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RELATIVISTIC TRANSPORT EQUATIONS FOR PLASMAS

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I. INTRODUCTION

In recent years increasing attention has been given to high temperature plasmas due to their importance in connection with controlled thermo-nuclear research. It is estimated that a minimum temperature of roughly 20 kev would be required for a self sustaining device employing the D-D reaction. Minimum temperature for a D-He^3 device would be even higher ($\sim$ 100 kev). Consideration of plasmas of still higher temperatures is not beyond possibility. Since the rest mass of an electron is $\sim$ 500 kev, one sees that in an operating thermo-nuclear device there would be a considerable number of electrons with kinetic energies comparable to the rest energy so that a relativistic description for electrons would be required. It is, therefore, of interest to formulate and study the balance relations appropriate to such high temperature systems. Several questions which have received no detailed consideration in the study of plasmas present themselves in the context of very high mean energy systems. In particular, the implications of the processes of pair creation and annihilation on irreversible behaviour and the nature of the thermodynamic state require careful consideration. We shall pay particular attention to these aspects in the present study.

In contrast to the non-relativistic case, it appears that the statistical mechanics of the irreversible processes in relativistic systems has been studied to a very limited extent. One of the earlier works is due to Eckart,(1) who has given a macroscopic relativistic theory of the simple fluid. A relativistic form of the Boltzmann equation for a simple gas was first presented by Taub(2). He derived the hydrodynamical equations and showed that some of the ad hoc restrictions imposed upon the macroscopic theory to make it consistent with the special theory of relativity were furnished from this Boltzmann formulation of the theory. A comprehensive, phenomenological description of the equilibrium properties of the relativistic gas was given by Synge[3].

A relativistic form of the collisionless Boltzmann equation was derived and its invariance under Lorentz transformations was proved by Clemmow and Willson[4]. A systematic approach to the derivation of transport equations for spinless charged particles in the presence of external electromagnetic fields was developed by Klimontovich[5, 6]. His approach is, essentially, a direct generalization of Bogoliubov's well-known method[7] to the relativistic case. He obtained a chain of equations for the relativistic distributions and showed that the first approximation to this chain is the covariant form of the Vlasov equation[5]. In the next order the retarded interactions of the charged particles are taken into account. By studying the pair correlation function, Klimontovich derived[6] the relativistic form of the Landau equation - an equation of the Fokker-Planck type. "Exact" relativistic Fokker-Planck coefficients for a uniform plasma including radiation were derived by Simon[8].

It should be noted that all of the attempts mentioned above are based upon classical rather than quantum considerations. Nevertheless, one expects that the classical theory would be satisfactory for the description of phenomena in which specific quantum effects are not important. However, there are several phenomena which can be investigated only through a quantum mechanical formulation of the problem. For instance, exchange effects which arise due to the indistinguishibility of the particles are specific quantum effects. Creation and annihilation of particles cannot be described classically and therefore require a quantum treatment. The difficulties faced by the classical theory of radiation in predicting the black-body spectrum is well-known. In fact, there seems to be no satisfactory way of treating a gas of photons within the classical formalism. In Synge's classical treatment(3) of the photon gas, for example, where photons were considered as material particles with vanishing rest mass, one is led, as was expected, to an equilibrium distribution which is different than that of the black-body spectrum. It is also true that calculations of the transition probabilities for many elementary interactions are most naturally carried out using the methods of the quantum mechanics. It is, therefore, important to find out in what way the calculated transition probabilities for certain processes enter the transport equations which describe systems of particles. To our knowledge, no derivation of transport equations including the short range interactions has been given for relativistic plasmas.

Another point could be mentioned in favor of a quantum mechanical formulation of the transport problem. We are here concerning ourselves with aspects of many body systems which depend crucially on electron dynamics. Therefore, it would seem most desirable that the statistical description be based upon the best microscopic theory of the electron, i.e., Dirac's theory. Then, for example, effects on the system arising from intrinsic magnetic moment of the electron will be accounted for both relativistically or non-relativistically. Furthermore, it is the second-quantized version of Dirac's theory that provides a natural framework for the description of processes involving the creation and destruction of electrons and positrons and their interactions with radiation fields. Finally, within this framework, the effects on the system due to particle statistics (Fermi-Dirac or Bose-Einstein) are naturally accounted for.

It is, therefore, our feeling that our understanding of the relativistic (as well as any) plasmas would be incomplete before an appropriate quantum mechanical description of such systems could be achieved and that any attempt which might shed some light upon this problem would be of interest. In the present study, we attempt in a systematic, self-contained manner to formulate transport equations appropriate to relativistic plasmas based upon a quantum mechanical formalism.

In quantum mechanics, the description of a system of many particles is conveniently achieved by the use of the second quantized formalism. In systems where the number of particles is conserved this is nothing but an alternative expression of the many-particle Schrödinger theory. However, in the relativistic systems where actual creation and annihilation of particles may occur it is necessary to account for the physical phenomena. Therefore, we shall employ this formalism in the present study. In the non-relativistic approximation, the second-quantized formalism has already been used both to derive the Boltzmann-Uehling-Uhlenbeck equation[9], and to obtain kinetic equations for the particle and photon transport in a fully ionized plasma[10]. We shall attempt in the present study to generalize the latter work to the relativistic case.

Another aspect of the problem under consideration is the introduction of the statistical concepts into the description of a system consisting of huge numbers of degrees of freedom consistent with the principles of quantum mechanics. In classical statistical mechanics, for the description of a system of N particles one introduces a distribution function which depends on the coordinates and the momenta of the particles and which satisfies the classical Liouville equation. A quantum mechanical distribution function (QMDF), as a function of the coordinates and the momenta of the particles, was first introduced by Wigner[11]. This function, in contrast to the classical distribution function, is not an observable quantity due to the impossibility of the simultaneous measurement of the coordinates and the momenta in quantum mechanics. However, it has striking similarities to the classical distribution function. For instance, the equation satisfied by QMDF reduces to the Liouville equation as $\hbar \to 0$. Including higher order terms in $\hbar$ one obtains quantum corrections to the classical equation. It should be stressed that although the QMDF itself is not an observable it can be used as an appropriate weight function for calculating the averages of the obervables pertaining to the system. A detailed study of Wigner's QMDF is given by Moyal[12] and others. We shall introduce a relativistic generalization of the Wigner distribution function appropriate to the study of spin 1/2 particles described by the Dirac equation.

In Chapter II, we introduce the formalism which we adopt for the dynamical and the statistical description of the system. Dynamical aspects are described via the conventional methods of the quantum field theory. For the statistical description we use the aforementioned invariant distribution function. Physical meaning and some properties of the distribution are discussed.

In Chapter III, the equation satisfied by the invariant distribution function is derived in the self-consistent field approximation. As $\hbar \to 0$ this equation reduces to the covariant form of the Vlasov equation (4, 5). To the first order in $\hbar$ it includes the spin effects on the transport. We derive a coupled set of equations for the invariant distribution function and the spin distributions ignoring the terms of $O(\hbar^2)$. The meaning of the higher order terms is not evident since to higher orders the resulting equations are no longer invariant under gauge transformations.

In Chapter IV, we derive a Boltzmann equation appropriate to a system of electrons, positrons, and photons in the absence of external electromagnetic fields.

In Chapter V, we discuss some aspects of equilibrium. An H-theorem is proved for the system in which creation and destruction of the particles may occur. Equilibrium distributions appropriate to this system are displayed.

In Chapter VI, we sketch a derivation of a Boltzmann-Vlasov equation within the formalism developed in this study.

Chapter VII is devoted to the conclusions.

Notation

We adopt the summation convention unless explicitly stated otherwise. Repeated Greek indices indicate summation from 1 to 4. Repeated Latin indices indicate summation from 1 to 3. A three-vector is indicated with a bar underneath (e.g. $\underline{A}$); a four-vector is indicated with no additional sign. The four-vector $A = \{A_\mu\}$, $(\mu = 1,2,3,4)$ has the spatial components $\underline{A} = \{A_j\}$, $(j = 1,2,3)$ and the imaginary time component $A_4 = iA_0$. Thus, for example, $\underline{p} \cdot \underline{x} = p_j x_j$ and $p \cdot x = p_\mu x_\mu = \underline{p} \cdot \underline{x} + p_4 x_4 = \underline{p} \cdot \underline{x} - p_0 x_0$.

Dirac matrices, γ_μ , satisfy the commutation relations $[\gamma_\mu, \gamma_\nu]_+ = 2\delta_{\mu\nu}$. We define $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$. We use the following explicit representation of the Dirac matrices

$$\gamma_j = \begin{pmatrix} 0 & -i\sigma_j \\ i\sigma_j & 0 \end{pmatrix} ; \qquad \gamma_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} ,$$

where σ_j are the two by two Pauli matrices.

Commutator brackets are defined by

$$[A, B]_{\pm} = AB \pm BA ,$$

where A, B are any two operators.

II. BASIC FORMULATION

We consider a system of interacting electrons, positrons and photons in the presence of external electromagnetic fields. This is taken, for instance, as an idealization of a fully ionized plasma at high temperatures. We assume that the temperature is high enough to require a relativistic description for electrons but not for the nuclei in the system. Our object herein is to derive transport equations which describe the system under consideration.

A. Dynamical Aspects

The dynamical description of the system will be based upon a Lagrangian for the system which is commonly adopted in the relativistic quantum field theory of the interacting electrons and which is discussed elswhere*. The Lagrangian density is given by

$$L(x) = L_e + L_\gamma + L_I ,$$

where L_e is the Lagrangian density for the free electron-positron field, L_γ is the Lagrangian density for the free electromagnetic field and L_I is the Lagrangian density for the interactions. Explicit expressions for these are**:

$$L_e = -\frac{\hbar c}{2} N \left\{ \bar{\psi}\, \gamma_\mu \frac{\partial \psi}{\partial x_\mu} - \frac{\partial \bar{\psi}}{\partial x_\mu} \gamma_\mu \psi \right\} - mc^2 N \left\{ \bar{\psi}\, \psi \right\} ,$$

$$L_\gamma = -\frac{1}{8\pi} N \left\{ \frac{\partial A_\mu}{\partial x_\nu} \frac{\partial A_\mu}{\partial x_\nu} \right\} ,$$

$$L_I = \frac{j_\mu A_\mu}{c} , \qquad (j_\mu \equiv iec\, N \left\{ \bar{\psi}\, \gamma_\mu \psi \right\}) \tag{2.1}$$

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* See any book on the relativistic quantum field theory.

** CGS Gaussian units are used throughout.

where ψ (and $\bar{\psi} = \psi^{\dagger}\gamma_4$) are four component wave operators describing the electron-positron field and $A_\mu = A_\mu^i + A_\mu^e$. A_μ^i stands for the internal electromagnetic fields which are quantized. A_μ^e stands for the external electromagnetic potentials which are presumed to be given and which satisfy the Lorentz condition, namely,

$$\frac{\partial A_\mu^e}{\partial x_\mu} = 0 . \tag{2.2}$$

The symbol N denotes the usual "normal product" which, acting upon a product of creation and destruction operators, re-orders them so that annihilation operators operate first with a factor of minus one being introduced for each transposition of the anticommuting field operators during the re-ordering.

The field equations in the Heisenberg representation are readily obtained using the variational principle with respect to $\bar{\psi}$, ψ and A_μ^i and are given by

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + \frac{mc}{\hbar}\right) \psi = \frac{ie}{\hbar c} \gamma_\mu A_\mu \psi ,$$

$$\bar{\psi} \left(\gamma_\mu \frac{\overleftarrow{\partial}}{\partial x_\mu} - \frac{mc}{\hbar}\right) = - \frac{ie}{\hbar c} \bar{\psi} \gamma_\mu A_\mu ,$$

$$\Box A_\mu^i = - \frac{4\pi}{c} j_\mu , \tag{2.3}$$

where

$$\Box \equiv \frac{\partial^2}{\partial x_\mu \partial x_\mu} .$$

The physically realizable states, Φ, of the system satisfy the condition that

$$(\Phi, \frac{\partial A^{i}_{\mu}}{\partial x_{\mu}} \Phi) = 0 \; . \qquad (2.4)$$

B. Statistical Aspects

A statistical description of the system will be obtained by the use of an invariant phase-space distribution function. This will be a relativistic generalization of the Wigner distribution(11) appropriate to the study of the spin $\frac{1}{2}$ particles described by the Dirac equation. However, before we introduce this function it seems desirable to review briefly the use of the phase-space distributions within the context of the quantum theory mainly to establish the terminology.

For simplicity, let us consider a system of N identical, non-interacting particles in a static external potential. The Schrödinger equation describing this system is given by

$$i\hbar \frac{\partial \Psi}{\partial t}(\underline{x}^1,\ldots,\underline{x}^N,t) = \left\{\sum_{\sigma=1}^{N}\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^{\sigma}_j \partial x^{\sigma}_j} + U(\underline{x}^{\sigma})\right]\right\}\Psi(\underline{x}^1,\ldots,\underline{x}^N,t), \qquad (2.5)$$

where Ψ is the wave function for the system from which all the possible physical information about the system may be obtained. If Ψ is given at a certain time, then one, in principle, can determine it at any later time using the above equation. Then, using Ψ, one could calculate all the observable quantities pertaining to the system. Specification of Ψ at a given time would require the knowledge of a complete set of constants of motion which could, in principle, be obtained performing a set of compatible measurements on the system. However, it is usually assumed that for reasons of complexity this information about the system is not available to us. One then sets up an ensemble to represent the physical system under consideration. In the present example, each of the systems in the ensemble would have the same number of particles. It is assumed that the state of each of the systems in the ensemble may be specified by giving the complete set of constants of motion. Then, one assumes that the average behaviour of the systems in the ensemble corresponds to the behaviour of the system under consideration. The expected value of an operator O is defined by

$$< O > = \sum_i w_i \int d^3x^1 \ldots d^3x^N \, \Psi_i^* (\underline{x}^1, \ldots, \underline{x}^N, t) \, O \, \Psi_i (\underline{x}^1, \ldots, \underline{x}^N, t) \ , \tag{2.6}$$

where Ψ_i is the wave function describing the i-th system in the ensemble and w_i is its statistical weight[17], and i runs over the systems in the ensemble.

Now, Wigner introduces a function of both coordinates and momenta of the particles by

$$\begin{aligned} & f_N (\underline{x}^1, \ldots, \underline{x}^N ; \ \underline{p}^1, \ldots, \underline{p}^N, t) \\ & = (\pi\hbar)^{-3N} \int \ldots \int d^3z^1 \ldots d^3z^N \, e^{-2i (\underline{p}^1 \cdot \underline{z}^1 + \ldots + \underline{p}^N \cdot \underline{z}^N)/\hbar} \\ & \cdot \sum_i w_i \, \Psi_i^* (\underline{x}^1 - \underline{z}^1, \ldots, \underline{x}^N - \underline{z}^N, t) \, \Psi_i (\underline{x}^1 + \underline{z}^1, \ldots, \underline{x}^N + \underline{z}^N, t) \ . \end{aligned} \tag{2.7}$$

It is easily seen that the integral of f_N over coordinates or momenta gives the ensemble averages of the usual probabilities in momentum or coordinate space, respectively, i.e.,

$$\int f_N\ (\underline{x}^1,\ldots,\underline{x}^N;\ \ \underline{p}^1,\ldots,\underline{p}^N,t)\ d^3p^1\ldots d^3p^N$$
$$= \sum_i w_i\ \left|\Psi_i(\underline{x}^1,\ldots,\underline{x}^N,t)\right|^2\ , \qquad (2.8)$$

$$\int f_N\ (\underline{x}^1,\ldots,\underline{x}^N;\ \ \underline{p}^1,\ldots,\underline{p}^N,t)\ d^3x^1\ldots d^3x^N$$
$$= \sum_i w_i\ \left|\Psi_i(\underline{p}^1,\ldots,\underline{p}^N,t)\right|^2\ , \qquad (2.9)$$

where $\Psi(\underline{p}^1,\ldots,\underline{p}^N,t)$ is the wave function in the momentum space. Properties (8) and (9) must, of course, be required of a sensible phase-space distribution. There is, however, a seemingly disturbing feature of the Wigner distribution. Although it is a real quantity it is not positive-definite. Therefore, it could not be interpreted as an observable probability distribution in the phase-space. In fact, one should not expect the existence of a physical probability distribution (in the fine-grained sense) which is a function of the eigenvalues of a set of non-commuting operators (here coordinates and momenta). It should be emphasized that one is usually interested in the phase-space distributions not for their own sake but for the purpose of employing them as appropriate weight functions to calculate the expected values of the observable quantities. It is in this sense that Wigner distribution proves to be a useful quantity. To be specific, let us consider the quantity

$$T = \int d^3x^1\ldots d^3x^N\ d^3p^1\ldots d^3p^N \left\{\sum_{\sigma=1}^{N}\frac{(p^\sigma)^2}{2m}\right\} f_N\ (\underline{x}^1,\ldots,\underline{x}^N;\ \underline{p}^1,\ldots,\underline{p}^N,t), \qquad (2.10)$$

which is precisely what we would have written down if we wished to calculate classically the total kinetic energy of the system. Using the definition (7) it is easily seen that Equation (10) may be written as

$$T = \sum_i w_i \int d^3x^1 \ldots d^3x^N \, \Psi_i^* (\underline{x}^1, \ldots, \underline{x}^N, t) \left\{ \sum_{\sigma=1}^{N} - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_j^\sigma \, \partial x_j^\sigma} \right\} \Psi_i (\underline{x}^1, \ldots, \underline{x}^N, t), \tag{2.11}$$

which is, by definition (6), the quantum-statistical average of the kinetic energy of the system. This suggests that quantum-statistical averages of the physical quantities pertaining to the system may be calculated by averaging, with respect to f_N, the corresponding classical quantities.

