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# STUDIES IN RADAR CROSS SECTIONS - XLV
# STUDIES IN NON-LINEAR MODELING - II.

FINAL REPORT ON
CONTRACT AF 19(604)-4993

by
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ABSTRACT

This report contains a collection of studies in the realm of non-linear modeling performed during the year 1960. It includes a discussion of the generality of non-linear modeling which displays that all second order ordinary differential equations arising from a conservative system can be locally modeled in a non-linear manner. Also included is a discussion of the problem of modeling the scalar wave equation in n-dimensions and a preliminary consideration of the effect of experimental errors on the applicability of non-linear modeling.

The problem of modeling a scalar scattering problem for one geometric configuration into a scalar scattering problem for a second geometric configuration is begun. Two cases are considered; (1) that of modeling a scalar scattering problem for an elliptical cylinder by one for a circular cylinder, and (2) that of modeling prolate spheroid problems into sphere problems.

# I

## INTRODUCTION AND SUMMARY

(J. W. Crispin Jr.)

This contract, AF 19(604)-4993, started on 1 January 1959 and ends on 31 December 1960 with the publication of this final report. The objective of this contract was the investigation of the application of non-linear modeling to Maxwell's equations and to the Navier-Stokes equation with the ultimate objective being to obtain an understanding of the phenomena of the interaction of electromagnetic energy with plasmas.

During the first year of this contract attention was directed toward the consideration of several non-linear models of equations of mathematical physics; this effort was summarized in [1]. In addition, during the first year, three studies, one on scattering from plasmas and two on basic electromagnetics were concluded with the publication of [2, 3, and 4.]

Another effort has also continued, resulting in the publication of [5]. This effort has been made by Professor D. A. Darling and involves the problem of obtaining diffraction and scattering solutions for bodies which are formed by the intersection of separable bodies (e.g. the body formed by the combination of a cone with either one or two spheres). To date the effort has been restricted to the Laplace equation and to scalar scattering; extension to the vector problem is clear.

During the last five months the efforts of the Radiation Laboratory in non-linear modeling have become split; the effort devoted to the study of the interaction of electromagnetic fields with plasmas via non-linear modeling techniques has become the subject matter of contract AF 19(604)-7428. The non-linear efforts conducted under AF 19(604) 4993 have been somewhat diversified with consideration being given to several important problems. One goal has become the solution of the "low cross section shape" problem via the process of non-linearly modeling a "low cross section shape problem" into a "large cross section shape problem"; the specific problem being considered at the present time is that of a prolate spheroid into a sphere. This problem has not been completely solved as yet but we feel we are well on the road to putting this modeling process in a form which will be extremely valuable in future laboratory studies. It is felt that the combination of this effort with the results of the extended work of Professor Darling, referred to above, will provide a strong tool for the study of low cross section missile shapes without resorting to special laboratory techniques.

Due to the diversity of the efforts conducted during the past year on this contract, this report is, in effect, a collection of papers on non-linear modeling. We include in this report all contributions of significance which have evolved since the publication of [1] with the exception of the work of Professor Darling which, as stated above, is covered in [5.]

In Section II we present a discussion of the generality of non-linear modeling; this work (by Professor R. K. Ritt) proves that all second order ordinary differential equations arising from a conservative system can be locally modeled in a non-linear manner.

Section III contains a discussion of the problem of modeling the scalar wave equation in n-dimensions. This work was performed by O. G. Ruehr and we might note that this is, in itself, of extreme importance in turbulence theory where high-dimensional solutions to the scalar wave equation play an important role.

Section IV is devoted to the consideration of the effect of experimental errors on the applicability of non-linear modeling. This analysis is devoted to the scalar wave case and is intended as an illustrative example of this effect. This question is of course of prime importance to the non-linear modeling concept when we think of experimental applications.

The remainder of this report is devoted to the consideration of the work which has been done during the past year on non-linear modeling of low radar cross section problems. In Section V the item of concern is the high-frequency forward scattering of a finite plasma disc; this we consider as a first step along these lines. Section VI is devoted to the consideration of the problem of scalar scattering by an elliptic cylinder in which this problem is modeled into a similar one for a circular cylinder. Sections VII and VIII are devoted to the consideration of modeling prolate spheroid problems into sphere problems; in Section VII this problem is considered as an extension of the work of Section VI (i.e. a perturbation method approach) while in Section VIII this fundamental problem is considered from a different point of view, that of using an expansion technique in which the expansions of the fields for the two bodies are related. This approach has the advantage of being applicable to all spheroids directly while the approach of Section VII is somewhat restricted by the magnitude of the eccentricity of the spheroid.

## REFERENCES

### SECTION I

[1] "Studies in Radar Cross Sections XXXVIII - Non-Linear Modeling of Maxwell's Equations," by J. E. Belyea, R. D. Low, and K.M. Siegel, University of Michigan, Radiation Laboratory Report No. 2871-4-T (December 1959).

[2] "Electromagnetic Scattering by High Density Meteor Trails" by H. Brysk, University of Michigan, Radiation Laboratory Report No. 2871-1-T (June 1959).

[3] "Scalar Diffraction by an Elliptic Cylinder," by N.D. Kazarinoff and R.K. Ritt, University of Michigan, Radiation Laboratory Report No. 2871-2-T (June 1959).

