac{\eta}{i} \qquad \text{for } H_o \text{ mode}\,. \tag{4.9}$$

The modes corresponding to $n \geq 1$ are coupled or hybrid EH modes. In general, the roots of u of the eigenvalue equation will be complex, so that the axial propagation h given by

$$h = k\left[1+\left(\frac{u}{ka}\right)^2\right]^{-1/2} \tag{4.10}$$

will also be complex. The mode will thus attenuate as it proceeds down the guide, that is, provided the restriction Im h > 0 is imposed.

APPENDIX A

PROPERTIES OF THE FUNCTIONS $\psi_\pi(\beta)$ AND $\psi(\beta)$

The meromorphic function $\psi_\pi(\beta)$ is defined as

$$\psi_\pi(\beta) = \exp\left\{-\frac{1}{8\pi}\int_0^\beta \frac{\pi \sin v - 2\sqrt{2}\,\pi \sin(v/2) + 2v}{\cos v} dv\right\}, \tag{A.1}$$

from which it will be observed that $\psi_\pi(\beta)$ is an even function of β whose logarithmic derivative is given by

$$\frac{\psi_\pi'(\beta)}{\psi_\pi(\beta)} = -\frac{1}{8}\frac{\sin\beta}{\cos\beta} + \frac{\sqrt{2}}{4}\frac{\sin(\beta/2)}{\cos\beta} - \frac{1}{4\pi}\frac{\beta}{\cos\beta}. \tag{A.2}$$

By means of the elementary integrals

$$\int_0^\beta \frac{\sin v}{\cos v} dv = -\ln(\cos\beta),$$

$$\sqrt{2}\int_0^\beta \frac{\sin(v/2)}{\cos v} dv = -\ln\left[\frac{\sqrt{2}\cos(\beta/2) - 1}{\sqrt{2}\cos(\beta/2) + 1}\cdot\frac{\sqrt{2}+1}{\sqrt{2}-1}\right],$$

we obtain the following alternative representations for $\psi_\pi(\beta)$:

$$\psi_\pi(\beta) = \left[\frac{\sqrt{2}\cos(\beta/2) + 1}{\sqrt{2}+1}\right]^{\frac{1}{2}} \frac{1}{(\cos\beta)^{1/8}} \exp\left\{-\frac{1}{4\pi}\int_0^\beta \frac{v\,dv}{\cos v}\right\}, \tag{A.3}$$

$$\psi_\pi(\beta) = \left[\frac{\sqrt{2}\cos(\beta/2) + 1}{\sqrt{2}+1}\right]^{\frac{1}{2}} \exp\left\{\frac{1}{8\pi}\int_0^\beta \frac{\pi \sin v - 2v}{\cos v} dv\right\}. \tag{A.4}$$

When $|\beta| < (\pi/2)$, the integral in (A.3) can be expanded as

$$\int_0^\beta \frac{v\,dv}{\cos v} = \frac{\beta^2}{2} + \frac{1}{2}\frac{\beta^4}{4} + \frac{5}{24}\frac{\beta^6}{6} + \frac{61}{720}\frac{\beta^8}{8} + \ldots + \frac{(-)^n E_{2n}}{(2n)!}\frac{\beta^{2n+2}}{2n+2} + \ldots, \tag{A.5}$$

where E_{2n} are the Euler numbers. When $\beta = i\infty$, we have (Gröbner and Hofreiter 1949)

$$\frac{1}{\pi}\int_0^{i\infty} \frac{v\,dv}{\cos v} = -\frac{1}{\pi}\int_0^\infty \frac{x\,dx}{\cosh x} = -\frac{2}{\pi}\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^2} = -b, \tag{A.6}$$

where b = $(2/\pi)$K with K = 0.9159656... (Catalan's constant). On the other hand, when $\beta = (\pi/2)$, we employ (A.4) with a change of integration variable $v = (\pi/2) - u$ to find

$$\psi_\pi(\pi/2) = \left[\frac{2}{\sqrt{2}+1}\right]^{\frac{1}{2}} \exp\left\{\frac{1}{8}\int_0^{\pi/2} \frac{\cos u - 1}{\sin u}\, du + \frac{1}{4\pi}\int_0^{\pi/2} \frac{u\, du}{\sin u}\right\}. \tag{A.7}$$