The equation satisfied by f_N is found making use of Equation (5) and is given by

$$\frac{\partial f_N}{\partial t} + \sum_{\sigma=1}^{N} \left\{ \frac{p_j^\sigma}{m} \frac{\partial f_N}{\partial x_j^\sigma} - \frac{2}{\hbar} U(\underline{x}^\sigma) \sin \left(\frac{\hbar}{2} \frac{\overleftarrow{\partial}}{\partial x_j^\sigma} \frac{\overrightarrow{\partial}}{\partial p_j^\sigma} \right) f_N \right\} = 0 , \tag{2.12}$$

where the sin function must be understood in terms of its power series expansion. It is seen that as $\hbar \rightarrow 0$ Equation (12) becomes identical with the Liouville equation for the corresponding classical system, namely,

$$\frac{\partial f_N}{\partial t} + \sum_{\sigma=1}^{N} \left\{ \frac{p_j^\sigma}{m} \frac{\partial f_N}{\partial x_j^\sigma} - \frac{\partial U(\underline{x}^\sigma)}{\partial x_j^\sigma} \frac{\partial f_N}{\partial p_j^\sigma} \right\} = 0 .$$

Higher order terms in $\hbar$ give quantum corrections to the classical Liouville equation.

We should point out that for calculating many quantities of interest the full knowledge of f_N is not required but one can use the reduced distribution functions which are obtained by integrating f_N over all but a few coordinates of interest. Reduced s-particle ($1 \leq s < N$) distribution functions are defined by

$$f_s(\underline{x}^1,\ldots,\underline{x}^s;\ \underline{p}^1,\ldots,\underline{p}^s,t)$$

$$= \int d^3x^{s+1}\ldots d^3x^N\ d^3p^{s+1}\ldots d^3p^N\ f_N\ (\underline{x}^1,\ldots,\underline{x}^N;\ \underline{p}^1,\ldots,\underline{p}^N,t)\ .$$

It is easily seen that the kinetic energy of the system may be expressed as

$$T = \int d^3x\ d^3p\ \frac{p^2}{2m}\ f_1(\underline{x},\underline{p},t)\ .$$

The equation satisfied by $f_1(\underline{x},\underline{p},t)$ is found by integrating Equation (12) over coordinates of particles $2,\ldots,N$ as

$$\frac{\partial f_1}{\partial t} + \frac{p_j}{m}\frac{\partial f_1}{\partial x_j} - \frac{2}{\hbar}\,U(\underline{x})\ \sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial x_j}\frac{\overrightarrow{\partial}}{\partial p_j}\right) f_1 = 0\ . \qquad (2.13)$$

A problem of considerably more interest, however, is the case in which particles are interacting via a two-body potential. It is found in that case that the equation satisfied by f_1 involves the doublet density, f_2, the equation satisfied by f_2 involves the triplet density, f_3 , etc. This gives one a quantum analog of Bogoliubov's chain equations.

So long as we are interested in but a few reduced distribution functions, a reformulation of the above problem using the second-quantized formalism* seems to be more convenient in that only the relevant distribu-

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* For a detailed discussion of this formalism the reader is referred to Reference(13), Chapter VI.

tions enter the discussion. In this formalism, the state function $\Psi(t)$ of the system satisfies, in the Schrödinger representation, the equation

$$i\hbar \frac{\partial \Psi}{\partial t} = \left\{ \int d^3x \, \psi^{\dagger}(\underline{x}) \left(-\frac{\hbar^2}{2m}\right) \nabla^2 \psi(\underline{x}) + \int d^3x \, U(\underline{x}) \, \psi^{\dagger}(\underline{x}) \psi(\underline{x}) \right\} \Psi(t) \,, \tag{2.14}$$

where $\psi(\underline{x})$, $\psi^{\dagger}(\underline{x})$ are operators satisfying the commutation relations

$$[\, \psi(\underline{x}),\ \psi^{\dagger}(\underline{x}')\,]_{-} = \delta(\underline{x}-\underline{x}') \qquad \text{for bosons,}$$

$$[\, \psi(x),\ \psi^{\dagger}(\underline{x}')\,]_{+} = \delta(\underline{x}-\underline{x}') \qquad \text{for fermions.}$$

The singlet distribution is defined by

$$f_1(\underline{x},\underline{p},t) = (\pi\hbar)^{-3} \int d^3z \, e^{-2i\underline{p}\cdot\underline{z}/\hbar} \sum_i w_i \Big(\Psi_i(t),\ \hat{\rho}(\underline{x} - \underline{z};\ \underline{x} + \underline{z})\ \Psi_i(t) \Big),$$

where $\hat{\rho}(\underline{x};\ \underline{x}\ ) = \psi^{\dagger}(\underline{x})\psi(\underline{x}\ )$ is the operator for the number density at $\underline{x}$. The equation satisfied by $f_1(x,p,t)$ may be obtained by using Equation (14) and the commutation relations to yield Equation (13) without recourse to the equation satisfied by the N-particle distribution. As we see, however, nothing new is gained by using the second-quantized formalism in this particular problem. So long as no actual creation and annihilation occurs in the system, this formalism is simply an alternative expression of the N-particle Schrödinger theory. However, in systems in which creation and annihilation of particles occurs so that the total number of a particular kind of particles in the system is no longer a constant of motion, this formalism is necessary.

It is seen that in the definition of the expected value of an operator given by Equation (6) two different averaging processes are involved. One is the quantum averaging with respect to the state of the i-th system in the ensemble and the other is the statistical averaging with respect to the systems in the ensemble. The point of view is taken sometimes that in the quantum mechanical formulation of the transport problem the concept of an ensemble need not be introduced at all. It was argued by van Kampen[14] that one pure quantum mechanical state corresponded to a classical ensemble. Osborn[15] has derived transport equations for particles and photons in plasmas without introducing an ensemble and has argued that the probabilistic nature of the problem was provided by the quantum mechanics itself.

We now proceed with the statistical formulation of our problem. We define an invariant phase-space distribution function for a pure state* by

$$f(x,p) = (\pi\hbar)^{-4} \int d^4z \; e^{-2ip\cdot x/\hbar} \left(\Phi, \; N\left\{ \bar{\psi}(x-z) \; \psi(x+z) \right\} \Phi \right), \tag{2.15}$$

where $d^4z = d^3z \, dz_0$, and Φ denotes the state vector describing the system in the Heisenberg representation. It is intended that $f(x,p)$ could be used for calculating the averages of certain physical quantities pertaining to the system in a fully covariant manner. It will be illustrative at this point to derive an expression which relates the quantum expectation of the

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* When the system is represented by an ensemble we have

$$f(x,p) = (\pi\hbar)^{-4} \int d^4z \; e^{-2ip\cdot x/\hbar} \sum_i w_i\left(\Phi_i, N\left\{ \bar{\psi}(x-z) \; \psi(x+z) \right\} \Phi_i \right) .$$

charge four-current to $f(x,p)$. This will also enable us to give a direct physical interpretation of the quantity $f(x,p)$. The quantum expectation of the charge four-current density is given by

$$J_\mu(x) = (\Phi, j_\mu \Phi) , \qquad (2.16)$$

where $j_\mu \equiv iec\, N \left\{ \bar{\psi}(x)\ \gamma_\mu\ \psi(x) \right\}$. It follows from the first two of Equations (3) that j_μ can be written as

$$j_\mu = j_\mu^o + j_\mu^p ,$$

where

$$j_\mu^o \equiv \frac{e}{2m} N \left\{ \bar{\psi} \left(\frac{\hbar}{i} \frac{\overrightarrow{\partial}}{\partial x_\mu} - \frac{e}{c} A_\mu\right) \psi - \bar{\psi} \left(\frac{\hbar}{i} \frac{\overleftarrow{\partial}}{\partial x_\mu} + \frac{e}{c} A_\mu\right) \psi \right\} , \qquad (2.17)$$

$$j_\mu^p \equiv \frac{e\hbar}{2m} \frac{\partial}{\partial x_\nu} N \left\{ \bar{\psi}\ \sigma_{\mu\nu}\ \psi \right\} , \qquad (2.18)$$

and where $\sigma_{\mu\nu} \equiv \frac{1}{2i} [\ \gamma_\mu, \gamma_\nu\ ]_-$. This particular decomposition of the total four-current operator into the "orbital" and the "polarization" four-currents as indicated by the superscripts o and p , respectively, was, apparently, first given by Gordon (see Reference 16, p.343). One may show that both j_μ^o and j_μ^p satisfy the continuity equations

$$\frac{\partial j_\mu}{\partial x_\mu} = 0 , \qquad \frac{\partial j_\mu^p}{\partial x_\mu} = 0 .$$

By this decomposition we separate from the total current the part due to spin which we do not expect to express solely in terms of $f(x,p)$.

Defining

$$\mathcal{A}_\mu(x) \equiv (\Phi, A_\mu \Phi) = A^e_\mu + (\Phi, A^i_\mu \Phi) ,$$

$$J^o_\mu(x) \equiv (\Phi, j^o_\mu \Phi) ,$$

it is a straightforward matter to show that the quantum expectation of the orbital four-current may be expressed as

$$J^o_\mu(x) = \frac{e}{m} \int d^4p \, (p_\mu - \frac{e}{c}\mathcal{A}_\mu) \, f(x,p) + g_\mu(x) , \qquad (2.19)$$

where

$$g_\mu(x) \equiv - \frac{e^2}{mc} \left\{ (\Phi, N\{\bar{\psi} A^i_\mu \psi\} \Phi) - (\Phi, A^i_\mu \Phi)(\Phi, N\{\bar{\psi}\psi\} \Phi) \right\} .$$

The term $g_\mu(x)$ appears as a correlation term in the expression for the orbital four-current.

We also note that $f(x,p)$ transforms like a scalar under Lorentz transformations as, will be shown in Appendix A.

III. DERIVATION OF THE TRANSPORT EQUATIONS FOR A RELATIVISTIC PLASMA IN THE "SELF-CONSISTENT FIELD" APPROXIMATION

We shall derive here an equation satisfied by $f(x,p)$. To do this, we take the first two of Equations (2.3) and let $x \to x + z$ in the first one and let $x \to x - z$ in the second one to obtain

$$\gamma_\mu \frac{\partial \psi(x+z)}{\partial x_\mu} + \frac{mc}{\hbar} \psi(x+z) = \frac{ie}{\hbar c} \gamma_\mu A_\mu(x+z)\, \psi(x+z) , \tag{3.1}$$

$$\frac{\partial \bar{\psi}(x-z)}{\partial x_\mu} \gamma_\mu - \frac{mc}{\hbar} \bar{\psi}(x-z) = -\frac{ie}{\hbar c} \bar{\psi}(x-z)\, \gamma_\mu A_\mu(x-z) . \tag{3.2}$$

We multiply Equation (1) from the left by $\bar{\psi}(x-z)$ and Equation (2) from the right by $\psi(x+z)$ and add the resulting equations to find*

$$\frac{\partial}{\partial x_\mu} \left\{ \bar{\psi}(x-z) \gamma_\mu \psi(x+z) \right\} - \frac{ie}{\hbar c} \bar{\psi}(x-z) \gamma_\mu \left\{ A_\mu(x+z) - A_\mu(x-z) \right\} \psi(x+z) = 0 . \tag{3.3}$$

Multiplying (3) by $(\pi\hbar)^{-4} e^{-2ip\cdot z/\hbar}$ and integrating over z and taking the expectation with respect to Φ one finds

$$\frac{\partial \mathcal{V}_\mu(x,p)}{\partial x_\mu} + \frac{e}{i\hbar c}(\pi\hbar)^{-4} \int d^4z\, e^{-2ip\cdot z/\hbar} \left(\Phi, \bar{\psi}(x-z)\, i \gamma_\mu \left\{ A_\mu(x+z) - A_\mu(x-z) \right\} \psi(x+z) \Phi \right) = 0 , \tag{3.4}$$

where

$$\mathcal{V}_\mu(x,p) \equiv (\pi\hbar)^{-4} \int d^4z\, e^{-2ip\cdot z/\hbar} \left(\Phi, \bar{\psi}(x-z)\; i\, \gamma_\mu\, \psi(x+z)\, \Phi \right) .$$

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* The method used here is the same as that of obtaining the continuity equation in the x-space, if one takes $z = 0$.

We note that if $g(x)$ is a function of x, then one may write

$$e^{-2ip\cdot z/\hbar}\left\{ g(x+z) \pm g(x-z) \right\}$$
$$= \frac{2}{2i}\, e^{-2ip\cdot z/\hbar} \begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial x} \right) g(x) \quad . \tag{3.5}$$

Using the relation (5) and recalling that $A_\mu = A_\mu^e + A_\mu^i$, Equation (4) may be written as

$$\frac{\partial \mathcal{V}_\mu}{\partial x_\mu} + \frac{2e}{\hbar c} A_\mu^e \sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial p} \right) \mathcal{V}_\mu$$
$$+ \frac{e}{\hbar c}(\pi\hbar)^{-4}\int d^4z\; e^{-2ip\cdot z/\hbar}\left(\Phi, \overline{\psi}(x-z)\gamma_\mu\left\{ A_\mu^i(x+z) - A_\mu^i(x-z)\right\}\psi(x+z)\Phi \right) = 0 \, . \tag{3.6}$$

Equation (6) may be further simplified in the "self-consistent field" approximation, that is, if one assumes

$$\left(\Phi, \overline{\psi}(x-z)\;\gamma_\mu A_\mu^i(x\pm z)\;\psi(x+z)\;\Phi\right) \tag{3.7}$$
$$\simeq \left(\Phi,\; A_\mu^i(x\pm z)\Phi\right)\left(\Phi, \overline{\psi}(x-z)\gamma_\mu\psi(x+z)\Phi\right) .$$

This amounts to ignoring the correlations between the electron-positron field and the quantized internal fields. Consequently, we are also ignoring here the correlations between the particles since they interact via the electromagnetic fields. Equation (7) expresses this approximation on a particular term. We shall make similar approximations on other terms in the following. We wish to note here that the correlation term $g_\mu(x)$ in the expression for the orbital four-current given by Equation (2.19) also vanishes in this approximation. Employing the approximation indicated by Equation (7) in Equation (6) we obtain

$$\frac{\partial \mathcal{V}_\mu}{\partial x_\mu} + \frac{2e}{\hbar c} \mathcal{A}_\mu \sin\left(\frac{\hbar}{2} \frac{\overleftarrow{\partial}}{\partial x} \frac{\overrightarrow{\partial}}{\partial p} \right) \mathcal{V}_\mu(x,p) = 0 \quad . \tag{3.8}$$

We now derive an expression which relates $\mathcal{V}_\mu(x,p)$ to $f(x,p)$. This may be obtained from Equations (1), (2) in the following way. Multiply Equation (1) from the left by $\bar{\psi}(x-z)\gamma_\nu$ and Equation (2) by $\gamma_\nu\, \psi(x+z)$ from the right; then subtract the latter equation from the first one. Making use of the relations

$$[\ \gamma_\mu,\ \gamma_\nu\ ]_+ = 2\ \delta_{\mu\nu}\ , \qquad [\ \gamma_\mu,\ \gamma_\nu\ ]_- = 2i\ \sigma_{\mu\nu}\ ,$$

one obtains after some manipulations,

$$\begin{aligned} &\frac{\partial}{\partial z_\nu} \left\{ \bar{\psi}(x-z)\ \psi(x+z) \right\} - i\ \frac{\partial}{\partial x_\mu} \left\{ \bar{\psi}(x-z)\, \sigma_{\mu\nu}\ \psi(x+z) \right\} \\ &+ \frac{2mc}{\hbar}\ \bar{\psi}(x-z)\ \gamma_\nu\ \psi(x+z) \\ &= \frac{ie}{\hbar c}\ \bar{\psi}(x-z) \left\{ A_\nu(x+z) + A_\nu(x-z) \right\} \psi(x+z) \\ &\quad + \frac{e}{\hbar c}\ \bar{\psi}(x-z)\ \sigma_{\mu\nu} \left\{ A_\mu(x+z) - A_\mu(x-z) \right\} \psi(x+z)\ . \end{aligned} \tag{3.9}$$

We apply to Equation (9) precisely the same operations and manipulations that we have used to derive Equation (8) from Equation (3).