[4] "Studies in Radar Cross Sections XXXVI - Diffraction of a Plane Wave by an Almost Circular Cylinder," by P. C. Clemmow and V.H. Weston, University of Michigan, Radiation Laboratory Report No. 2871-3-T (September 1959).

[5] "Some Relations Between Potential Theory and the Wave Equation" by D. A. Darling, University of Michigan, Radiation Laboratory Report No. 2871-5-T (December 1960).

II

# A NOTE ON THE THEORY OF MODELING

(R. K. Ritt)

## 1. Introduction

In [1] it was shown that if two physical systems could be assigned trajectories in phase space of the form $x = e^{At}x_0$, where $e^{At}$ is a one-parameter semigroup, then, at least for small values of t, there existed a one-one correspondence between the trajectories of the two systems, and that this correspondence would be extended to the entire trajectories, subject to restrictions imposed by the spectrum of the operators A; it is even possible to change the time scale in one of the systems, without losing any generality. When this correspondence is established, the systems are said to be models of each other, and the restrictions mentioned are called the similitude conditions; they are generalizations of the classical similitude conditions, which assume that the correspondence between the phase spaces is a linear one. In [2], the extension of this theory to certain electromagnetic problems was accomplished, and it was shown that the modeling could be carried out in certain systems whose equations of evolution were non-linear. In particular, for systems involving one degree of freedom whose equation is of the form

$$\frac{d^2q}{dt^2} + f(q) = 0 \quad ,$$

the correspondence was exhibited explicitly, in terms of quadrature.

In the present note, the last of the above-mentioned results will be rederived in the setting of general dynamical systems, and the possibility of extending the result to systems with more than one degree of freedom will be analyzed. The theory will be a local one, i.e. we shall discuss only the initial part of the trajectory so that the question of similitude conditions will be ignored. This is a question which, at the moment, appears to be quite difficult to solve in the general case.

## 2. Systems with one degree of freedom.

Let (q, p) be the canonical coordinates of the system, which will be assumed to be conservative, and let h(p, q) be its Hamiltonian. Let e be the energy of the system, determined by the initial conditions. Then assume that initially, $\partial h/\partial p \neq 0$, so that, at least for small values of t, the equation $h(p, q) = e$ can be solved for p, obtaining

$$p = p(e, q) \quad . \tag{1}$$

Then the equation of motion is of the form:

$$\frac{dq}{dt} = \frac{\partial h}{\partial p} \quad , \tag{2}$$

and replacing the p which may appear in the right member of (2), we obtain

$$\frac{dq}{dt} = \psi\,(e, q) \quad , \tag{3}$$

in which $\psi$ (e, q) is not zero for t sufficiently small. If a second system has canonical coordinates (Q, P), and whose time scale, T, is given as $T = \lambda t$, and whose energy is E, an analogous discussion leads to

$$\frac{dQ}{dt} = \lambda \ \Psi(E, Q) \tag{4}$$

From (3) and (4), we obtain

$$\lambda \ \Psi(E, Q)\, dq - \psi(e, q)\, dQ = 0 \tag{5}$$

which is a nonsingular differential equation which can be solved by quadrature. This is essentially the result in [2] we have mentioned.

3. Systems with more than one degree of freedom.

Let there be two such systems, with canonical coordinates $(q_1, \ldots, q_n; p_1, \ldots, p_n)$ and $(Q_1, \ldots, Q_n; P_1, \ldots, P_n)$. If we were to admit, among our admissible modeling functions, all those of the form

$$(q, p) = F(Q, P) \quad , \tag{6}$$

in which the parentheses represent points in the 2n dimensional phase space, the question of existence would be trivial. Instead, what we shall investigate is the existence of modeling functions of the form

$$q = F(Q) \quad . \tag{7}$$

In general, the contact transformations which provide solutions of (6) do not separate to give solutions of the form (7); as we shall see the restrictions are severe, but they have the virtue of being capable of an explicit formulation.

Let $h_j(p, q)$ and $H_j(p, q)$, $j = 1, 2, \ldots, n$ be n independent integrals of the systems, which are presumed to be known; $h_1$ and $H_1$ are the hamiltonians of the systems. Let us assume that the initial conditions are such that the Jacobians (with respect to p and P) of the equations:

$$h_j(p, q) = e_j \quad , \quad H_j(P, Q) = E_j \tag{8}$$

do not vanish initially. Then the two sets of equations (8) can be solved for p and q, giving

$$p_j = p_j(e, q) \quad , \quad P_j = P_j(E, Q) \quad , \quad j = 1, \ldots, n \quad . \tag{9}$$

If the time scales are as in § 2, the equations of motion are

$$\frac{dq_j}{dt} = \frac{\partial h_1}{\partial p_j} \quad , \quad \frac{dQ_j}{dt} = \lambda \frac{\partial H_1}{\partial P_j} \quad , \quad j = 1, \ldots, n \quad , \tag{10}$$

and when the values (9) are substituted into the right members of (10), we obtain:

$$\frac{dq_j}{dt} = \psi_j(e, q) \ ; \ \frac{dQ_j}{dt} = \lambda \Psi_j(E, Q) \ .$$