The first integral is elementary and the second integral is (Gröbner and Hofreiter 1949)

$$\frac{1}{\pi}\int_0^{\pi/2} \frac{u\, du}{\sin u} = \frac{2}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = b. \tag{A.8}$$

We obtain then

$$\psi_\pi(\pi/2) = \left[\frac{2^{\frac{1}{2}}}{\sqrt{2}+1}\, e^{b/2}\right]^{\frac{1}{2}}. \tag{A.9}$$

It is easy to verify the following fundamental identity (Maliuzhinets 1958)

$$\psi_\pi(\beta + \tfrac{1}{2}\pi)\psi_\pi(\beta - \tfrac{1}{2}\pi) = [\psi_\pi(\pi/2)]^2 \cos(\beta/4), \tag{A.10}$$

and by successive application of (A.10) one obtains

$$\psi_\pi(\beta + \pi)\psi_\pi(\beta - \pi) = \frac{1}{2}\frac{[\psi_\pi(\pi/2)]^4}{[\psi_\pi(\beta)]^2}[\cos(\beta/2) + \cos(\pi/4)], \tag{A.11}$$

$$\psi_\pi\left(\beta + \frac{3\pi}{2}\right)\psi_\pi\left(\beta - \frac{3\pi}{2}\right) = \tfrac{1}{2}[\psi_\pi(\pi/2)]^2 \frac{\cos(\beta/2)}{\cos(\beta/4)}. \tag{A.12}$$

From this last equation we observe that the zeros of $\psi_\pi(\beta)$ which are closest to the point $\beta = 0$ and the corresponding poles are the points $\beta = \pm(5\pi/2)$ and $\beta = \pm(7\pi/2)$, respectively. From Eq. (A.10) one also derives

$$\frac{\psi_\pi(\beta + \pi)}{\psi_\pi(\beta - \pi)} = \frac{\cos(\frac{1}{4}\beta + \frac{1}{8}\pi)}{\cos(\frac{1}{4}\beta - \frac{1}{8}\pi)}, \tag{A.13}$$

$$\frac{\psi_\pi(\beta + 2\pi)}{\psi_\pi(\beta - 2\pi)} = \cot(\tfrac{1}{2}\beta + \tfrac{1}{4}\pi). \tag{A.14}$$

The function $\psi(\beta)$ is expressed in terms of the function $\psi_\pi(\beta)$ by the product:

$$\psi(\beta) = \psi_\pi(\beta + \pi + \alpha_1)\psi_\pi(\beta + \pi - \alpha_1)\psi_\pi(\beta - \pi - \alpha_2)\psi_\pi(\beta - \pi + \alpha_2), \tag{A.15}$$

from which, by means of (A.14), we derive

$$\frac{\psi(\pi+\beta)}{\psi(\pi-\beta)}=\frac{\psi_\pi(\beta+\alpha_1+2\pi)\psi_\pi(\beta-\alpha_1+2\pi)}{\psi_\pi(\beta+\alpha_1-2\pi)\psi_\pi(\beta-\alpha_1-2\pi)}$$

$$=\cot\left(\frac{1}{2}\beta+\frac{1}{2}\alpha_1+\frac{1}{4}\pi\right)\cot\left(\frac{1}{2}\beta-\frac{1}{2}\alpha_1+\frac{1}{4}\pi\right)=\frac{\cos\alpha_1-\sin\beta}{\cos\alpha_1+\sin\beta}\ , \tag{A.16}$$

and similarly

$$\frac{\psi(-\pi-\beta)}{\psi(-\pi+\beta)}=\frac{\cos\alpha_2-\sin\beta}{\cos\alpha_2+\sin\beta}\ . \tag{A.17}$$

Another identity of interest may be derived through the application of (A.13); in particular,

$$\frac{\psi(\pi-\beta)}{\psi(-\pi+\beta)}=\frac{\left[\cos\left(\frac{\beta}{2}+\frac{\pi}{4}\right)-\cos\left(\frac{\alpha_1}{2}\right)\right]\left[\cos\left(\frac{\beta}{2}-\frac{\pi}{4}\right)+\cos\left(\frac{\alpha_2}{2}\right)\right]}{\left[\cos\left(\frac{\beta}{2}-\frac{\pi}{4}\right)+\cos\left(\frac{\alpha_1}{2}\right)\right]\left[\cos\left(\frac{\beta}{2}+\frac{\pi}{4}\right)-\cos\left(\frac{\alpha_2}{2}\right)\right]}\ , \tag{A.18}$$

from which it follows that

$$\frac{\psi(\pi-\beta)}{\psi(-\pi+\beta)}\cdot\frac{\psi(-\pi-\beta)}{\psi(\pi+\beta)}=\frac{\sin\beta+\cos\alpha_1}{\sin\beta-\cos\alpha_1}\cdot\frac{\sin\beta-\cos\alpha_2}{\sin\beta+\cos\alpha_2} \tag{A.19}$$

in agreement with the results in (A.16) and (A.17).