The result is

$$mc\,\mathcal{V}_\nu(x,p) = \left\{ p_\nu - \frac{e}{c}\mathcal{A}_\nu \cos\left(\frac{\hbar}{2} \frac{\overleftarrow{\partial}}{\partial x} \frac{\overrightarrow{\partial}}{\partial p} \right) \right\} f(x,p) \qquad (3.10)$$

$$+ \left\{ \frac{\hbar}{2} \frac{\partial}{\partial x_\mu} + \frac{e}{c}\mathcal{A}_\mu \sin\left(\frac{\hbar}{2} \frac{\overleftarrow{\partial}}{\partial x} \frac{\overrightarrow{\partial}}{\partial p} \right) \right\} \mathcal{O}_{\nu\mu}(x,p) \ ,$$

where

$$\mathcal{O}_{\nu\mu}(x,p) \equiv (\pi\hbar)^{-4} \int d^4z\ e^{-2ip\cdot z/\hbar} \left(\Phi, \bar{\psi}(x-z)\ \sigma_{\nu\mu}\ \psi(x+z)\ \Phi \right) \ .$$

Keeping only the terms to the first order in $\hbar$, one obtains from Equations (8), (10) the following equations:

$$\left(\frac{\partial}{\partial x_\mu} + \frac{e}{c} \frac{\partial \mathcal{A}_\mu}{\partial x_\nu} \frac{\partial}{\partial p_\nu} \right) \mathcal{V}_\mu(x,p) = 0 \ , \qquad (3.11)$$

$$mc\,\mathcal{V}_\nu(x,p) = (p_\nu - \frac{e}{c}\mathcal{A}_\nu) f(x,p) + \frac{\hbar}{2}\left(\frac{\partial}{\partial x_\mu} + \frac{e}{c} \frac{\partial \mathcal{A}_\mu}{\partial x_\rho} \frac{\partial}{\partial x_\rho} \right) \mathcal{O}_{\nu\mu}(x,p) \ . \qquad (3.12)$$

Introduce a change of variables from $\{x,p\}$ to $\{x,\pi\}$ by the relation $p - \frac{e}{c}\mathcal{A} = \pi$, and define $f(x,\pi)$ by

$$f(x,\pi)\ d^4x\ d^4\pi = f(x,p) d^4x\ d^4p$$

and similarly for $\mathcal{V}_\mu(x,\pi)$ and $\mathcal{O}_{\mu\nu}(x,\pi)$. Noting that under this change of variables

$$\left\{ \frac{\partial}{\partial x_\mu} + \frac{e}{c} \frac{\partial \mathcal{A}_\mu}{\partial x_\nu} \frac{\partial}{\partial p_\nu} \right\} \rightarrow \left\{ \frac{\partial}{\partial x_\mu} + \frac{e}{c} \left(\frac{\partial \mathcal{A}_\mu}{\partial x_\nu} - \frac{\partial \mathcal{A}_\nu}{\partial x_\mu} \right) \frac{\partial}{\partial \pi_\nu} \right\} ,$$

and defining

$$F_{\nu\mu} \equiv \frac{\partial \mathcal{A}_\mu}{\partial x_\nu} - \frac{\partial \mathcal{A}_\nu}{\partial x_\mu} , \tag{3.13}$$

we find from Equations (10), (11)

$$\left\{ \frac{\partial}{\partial x_\mu} + \frac{e}{c} F_{\nu\mu} \frac{\partial}{\partial \pi_\nu} \right\} \mathcal{V}_\mu(x,\pi) = 0 , \tag{3.14}$$

$$mc\mathcal{V}_\nu(x,\pi) = \pi_\nu f(x,\pi) + \frac{\hbar}{2} \left\{ \frac{\partial}{\partial x_\mu} + \frac{e}{c} F_{\rho\mu} \frac{\partial}{\partial \pi_\rho} \right\} \sigma_{\nu\mu}(x,\pi) . \tag{3.15}$$

Inserting Equation (15) into Equation (14) and using the anti-symetry of $F_{\nu\mu}$ and $\sigma_{\nu\mu}$ it follows that

$$\left\{ \pi_\nu \frac{\partial}{\partial x_\nu} + \frac{e}{c} F_{\mu\nu} \pi_\nu \frac{\partial}{\partial \pi_\mu} \right\} f(x,\pi) + \frac{eh}{4c} \left\{ \frac{\partial F_{\sigma\mu}}{\partial x_\nu} + \frac{\partial F_{\nu\sigma}}{\partial x_\mu} \right\} \frac{\partial \sigma_{\nu\mu}(x,\pi)}{\partial \pi_\sigma} = 0 . \tag{3.16}$$

By definition (13) one finds that

$$\frac{\partial F_{\sigma\mu}}{\partial x_\nu} + \frac{\partial F_{\nu\sigma}}{\partial x_\mu} + \frac{\partial F_{\mu\nu}}{\partial x_\sigma} = 0 . \tag{3.17}$$

Using Equation (17), Equation (16) reduces to*

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* It should be noted that if terms of $O(\hbar^2)$ and higher were kept, the equation for $f(x,\pi)$ is no longer invariant under gauge transformations, that is, the transformations defined by

$$A_\mu \to A'_\mu = A_\mu + \frac{\partial \Lambda}{\partial x_\mu} ,$$

where Λ satisfies the equation $\Box \Lambda = 0$. That this should be expected may be seen from the very definition of $f(x,\pi)$, as will be shown in Appendix C.

$$\pi_\nu \frac{\partial f}{\partial x_\nu} + \frac{e}{c} F_{\nu\mu}\pi_\mu \frac{\partial f}{\partial \pi_\nu} + \frac{e\hbar}{4c} \frac{\partial F_{\mu\nu}}{\partial x_\sigma} \frac{\partial \mathfrak{S}_{\mu\nu}}{\partial \pi_\sigma} = 0 \quad , \tag{3.18}$$

where $f = f(x,\pi)$, $\mathfrak{S}_{\mu\nu} = \mathfrak{S}_{\mu\nu}(x,\pi)$.

We note that, without the last term, Equation (18) corresponds to the usual Vlasov equation in its covariant form[4,5]. The last term accounts for the interaction of the spin moments of the particles with the electromagnetic fields. It is observed, however, that with this additional term, Equation (18) is not simply an equation for f but involves the quantity $\mathfrak{S}_{\mu\nu}$. In order to obtain a closed set of equations we must find an equation satisfied by $\mathfrak{S}_{\mu\nu}$.

We shall not enter any more of the derivations here since they are somewhat cumbersome. The details are given in Appendix B. However, we shall quote some of the results. We have found it convenient to introduce the quantity

$$S_\mu(x,p) = (\pi\hbar)^{-4} \int d^4z \; e^{-2ip\cdot z/\hbar} \Big(\Phi, N\Big\{\psi(x-z)\, i\gamma_5\gamma_\mu \psi(x+z)\Big\}\, \Phi\Big) .$$

We note that $\mathfrak{S}_{\mu\nu}$ may be expressed in terms of S_μ as (see Appendix B, Equation (13))

$$mc\,\mathfrak{S}_{\mu\nu}(x,\pi) = i\epsilon_{\mu\nu\rho\sigma}\pi_\rho S_\sigma(x,\pi) + O(\hbar) \;,$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the completely antisymmetic Levi-Civita tensor density, and $S_\mu(x,\pi)$ is defined by

$$S_\mu(x,\pi)d^4x \; d^4\pi = S_\mu(x,p)d^4x \; d^4p \; .$$

Using Equation (19), Equation (18) may be written as

$$\pi_\nu \frac{\partial f}{\partial x_\nu} + \frac{e}{c} F_{\nu\mu}\pi_\mu \frac{\partial f}{\partial \pi_\nu} + \frac{e\hbar}{2mc^2} \frac{\partial F^+_{\nu\mu}}{\partial x_\sigma} \pi_\mu \frac{\partial S_\nu}{\partial x_\sigma} = 0 , \tag{3.20}$$

where $F^+_{\nu\mu} \equiv \frac{1}{2i} \epsilon_{\nu\mu\rho\sigma} F_{\rho\sigma}$. We find a subsidiary condition to Equation (20) as (see Appendix B, Equation (20))

$$(\pi^2 + m^2c^2)\ f(x,\pi) - \frac{e\hbar}{mc^2} F^+_{\mu\nu}\ \pi_\nu S_\mu(x,\pi) = 0 . \tag{3.21}$$

Finally, the transport equation satisfied by $S_\mu(x,\pi)$ and its subsidiary condition are given by (Appendix B, Equations (23) and (24))

$$\left\{\pi_\mu \frac{\partial}{\partial x_\mu} + \frac{e}{c} F_{\rho\mu}\pi_\mu \frac{\partial}{\partial \pi_\rho}\right\} S_\nu = \frac{e}{c} F_{\nu\mu}\ S_\mu , \tag{3.22}$$

$$(\pi^2 + m^2c^2)\ S_\nu = 0 , \tag{3.23}$$

where the terms of $O(\hbar)$ are ignored.

Equations (20) and (22) form the coupled set of equations between $f(x,\pi)$ and $S_\mu(x,\pi)$ which we had aimed to obtain. Coupled with these equations are the Maxwell's equations which are satisfied between the expected values of the corresponding field operators. It follows from equations (2.3), (2.4) and (2.21) that for the internal fields we have

$$\frac{\partial F^i_{\mu\nu}}{\partial x_\mu} = - \frac{4\pi}{c} J_\nu(x) , \tag{3.24}$$

$$\frac{\partial F^{i+}_{\mu\nu}}{\partial x_\mu} = 0 . \tag{3.25}$$

We can express $J_\nu(x)$ in terms of $f(x,\pi)$ and $\sigma_{\mu\nu}(x,\pi)$ as

$$J_\nu(x) \simeq \frac{e}{m}\int d^4\pi \ \pi_\nu f(x,\pi) + \frac{e\hbar}{2m}\frac{\partial}{\partial x_\mu}\int d^4\pi \ \sigma_{\nu\mu}(x,\pi) \ , \tag{3.26}$$

where the correlations are ignored.

A. <u>Connection with the Usual Vlasov Equation</u>

We have noted previously that as $\hbar \to 0$ our Equation (18) corresponds to the usual Vlasov equation. However, there is still a slight difference because in Equation (18) all four momentum variables π_μ appear as independent variables whereas in the conventional form the distribution function depends only on the first three components π_j .

As $\hbar \to 0$, Equation (18) and (21) reduces to

$$\pi_\mu \frac{\partial f}{\partial x_\mu} + \frac{e}{c} F_{\nu\mu} \ \pi_\mu \frac{\partial f}{\partial \pi_\nu} = 0 \ , \tag{3.27}$$

$$(\pi^2 + m^2c^2) \ f = 0 \ . \tag{3.28}$$

Equation (28) expresses the fact that $f(x,\pi)$ vanishes except when $(\pi^2 + m^2c^2) = 0$ which is to be expected. In other words, it implies that $f(x,\pi)$ must be of the form

$$f(x,\pi) = \frac{mc^2}{E_\pi}\left\{\delta(\pi_0 - \frac{E_\pi}{c})F_e(x,\underline{\pi}) + \delta(\pi_0 + \frac{E_\pi}{c})F_p(x,-\underline{\pi})\right\} \ , \tag{3.29}$$

where $E_\pi = \sqrt{m^2c^4 + |\underline{\pi}|^2c^2}$, and F_e , F_p are arbitrary functions of their arguments.

Inserting this into Equation (27) and integrating with respect to π_o over the infinitesimal intervals about $\pi_o = \frac{E_\pi}{c}$ and $\pi_o = -\frac{E_\pi}{c}$ one obtains

$$\frac{E_\pi}{mc^2}\frac{\partial F_e}{\partial t} + \frac{\pi_j}{m}\frac{\partial F_e}{\partial x_j} + e\left\{\frac{1}{mc}[\underline{\pi} x \underline{H}]_j + \frac{E_\pi}{mc^2} E_j\right\}\frac{\partial F_e}{\partial \pi_j} = 0 , \tag{3.30}$$

$$\frac{E_\pi}{mc^2}\frac{\partial F_p}{\partial t} + \frac{\pi_j}{m}\frac{\partial F_p}{\partial x_j} - e\left\{\frac{1}{mc}[\underline{\pi} x \underline{H}]_j + \frac{E_\pi}{mc^2} E_j\right\}\frac{\partial F_p}{\partial \pi_j} = 0 , \tag{3.31}$$

where $F_e = F_e(x,\underline{\pi})$, $F_p = F_p(x,\underline{\pi})$, $e = -|e|$.

It is clear that F_e and F_p are interpretable as distribution functions for electrons and positrons, respectively. That is, $F_e(x,\underline{\pi})d^3x\ d^3\pi$ is the expected number of electrons in d^3x about $\underline{x}$, in $d^3\pi$ about π at time t. In the covariant language(5),

$$f(x,\pi)\ |\pi_\mu d\sigma_\mu/mc|\, d^4\pi \tag{3.32}$$

determines the probability that the world-line of a particle intersects the hypersurface element $d\sigma_\mu$ and that the four-momentum has a value in $d^4\pi$ about π. If one chooses $d\sigma_\mu$ in the direction of the time axis, then Equation (32) reduces to

$$f(x,\pi)\ (\pi_4/imc)\ d^3x\ d^4\pi\ , \tag{3.33}$$

which is interpreted as the expected number of electrons in d^3x about $\underline{x}$, in $d^4\pi$ about π at time t. The expectation of any quantity

$Q(x,\pi)$ is, therefore, defined by

$$< Q >_{(t)} = \int Q(x,\pi)\ f(x,\pi)(\pi_4/imc)d^3x\ d^4\pi \ . \tag{3.34}$$

IV. DERIVATION OF THE RELATIVISTIC BOLTZMANN EQUATION

We shall consider here an infinite, homogenous plasma consisting of electrons, positrons and photons in the absence of any external fields.

The derivation will proceed in two stages:

1. Derivation of a Pauli type equation (Master equation).
2. Deduction of the Boltzmann equation from the Pauli equation.

A. Derivation of the "Master" Equation

In the Schrödinger representation the time-dependent state vector of the system satisfies the equation

$$i\hbar \frac{\partial}{\partial t} \mid F(t) > \;=\; H \mid F(t) > , \tag{4.1}$$

where H is the Hamiltonian appropriate to the system.

The Hamiltonian can be written in the form

$$H = H_0 + H_I , \tag{4.2}$$

where H_0 is the Hamiltonian of the free-fields and H_I is the interaction Hamiltonian. In the present problem, H_I is responsible for the self interactions as well as the interactions between particles. Therefore, one must be careful in the application of the perturbation theory. However, we shall not be concerned with these questions here since they have been discussed elsewhere. For a detailed discussion of the formal theory of scattering and its applications in the quantum field theory the reader is referred to reference[13].

The density operator $\hat{\rho}(t)$ is defined for a pure-state by

$$\hat{\rho}(t) \equiv |F(t)\rangle \langle F(t)| \quad . \tag{4.3}$$

If the state of the system is represented by an ensemble of the systems, instead of this one has

$$\hat{\rho}(t) \equiv \sum_i w_i \; |F_i(t)\rangle \langle F_i(t)| \; , \tag{4.4}$$

where i runs over the systems in the ensemble and w_i indicates their statistical weights.

One finds that in either case the density operator satisfies the equation

$$i\hbar \frac{d\hat{\rho}(t)}{dt} = [\, H, \hat{\rho}(t)]_- \quad . \tag{4.5}$$

We denote by $|n\rangle$ a complete set of state vectors in the occupation-number space of the unperturbed Hamiltonian which has the properties that

$$\begin{aligned} H_0|n\rangle &= E_n|n\rangle \; , \\ \langle n|m\rangle &= \delta_{nm} \; , \\ \sum_n |n\rangle \langle n| &= I \quad . \end{aligned} \tag{4.6}$$

We expand the state $|F(t)\rangle$ into this set of states as

$$|F(t)\rangle = \sum_n c_n(t) \; |n\rangle \; . \tag{4.7}$$

Thus, the density operator may be written as

$$\hat{\rho}(t) = \sum_{n,m} c_n(t) \; c_m^*(t) \; |n\rangle \langle m| \text{ for a pure-state,}$$

$$\hat{\rho}(t) = \sum_{n,m} \sum_{i} w_i c_n^i(t)\; c_m^{i*}(t)\; |n\rangle\langle m| \text{ for a mixed-state.} \qquad (4.8)$$

The density matrix in the $|n\rangle$ representation is given by

$$\begin{aligned} \rho_{nm}(t) &= \langle n|\hat{\rho}(t)|m\rangle = c_n(t)\; c_m^{*}(t) \qquad \text{for a pure-state,} \\ &= \sum_i w_i c_n^i(t)\; c_m^{i*}(t) \text{ for a mixed-state} \end{aligned} \qquad (4.9)$$

so that the diagonal elements of the density matrix, $\rho_{nn}(t)$, gives the probability of finding the system in the state $|n\rangle$ at time t.

We shall be interested in finding an expression for the time rate of change of the diagonal elements of the density matrix.

Using the first one of the properties (7) one finds from Equation (5) that

$$\frac{\partial}{\partial t}\rho_{nn} = \frac{2}{\hbar}\,\mathrm{Im}\left\{ \langle n|H_I\hat{\rho}(t)|n\rangle \right\} . \qquad (4.10)$$

Let us consider the "incoming" wave eigenstates, $|n^+\rangle$, which satisfy the Lippmann-Schwinger equation (see Reference 13, p.315), namely,

$$|n^+\rangle = |n\rangle + \lim_{\epsilon\to 0+} \frac{1}{E_n - H_0 + i\epsilon} H_I\; |n^+\rangle . \qquad (4.11)$$

One verifies that

$$\begin{aligned} H\; |n^+\rangle &= E_n|n^+\rangle , \\ \langle n^+|m^+\rangle &= \delta_{nm} , \\ \sum_n |n^+\rangle\langle n^+| &= I - \sum_\beta |\beta\rangle\langle\beta| , \end{aligned} \qquad (4.12)$$

where $|\beta\rangle$ denotes the set of bound states of the total Hamiltonian H.