From this equation is obtained the system of partial differential equations

$$\frac{\partial q_j}{\partial Q_k} = \frac{\psi_j(e, q)}{\lambda \Psi_k(E, Q)} \quad . \tag{12}$$

The existence of solutions for (12) is equivalent to the satisfaction of the compatability conditions. Now

$$\frac{\partial}{\partial Q_\ell}\left(\frac{\partial q_j}{\partial Q_k}\right) = \frac{1}{\lambda^2 \Psi_k \Psi_\ell} \sum_{r=1}^{n} \frac{\partial \psi_j}{\partial q_r} \psi_r + \frac{\psi_j}{\lambda} \frac{\partial}{\partial Q_\ell}\left(\frac{1}{\Psi_k}\right). \tag{13}$$

Because of the symmetry of the first term of the right member of (13), the necessary and sufficient condition that equation (12) have solutions, is that

$$\frac{\partial}{\partial Q_k}\left(\frac{1}{\Psi_\ell}\right) = \frac{\partial}{\partial Q_\ell}\left(\frac{1}{\Psi_k}\right) \quad , \quad k, \ell = 1, \ldots, n \quad . \tag{14}$$

4. Discussion of the result.

In general, the condition (14), does not hold. And of course, any attempt to test a particular system depends upon the knowledge of n independent integrals; so that, practically, especially if n is large, the question of whether a modeling function of the type (7) exists is a difficult one to settle. However, an interesting theoretical result arises from the above discussion; namely, that the existence of a modeling function of the type (7), depends only upon the nature of the (Q, P) system.

## REFERENCES

## SECTION II

[1] Ritt, R. K. "The Modeling of Physical Systems," IRE Trans. on Antennas and Propagation, Vol. AP-4, No. 3, July 1958.

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# III

## MODELING OF THE HELMHOLTZ EQUATION IN N DIMENSIONS

(O. G. Ruehr)

### Introduction

This chapter is concerned with the non-linear modeling of one Helmholtz equation by another in a space of n dimensions. We speak of modeling differential equations rather than physical systems since the equations together with boundary conditions can be considered as representatives of the physical systems. In particular, we are interested here in determining what can be said about modeling functions and similitude conditions [1] from the differential equations themselves without bringing in boundary conditions.

Suppose two functions of position $\phi(\vec{x})$ and $\psi(\vec{x})$ satisfy Helmholtz equations:

$$\nabla^2 \phi + K_1^2 \phi = 0 \tag{1}$$

$$\nabla^2 \psi + K_2^2 \psi = 0 \tag{2}$$

We would like to model equation (1) by equation (2) with a modeling function of the form $\phi = \phi(\psi)$. That is, given a value of $\psi$ at a point we want to determine $\phi$ at that point as a function of $\psi$ only (and not of position). To put it another way, we want to find all single-valued transformations $\phi = \phi(\psi)$ between the solutions of equations (1) and (2). Conditions on $K_1$ and $K_2$ for which such modeling functions exist are called similitude conditions. It is clear that, if $K_1 = K_2$, $\phi = A\psi$, where A is constant, is such a transformation (linear modeling). Let $\vec{c}$ be a constant unit vector and let $\vec{r}$ be the n dimensional radius vector. Then particular solutions of (1) and (2) are given by

$$\phi = ae^{iK_1(\vec{r}\cdot\vec{c})} \tag{3}$$

$$\psi = be^{iK_2(\vec{r}\cdot\vec{c})} \tag{4}$$

Examination of these solutions shows that the following transformation is a modeling function when it is single-valued, i.e. when $\frac{K_1}{K_2}$ is an integer.

$$\phi = a(\psi/b)^{K_1/K_2} \tag{5}$$

Here we have an example of non-linear modeling function in n-dimensions with the similitude condition that $K_1/K_2$ must be an integer. The objective of this chapter is to characterize such modeling functions. It is clear that by a similarity (change of scale) transformation, $\vec{x} = K_1\,\vec{y}$, we can deal with the following system:

$$K = K_2/K_1 \tag{6}$$

$$\nabla^2\phi + K^2\phi = 0 \tag{7}$$

$$\nabla^2\psi + \psi = 0 \tag{8}$$

$$\phi = \phi(\psi)$$

Here, the $\nabla$ operators refer, of course, to the new dimensionless coordinates, $\vec{y}$.

Theorem I:

Suppose $\phi$ and $\psi$ satisfy equation (6), (7), and (8). Then $(\nabla\psi)^2$ must be a function of $\psi$ and the following linear ordinary differential equation is satisfied by non-linear modeling functions $\phi(\psi)$:

$$F(\psi)\frac{d^2\phi}{d\psi^2} - \psi\frac{d\phi}{d\psi} + K^2\phi = 0 \tag{9}$$

$$(\nabla\psi)^2 = f(\psi) \tag{10}$$

Proof:

Operate on equation (8) with the gradient $\nabla$:

$$\nabla\phi = \frac{d\phi}{d\psi} \cdot \nabla\psi \tag{11}$$

$$\nabla^2\phi = \frac{d^2\phi}{d\psi^2}(\nabla\psi)^2 + \frac{d\phi}{d\psi}\nabla^2\psi \tag{12}$$

Substitute from equation (6) and (7) and rearrange:

$$(\nabla\psi)^2\frac{d^2\phi}{d\psi^2} - \psi\frac{d\phi}{d\psi} + K^2\phi = 0 \tag{13}$$

Since all quantities appearing in (13) other than $(\nabla\psi)^2$ are functions of $\psi$ it follows that $(\nabla\psi)^2 = f(\psi)$. Note that $\frac{d^2\phi}{d\psi^2} \neq 0$ by the assumption that we are dealing with non-linear modeling functions.