If $\alpha_1=\alpha_2$ (in which case the two surface impedances are equal), Eq.(A.18) reduces to

$$\psi(\pi-\beta)=\psi(-\pi+\beta)\ . \tag{A.20}$$

In this instance $\psi(\beta)$ is related to the "split" functions $K_+(k\cos\beta)$ and $L_+(k\cos\beta)$ of Senior 1952) as follows:

$$K_+(k\cos\beta)=\frac{\cos(\beta/2)}{2\psi(\beta)}\left[\psi_\pi\left(\frac{\pi}{2}\right)\right]^4\sqrt{\cos\alpha_1}\qquad\text{with }\cos\alpha_1=\frac{1}{\eta}\ , \tag{A.21}$$

$$L_+(k\cos\beta)=\frac{\cos(\beta/2)}{2\psi(\beta)}\left[\psi_\pi\left(\frac{\pi}{2}\right)\right]^4\sqrt{\cos\alpha_1}\qquad\text{with }\cos\alpha_1=\eta\ ,$$

where the constant η represents the complex impedance of the half-plane . Senior's functions may be defined by the equations

$$K_+(k\cos\beta)K_+(-k\cos\beta) = \frac{\sin\beta}{1+\eta\sin\beta}\ , \tag{A.23}$$

$$L_+(k\cos\beta)L_+(-k\cos\beta) = \frac{\eta\sin\beta}{\eta+\sin\beta}\ , \tag{A.24}$$

whereas $\psi(\beta)$ with $\alpha_1=\alpha_2$ satisfies the relation [employ (A.10) and (A.12)]

$$\psi\left(\frac{\pi}{2}-\beta\right)\psi\left(\frac{\pi}{2}+\beta\right) = \frac{1}{8}\left[\psi_\pi\left(\frac{\pi}{2}\right)\right]^8(\cos\beta+\cos\alpha_1)\ . \tag{A.25}$$

It is easily verified that (A. 25) is consistent with (A.21) through (A. 24).

For $\alpha_1=\alpha_2=(\pi/2)$, corresponding to a perfect conductor with H polarization, we have

$$\psi(\beta) = \frac{1}{2}\left[\psi_\pi\left(\frac{\pi}{2}\right)\right]^4\cos(\beta/2)\ ; \tag{A.26}$$

on the other hand, for $\alpha_1=\alpha_2 \to \pm i\infty$, corresponding to a perfect conductor with E polarization, one finds asymptotically

$$\psi(\beta) \sim \frac{1}{4}\left[\psi_\pi\left(\frac{\pi}{2}\right)\right]^4 e^{|\alpha_1|/2}\ . \tag{A.27}$$

These limiting expressions are valuable when checking the results of calculations against known perfectly conducting results.

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13. ABSTRACT

As part of an investigation concerning radar scattering by a rectangular duct lined with absorbing materials, some problems associated with the diffraction of plane electromagnetic waves incident obliquely (with respect to the edge) on an absorbing half plane are considered. The half plane is initially assumed to be governed by an impedance boundary condition wherein the impedances on the upper and lower surfaces of the diffracting screen may differ from one another; however, the investigation then reveals that a rigorous solution appears possible only in the case for which both impedances are identical. This means that in order to apply ray optical techniques to estimate the field generated at the mouth of the duct, it is necessary to consider a duct which is coated on both the interior and exterior surfaces with absorbing materials manifesting closely similar electrical properties. In connection with the modes sustained by a rectangular waveguide whose four walls obey an impedance boundary condition, the boundary conditions are found to be inseparable and no explicit results for the modes are available at present. On the other hand, closed form solutions can be obtained for the modes that exist in a rectangular guide in which two parallel walls are absorbing while the remaining two walls are perfectly conducting. These modes are studied as a preliminary to the more difficult fully-lined duct problem.

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