It is expected that the bound states would have only a small effect on the transport in a relativistic plasma. Thus, ignoring the bound states of the Hamiltonian, one may take

$$\sum_n |n^+\rangle \langle n^+| \simeq I \; .$$

Using the set $|n^+\rangle$ we write Equation (10) as

$$\frac{\partial}{\partial t} \rho_{nn}(t) \simeq \frac{2}{\hbar} \operatorname{Im} \left\{ \sum_{m,p} \langle n|H_I|m^+\rangle \langle m^+|\hat{\rho}(t)|p^+\rangle \langle p^+|n\rangle \right\} \tag{4.13}$$

In view of Equation (4.11) we write

$$\langle m^+|\hat{\rho}(t)|p^+\rangle = \rho_{mp} + a_{mp} \; , \tag{4.14}$$

which also defines a_{mp} . Note that a_{mp} vanishes as $H_I \to 0$. We shall ignore the contribution of these terms in the present analysis.

Finally, we employ the so-called "random phase approximation". One argues that the major contribution to the time rate of change of $\rho_{nn}(t)$ comes from the diagonal elements of the density matrix, and that the effects of the off diagonal elements tend to cancel out when averaged over small intervals of time. When the system is represented by an ensemble, it is presumed that the ensemble is set up with "random a priori phases" such that off diagonal elements of the density matrix vanish at time t . In any case, this quick elimination of the off diagonal elements is far from being satisfactory. Nevertheless, this assumption is usually made in the derivation of the transport equations and a detailed analysis of this point is beyond the limits of the present study. Ignoring the contribution of the off diagonal elements one finds

$$\frac{\partial}{\partial t}\rho_{nn}(t) \simeq \frac{2}{\hbar}\sum_{m}\rho_{mm}(t)\, w_{m\to n} \,. \tag{4.15}$$

The quantity

$$w_{m\to n} \equiv \frac{2}{\hbar}\, \mathrm{Im}\left\{ \langle n|H_I|m^+\rangle \langle m^+|n\rangle \right\} , \tag{4.16}$$

appearing in Equation (15) is equal to the increase per unit time of the probability that a system initially in the state $|m\rangle$ will be found in $|n\rangle$ (see Reference 13, p.324) and may, alternatively, be written as

$$w_{m\to n} = \frac{2}{\hbar}\,\delta_{mn}\, \mathrm{Im}\,\{R_{nm}\} + \frac{2\pi}{\hbar}\,\delta(E_m - E_n)\,|R_{nm}|^2 , \tag{4.17}$$

where

$$R_{nm} = H_{I\,nm} + \sum_{p} \frac{H_{I\,np}\, H_{I\,pm}}{E_m - E_p + i\epsilon} + \ldots \,. \tag{4.18}$$

It may be shown that (see Reference 13, p.325)

$$w_{m\to m} = -\sum_{n\neq m} w_{m\to n} \tag{4.19}$$

which expresses the fact that the rate of transition out of the state $|m\rangle$ must be compensated by a decrease in the amplitude of the state $|m\rangle$. Using Equation (19) one finds from Equation (15) that

$$\frac{\partial}{\partial t}\rho_{nn}(t) \simeq \sum_{\substack{m \\ m\neq n}} \left\{\rho_{mm}(t)\, w_{m\to n} - \rho_{nn}(t)\, w_{n\to m}\right\}, \tag{4.20}$$

where

$$W_{m\to n} = \frac{2\pi}{\hbar}\,\delta(E_m - E_n)\;|R_{nm}|^2 \qquad (m \neq n) .$$

Equation (20) is commonly referred to as Pauli or "master" equation.

B. Deduction of the Boltzmann Equation from the "Master" Equation

For convenience, we shall normalize the system in a large but finite box of volume V with the periodic boundary conditions. We denote by $N_i(\underline{k})$ the operator representing the number of particles of i-th kind in volume V with momentum $\underline{k}$. The subscript i refers here to electrons or positrons of spin up or down and to photons of either polarization. $N_i(\underline{k})$ has the property that it commutes with the Hamiltonian of the free fields, i.e.,

$$[\, N_i(\underline{k}), H_0 \,] = 0 . \tag{4.21}$$

The set of states $|n\rangle$ which has been introduced previously diagonalizes both H_0 and $N_i(\underline{k})$ simultaneously so that in addition to properties (6) one has

$$N_i(\underline{k})\;|n\rangle = n_i(\underline{k})\;|n\rangle , \tag{4.22}$$

where $n_i(\underline{k})$ is the number of particles of i-th kind with momentum $\underline{k}$ in the state $|n\rangle$.

We define

$$f_i(\underline{k},t) \equiv \frac{1}{V}\,(F(t),\, N_i(\underline{k})\, F(t)\,), \tag{4.23}$$

so that $f_i(\underline{k},t)$ gives the expected number of particles of the i-th kind with momentum $\underline{k}$, per unit volume, at time t . Using the expansion (7) one finds

$$f_i(\underline{k},t) = \frac{1}{V} \sum_n P_n(t)\ n_i(\underline{k}) \ , \tag{4.24}$$

where $P_n(t) \equiv \rho_{nn}(t)$.

It follows from Equations (20) and (24) that

$$\frac{\partial f_i(\underline{k},t)}{\partial t} = \sum_{\substack{n,m \\ m \neq n}} P_n(t)\ \Gamma_{n \to m} \left\{ m_i(\underline{k}) - n_i(\underline{k}) \right\} \ , \tag{4.25}$$

where

$$\Gamma_{n \to m} \equiv \frac{w_{n \to m}}{V} = \frac{1}{V} \frac{2\pi}{\hbar}\ \delta(E_m - E_n)\ |R_{mn}|^2 \ . \quad (n \neq m) \tag{4.26}$$

It should be emphazised that the normalization in a finite box of volume V is introduced only for convenience; we shall be interested, eventually, in the limit of the Equation (25) as $V \to \infty$.

The scattering amplitude, S_{mn}, (see Reference 13, p.323) is related to R_{mn} by

$$S_{mn} = -\ 2\pi i\ \delta(E_m - E_n)\ R_{mn} \ . \quad (m \neq n) \tag{4.27}$$

Noting that

$$\lim_{T \to \infty} \frac{1}{T} \left| \int_{-T/2}^{+T/2} e^{i(E_m - E_n)t/\hbar}\ dt \right|^2 = 2\pi\hbar\ \delta(E_m - E_n) \ ,$$

one obtains an alternative expression for $\Gamma_{n \to m}$ as

$$\Gamma_{n \to m} = \lim_{\substack{V \to \infty \\ T \to \infty}} \frac{1}{VT}\ |S_{mn}|^2_{VT}$$

where $\Gamma_{n\to m}$ is interpreted as the transition probability per unit space-time volume that a system initially in the state $|n\rangle$ will be found in the state $|m\rangle$.

In the following we shall restrict ourselves to the second order* perturbation analysis for the calculation of the transition probabilities. We hope it is understood at this point that our purpose is not to re-calculate the already calculated transition probabilities for several collision processes but to indicate the manner in which they enter into a Boltzmann type equation. The S-Matrix, in this approximation, is given by**

$$S^{(2)} = \frac{1}{2!} \left(\frac{e}{\hbar c}\right)^2 \iint d^4x_2 d^4x_1 \; T\left\{N(\bar{\psi}\gamma_\nu A_\nu \psi)_{x_2} \; N(\bar{\psi}\gamma_\mu A_\mu \psi)_{x_1}\right\} ,$$

where T denotes Wick's chronological product, which acting on a set of operators re-orders them such that the earlier time operators operate first with a factor of minus one being introduced for each transposition of the anticommuting fermion operators during the re-ordering. In the present case, no change in sign will result since the fermion operators appear in pairs in each term. The momentum space expansions of the fields in terms of their

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* This is the first non-vanishing order in the present analysis.

** For a systematic and concise discussion of the processes described by $S^{(2)}$ the reader is referred to Reference 18, Chapter 14.

positive and negative frequency parts are given by

$$\psi^{(+)}(x) = \frac{1}{\sqrt{V}} \sum_{\underline{p}} \left(\frac{mc^2}{E_{\underline{p}}}\right)^{1/2} \sum_{s=1}^{2} c_s(\underline{p})\, u^{(s)}(\underline{p})\, e^{ip\cdot x/\hbar} \quad ,$$

$$\psi^{(-)}(x) = \frac{1}{\sqrt{V}} \sum_{\underline{q}} \left(\frac{mc^2}{E_{\underline{p}}}\right)^{1/2} \sum_{r=1}^{2} d_r^{\dagger}(\underline{q})\, u^{(r+2)}(\underline{q})\, e^{-iq\cdot x/\hbar} \quad ,$$

$$\bar{\psi}^{(-)}(x) = \frac{1}{\sqrt{V}} \sum_{\underline{p}} \left(\frac{mc^2}{E_{\underline{p}}}\right)^{1/2} \sum_{s=1}^{2} c_s^{\dagger}(\underline{p})\, \bar{u}^{(s)}(\underline{p})\, e^{-ip\cdot x/\hbar} \quad ,$$

$$\bar{\psi}^{(+)}(x) = \frac{1}{\sqrt{V}} \sum_{\underline{q}} \left(\frac{mc^2}{E_{\underline{q}}}\right)^{1/2} \sum_{r=1}^{2} d_r(\underline{q})\, \bar{u}^{(r+2)}(\underline{q})\, e^{iq\cdot x/\hbar} \quad ,$$

$$A_{\mu}^{(+)}(x) = \frac{1}{\sqrt{V}} \sum_{\underline{k}} \left(\frac{2\pi\hbar c^2}{\omega}\right)^{1/2} \sum_{m=1}^{4} a_m(\underline{k})\, \epsilon_{\mu}^{(m)}(\underline{k})\, e^{ik\cdot x} \quad ,$$

$$A_{\mu}^{(-)}(x) = \frac{1}{\sqrt{V}} \sum_{\underline{k}} \left(\frac{2\pi\hbar c^2}{\omega}\right)^{1/2} \sum_{m=1}^{4} a_m^{\dagger}(\underline{k})\, \epsilon_{\mu}^{(m)*}(\underline{k})\, e^{-ik\cdot x} \quad , \qquad (4.28)$$

where the terms have the same meanings as in Reference 18, and $\omega = c|\underline{k}|$.

Boltzmann Equation for Electrons

The processes which effect the electron transport to the second order in perturbation analysis may be summarized as follows:

1. Electron-electron scattering,
2. Electon-photon scattering,
3. Electron-positron scattering,
4. Two quantum pair annihilation or creation,
5. Pair annihilation or creation in the field of an electron or positron.

To illustrate the method of deducing the Boltzmann equation from Equation (25) we shall consider in some detail the first two of the above processes.

Electron-Electron Scattering (Möller Scattering)

The part of $S^{(2)}$ which describes this process is given by

$$S^{(2)}_{e-e} = \frac{1}{2!} \left(\frac{e}{\hbar c}\right)^2 \iint d^4x_1 d^4x_2 \, N \left\{ (\bar{\psi}^{(-)} \gamma_\nu \underbrace{A_\nu \psi^{(+)})_{x_2} (\bar{\psi}^{(-)} \gamma_\mu A_\mu}} \psi^{(+)})_{x_1} \right\},$$

where

$$\underbrace{A_\nu(x_2) A_\mu}(x_1) = 2\pi\hbar c \, \delta_{\nu\mu} D_F(x_2 - x_1) ,$$

$$D_F(x) = -\frac{2i}{(2\pi)^4} \int d^4k \, \frac{e^{ik\cdot x}}{k^2 - i\epsilon} .$$

Defining,

$$\Gamma^{(e-e)}_{n \to m} = \frac{1}{VT} \left| \langle m | \, S^{(2)}_{e-e} \, | n \rangle \right|^2 ,$$

one can show that

$$\Gamma^{(e-e)}_{n \to m} = (2\pi e^2 \hbar)^2 \frac{VT}{V^4} \sum_{\substack{\underline{p}_2', \underline{p}_1', \underline{p}_2, \underline{p}_1 \\ s_2', s_1', s_2, s_1}} \frac{m^4 c^8}{E_{\underline{p}_2'} E_{\underline{p}_1'} E_{\underline{p}_2} E_{\underline{p}_1}} \left| A^{e-e} \right|^2 \tag{4.29}$$

$$\cdot \{1 - n_e(1')\}\{1 - n_e(2')\} n_e(2) \, n_e(1) \langle m | \ldots, n_e(1')+1, \ldots, n_e(2')+1, \ldots, n_e(2)-1, \ldots, n_e(1)-1, \ldots \rangle$$

where $n_e(i) \equiv n_e(\underline{p}_i, s_i)$, δ_p denotes the four-dimensional Kronecker delta and

$$A^{e-e} \equiv A^{e-e}\,(2';1' \mid 2;1)$$

$$\equiv \left\{ \frac{\left(\bar{u}^{(s_2')}(\underline{p}_2')\gamma_\mu\, u^{(s_2)}(\underline{p}_2)\right)\left(\bar{u}^{(s_1')}(\underline{p}_1')\gamma_\mu u^{(s_1)}(\underline{p}_1)\right)}{(p_1'-p_1)^2} \right. \tag{4.30}$$

$$\left. - \frac{\left(\bar{u}^{(s_1')}(\underline{p}_1')\gamma_\mu\, u^{(s_2)}(\underline{p}_2)\right)\left(\bar{u}^{(s_2')}(\underline{p}_2')\gamma_\mu u^{(s_1)}(\underline{p}_1)\right)}{(p_2'-p_1)^2} \right\} .$$

It may be shown that A^{e-e} possesses the following symmetry properties:

$$A^{e-e}(2';1' \mid 2;1) = - A^{e-e}(1';2' \mid 2;1)$$
$$= A^{e-e*}(2;1 \mid 2';1') . \tag{4.31}$$

It follows from Equation (25) that the contribution to the electron transport due to the electron-electron collisions is given by

$$\frac{\partial f_e}{\partial t}\,(\underline{p},s,t) = \sum_{n,m} P_n(t)\; \Gamma^{e-e}_{n\rightarrow m} \left\{m_e(\underline{p},s) - n_e(\underline{p},s)\right\} . \tag{4.32}$$

Substitution of (29) into Equation (32) yields

$$\frac{\partial f_e}{\partial t} = (2\pi e^2\hbar)^2 \frac{VT}{V^4} \sum_n P_n(t) \sum_{\substack{\underline{p}_2', \underline{p}_1', \underline{p}_1 \\ s_2', s_1', s_1}} \frac{m^4c^8}{E_{2'}E_{1'}E_1E_{\underline{p}}} \delta_{p_2'+p_1'-p-p_1}$$

$$\cdot \left|A^{e-e}(2';1'|\underline{p},s;1)\right|^2 \Big\{\{1-n_e(\underline{p},s)\}\{1-n_e(1)\}n_e(2')n_e(1')$$

$$- \{1-n_e(2')\}\{1-n_e(1')\}\ n_e(\underline{p},s)n_e(1)\Big\} \quad . \qquad (4.33)$$

In obtaining this equation we have carried out the summation over the states $|m\rangle$ and made use of the properties given by (31).

We now approximate the average of the products of the occupation numbers, n_e , by the product of the averages, namely, we assume

$$\sum_n P_n(t)n_e(i)\ldots n_e(j)\ldots \simeq \{\sum_n P_n(t)n_e(i)\}\ldots\{\sum_n P_n(t)n_e(j)\}\ldots \quad . \qquad (4.34)$$

Making use of this approximation in Equation (33) one obtains

$$\frac{\partial f_e}{\partial t} = \frac{(4\pi e^2\hbar)^2}{2} \frac{VT}{V^4} \sum_{\substack{\underline{p}_2', \underline{p}_1', \underline{p}_1 \\ s_2', s_1', s_1}} \delta_{p_2'+p_1'-p-p_1} \left| A^{e-e}(2';1' \mid \underline{p},s;1) \right|^2$$

$$\cdot \Big\{\{1-Vf_e\}\ \{1-Vf_e(1)\}\ Vf_e(2')\ Vf_e(1')$$

$$- \ \{1-Vf_e(2')\}\ \{1-Vf_e(1')\}\ Vf_e\ Vf_e(1)\Big\} \frac{m^4c^8}{E_{2'}E_{1'}E_1E_{\underline{p}}} \qquad (4.35)$$

where $f_e \equiv f_e(\underline{p},s,t)$.

Transition to the continuum variables in the momentum space is achieved by defining

$$\sum_{\underline{p}\epsilon d^3p} f_e(\underline{p},s,t) = F_e(\underline{p},s,t)\, d^3p \ , \tag{4.36}$$

and noting that one must replace

$$\sum_{\underline{p}\epsilon d^3p} 1 \rightarrow \frac{V}{(2\pi\hbar)^3}\, d^3p \ ,$$

$$V\,\delta_{\underline{p}} \rightarrow (2\pi\hbar)^3\, \delta(\underline{p}) \ ,$$

$$T\,\delta_{E_{\underline{p}}} \rightarrow (2\pi\hbar)\, \delta(E_{\underline{p}}) \ . \tag{4.37}$$

In the continuumvariables, Equation (35) becomes

$$\frac{E_{\underline{p}}}{mc^2}\frac{\partial F_e(\underline{p},s,t)}{\partial t} = \frac{1}{2}(2e^2)^2 \rho^{-2} \sum_{s_2',s_1',s_1} \int d^3p_2' d^3p_1' d^3p_1 \frac{m^3c^6}{E_{2'}E_{1'}E_1}$$

$$\cdot\, \delta(\underline{p}_2'+\underline{p}_1'-\underline{p}-\underline{p}_1)\, \delta(E_{2'}+E_{1'}-E_{\underline{p}}-E_1) \left| A^{e-e}(2';1' \mid \underline{p},s;1) \right|^2$$

$$\cdot \Big\{ \{\rho-F_e(\underline{p},s,t)\} \{\rho-F_e(1)\} F_e(2')\, F_e(1')$$

$$- \{\rho-Fe(2')\} \{\rho-Fe(1')\}\, Fe(\underline{p},s,t)\, Fe(1) \Big\} , \tag{4.38}$$

where $\rho \equiv (2\pi\hbar)^{-3}$.