This result was quoted in part by Belyea, Low, and Siegel [2] although they did not utilize the fact that $(\nabla\psi)^2 = f(\psi)$ which we will find to be very helpful. Clearly Theorem I allows us to concentrate on determining the form of $f(\psi)$. To the extent that we can characterize $f(\psi)$ from equations (7) and (10) we can determine $\phi(\psi)$. Before discussing the general case we examine the problem in one and in two dimensions.

## One Dimensional Case

This case has been treated completely by Siegel, Ritt, and others [1], [2]. Here $\phi$ and $\psi$ are functions of one dimensionless variable x:

$$\frac{d^2\phi}{dx^2} + K^2\phi = 0 \tag{14}$$

$$\frac{d^2\psi}{dx^2} + \psi = 0 \tag{15}$$

The assumption $\phi = \phi(\psi)$ using Theorem I yields

$$\left(\frac{d\psi}{dx}\right)^2 = f(\psi) \tag{16}$$

Differentiate (16) with respect to x:

$$2\,\frac{d\psi}{dx}\,\frac{d^2\psi}{dx^2} = \frac{df}{d\psi}\,\frac{d\psi}{dx} \tag{17}$$

From (15) and (17) we obtain (assuming $\psi \neq 0$):

$$\frac{df}{d\psi} = -2\psi \tag{18}$$

Thus we find that $f(\psi)$ satisfies an ordinary differential equation. This is to be expected, of course, in the unidimensional case since the single independent variable x can be eliminated, in principle, between two equations. Equation (18) can be integrated immediately:

$$f = c^2 - \psi^2 \quad , \qquad \text{c arbitrary} \tag{19}$$

From theorem I we have:

$$(c^2 - \psi^2) \frac{d^2 \phi}{d\psi^2} - \psi \frac{d\phi}{d\psi} + K^2 \phi = 0 \tag{20}$$

This is the equation obtained by Ritt. As pointed out by him, the single-valued solutions of (20) when $c \neq 0$ exist only when K is an integer. These are the Tschebyscheff polynomials [3]:

$$\phi = T_K \left( \frac{\psi}{c} \right) \tag{21}$$

When c = 0 we have the solutions

$$\phi = \psi^K \tag{22}$$

Here also we have the similitude condition, K = integer, necessary for single-valuedness. Equation (22) corresponds to equation (18) for the one dimensional case. Thus we have a complete characterization of the modeling functions in the one dimensional case. Indeed, the results (21) and (22) can be obtained simply by eliminating the independent variable between the general solutions of (14) and (15) and then imposing the condition of single-valuedness. This, of course, can not be done in higher dimensions since there we are dealing with partial differential equations. Some of the ideas of this case do occur later however.

Two Dimensional Case

With the aid of Theorem I, the essential question in extending the discussion to two dimensions is the following. Suppose $\psi(x, y)$ satisfies the two dimensional Helmholtz equation:

$$\psi_{xx}+\psi_{yy}+\psi = 0 \tag{23}$$

and suppose further that:

$$\psi_x^2 +\psi_y^2 = f(\psi) \tag{24}$$

What conditions must f then satisfy as a function of $\psi$? It will be shown in this section that f must satisfy a second order non-linear ordinary differential equation. The solution of this differential equation as a result of Theorem I will aid in determining modeling functions and similitude conditions.

Differentiate equation (24) with respect to x and to y:

$$\psi_x\psi_{xx}+\psi_y\psi_{yx} = 1/2\ f'\ \psi_x \tag{25}$$

$$\psi_x\psi_{xy}+\psi_y\psi_{yy} = 1/2\ f'\psi_y \tag{26}$$

(primes denote differentiation with respect to $\psi$). Write equation (25) and (26) as homogeneous algebraic equations for $\psi_x$ and $\psi_y$ :

$$\psi_x(\psi_{xx} - \frac{f'}{2}) + \psi_y \psi_{yx} = 0 \tag{27}$$

$$\psi_x \psi_{xy} + \psi_y(\psi_{yy} - \frac{f'}{2}) = 0 \tag{28}$$

Now suppose $\psi$ is not identically zero. Then not both $\psi_x$ and $\psi_y$ can be zero since a non-zero constant is not a solution of (23). Hence the determinant of the coefficients in the system (27), (28) must vanish:

$$(\psi_{xx} - \frac{f'}{2})(\psi_{yy} - \frac{f'}{2}) - \psi_{xy}^2 = 0 \tag{29}$$

Differentiate equations (23) and (25) with respect to x and (23) and (26) with respect to y:

$$\psi_{xxx} + \psi_{yyx} + \psi_x = 0 \tag{30}$$

$$\frac{f''}{2}\psi_x^2 + \frac{f'}{2}\psi_{xx} = \psi_{xx}^2 + \psi_x \psi_{xxx} + \psi_{xy}^2 + \psi_y \psi_{yxx} \tag{31}$$

$$\psi_{xxy} + \psi_{yyy} + \psi_y = 0 \tag{32}$$

$$\frac{f''}{2}\psi_y^2 + \frac{f'}{2}\psi_{yy} = \psi_{yy}^2 + \psi_y \psi_{yyy} + \psi_{xy}^2 + \psi_x \psi_{xyy} \tag{33}$$