If we assume isotropic spin distributions, that is,

$$F_e(\underline{p},s,t) = \frac{1}{2} F_e(\underline{p},t), \quad (s = 1,2)$$

then

$$\frac{E_{\underline{p}}}{mc^2}\frac{\partial F_e}{\partial t}(\underline{p},t) = \frac{1}{2}(2e^2)^2\frac{\rho^2}{4}\int d^3p_2' d^3p_1' d^3p_1 \frac{m^3c^6}{E_{2'}E_{1'}E_1}\delta(\underline{p}_2' + \underline{p}_1' - \underline{p} - \underline{p}_1)$$

$$\cdot\,\delta(E_{2'} + E_{1'} - E_{\underline{p}} - E_1)\, B^{e-e}(\underline{p}_2',\underline{p}_1' \mid \underline{p},\underline{p}_1)$$

$$\cdot\Big\{\{2\rho - F_e(\underline{p},t)\}\{2\rho - F_e(1)\}F_e(2')F_e(1')$$

$$-\{2\rho - F_e(2')\}\{2\rho - F_e(1')\}F_e(\underline{p},t)F_e(1)\Big\}, \qquad (4.39)$$

where

$$B^{e-e}(\underline{p}_2',\underline{p}_1' \mid \underline{p},\underline{p}_1) = \frac{1}{4}\sum_{s_2',s_1',s,s_1}\left| A^{e-e}(2';1' \mid \underline{p},s;1)\right|^2 .$$

It is noted that for the explicit calculation of the collision terms we have specialized to a particular frame of reference by working in the Schrödinger representation. However, the covariance of the resulting Equation (38) may be seen if one observes that for a homogenous system $\frac{E_{\underline{p}}}{mc^2}\frac{\partial}{\partial t} = \frac{p_\mu}{m}\frac{\partial}{\partial x_\mu}$, and that $\frac{d^3p}{E_{\underline{p}}}$ is an invariant, and also that $|A^{e-e}|^2$ is expressed explicitly in a covariant form by Equation (30).

A similar derivation concerning the contributions to the electron transport due to the electron-photon scattering and the two quantum pair creation and annihilation will be given in Appendix D.

V. SOME ASPECTS OF EQUILIBRIUM

Previously, we have derived an equation describing the electron transport which has the form

$$\frac{\partial F_e(\underline{p},s,t)}{\partial t} = (\partial F_e/\partial t)_{e-e} + (\partial F_e/\partial t)_{c.s.} + \cdots\cdots .$$

In this section we shall discuss briefly the implications of this equation for equilibrium systems. In particular, we shall prove an H-theorem for electron-photon systems. For the present, we ignore the positrons so that only the terms describing electron-electron and electron-photon scattering need be considered.

Before proceeding to the proof of an H-theorem, we need a transport equation for photons also. This, however, may be written down immediately in view of the derivation leading to Equation (D-4). One has

$$\frac{\partial \chi}{\partial t}(\underline{\kappa},m,t) = (\partial\chi/\partial t)_{c.s.} + \cdots\cdots ,$$

where

$$\left(\frac{\partial\chi}{\partial t}\right)_{c.s.} = (e^2c)^2\rho^{-2}\sum_{s',s,m'}\int d^3p'd^3p\, d^3\kappa'\,\frac{m^2c^4}{\varpi'\varpi\, E_{\underline{p}}E_{\underline{p}'}}\,\delta(\underline{p}'+\underline{\kappa}'-\underline{p}-\underline{\kappa})$$

$$\cdot\ \delta\,(E_{\underline{p}'}+\varpi'-E_{\underline{p}}-\varpi)\ \left|A^{c.s.}(\underline{p}',s';\underline{\kappa}',m'|\,\underline{p},s;\underline{\kappa},m)\right|^2$$

$$\cdot\left\{\{\rho-F_e\}\{\rho+\chi\}F_e'\chi' - \{\rho-F_e'\}\{\rho+\chi'\}\,F_e\chi\right\}, \qquad (5.1)$$

where $\varpi \equiv \hbar\omega$, $\underline{\kappa} \equiv \hbar\underline{k}$. Equation (1) gives the time rate of change of the photon distribution due to the collisions with electrons. Corresponding expression to this, in the non-relativistic approximation for electrons, has been derived previously[19].

Let us define a quantity S(t) by[20]

$$S(t) = -k \sum_{s} \int d^3p \left\{ F_e \ln(F_e/\rho - F_e) - \rho \ln(\rho/\rho - F_e) \right\}$$
$$- k \sum_{m} \int d^3k \left\{ \chi \ln(\chi/\rho + \chi) + \rho \ln(\rho/\rho + \chi) \right\}, \qquad (5.2)$$

where $F_e \equiv F_e(\underline{p}, s, t)$, $\chi \equiv \chi(\underline{\kappa}, m, t)$.

Using Equations (4.38), (D-4) and (1) it is a straightforward matter to show that

$$dS/dt \geq 0 ,$$

and that

$$dS/dt = 0 ,$$

if and only if the following conditions are satisfied simultaneously, that is,

$$\ln \frac{F_e(1)}{\rho - F_e(1)} + \ln \frac{F_e(2)}{\rho - F_e(2)} = \ln \frac{F_e(1')}{\rho - F_e(1')} + \ln \frac{F_e(2')}{\rho - F_e(2')} \qquad (5.3)$$

for all $s_1, s_2, s_1', s_2' = 1, 2$ provided

$$\underline{p}_1 + \underline{p}_2 = \underline{p}_1' + \underline{p}_2' , \qquad (5.4)$$

$$E_{\underline{p}_1} + E_{\underline{p}_2} = E_{\underline{p}_1'} + E_{\underline{p}'_2} ; \qquad (5.5)$$

and

$$\ln \frac{F_e}{\rho - F_e} + \ln \frac{\chi}{\rho + \chi} = \ln \frac{F_e'}{\rho - F_e'} + \ln \frac{\chi'}{\rho + \chi'} \qquad (5.6)$$

for all $s, m, s', m' = 1, 2$ provided

$$\underline{p} + \underline{\kappa} = \underline{p}' + \underline{\kappa}' , \qquad (5.7)$$

$$E_{\underline{p}} + \varpi = E_{\underline{p}'} + \varpi' . \qquad (5.8)$$

It is interesting to note that an examination of the quantity $\frac{dS(t)}{dt}$ shows that it is a relativistically invariant quantity. Therefore, if it vanishes in one frame of reference, then it vanishes in any other Lorentz frame. If we call $S(t)$ the entropy per unit volume we may state that the time rate of change of the entropy per unit volume is the same in all Lorentz frames.

One finds that the Equations (3) and (4) under the constraints (4), (5), (7) and (8) admit the solutions

$$F(\underline{p}, s) = \frac{\rho}{B_e \; e^{-\alpha_\mu p_\mu} + 1} , \quad (s = 1,2)$$

and

$$\chi(\underline{\kappa}, m) = \frac{\rho}{C_\gamma \; e^{-\alpha_\mu \kappa_\mu} - 1} , \quad (m=1,2)$$

where B_e, C_γ, α_μ are, so far, arbitrary constants independent of $\underline{p}$.

In order to be interpretable as physical photon density, $\chi(\underline{\kappa}, m)$ must be non-negative for all values of $\underline{\kappa}$. This restricts α_μ

to be a time like vector. Therefore, one can always make a Lorentz transformation so that time axis coincides with α_μ . In this frame of reference, electron and photon densities are expressed as

$$F_e(\underline{p},s) = \frac{\rho}{B_e\ e^{\beta E_{\underline{p}}} + 1}\ , \quad (s = 1,2)$$

$$\chi(\underline{\kappa},m) = \frac{\rho}{C_\gamma\ e^{\beta\varpi} - 1}\ , \quad (m = 1,2)$$

where β is a positive constant. Note that non-negativeness of $\chi(\underline{\kappa},m)$ also implies that $C_\gamma \geq 1$. The common constant β is identified with $1/kT$ where T is the temperature of the system. There remains, however, two more constants, namely, the constants B_e and C_γ to be determined. The only additional knowledge is the number of electrons per unit volume in the system which determines the constant B_e . It is seen that in this hypothetical system which consists only of electrons and photons and which interact only through electron-electron and electron-photon scattering there is no way of determining the constant C_γ . That means that equilibrium photon distribution is not completely determined. However, this is not the actual system under consideration. We shall now show that in the physical system which consists of electrons, positrons and photons this arbitrariness is removed by the consideration of all interactions which are described to the second order in the perturbation analysis.

Let us consider the effect of the two-quantum pair creation and annihilation on the equilibrium distributions. The system now includes electrons, positrons and photons. The additional condition to be satisfied by the equilibrium distributions is found to be

$$\ln \frac{F_e}{\rho - F_e} + \ln \frac{F_p}{\rho - F_p} = \ln \frac{\chi_2}{\rho + \chi_2} + \ln \frac{\chi_1}{\rho + \chi_1} \quad , \qquad (5.9)$$

$(F_p \equiv F_p(\underline{q}, r))$ for all $s, r, m_i = 1,2$, provided

$$\underline{p} + \underline{q} = \underline{\kappa}_2 + \underline{\kappa}_1 \quad ,$$

$$E_{\underline{p}} + E_{\underline{q}} = \varpi_2 + \varpi_1 \quad .$$

Equilibrium distributions are

$$F_e(\underline{p}, s) = \frac{\rho}{B_e \, e^{\beta E_{\underline{p}}} + 1} \quad , \; (s = 1,2)$$

$$F_p(\underline{q}, r) = \frac{\rho}{B_p \, e^{\beta E_{\underline{p}}} + 1} \quad , \; (r = 1,2)$$

$$\chi(\underline{\kappa}, m) = \frac{\rho}{C_\gamma \, e^{\beta \varpi} - 1} \quad , \; (m = 1,2) \qquad (5.10)$$

with

$$B_e B_p = C_\gamma^2 \quad . \qquad (5.11)$$

Finally, from consideration of the pair creation and annihilation in the field of an electron one finds that the equilibrium distributions must also satisfy the condition

$$\ln \frac{F_e'}{\rho - F_e'} = \ln \frac{F_e(1)}{\rho - F_e(1)} + \ln \frac{F_e(2)}{\rho - F_e(2)} + \ln \frac{F_p}{\rho - F_p} \quad ,$$

for all $s', s_i, r = 1,2$ provided that

$$\underline{p}' = \underline{p}_1' + \underline{p}_2' + \underline{q} \ ,$$

$$E_{\underline{p}'} = E_{\underline{p}_1'} + E_{\underline{p}_2'} + E_{\underline{q}} \ .$$

The distributions (10) satisfy this condition only if

$$B_e = B_e^2 B_p \ . \qquad (5.12)$$

From Equations (11) and (12) and recalling that $C_\gamma \geq 1$ one finds $C_\gamma = 1$, $B_e B_p = 1$. In the thermodynamical language this corresponds to the fact that the chemical potential of the photon gas is equal to zero and that the sum of the chemical potentials of the electrons and positrons vanish (see Reference 28, p.325). In the reference cited an identical conclusion is drawn strictly from thermodynamical considerations. First it was established that the chemical potential of the photon gas should vanish because of the requirement that the free energy of the photon gas should be minimum, for given temperature and volume, as a function of the number of photons in the system. Then, considering the pair creation (or annihilation) as a chemical reaction it was concluded that the sum of the chemical potentials of the electrons and the positrons must be equal to that of the photon gas. Here, strictly from a consideration of the kinetic equations describing the system we have shown that the system approaches to this equilibrium state irreversibly.

It is noted that Equations (11) and (12) together with the knowledge of the net charge per unit volume in the system is sufficient to determine all the constants C_γ, B_e, B_p for a given temperature. If the net charge associated with the electron-positron field* per cm^3 is en, then one can determine B_e (or B_p) from the relation

$$\sum_{s=1}^{2} \int d^3p \; F_e(\underline{p},s) - \sum_{r=1}^{2} \int d^3q \; F_p(\underline{q},r) = n \quad , \tag{5.13}$$

or, equivalently,

$$4\pi \int dp \; |\underline{p}| \left\{ \frac{1}{B_e \; e^{\beta E_{\underline{p}}} + 1} - \frac{1}{B_e^{-1} \; e^{\beta E_{\underline{p}}} + 1} \right\} = \frac{n}{2\rho} \quad . \tag{5.14}$$

The implication of this result is important particularly for very high temperature equilibrium systems ($kT \sim mc^2$ or greater). One finds that,[28] while the density of the electron-positron pairs is negligibly small for non-relativistic systems, it already exceeds by a great deal the usual densities of atomic electrons for $kT \sim mc^2$. For extreme relativistic temperatures, the density of the electron positron pairs increases with $(kT/\hbar c)^3$.

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* Over all charge neutrality is assumed to be maintained by a uniform background of ions.

VI. BOLTZMANN-VLASOV EQUATION FOR A RELATIVISTIC PLASMA

In the derivation of the self-consistent field equations we have ignored the correlations completely. In this way, we were able to obtain the usual Vlasov's equation for a relativistic plasma with additional spin-dependent terms. It is usually assumed that the Vlasov equation, embodying in itself the self-consistent internal fields is an appropriate starting point for the study of the collective behaviour of plasmas. Starting with the Vlasov's equation [Equation (3.30)], coupled with the Maxwell's equations, the linear oscillations in a relativistic plasma has been studied to some extent by several authors.[21-24] The absence of the correlation terms in the Vlasov's equation reflects itself in the fact that this equation does not have any indication of how the equilibrium is attained in the system and what the equilibrium distributions should be.

On the other hand, employing a field-theoretic formalism we have also derived a set of Boltzmann type equations for a spatially homogenous, relativistic plasma in the absence of external fields. We have shown that, in the sense of certain approximations, these equations imply unique equilibrium distributions for the constituents of the plasma.

A question now arises: what is the connection between the correlation terms that we have ignored in the derivation of the Vlasov equation and the collision terms of the Boltzmann equation? To acquire some insight into this question let us return to Equation (3.6) where the correlations have not yet been discarded. We have

$$\frac{\partial \mathcal{V}_\mu}{\partial x_\mu} + \frac{2e}{\hbar c} A_\mu^e \sin\left(\frac{\hbar}{2} \frac{\overleftarrow{\partial}}{\partial x} \frac{\overrightarrow{\partial}}{\partial p}\right) \mathcal{V}_\mu$$

$$+ \frac{e}{\hbar c} (\pi\hbar)^{-4} \int d^4z \, e^{-2ip\cdot z/\hbar} (\Phi, N\{\overline{\psi}(x-z)\gamma_\mu(A_\mu^i(x+z) - A_\mu^i(x-z))\psi(x+z)\}\Phi) = 0. \tag{6.1}$$

We have seen in Chapter III that if one uses the "self-consistent" field approximation on the last term in Equation (1), then one is led to the Vlasov equation. We wish to investigate here in what sense one may incorporate the collision terms with the Vlasov's equation. We introduce, on physical grounds, a decomposition of the internal potentials into two parts[10]

$$A_\mu^i = A_\mu^s + A_\mu^c , \tag{6.2}$$

where it is presumed that A_μ^s is responsible for the "self-consistent" fields and A_μ^c is responsible for the interactions which require a "collision" description. We shall employ the self-consistent field approximation in the same sense as before on all terms involving A_μ^s but not on A_μ^c.