Now add equations (31) and (33) using (23), (24), (30), and (32):

$$\frac{ff''}{2} - \psi\frac{f'}{2} = \psi_{xx}^2 + \psi_{yy}^2 + 2\psi_{xy}^2 - f \tag{34}$$

Square equation (23) and expand equation (29).

$$\psi_{xx}^2 + 2\psi_{xx}\psi_{yy} + \psi_{yy}^2 = \psi^2 \tag{35}$$

$$\psi_{xx}\psi_{yy} + \frac{f'}{2}\psi + \left(\frac{f'}{2}\right)^2 - \psi_{xy}^2 = 0 \tag{36}$$

Substituting from (35) and (36) in the right side of (34) we have finally eliminated all partial derivatives of $\psi$ with respect to x and y:

$$\frac{ff''}{2} - \psi\frac{f'}{2} = \psi^2 - f + \psi f' + \frac{(f')^2}{2} \tag{37}$$

After simplification and factoring the differential equation for f has the form:

$$f(f''+2) = (f'+2\psi)(f'+\psi) \tag{38}$$

From Theorem I we have:

$$f\phi'' - \psi\phi' + K^2\phi = 0 \tag{39}$$

Corresponding to the one-dimensional case we would like to know for what numbers K (similitude conditions) are there single-valued solutions $\phi(\psi)$, (modeling function), of (38) and (39). Because of the difficulty in finding a simple expression for the general solution[+] of (38) this question has not yet been answered. However, using the parametric solution of (38) discussed in appendix B, the problem can be seen to be equivalent to the following problem in one dimension (modeling of ordinary differential equations). Let $\psi = \psi(t)$,

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[+] The equation has movable essential singularities. See Ince [4].

$f = f(t) = \dot{\psi}^2$. Then:

$$\ddot{\psi} + \frac{\dot{\psi}}{t} + \psi = 0 \tag{40}$$

$$\ddot{\phi} + \frac{\dot{\phi}}{t} + K^2 \phi = 0 \tag{41}$$

Cast in this form what remains is to find those values of K for which t can be eliminated between the solutions of (40) and (41) to yield a single-valued function $\phi = \phi(\psi)$. In the next section we see that this reduction can be accomplished in a general n-dimensional space.

## n-Dimensional Case

Again with the aid of Theorem 1 our task here is first to determine conditions of $f(\psi)$. We find that in the general case, as in the one and two dimensional cases, f must satisfy a non-linear ordinary differential equation. The order of the equation is the dimension n. Again this result is obtained by a process of chain rule differentiations; however, as the dimension increases the expressions become very complicated. Theorem 2 below greatly facilitates the elimination of independent variables by allowing the application of algebraic results from the theory of matrices. For each point in n space the following matrix is defined:

$$\left[\psi\right] = \psi_{ij} = \frac{\partial^2 \psi}{\partial x_i \, \partial x_j} \tag{42}$$

Denote the eigenvalues of $\left[\psi\right]$ by $r_K$. It is well known that the trace of $\left[\psi\right]$ is an algebraic invariant [5]. Moreover, from the canonical form it is easily shown that the trace of the p'th power of $\left[\psi\right]$ is also an invariant given by:

$$\alpha_p = \text{trace}\,[\psi]^p = \sum_{K=1}^{n} r_K^p \qquad 1 \leq p \leq n \tag{43}$$

The following theorem, which is proved in appendix A, provides the link between $f(\psi)$ and its derivatives and the invariants $\alpha_p$ and $r_K$. (Primes in all cases denote derivatives with respect to $\psi$.)

Theorem II:

Let $\psi$ be a non zero function of position in n-space satisfying the following conditions and $[\psi]$, $\alpha_p$, and $r_K$ be as defined above:

$$\nabla^2 \psi + \psi = 0 \tag{44}$$

$$(\nabla \psi)^2 = f(\psi) \tag{45}$$

Then one of the eigenvalues of $[\psi]$, say $r_1$, is a function of $\psi$ only given by

$$r_1 = \frac{f'}{2} \tag{46}$$

Moreover, the invariants $\alpha_p$ are all functions of $\psi$ only described in terms of f and its derivatives in the following recursive manner:

$$\alpha_1 = -\psi \tag{47}$$

$$\alpha_p = f\left[r_1^{p-2}\,\frac{f''}{2} - \frac{\alpha'_{p-1}}{p-1}\right] + r_1 \alpha_{p-1}, \quad 2 \leq p \leq n \tag{48}$$

Proof: See appendix A

Now consider the defining equations (43) for $\alpha_p$ in terms of the eigenvalues $r_K$. By eliminating the n-1 quantities $r_K$ for $K > 1$ from these n equations we could obtain a single relation on $\alpha_p$ and $r_1$. But from equations (46), (47), and (48) $\alpha_p$ and $r_1$ are known in terms of f and its derivatives. The relation then becomes a differential equation for f. The discussion is simplified by the introduction of the invariants $a_p$.

$$a_p = \alpha_p - r_1^p = \sum_{K=2}^{n} r_K^p, \quad 1 \leq p \leq n \tag{49}$$

Theorem III:

Under the hypothesis of Theorem II, $f(\psi)$ is characterized by the following system of algebraic and differential equations.

$$a_p = \sum_{K=2}^{n} r_K^p, \qquad 1 \leq p \leq n \tag{50}$$

$$a_1 = -\psi - \frac{f'}{2} \tag{51}$$

$$a_p = \frac{f'}{2}\, a_{p-1} - f\, \frac{a'_{p-1}}{(p-1)}, \qquad 2 \leq p \leq n \tag{52}$$

Note: We have 2n equations for the 2n quantities $a_p$, $r_K$, and f in terms of $\psi$.