Introducing this decomposition into Equation (1) one finds

$$\frac{\partial \mathcal{V}_\mu}{\partial x_\mu} + \frac{2e}{\hbar c} \mathcal{A}_\mu \sin\left(\frac{\hbar}{2} \frac{\overleftarrow{\partial}}{\partial x} \frac{\overrightarrow{\partial}}{\partial p}\right) \mathcal{V}_\mu = I , \tag{6.3}$$

where

$$\mathcal{A}_\mu \equiv A_\mu^e + (\Phi, A_\mu^s \Phi),$$

$$I \equiv - \frac{e}{\hbar c}(\pi\hbar)^{-4} \int d^4z \, e^{-2ip\cdot z/\hbar} (\Phi, N\{\overline{\psi}(x-z)\gamma_\mu(A_\mu^c(x+z) - A_\mu^c(x-z))\psi(x+z)\}\Phi).$$

To avoid additional complications we ignore all the terms of order $\hbar$ on the left hand side of Equation (1) so that

$$\frac{\partial \mathcal{V}_\mu}{\partial x_\mu} + \frac{e}{c}\frac{\partial \mathcal{A}_\mu}{\partial x_\nu}\frac{\partial \mathcal{V}_\mu}{\partial p_\nu} \simeq I . \tag{6.4}$$

From Equation (3.10), ignoring again the terms of order $\hbar$, we find

$$mc\mathcal{V}_\nu \simeq (p_\nu - \frac{e}{c}\mathcal{A}_\nu)\ f. \tag{6.5}$$

Changing variables from $\{x,p\}$ to $\{x,\pi\}$ by the relation $p - \frac{e}{c}\mathcal{A} = \pi$ and inserting the relation (5) into Equation (4) one obtains

$$Df(x,\pi) = I, \tag{6.6}$$

where

$$D = \pi_\mu \frac{\partial}{\partial x_\mu} + \frac{e}{c} F_{\nu\mu}\pi_\mu \frac{\partial}{\partial \pi_\nu}$$

$$I = -\frac{em}{\hbar}(\pi\hbar)^{-4}\int d^4z\ e^{-2i(\pi + \frac{e}{c}\mathcal{A})\cdot z/\hbar}(\Phi, N\{\bar{\psi}(x-z)\gamma_\mu \cdot (A_\mu^c(x+z)-A_\mu^c(x-z))\psi(x+z)\}\Phi).$$

Integrating Equation (6) over the variable π_0 and making use of Equation (3.29) it follows that

$$\{D_e F_e(x,\underline{\pi}) + D_p F_p(x,-\underline{\pi})\} = I , \tag{6.7}$$

where

$$D_e = \frac{c^2}{E_\pi}\{\pi_\mu \frac{\partial}{\partial x_\mu} + \frac{e}{c} F_{j\mu}\pi_\mu \frac{\partial}{\partial \pi_j}\}_{\pi_0 = E_\pi/c},$$

$$D_p = \frac{c^2}{E_\pi}\{\pi_\mu \frac{\partial}{\partial x_\mu} + \frac{e}{c} F_{j\mu}\pi_\mu \frac{\partial}{\partial \pi_j}\}_{\pi_0 = -E_\pi/c},$$

$$I = -\frac{e}{\hbar}(\pi\hbar)^{-3}\int d^3z\ e^{-2i(\underline{\pi}+\frac{e}{c}\underline{\mathcal{A}})\cdot \underline{z}/\hbar} \cdot (\Phi, N\{\bar{\psi}(\underline{x}-\underline{z},t)\gamma_\mu(A_\mu^c(\underline{x}+\underline{z},t)-A_\mu^c(\underline{x}-\underline{z},t))\psi(\underline{x}+\underline{z},t)\}\Phi).$$

Let us now examine the term I more closely. We first note that it may be expressed in the Schrödinger representation as

$$I = -\frac{e}{\hbar}(\pi\hbar)^{-3}\int d^3z\; e^{-2i(\underline{\pi}+\frac{e}{c}\underline{\mathcal{A}})\cdot\underline{z}/\hbar}$$
$$(F(t)\ast N\{\overline{\psi}(\underline{x}-\underline{z})\gamma_\mu(A_\mu^c(\underline{x}+\underline{z})-A_\mu^c(\underline{x}-\underline{z}))\psi(\underline{x}+\underline{z})\}F(t)).$$

Using the commutation relations of the field operators one can show that

$$I = \frac{1}{i\hbar}(F(t),\ [\hat{f}(x,\underline{\pi}),H_I^c]_-\ F(t))\ , \tag{6.8}$$

where

$$H_I^c \equiv ie\int d^3x\ N\{\overline{\psi}(\underline{x})\gamma_\mu A_\mu^c(\underline{x})\psi(\underline{x})\},$$

$$\hat{f}(x,\underline{\pi}) \equiv (\pi\hbar)^{-3}\int d^3z\ e^{-2i(\underline{\pi}+\frac{e}{c}\underline{\mathcal{A}})\cdot\underline{z}/\hbar}\ N\{\psi^\dagger(\underline{x}-\underline{z})\psi(\underline{x}+\underline{z})\}.$$

Let us fix our attention to a volume V about the space point $\underline{x}$. We define the coarse-grained distribution, $\overline{F}_e(x,\underline{\pi})$, by the relation

$$\overline{F}_e(x,\underline{\pi}) = \frac{1}{V}\int_{\underline{x}\in V} d^3x\ F_e(x,\underline{\pi}).$$

Assuming that F_e, F_p and $F_{j\mu}$ are slowly varying over the dimensions of the volume V, one obtains by integrating Equation (7) over the volume V

$$\{D_e\overline{F}_e(x,\underline{\pi}) + D_p\overline{F}_p(x,-\underline{\pi})\} = \frac{1}{i\hbar}\frac{1}{V}(F(t),\ [\int_{\underline{x}\in V} d^3x\ \hat{f}(x,\underline{\pi}),H_I^c]_-\ F(t)). \tag{6.9}$$

The right hand side of Equation (9) gives the effect on transport of the collisions which occur in the presence of the external and the self-consistent fields. One notes that the self-consistent fields are not really known until the whole problem is solved. Also, without an explicit knowledge of the external fields, the states of the particles which participate in the collisions would not be available. Therefore, we specialize here to a case where the external and the self-consistent fields do not effect the collisions appreciably. In other words, we assume that during the collisions we may consider the particles to be isolated from the rest of the system so that initial and final states of the particles may be taken as the free-particle solutions of the Dirac equation. Thus, using the expansions given by (4 .28) with $x_4 = 0$ (in the Schrödinger representation) and ignoring here the term $\frac{e}{c}\mathcal{A}$ compared to $\underline{\pi}$ for consistency, one finds that

$$\int_{\underline{x}\in V} d^3x \, \hat{f}(x,\underline{\pi}) \simeq \sum_{s=1,2} N_e(\underline{\pi},s) - \sum_{r=1,2} N_p(-\underline{\pi},r), \qquad (6.10)$$

where

$$N_e(\underline{\pi},s) = C_s^\dagger(\underline{\pi})C_s(\underline{\pi}), \quad N_p(\underline{\pi},r) = d_r^\dagger(\underline{\pi})d_r(\underline{\pi}).$$

It is noted that $N_e(\underline{\pi},s)$ is the operator for the number of electrons in the volume V, with momentum $\underline{\pi}$ and spin s; and N_p is similarly for positrons. Inserting (10) into Equation (9) and identifying the contributions to the electron and positron transport separately we have

$$D_e\overline{F}_e(x,\underline{\pi}) = \frac{1}{i\hbar}\,\frac{1}{V}(F_V(t),\,[\sum_{s=1,2} N_e(\underline{\pi},s),H_I^c]_-\,F_V(t)), \qquad (6.11)$$

$$D_p\overline{F}_p(x,-\underline{\pi}) = -\frac{1}{i\hbar}\,\frac{1}{V}(F_V(t),[\sum_{r=1,2} N_p(-\underline{\pi},r),H_I^c]_-\,F_V(t)), \qquad (6.12)$$

where $F_V(t)$ is a state vector in the Schrödinger representation which approximately describes the system in volume V during the time of the collision.* In the sense of these approximations, the state vector $|F_V(t)>$ satisfies the equation

$$i\hbar \frac{\partial}{\partial t} |F_V(t)> = H_V|F_V(t)>,$$

where H_V is the total Hamiltonian in V in the absence of the external and the self-consistent fields. Noting that N_e, N_p commute with the Hamiltonian of the free-fields we may re-write Equations (11) and (12) as

$$D_e\bar{F}_e(x,\underline{\pi}) = \sum_{s=1,2} \frac{\partial F_e}{\partial t} (\underline{\pi},s,t) \quad , \qquad (6.13)$$

$$D_p\bar{F}_p(x,-\underline{\pi}) = - \sum_{r=1,2} \frac{\partial F_p}{\partial t} (-\underline{\pi},r,t) , \qquad (6.14)$$

where

$$F_e(\underline{\pi},s,t) \equiv \frac{1}{V}(F_V(t),\ N_e(\underline{\pi},s)F_V(t)) \ ,$$

$$F_p(\underline{\pi},r,t) \equiv \frac{1}{V}(F_V(t),\ N_p(\underline{\pi},r)F_V(t)) \ .$$

Equation (14) may also be written as

$$D'_p\bar{F}_p(x,-\underline{\pi}) = \sum_{r=1,2} \frac{\partial F_p}{\partial t} (\underline{\pi},r,t) \quad , \qquad (6.15)$$

where D'_p differs from D_e only by a change in the sign of e, as would be expected.

Explicit forms of the collision integrals arising from the right hand sides of Equations (13) and (15) have been given in Chapter IV.

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* The assumption which is being made here is that the state vector $|F(t)>$ may be factorized as $|F(t)> = |F_V(t)>|F_0(t)>$, where V and 0 refer to the volume V and to the outside of volume V, respectively, and states are normalized separately to unity. Since N_e, N_p represent only the particles in V, $|F_0(t)>$ may be omitted here.

These equations will be called Boltzmann-Vlasov equations since they include both the self consistent fields of the Vlasov equation and the collision integrals of the Boltzmann equation. They are coupled due to the collision terms describing creation and annihilation processes.

In this section, we have given a systematic (although quite intuitive) derivation of a set of Boltzmann-Vlasov equations for electrons and positrons in a relativistic plasma taking pair creation and annihilation into account. Also, by this "unified" derivation we have shown the connection between the two problems considered separately in Chapters III and IV, namely the derivations of the Vlasov and the Boltzmann equations. Lastly, we have tried to indicate qualitatively the restrictions on the validity of the Boltzmann-Vlasov equations by indicating the assumptions needed to be made in such a derivation.

We wish to make a few more comments on the validity of a Boltzmann-Vlasov description of a plasma, not necessarily a relativistic plasma. If one considers the Vlasov equations (where all correlations are ignored) as the "lowest order" in an approximate description of a plasma, then the collision terms appear as corrections to the Vlasov equation. Objection, however, may be made to the validity of this description, especially for dilute plasmas. The point is that the collision terms describing the interactions between the charged particles are of binary character and in a dilute plasma these encounters would be quite rare, therefore, less important compared to the simultaneous interactions of many particles due to the infinite extension of the Coulomb interactions. Recently, this problem has been discussed successfully for the non-relativistic systems interacting via the Coulomb forces by both Balescu[25] and Guernsey[26]. Using e^2n, where n is the density of electrons, as a parameter of smallness and summing explicity the contributions to all orders in this parameter, Balescu was able to derive an equation appropriate to the description of a plasma which has also been obtained indepently by Guernsey. It was shown that[26] this equation could be cast into an equation of the Fokker-Planck type.* An equation of this type may also be derived from the Boltzmann integral for charged particles by considering the collisions of small momentum transfer and by introducing an appropriate cut-off procedure to avoid divergences. This equation is usually referred to as Landau Equation[27]. The difference between the two Fokker-Planck type equations is reflected in the coefficients. Whereas the coefficients a_j, b_{jk} in the Landau equation are the first and second moments of the momentum change in a binary collision, the coefficients in the Fokker-Planck equation given by Guernsey do not have such simple interpretation and are of more general validity. In the latter case, the coefficients reflect the non-locality of the interactions. In this connection, the work by Lenard[28] should also be mentioned. Starting with an integral equation proposed by Bogoliubov to determine the time development of the velocity distribution for a spatially uniform plasma, he has shown that the time derivative of the velocity distribution can be expressed, exactly, in terms of the distribution itself. The resulting equation is again of the Fokker-Planck type.

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* By Fokker-Planck equation we mean here an equation of the form

$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial p_j} (a_j F) + \frac{1}{2} \frac{\partial^2}{\partial p_j \partial p_k} (b_{jk} F),$$

where coefficients a_j, b_{jk} themselves may depend on F.

He has further shown that by means of an excellent approximation it could be reduced to that derived by Landau. This indicates that binary collision description although restricted in general terms does contain sufficient information for many purposes.

Finally, we note that our Equation (4.39) which describes the electron-electron collisions for relativistic systems could also be reduced into a Fokker-Planck form. Since the retarded interactions as well as the Coulomb interactions are accounted for in the transition probability appearing in this equation, we expect that the resulting Fokker-Planck equation (when specialized to non-degenerate systems) should correspond to that derived by Klimontovich[6] through the study of the pair-correlation function for a relativistic plasma.

VII. CONCLUSIONS

In the present study, we have formulated transport equations appropriate to high temperature plasmas based upon second-quantized Dirac theory of electrons. For the study of the statistical aspects of the problem we have introduced an invariant phase space distribution, in analogy with the Wigner distribution function,[11] which is suitable to the study of spin 1/2 particles described by the Dirac equation. In Chapter III, we have derived a coupled set of equations between this function and the spin distributions which reduces to the covariant form of the Vlasov equation as $\hbar \rightarrow 0$. In Chapter IV, we have derived a Boltzmann equation for electrons in a system consisting of electrons, positrons and photons which to our knowledge has not previously been given. We have used the S-Matrix formalism and restricted ourselves to the lowest non-vanishing order in the perturbation expansion. Employing the results of this chapter, in Chapter V we have discussed the equilibrium properties of this system. An H-theorem was proved and the equilibrium solutions were displayed. The results are in agreement with those obtained strictly from thermodynamical considerations.[28] This question also does not seem to have been examined previously from the point of view of the quantum electrodynamics. We have obtained here the kinetic equations describing the system and its approach to equilibrium based upon a field theoretic formalism and have shown that these equilibrium distributions are attained irreversibly. The kinetic equations for relativistic systems based upon a classical formalism had necessarily ignored such processes as pair creation and annihilation which, as we have shown in Chapter V,is necessary to consider for a complete description of the system.

The interesting feature of this result was that for very high temperature systems $(kT \sim mc^2)$ the density of the electron-positron pairs in the system is very large compared to the usual densities of the atomic electrons in the system. This, for instance, would have a profound effect upon the propagation of the electromagnetic waves in such systems. In the study of the plasma oscillations in relativistic systems, reference is made sometimes(23,24) to extreme-relativistic limits without a consideration of the electron-positron pairs which would be present in such systems. Therefore, caution would be necessary in interpreting these results from the point of view of actual physical systems at extreme relativistic temperatures. Finally, in Chapter VI, we have sketched the derivation of a Boltzmann-Vlasov equation for an electron-positron system taking pair creation and annihilation into account.

We would like to point out that in the present study we have restricted ourselves with a lowest order description of the system in that the "dressing" of particles or photons by the collective effects of the medium were not taken into account. It has, however, been shown in the study of the photon transport in non-relativistic systems(19) that the consideration of the first order collective effects in a plasma leads one to a modified black-body spectrum effected by the plasma oscillations. Therefore, a refinement on the present study from this point of view is suggestive. In Appendix E we present a quantum dispersion relation for transverse oscillations in a relativistic plasma. In the non-relativistic limit, this reduces to that given by Bohm and Pines[29] with an additional term due to spin.

APPENDIX A

INVARIANCE OF f(x,p) UNDER LORENTZ TRANSFORMATIONS

Recall the definition

$$f(x,p) = (\pi\hbar)^{-4} \int d^4z\, e^{-2ip\cdot z/\hbar} (\Phi, N\{\bar{\psi}_\alpha(x-z)\psi_\alpha(x+z)\}\Phi), \quad \text{(A-1)}$$

where we have indicated the summation over the spinor indices explicitly.

We wish to prove that under the simultaneous Lorentz transformations on x and p, namely,

$$x \to x' = Lx, \quad p \to p' = Lp, \quad \text{(A-2)}$$

the following statement holds true

$$f'(x',p') = f(x,p) \;, \quad \text{(A-3)}$$

where $f'(x',p')$ is given by

$$f'(x',p') = (\pi\hbar)^{-4} \int d^4z\, e^{-2ip'\cdot z/\hbar} (\Phi', N\{\bar{\psi}_\alpha(x'-z)\psi_\alpha(x'+z)\}\Phi'), \quad \text{(A-4)}$$

where Φ' denotes the transformed state-vector.

Proof:

We first note that the transformed state vector may be written as*

$$\Phi' = U(L)\Phi, \quad \text{(A-5)}$$

* See, for instance, Reference 13, p. 651.

where $U(L)$ is a unitary transformation such that

$$U(L)\ \psi_\alpha(x)\ U^{-1}(L) = S^{-1}_{\alpha\beta}(L)\ \psi_\beta(Lx), \tag{A-6}$$

$$U(L)\ \bar{\psi}_\alpha(x)\ U^{-1}(L) = \bar{\psi}_\beta(Lx)\ S_{\beta\alpha}(Lx), \tag{A-7}$$

where $S_{\alpha\beta}(L)$ is the spinor representation of the homogenous Lorentz group.

Following statements follow from above and from each other and need no further comment.

$$\begin{aligned}\bar{\psi}_\alpha(x'-z)\ \psi_\alpha(x'+z) &= \bar{\psi}_\alpha(Lx-z)\ \psi_\alpha(Lx+z)\\ &= \bar{\psi}_\alpha(L(x-L^{-1}z))\ \psi_\alpha(L(x-L^{-1}z))\\ &= \bar{\psi}_\alpha(L(x-L^{-1}z))\ S_{\alpha\beta}(L)\ S^{-1}_{\beta\gamma}(L)\ \psi_\gamma(L(x+L^{-1}z))\\ &= U(L)\ \bar{\psi}_\beta(x-L^{-1}z)\ U^{-1}(L)U(L)\ \psi_\beta(x+L^{-1}z)\ U^{-1}(L)\\ &= U(L)\ \bar{\psi}_\beta(x-L^{-1}z)\ \psi_\beta(x+L^{-1}z)\ U^{-1}(L).\end{aligned} \tag{A-8}$$

From Equations (5) and (8) one finds

$$f'(x',p') = (\pi\hbar)^{-4} \int d^4z\ e^{-2iLp\cdot z/\hbar}\ (\Phi,N\{\psi_\beta(x-L^{-1}z)\ \psi_\beta(x+L^{-1}z)\}\Phi). \tag{A-9}$$

We note that $Lp\cdot z = p\cdot L^{-1}z$. Then, we perform a change of variables from $z \to \tau = L^{-1}z$ and note that the Jacobian of this transformation is unity owing to the fact that $Det(L) = 1$. It follows that

$$f'(x',p') = (\pi\hbar)^{-4} \int d^4\tau\ e^{-2ip\cdot\tau/\hbar}\ (\Phi,N\{\bar{\psi}_\beta(x-\tau)\ \psi_\beta(x+\tau)\}\Phi). \tag{A-10}$$

From a comparison of Equations (10) and (1), Equation (3) follows. Q.E.D.