Proof:

Equation (50) is already established (by definition). Equation (51) follows from (46) and (47) and (49) for p = 1. Equation (52) is obtained simply by substituting for $\alpha_p$ from (49) into (48):

$$a_p + r_1^p = f\frac{f''}{2} r_1^{p-2} - \frac{f}{p-1} \frac{d}{d\psi}(a_{p-1} + r_1^{p-1}) + r_1 a_{p-1} + r_1^p \tag{53}$$

$$a_p = r_1 a_{p-1} - \frac{f}{p-1} a'_{p-1} + \frac{ff''}{2} r_1^{p-2} - f r_1^{p-2} r_1' \tag{54}$$

Since, from (46), $r_1 = \frac{f'}{2}$ and $r_1' = \frac{f''}{2}$ , the last two terms of (54) cancel yielding (52).

Before proceeding to a solution of these equations let us examine them in some simple cases. When n = 1 we have from (50) and (51):

$$a_1 = -\psi - \frac{f'}{2} = 0 \tag{55}$$

This agrees with the result in equation (46). For n = 2 we have:

$$a_1 = r_2 \tag{56}$$

$$a_2 = r_2^2 \tag{57}$$

$$a_1 = -\psi - \frac{f'}{2} \tag{58}$$

$$a_2 = \frac{f'}{2}\left\{-\psi - \frac{f'}{2}\right\} - f\left\{-1 - \frac{f''}{2}\right\} \tag{59}$$

Combining these equations we obtain:

$$\left(\frac{f'}{2} + \psi\right)^2 = -\frac{f'}{2}\left(\frac{f'}{2} + \psi\right) + f\left(\frac{f''}{2} + 1\right) \tag{60}$$

$$f\,(f'' + 2) = (f' + 2\psi)(f' + \psi) \tag{61}$$

This is equation (57). For n = 3 the corresponding equation for f is:

$$f^2 f''' - \frac{5}{2} f f' f'' - 4\psi f f'' + \frac{11}{2} \psi (f')^2 + \frac{3}{2} (f')^3 - 4 f f' + 6\psi^2 f' - 6\psi f + 2\psi^3 = 0 \qquad (62)$$

Since we could not obtain an explicit solution for $f(\psi)$ from equation (61) in the two dimensional case we cannot expect to solve (62) and corresponding equations in higher dimensions directly. It is important, however, to note that such equations exist and can be found explicitly. Going back to Theorem I, then, we can see that our original problem reduces to finding single-valued solutions $\phi = \phi(\psi)$ of systems of two simultaneous differential equations one is of the type (55), (61), (62) etc. for $f(\psi)$ and the other is:

$$f \phi'' - \psi \phi' + K^2 \phi = 0 \qquad (63)$$

Following the procedure used in the case n = 2 we seek to transform the system to a pair of linear equations after finding a parametric representation for $f(\psi)$. Using the fact that for each n we must obtain all solutions for previous cases the general solution for $f(\psi)$ can be derived. We prefer to dispense with a lengthy derivation and state the results in Theorem IV, which is proved relatively easily.

<u>Theorem IV:</u>

A general parametric solution, $f = f(t)$, $\psi = \psi(t)$ of the system of equations described in Theorem III is given by:

(dots are derivatives with respect to t)

$$f = \dot{\psi}^2 \tag{64}$$

$$r_K = \frac{\dot{\psi}}{t - c_K}, \qquad c_K \text{ arbitrary constants, } K = 2, \ldots, n \tag{65}$$

$$r_1 = \frac{1}{2} \frac{df}{d\psi} = \ddot{\psi} \tag{66}$$

where $\psi$ is a general solution of the linear equation:

$$\ddot{\psi} + \dot{\psi} \sum_{K=2}^{n} \frac{1}{t - c_K} + \psi = 0 \tag{67}$$

Note that we obtain n + 1 arbitrary constants, n-1 from (65) and 2 from (67). This is one in excess of that expected for an n'th order differential equation; however, a change of parameter $\tau = t - c_1$ changes nothing and would remove the excess constant.

Proof:

Equation (51) of Theorem III is checked directly

$$a_1 = \sum_{K=2}^{n} r_K = \sum_{K=2}^{n} \frac{\dot{\psi}}{t - c_K} = -\psi - \frac{f'}{2} = -\psi - \ddot{\psi} \tag{68}$$

$$\ddot{\psi} + \dot{\psi} \sum_{K=2}^{n} \frac{1}{t - c_K} + \psi = 0 \tag{69}$$

Thus (51) follows from (67). We now need only to substitute (50) into (52) and check (52)

$$\sum_{K=2}^{n} r_K^{p} = \frac{f'}{2} \sum_{K=2}^{n} r_K^{p-1} - \frac{f}{p-1} \frac{d}{d\psi} \sum_{K=2}^{n} r_K^{p-1} \tag{70}$$

From (64), (65), and (66):

$$\sum_{K=2}^{n} \frac{\dot{\psi}^{p}}{(t-c_K)^{p}} = \ddot{\psi} \sum_{K=2}^{n} \frac{\dot{\psi}^{p-1}}{(t-c_K)^{p-1}} - \frac{\dot{\psi}}{p-1} \frac{d}{dt} \sum_{K=2}^{n} \frac{\dot{\psi}^{p-1}}{(t-c_K)^{p-1}} \tag{71}$$

$$= \ddot{\psi} \sum_{K=2}^{n} \frac{\dot{\psi}^{p-1}}{(t-c_K)^{p-1}} - \dot{\psi} \left[ \sum_{K=2}^{n} \left\{ \frac{\dot{\psi}^{p-2}\ddot{\psi}}{(t-c_K)^{p-1}} - \frac{\dot{\psi}^{p-1}}{(t-c_K)^{p}} \right\} \right] \tag{72}$$

Hence equation (72) is an identity and the theorem follows. We now complete the reduction of the modeling problem to one in linear ordinary differential equations by introducing the parameter t into equation (63).

<u>Theorem V:</u>

A general parametric solution for $\phi(\psi)$ is given by general solutions for $\phi$ and $\psi$ respectively of the following differential equations:

$$\ddot{\phi} + \dot{\phi} \sum_{j=2}^{n} \frac{1}{t-c_j} + K^2 \phi = 0 \tag{73}$$

$$\ddot{\psi} + \dot{\psi} \sum_{j=2}^{n} \frac{1}{t-c_j} + \psi = 0 \tag{74}$$

<u>Proof:</u>

Introduce the parameter t of Theorem IV into equation (63)

$$\phi' = \dot{\phi}/\dot{\psi}, \quad \phi'' = \ddot{\phi}/\dot{\psi}^2 - \frac{\dot{\phi}\ddot{\psi}}{\dot{\psi}^3} \tag{75}$$

$$f\phi'' = \dot{\psi}^2\, \phi'' = \ddot{\phi} - \dot{\phi}/\dot{\psi}\, \ddot{\psi} \tag{76}$$

From (63):

$$\ddot{\phi} - \frac{\dot{\phi}}{\dot{\psi}}\, \ddot{\psi} - \frac{\psi\dot{\phi}}{\dot{\psi}} + K^2 \phi = 0 \tag{77}$$

Using (74) we obtain (73). Equation (74), of course, follows from Theorem IV.

It is interesting to note that by taking $c_n = \infty$ we reduce to the $n-1$ dimensional case. The corresponding eigenvalue $r_n$ becomes zero as it should. Thus each case includes all the previous cases. To complete the determination of modeling functions it is necessary to find those values of K for which t can be eliminated between solutions of (73) and (74) so that $\phi(\psi)$ is single-valued. We can, of course, refer our problem back to the original coordinates (equations (52), (53), and (54) of Section I) by letting $t = K_1\tau$ and recalling that $K = K_2/K_1$. Equations (73) and (74) become

$$\ddot{\phi} + \dot{\phi} \sum_{j=2}^{n} \frac{1}{\tau - d_j} + K_2^2\, \phi = 0 \tag{78}$$

$$\ddot{\psi} + \dot{\psi} \sum_{j=2}^{n} \frac{1}{\tau - d_j} + K_1^2\, \psi = 0 \tag{79}$$

Here the dots denote derivatives with respect to $\tau$ and new arbitrary constants are $d_j = c_j/K_1$.

# APPENDIX A

Note: Subscripts on $\psi_i$ denote derivatives with respect to $x_i$ using the summation convention with repeated indices. Subscripts on other quantities are not derivatives.

## Proof of Theorem II

THEOREM II

Assume:

$$\psi \not\equiv 0 \tag{A-1}$$

$$\psi_i^2 = f(\psi) \tag{A-2}$$

$$\psi_{ii} + \psi = 0 \tag{A-3}$$

Denote:

$$r_i = \text{eigenvalues of: } \left[\psi\right] = \psi_{ij} \tag{A-4}$$

$$\alpha_i = \text{trace of} \left[\psi\right]^i = \sum_{k=1}^{n} r_k^i \tag{A-5}$$

Then:

$$\alpha_1 = -\psi \tag{A-6}$$

$$r_1 = \frac{f'}{2} \tag{A-7}$$

$$\alpha_i = f\left[r_1^{i-2}\frac{f''}{2} - \frac{\alpha'_{i-1}}{i-1}\right] + r_1\alpha_{i-1}, \quad 2 \leq i \leq n \tag{A-8}$$

PROOF

Equation (A-6) follows from (A-3) and (A-5):

$$-\psi = \psi_{ii} = \text{trace}\left[\psi\right] = \alpha_1 \tag{A-9}$$

Equation (A-7) is obtained from (A-2) by differentiation with respect to $x_j$.

$$2\psi_i\psi_{ij} = f'\,\psi_j \tag{A-10}$$

$$\psi_i\left[\psi_{ij} - \frac{f'}{2}\,\delta_{ij}\right] = 0 \tag{A-11}$$

Since $\psi \not\equiv 0$, it follows from (A-3) that not all $\psi_i$ are zero, hence:

$$\det\left[\psi_{ij} - \frac{f'}{2}\,\delta_{ij}\right] = 0 \tag{A-12}$$

Thus $\frac{f'}{2}$ is an eigenvalue of $\left[\psi\right]$, say $r_1$.