APPENDIX B

DETAILS OF THE DERIVATION OF THE SELF-CONSISTENT FIELD EQUATIONS

It is convenient for the present purpose to obtain from the Equations (2.3) a set of equations by multiplying the first one from the left and the second one from the right by the Dirac Matrices

$$I,\ v_\mu = i\gamma_\mu,\ \sigma_{\mu\nu} = \frac{1}{2i}[\gamma_\mu,\gamma_\nu]_-,\quad s_\mu = i\gamma_5\gamma_\mu,\quad \tau = i\gamma_5 .$$

The reason for the selection of these particular matrices is that they form a complete set in the algebra of the Dirac Matrices. To suppress the notation we define

$$\overrightarrow{D}_\mu = \frac{\overrightarrow{\partial}}{\partial x_\mu} - \frac{ie}{\hbar c} A_\mu(x),\quad \overleftarrow{D}_\mu = \frac{\overleftarrow{\partial}}{\partial x_\mu} + \frac{ie}{\hbar c} A_\mu(x) .$$

After simple manipulations one obtains the following set of equations:

$$\begin{aligned}
&\{\gamma_\mu\overrightarrow{D}_\mu + M\}\ \psi = 0,\\
\bar{\psi}\ &\{\gamma_\mu\overleftarrow{D}_\mu - M\} = 0,\\[1em]
&\{\overrightarrow{D}_\nu - i\sigma_{\mu\nu}\overrightarrow{D}_\mu + M\ \gamma_\nu\}\ \psi = 0,\\
\bar{\psi}\ &\{\overleftarrow{D}_\nu + i\sigma_{\mu\nu}\overleftarrow{D}_\mu - M\ \gamma_\nu\} = 0,\\[1em]
&\{s_{\sigma,\mu\nu}\overrightarrow{D}_\sigma + i(\gamma_\mu\overrightarrow{D}_\nu - \gamma_\nu\overrightarrow{D}_\mu) + M\ \sigma_{\mu\nu}\}\ \psi = 0,\\
\bar{\psi}\ &\{s_{\sigma,\mu\nu}\overleftarrow{D}_\sigma + i(\gamma_\mu\overleftarrow{D}_\nu - \gamma_\nu\overleftarrow{D}_\mu) - M\ \sigma_{\mu\nu}\} = 0,\\[1em]
&\{\tau\ \overrightarrow{D}_\nu + \gamma_5\sigma_{\mu\nu}\overrightarrow{D}_\mu + M\ s_\nu\}\ \psi = 0,\\
\bar{\psi}\ &\{-\tau\ \overleftarrow{D}_\nu + \gamma_5\sigma_{\mu\nu}\overleftarrow{D}_\mu - M\ s_\nu\} = 0,
\end{aligned} \qquad \text{(B-1)}$$

$$\{s_\mu \overrightarrow{D}_\mu + M\,\tau\}\ \psi = 0,$$

$$\overline{\psi}\ \{s_\mu \overleftarrow{D}_\mu + M\,\tau\} \quad = 0,$$

where $M = mc/\hbar$, $s_{\sigma,\mu\nu} = \frac{1}{2}[\gamma_\sigma, \sigma_{\mu\nu}]_+$, $\psi = \psi(x)$, $\overline{\psi} = \overline{\psi}(x)$.

We now take any pair of the Equations (1), let $x \to x + z$ in the first one and $x \to x - z$ in the second one, then multiply the first one from the left by $\overline{\psi}(x-z)$ and the second one from the right by $\psi(x+z)$. By adding or subtracting the resulting equations and introducing further the notation

$$\Pi^+_\mu = \frac{1}{2M}\{\overrightarrow{D}^+_\mu + \overleftarrow{D}^-_\mu\},\quad \Pi^-_\mu = \frac{1}{2iM}\{\overrightarrow{D}^+_\mu - \overleftarrow{D}^-_\mu\},$$

where

$$\overrightarrow{D}^+_\mu = \frac{\partial}{\partial x_\mu} - ie\ A_\mu(x+z),\quad \overleftarrow{D}^-_\mu = \frac{\overleftarrow{\partial}}{\partial x_\mu} + ie\ A_\mu(x-z),$$

one obtains

$$\overline{\psi}\quad v_\mu \Pi^+_\mu \psi = 0,$$

$$\overline{\psi}\ \{v_\mu \Pi^-_\mu + 1\}\ \psi = 0,$$

$$\overline{\psi}\ \{\Pi^+_\nu + \sigma_{\mu\nu}\Pi^-_\mu\}\ \psi = 0,$$

$$\overline{\psi}\ \{\Pi^-_\nu - \sigma_{\mu\nu}\Pi^+_\mu - v_\nu\}\ \psi = 0, \qquad \text{(B-2)}$$

$$\overline{\psi}\ \{i\ s_{\sigma,\mu\nu}\ \Pi^+_\sigma + (v_\mu \Pi^-_\nu - v_\nu \Pi^-_\mu)\}\ \psi = 0,$$

$$\overline{\psi}\ \{i\ s_{\sigma,\mu\nu}\ \Pi^-_\sigma - (v_\mu \Pi^+_\nu - v_\nu \Pi^+_\mu) + \sigma_{\mu\nu}\}\ \psi = 0,$$

$$\overline{\psi}\,\{\tau\,\Pi_\nu^- - i\gamma_5\sigma_{\mu\nu}\Pi_\mu^+\}\,\psi = 0,$$

$$\overline{\psi}\,\{\tau\,\Pi_\nu^+ + i\gamma_5\sigma_{\mu\nu}\Pi_\mu^- + s_\nu\}\,\psi = 0,$$

$$\overline{\psi}\,\{s_\mu\Pi_\mu^+ + \tau\}\,\psi = 0,$$

$$\overline{\psi}\;\; s_\mu\Pi_\mu^-\,\psi = 0,$$

where $\overline{\psi} = \overline{\psi}(x-z),\quad \psi = \psi(x+z)$.

We give a generic definition of phase-space distributions associated with the Dirac matrices by

$$\mathcal{D}(x,p) = \langle D \rangle = (\pi\hbar)^{-4} \int d^4z\; e^{-2ip\cdot z/\hbar}(\Phi, N\{\overline{\psi}(x-z)\; D\; \psi(x+z)\}\;\Phi), \tag{B-3}$$

where D stands for any one of the matrices I, v_μ, $\sigma_{\mu\nu}$, s_μ or τ. Thus, we introduce the notation

$$\langle I \rangle = f(x,p),\; \langle v_\mu \rangle = \mathcal{V}_\mu(x,p),\; \langle \sigma_{\mu\nu} \rangle = \mathcal{O}_{\mu\nu}(x,p),$$

$$\langle s_\mu \rangle = S_\mu(x,p),\quad \langle \tau \rangle = \mathcal{T}(x,p).$$

We now obtain two relations which are useful in deriving the transport equations from Equations (2). Let us consider the quantities

$$Q_\mu^\pm(x,p) = (\pi\hbar)^{-4} \int d^4z\; e^{-2ip\cdot z/\hbar}(\Phi, N\{\overline{\psi}(x-z)\; D\; \Pi_\mu^\pm\; \psi(x+z)\}\;\Phi). \tag{B-4}$$

Recalling the definitions of $\Pi_\mu^\pm$ and making use of the Equation (3.5) one obtains after straightforward manipulations that

$$Q^{\pm}(x,p) = \frac{1}{mc}\left\{ \begin{matrix} \frac{\hbar}{2}\frac{\partial}{\partial x_\mu} + \frac{e}{c} A^e_\mu \sin(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial p}) \\ p_\mu - \frac{e}{c} A^e_\mu \cos(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial p}) \end{matrix} \right\} \mathcal{D}(x,p)$$

(B-5)

$$- \frac{e}{2mc^2} \left\{ \begin{matrix} i \\ 1 \end{matrix} \right\} (\pi\hbar)^{-4} \int d^4z\, e^{-2ip\cdot z/\hbar} (\Phi, N\{\overline{\psi}(x-z) D(A^i_\mu(x+z)$$

$$\mp A^i_\mu(x-z))\psi(x+z)\} \Phi).$$

Employing the "self-consistent field" approximation in the sense that

$$(\Phi, N\{\overline{\psi}(x-z) D A^i_\mu(x\pm z)\psi(x+z)\}\Phi) \simeq (\Phi, A^i_\mu(x\pm z)\Phi)(\Phi, N\{\overline{\psi}(x-z) D\psi(x+z)\}\Phi),$$

one gets from Equation (5)

$$Q^{\pm}(x,p) \simeq \frac{1}{mc}\left\{ \begin{matrix} \frac{\hbar}{2}\frac{\partial}{\partial x_\mu} + \frac{e}{c} \mathcal{A}_\mu \sin(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial p}) \\ p_\mu - \frac{e}{c} \mathcal{A}_\mu \cos(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial p}) \end{matrix} \right\} \mathcal{D}(x,p), \qquad \text{(B-6)}$$

where

$$\mathcal{A}_\mu(x) = A^e_\mu + (\Phi, A^i_\mu \Phi) .$$

Ignoring the terms of $O(\hbar^2)$ and changing variables from $\{x,p\}$ to $\{x,\pi\}$ by the relation $p - \frac{e}{c}\mathcal{A} = \pi$ we obtain from Equation (6)

$$Q^{\pm}(x,\pi) \simeq \left\{ \begin{matrix} \frac{h}{2mc}(\frac{\partial}{\partial x_\mu} + \frac{e}{c} F_{\nu\mu} \frac{\partial}{\partial \pi_\nu}) \\ \frac{1}{mc} \quad p_\mu \end{matrix} \right\} \mathcal{D}(x,\pi) , \qquad \text{(B-7)}$$

where $F_{\nu\mu} = \frac{\partial \mathcal{A}_\mu}{\partial x_\nu} - \frac{\partial \mathcal{A}_\nu}{\partial x_\mu}$ and $\mathcal{D}(x,\pi)$ is defined by the relation

$$\mathcal{D}(x,\pi)\, d^4x d^4\pi = \mathcal{D}(x,p)\, d^4x d^4p .$$

We now multiply Equations (2) by $(\pi\hbar)^{-4}e^{2ip\cdot z/\hbar}$ and integrate over z and take the expectation with respect to Φ. Using Equation (7) and the relations

$$s_{\sigma,\mu\nu} = s_\rho \; \epsilon_{\rho\sigma\mu\nu} \;,$$

$$i\gamma_5\sigma_{\mu\nu} = \frac{1}{2i}\,\epsilon_{\mu\nu\rho\sigma}\sigma_{\rho\sigma} \equiv \sigma^+_{\rho\sigma} \;,$$

which may be verified, we obtain the following set of equations:

$$\hat{T}_\mu \mathcal{V}_\mu = 0, \tag{B-8}$$

$$\pi_\mu \mathcal{V}_\mu + mcf = 0, \tag{B-9}$$

$$\pi_\mu \boldsymbol{\sigma}_{\mu\nu} = -\frac{\hbar}{2}\hat{T}_\nu f \;, \tag{B-10}$$

$$mc\mathcal{V}_\nu = \pi_\nu f + \frac{\hbar}{2}\hat{T}_\mu \boldsymbol{\sigma}_{\nu\mu}, \tag{B-11}$$

$$\pi_\mu \mathcal{V}_\nu - \pi_\nu \mathcal{V}_\mu = \frac{i\hbar}{2}\,\epsilon_{\mu\nu\rho\sigma}\hat{T}_\sigma S_\rho, \tag{B-12}$$

$$mc\boldsymbol{\sigma}_{\mu\nu} = -i\,\epsilon_{\mu\nu\rho\sigma}S_\rho\pi_\sigma + \frac{\hbar}{2}\{\hat{T}_\nu \mathcal{V}_\mu - \hat{T}_\mu \mathcal{V}_\nu\} \;, \tag{B-13}$$

$$\pi_\nu \mathcal{T} - \frac{\hbar}{2}\hat{T}_\mu \boldsymbol{\sigma}^+_{\mu\nu} = 0, \tag{B-14}$$

$$mcS_\nu = \boldsymbol{\sigma}^+_{\nu\mu}\pi_\mu + \frac{\hbar}{2}\hat{T}_\nu \mathcal{T}, \tag{B-15}$$

$$\frac{\hbar}{2mc}\hat{T}_\mu S_\mu + \mathcal{T} = 0, \tag{B-16}$$

$$\pi_\mu S_\mu = 0, \tag{B-17}$$

where we have omitted the arguments (x,π), and defined

$$\hat{T}_\mu = \frac{\partial}{\partial x_\mu} + \frac{e}{c}F_{\nu\mu}\frac{\partial}{\partial \pi_\nu} \;.$$

It is observed that Equations (8) - (17) are not all independent. We have written them all because they make a convenient reference.

Transport Equation for $f(x,\pi)$

From Equations (8) and (11) one obtains

$$\pi_\nu \hat{T}_\nu f + \frac{\hbar}{2} \hat{T}_\nu \hat{T}_\mu \sigma_{\nu\mu} = 0 \ . \tag{B-18}$$

Noting that $\sigma_{\nu\mu} = -\sigma_{\mu\nu}$ and that

$$[\hat{T}_\nu, \hat{T}_\mu]_- = \frac{e}{c} \frac{\partial F_{\nu\mu}}{\partial x_\rho} \frac{\partial}{\partial \pi_\rho} \ ,$$

it follows that

$$\pi_\mu \hat{T}_\mu f + \frac{e\hbar}{4c} \frac{\partial F_{\mu\nu}}{\partial x_\rho} \frac{\partial \sigma_{\mu\nu}}{\partial \pi_\rho} = 0 \ . \tag{B-19}$$

This is the same as Equation (3 .18) in the main context.

A subsidiary condition to Equation (19) is obtained by using Equations (9), (10), (11) and noting that $[p_\mu, \hat{T}_\nu]_- = \frac{e}{c} F_{\nu\mu}$ as

$$(\pi^2 c^2 + m^2 c^4) f - \frac{1}{2} e\hbar c \ F_{\mu\nu} \sigma_{\mu\nu} = 0. \tag{B-20}$$

Transport Equation for $S_\mu(x,\pi)$

From Equations (14) and (16) one finds

$$\pi_\nu \hat{T}_\mu S_\mu + mc \ \hat{T}_\mu \sigma^+_{\mu\nu} = 0. \tag{B-21}$$

Also, it follows from Equation (13) that

$$mc \, \sigma^+_{\mu\nu} = \pi_\mu S_\nu - \pi_\nu S_\mu . \tag{B-22}$$

Combining Equations (21) and (22) one obtains

$$\pi_\mu \hat{T}_\mu S_\nu = \frac{e}{c} F_{\nu\mu} S_\mu , \tag{B-23}$$

which is the transport equation for $S_\mu(x,\pi)$. Subsidiary condition to Equation (23) is obtained by combining the Equations (15), (17) and (22) as

$$(\pi^2 + m^2c^2)\ S_\nu = 0. \tag{B-24}$$

It should be noted that we have ignored the terms of order $\hbar$ in equations for the spin density since the spin dependent terms in Equations (19) and (20) appear already multiplied by $\hbar$.

APPENDIX C

GAUGE INVARIANCE OF $f(x,\pi)$

We show here that $f(x,\pi)$ is invariant under gauge transformation to order $(\hbar^2)$. By definition,

$$f(x,\pi) = (\pi\hbar)^{-4} \int d^4z \; e^{-2i(\pi + \frac{e}{c}\mathcal{A})\cdot z/\hbar}(\Phi, \bar{\psi}(x-z)\psi(x+z) \; \Phi). \qquad \text{(C-1)}$$

change dummy variable $z \rightarrow \tau$ by the relation $z = \hbar\tau$ so that

$$f(x,\pi) = \pi^{-4} \int d^4\tau \; e^{-2i(\pi + \frac{e}{c}\mathcal{A})\cdot\tau}(\Phi, \bar{\psi}(x-\hbar\tau)\psi(x+\hbar\tau) \; \Phi). \qquad \text{(C-2)}$$

gauge transformations of the second kind is defined by

$$\begin{aligned} A_\mu &\rightarrow A'_\mu = A_\mu + \frac{\partial\Lambda}{\partial x_\mu} \\ \psi &\rightarrow \psi' = \psi \; e^{-ie\Lambda/\hbar c} \\ \bar{\psi} &\rightarrow \bar{\psi}' = \bar{\psi}' e^{ie\Lambda/\hbar c} \end{aligned} \qquad \text{(C-3)}$$

where Λ satisfies the equation $\Box\Lambda = 0$. Under these transformations,

$$f(x,\pi) \rightarrow f'(x,\pi) = \pi^{-4} \int d^4\tau \; e^{-2i(\pi + \frac{e}{c}\mathcal{A})\cdot\tau}(\Phi, \bar{\psi}'(x-\hbar\tau)\psi'(x+\hbar\tau) \; \Phi). \qquad \text{(C-4)}$$

Inserting the relations (C-3) into (C-4) one finds that

$$f'(x,\pi) = \pi^{-4} \int d^4\tau \; e^{-2i(p + \frac{e}{c}\mathcal{A})\cdot\tau} \; e^{iQ}(\Phi, \bar{\psi}(x-\hbar\tau)\psi(x+\hbar\tau) \; \Phi), \qquad \text{(C-5)}$$

where

$$Q \equiv \frac{ie}{hc}(\Lambda(x+\hbar\tau) - \Lambda(x-\hbar\tau)) - 2i \frac{e}{c} \tau_\mu \frac{\partial\Lambda}{\partial x_\mu} \; . \qquad \text{(C-6)}$$

Expanding Λ in the expression for Q into Taylor series about x it is easily seen that Q is of $O(\hbar^2)$.

APPENDIX D

CONTRIBUTIONS TO $(\frac{\partial F_e}{\partial t})$ OF COMPTON SCATTERING AND PAIR PRODUCTION

1. Electron-Photon Scattering (Compton Scattering)

The part of $S^{(2)}$ which describes this process is given by

$$S^{(2)}_{c.s.} = S_1 + S_2,$$

where

$$S_1 = (\frac{e}{\hbar c})^2 \iint d^4x_1 d^4x_2 N\{(\bar{\psi}^{(-)}\gamma_\nu A_\nu^{(-)} \underline{\psi)_{x_2}(\bar{\psi}}\gamma_\mu A_\mu^{(+)}\psi^{(+)})_{x_1}\},$$

$$S_2 = (\frac{e}{\hbar c})^2 \iint d^4x_1 d^4x_2 N\{\bar{\psi}^{(-)}\gamma_\nu A_\nu^{(+)} \underline{\psi)_{x_2}(\bar{\psi}}\gamma_\mu A_\mu^{(-)}\psi^{(+)})_{x_1}\},$$

and

$$\underline{\psi(x_2)\bar{\psi}}(x_1) = -\frac{\hbar}{2} S_F(x_2-x_1) ,$$

$$S_F(x) = \frac{2}{(2\pi\hbar)^4} \int d^4q \frac{e^{iqx/\hbar}}{\not{q} - imc} , \quad (\not{q} \equiv \gamma_\mu q_\mu) .$$

Defining

$$\Gamma^{c.s.}_{n\to m} = \frac{1}{VT} |< m| \; S^{(2)}_{c.s.} \; |n >|^2,$$

one can show that

$$\Gamma^{c.s.}_{n\to m} = (4\pi e^2 c)^2 \frac{VT}{V^4} \sum_{\substack{\underline{p}';\underline{p} \\ s';s}} \sum_{\substack{\underline{k}';\underline{k} \\ m';m}} \frac{m^2c^4}{4\,\omega'\omega E_{\underline{p}'} E_{\underline{p}}} \delta_{p' + \hbar k' - p - \hbar k} |A^{c.s.}|^2$$

$$\cdot \{1-n_e(\underline{p}';s')\}\{1+n_\gamma(\underline{k}';m')\}n_\gamma(\underline{k},m)n_e(\underline{p},s) \quad < m|..,n_e(\underline{p}';s') \qquad \text{(D-1)}$$

$$+ 1,..,n_e(\underline{p},s) - 1,..;..,n_\gamma(\underline{k}';m') + 1,..,n_\gamma(\underline{k},m) - 1,...> ,$$

where $n_\gamma(\underline{k},m)$ denotes the number of photons of momentum $\underline{k}$ and polarization m in the state $|n>$, and

$$A^{c.s.} \equiv A^{c.s.}(\underline{p}';s';\hbar\underline{k}'m'|\underline{p},s;\hbar\underline{k},m)$$

$$\equiv \bar{u}^{(s')}(\underline{p}') \{\not\epsilon^{(m')}(\underline{k}') \frac{1}{(\not p+\hbar\not k)-imc} \not\epsilon^{(m)}(\underline{k}) \tag{D-2}$$

$$+ \not\epsilon^{(m)}(\underline{k}) \frac{1}{(\not p+h\not k')-imc} \not\epsilon^{(m')}(\underline{k}')\} u^{(s)}(\underline{p})$$

$A^{c.s.}$ has the following symmetry property

$$A^{c.s.}(\underline{p}';s';\hbar\underline{k}';m'|\underline{p},s;\hbar\underline{k},m) = -A^*(\underline{p},s;\hbar\underline{k},m|\underline{p}';s';\hbar\underline{k}';m').$$

Note that in Equation (1) and thereafter sums over photon polarizations m, m' are restricted to 1,2. This is because the terms with m, m' = 3, 4 do not give any contribution due to the fact that initial and final states $|n>$ and $|m>$ do not contain[18] any photons with polarizations 3 and 4. The rest of the derivation is quite similar to that of the Möller scattering. Therefore, we shall be more brief here.

One finds [Corresponding to Equation (4.35)]

$$\frac{\partial f_e}{\partial t}(\underline{p},s,t) = (4\pi e^2 c)^2 \frac{VT}{V^4} \sum_{\underline{p}';s'} \sum_{\substack{\underline{k}';\underline{k} \\ m';m}} \frac{m^2c^4}{4\,\omega'\omega\, E_{\underline{p}}E_{\underline{p}'}} \delta_{p'+\hbar k'-p-\hbar k}$$

$$\cdot\ |A^{c.s.}(\underline{p}'s';\hbar\underline{k}'m'|\underline{p},s;\hbar\underline{k},m)|^2 \tag{D-3}$$

$$\cdot\ \{\{1-Vf_e\}\{1+Vf_\gamma\}\ Vf'_\gamma Vf'_e - \{1-Vf'_e\}\{1+Vf'_\gamma\}\ Vf_\gamma Vf_e\}$$

where

$$f_\gamma(\underline{k},m,t) \equiv \frac{1}{V} \sum_n P_n(t) n_\gamma(\underline{k},m) .$$

Transition to the continuum variables is achieved by defining, in addition to (4.36) and (4.37),

$$\sum_{\underline{k}\epsilon d^3k} f_\gamma(\underline{k},m,t) = \chi(\underline{\kappa},m,t)d^3\kappa ,$$

where $\underline{\kappa} = \hbar k$, and noting that

$$\sum_{\underline{k}\epsilon d^3k} 1 \rightarrow \frac{V}{(2\pi\hbar)^3} d^3\kappa .$$

One obtains, in the continuum variables,

$$\frac{\partial F_e}{\partial t}(\underline{p},s,t) = (e^2c)^2 \rho^{-2} \sum_{s',m',m} \int d^3p'd^3\kappa'd^3\kappa \frac{m^2c^4}{\varpi'\varpi E_{\underline{p}'}E_{\underline{p}}} \delta(\underline{p}'+\underline{\kappa}'-\underline{p}-\underline{\kappa})$$

$$\cdot \delta(E'+\varpi'-E-\varpi)\left|A^{c.s.}(\underline{p}',s';\underline{\kappa}',m'|\underline{p},s;\underline{\kappa},m)\right|^2 \qquad \text{(D-4)}$$

$$\left\{\{\rho-F_e\}\{\rho+\chi\} F_e'\chi' - \{\rho-F_e'\}\{\rho+\chi'\} F_e\chi\right\}$$

where $\varpi = \hbar\omega$, $\rho \equiv (2\pi\hbar)^{-3}$.

If we assume isotropic distributions in spins and polarizations, that is,

$$F_e(\underline{p},s,t) = \frac{1}{2} F_e(\underline{p},t) , \qquad (s=1,2)$$

$$\chi(\underline{\kappa},m,t) = \frac{1}{2} \chi(\underline{\kappa},t), \qquad (m=1,2)$$

then

$$\frac{\partial F_e}{\partial t}(\underline{p},t) = (e^2c)^2 \frac{\rho^{-2}}{4} \int d^3p'd^3\kappa'd^3\kappa \frac{m^2c^4}{\varpi'\varpi E_{\underline{p}'}E_{\underline{p}}} \delta(\underline{p}'+\underline{\kappa}'-\underline{p}-\underline{\kappa})$$

$$\cdot \delta(E'+\varpi'-E-\varpi) \cdot B^{c.s.}(\underline{p}',\underline{\kappa}'|\underline{p},\underline{\kappa}) \qquad \text{(D-5)}$$

$$\cdot \left\{\{2\rho-F_e\}\{2\rho+\chi\} F_e'\chi' - \{2\rho-F_e'\}\{2\rho-\chi'\} F_e\chi\right\} ,$$

where

$$B^{c.s.}(\underline{p}',\underline{\kappa}'|\underline{p},\underline{\kappa}) = \frac{1}{4} \sum_{s,s',m,m'} |A^{c.s.}(\underline{p}',s';\underline{\kappa}'m'|\underline{p},s;\underline{\kappa},m)|^2.$$

2. Two-Quantum Pair-Creation and Annihilation

The parts of $S^{(2)}$ which describe these processes which are the inverse of each other are given by

$$S^{(2)}_{p.a.} = \left(\frac{e}{\hbar c}\right)^2 \iint d^4x_1 d^4x_2 N\,\{(\bar{\psi}^{(+)}\gamma_\nu A_\nu^{(-)}\underline{\psi})_{x_2}(\underline{\bar{\psi}}\gamma_\mu A_\mu^{(-)}\psi^{(+)})_{x_1}\},$$

$$S^{(2)}_{p.c.} = \left(\frac{e}{\hbar c}\right)^2 \iint d^4x_1 d^4x_2 N\,\{(\psi^{(-)}\gamma_\nu A_\nu^{(+)}\underline{\psi})_{x_2}(\underline{\psi}\gamma_\mu A_\mu^{(+)}\psi^{(-)})_{x_1}\},$$

where

$$\underline{\psi(x_2)\psi(x_1)} = -\frac{\hbar}{(2\pi\hbar)^4}\int d^4\kappa\, \frac{e^{i\kappa\cdot(x_2-x_1)/\hbar}}{\not{\kappa} - imc}.$$

A similar analysis as before yields

$$\frac{\partial F_e}{\partial t}(\underline{p},s,t) = \frac{1}{2}(e^2c)^2\rho^{-2} \sum_{r,m_2,m_1} \int d^3q\,d^3\kappa_2 d^3\kappa_1 \frac{m^2c^4}{\varpi_2\varpi_1\, E_{\underline{q}}E_{\underline{p}}}$$

$$\cdot\ \delta(\underline{\kappa}_2+\underline{\kappa}_1 - \underline{q}-\underline{p})\,\delta(\varpi_2+\varpi_1-E_{\underline{p}}-E_{\underline{q}})\ |A|^2$$

$$\cdot\ \{\{\rho-F_e(\underline{p},s,t)\}\{\rho-F_p(\underline{q},r,t)\}\ X(\underline{\kappa}_2,m_2,t)\ X(\underline{\kappa}_1,m_1,t)$$

$$-\ \{\rho+X(\underline{\kappa}_2,m_2,t)\}\{\rho+X(\underline{\kappa}_1,m_1,t)\}\ F_e(\underline{p},s,t)F_p(\underline{p},s,t)\}, \qquad \text{(D-6)}$$

where

$$|A|^2 \equiv |A^{p.a.}|^2 = |A^{p.c.}|^2,$$

and

$$A^{p.a.} = A^{p.a.}(\underline{\kappa}_2,m_2;\underline{\kappa}_1,m_1|\underline{p},s;\underline{q},r)$$

$$= \bar{u}^{(r+2)}(\underline{q})\ \{\not{e}^{(m_2)}(\underline{\kappa}_2)\ \frac{1}{(\not{p}-\not{\kappa}_1)-imc}\ \not{e}^{(m_1)}(\underline{\kappa}_1)$$

$$+ \not{e}^{(m_1)}(\underline{\kappa}_1)\ \frac{1}{(\not{p}-\not{\kappa}_2)-imc}\ \not{e}^{(m_2)}(\underline{\kappa}_2)\}\ u^{(s)}(\underline{p}) \quad ,$$

$$A^{p.c.} = A^{p.c.}(\underline{p},s;\underline{q},r|\underline{\kappa}_2,m_2;\underline{\kappa}_1,m_1)$$

$$= \bar{u}^{(s)}(\underline{p})\ \{\not{e}^{(m_2)}(\underline{\kappa}_2)\ \frac{1}{(\not{p}-\not{\kappa}_2)-imc}\ \not{e}^{(m_1)}(\underline{\kappa}_1)$$

$$+ \not{e}^{(m_1)}(\underline{\kappa}_1)\ \frac{1}{(\not{p}-\not{\kappa}_1)-imc}\ \not{e}^{(m_2)}(\underline{\kappa}_2)\}\ u^{(r+2)}(\underline{q}) \quad .$$

The symmetry relation between $A^{p.a.}$ and $A^{p.c.}$ which has been used in deriving the Equation (6) is noted as

$$A^{p.c.}(\underline{p},s;\underline{q},r|\underline{\kappa}_2,m_2;\underline{\kappa}_1,m_1) = -A^{*p.a.}(\underline{\kappa}_2,m_2;\underline{\kappa}_1,m_1|\underline{p},s;\underline{q},r).$$

APPENDIX E

A QUANTUM DISPERSION RELATION FOR TRANSVERSE OSCILLATIONS IN A RELATIVISTIC PLASMA

In the previous analysis we have assumed that photons propagate with the velocity c between successive events so that $\omega = c|\underline{k}|$. It is known that[30] the energy-momentum relation for photons traveling in a medium differs from this relation due to the collective effects of the medium. We wish to present here a quantum dispersion relation which determines ω as a function of $\underline{k}$ appropriate to a fully ionized relativistic plasma in the absence of any external fields.

We use here a procedure developed by Mead[31] in the study of the quantum theory of the refractive index. The details of the calculations is of little interest since it only involves a straightforward perturbation calculation to second-order. We find that the modified energy-momentum relation for transverse photons (i.e., m = 1, 2) is determined by the dispersion relation

$$c^2 k_m \cdot k_m = \omega_p^2 \frac{mc^2}{2} \int \frac{d^3p}{E_{\underline{p}}} F_e(\underline{p}) \tag{E-1}$$

$$\cdot \Big\{ \Big[\frac{2(\epsilon^{(m)}(\underline{k})\cdot p)^2 - (p\cdot q_3 + m^2c^2)}{hk_m\cdot q_3 + (p\cdot q_3 + m^2c^2)} - \frac{2(\epsilon^{(m)}(\underline{k})\cdot p)^2 - (p\cdot q_2 + m^2c^2)}{hk_m\cdot q_2 - (p\cdot q_2 + m^2c^2)} \Big]$$

$$+ \Big[\frac{2(\epsilon^{(m)}(\underline{k})\cdot p)^2 + (p\cdot q_4 - m^2c^2)}{hk_m\cdot q_4 - (p\cdot q_4 - m^2c^2)} - \frac{2(\epsilon^{(m)}(\underline{k})\cdot p)^2 + (p\cdot q_1 - m^2c^2)}{hk_m\cdot q_1 + (p\cdot q_1 - m^2c^2)} \Big] \Big\}$$

where $\epsilon^{(m)}(\underline{k}) \equiv (\underline{k}/|\underline{k}| \; ; 0)$, $k_m \equiv (\underline{k} \; ; i\omega_m/c)$, $\omega_p^2 \equiv \frac{4\pi Ne^2}{mV}$,

and

$$\underline{q}_1 = -(\underline{p}+\hbar\underline{k}), \quad \underline{q}_3 = \underline{p}+\hbar\underline{k}$$

$$\underline{q}_2 = (\underline{p}-\hbar\underline{k}), \quad \underline{q}_4 = -(\underline{p}-\hbar\underline{k})$$

and where $F_e(\underline{p})$ is normalized to unity.

In the non-relativistic limit the relation (1) reduces to

$$\omega_m^2 = c^2|\underline{k}|^2 + \omega_p^2 \left\{1 + \int d^3v\ f_e(\underline{v})\ \frac{(\underline{\epsilon}^{(m)}\cdot\underline{v})k^2 + (\frac{\hbar k^2}{2m})^2}{(\underline{k}\cdot\underline{v}-\omega_m)^2 - (\frac{\hbar k^2}{2m})^2}\right\} . \qquad \text{(E-2)}$$

It is observed that this relation differs from that given by Bohm and Pines[30] with the appearance of an additional term, $(\hbar k^2/2m)^2$, in the nominator of the integrand. This additional term arises due to the inclusion of the spins of the electrons in the analysis.*

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* After the work presented here was completed, we found that in a recent article by P. Burt and H. D. Wahlquist[29] spin and exchange corrections to the plasma dispersion relations have been calculated by a different procedure. They have employed a quantum Vlasov equation based upon the non-relativistic Pauli equation in order to include the spin effects. Their result for the transverse dispersion relation including spin correction agrees with our Equation (E-2). Exchange corrections do not appear in our dispersion relation given by (E-1). In order to include the exchange effects in the transverse dispersion relation we must go to the fourth order in the perturbation analysis. In fact, the exchange correction given in the aforementioned paper is proportional to $\omega_p^4 \sim e^4$.

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