To prove (A-8) we consider the case i = 2 separately. Differentiate (A-10) with respect to $x_k$, let k = j, and sum over j:

$$2\psi_{ik}\psi_{ij} + 2\psi_i\psi_{ijk} = f''\psi_j\psi_k + f'\,\psi_{jk} \tag{A-13a}$$

$$2\psi_{ij}^{\;2} + 2\psi_i\psi_{ijj} = f''\psi_j^{\;2} + f'\,\psi_{jj} \tag{A-13b}$$

Interpret and simplify using (A-2) - (A-5):

$$\psi_{ij}^{\;2} = \text{trace}\left[\psi\right]^2 = \alpha_2 \tag{A-14}$$

$$\psi_i \psi_{ijj} = \psi_i(-\psi_i) = -f \tag{A-15}$$

$$\alpha_2 = f(\frac{f''}{2} + 1) - \psi \frac{f'}{2} = f(\frac{f''}{2} - \alpha_1') + r_1 \alpha_1 \tag{A-16}$$

Equation (A-16) is (A-8) for i = 2. For $3 \leq i \leq n$ consider:

$$\alpha_{i-1} = \text{trace} \left[\psi\right]^{i-1} = \psi_{s_1 s_2} \psi_{s_2 s_3} \cdots \psi_{s_{i-1} s_1} \tag{A-17}$$

Differentiate with respect to $x_r$ and multiply by $\psi_r$ (summing over r):

$$\alpha_{i-1}' \psi_r^2 = (i-1)\psi_{s_1 s_2} \cdots \psi_{s_{i-2} s_{i-1}} \psi_{s_{i-1} s_1 r} \psi_r \tag{A-18}$$

Substitute from (A13a) to remove triple subscript:

$$f\alpha_{i-1}' = (i-1)\psi_{s_1 s_2} \cdots \psi_{s_{i-2} s_{i-1}} \cdot \left[\frac{f''}{2} \psi_{s_1} \psi_{s_{i-1}} + \frac{f'}{2} \psi_{s_1 s_{i-1}} - \psi_{rs_1} \psi_{rs_{i-1}}\right] \tag{A-19}$$

From (A-5):

$$\frac{f\alpha_{i-1}'}{(i-1)} = \frac{f''}{2} \left[\psi_{s_1 s_2} \cdots \psi_{s_{i-2} s_{i-1}} \psi_{s_1} \psi_{s_{i-1}}\right] + \frac{f'}{2} \alpha_{i-1} - \alpha_i \tag{A-20}$$

Equation (A-8) follows by transpositions subject only to the following lemma.

LEMMA

Under the conditions of the theorem for $3 \leq i \leq n$:

$$\psi_{s_1 s_2} \cdots \psi_{s_{i-2} s_{i-1}} \psi_{s_{i-1}} \psi_{s_1} = f r_i^{i-2} \tag{A-21}$$

PROOF OF LEMMA

For $i = 3$, multiply equation (A-10) by $\frac{1}{2} \psi_j$ and sum:

$$\psi_{ij} \psi_i \psi_j = \frac{f'}{2} \psi_j^2 = \frac{ff'}{2} = fr_1 \tag{A-22}$$

With a change of indices (A-22) becomes (A-21) for i = 3. We proceed by finite induction. Assume (A-21) for i = n - 1. Substituting using equation (A-10) for $\psi_{s_{i-2} s_{i-1}} \psi_{s_{i-1}}$ using equation (A-10) we have:

$$\psi_{s_1 s_2} \cdots \psi_{s_{i-2} s_{i-1}} \psi_{s_{i-1}} \psi_{s_1} = \psi_{s_1 s_2} \cdots \psi_{s_{i-3} s_{i-2}} r_1 \psi_{s_{i-2}} \psi_{s_1} \tag{A-23}$$

The lemma follows by applying the induction hypothesis to the right side of (A-23).

# APPENDIX B

## SOLUTION OF THE DIFFERENTIAL EQUATION FOR f IN TWO DIMENSIONS

From equations (38) and (39) we have the system:

$$f\phi'' - x\phi' + K^2\phi = 0 \tag{B-1}$$

$$f(f''+2) = (f'+2x)(f'+x) \tag{B-2}$$

for $\phi$ and f as functions of the independent variable x. We will develop a parametric solution $\phi = \phi(t)$, $f = f(t)$, $x = x(t)$, for this system. (B-2) can be written as

$$\frac{d}{dx} \ln\left\{\frac{f'+2x}{f}\right\} = \frac{x}{f} \tag{B-3}$$

Now we put (B-1) in self-adjoint form: Let

$$u = e^{-\int \frac{x}{f}} \qquad u' = -\frac{x}{f}u \tag{B-4}$$

$$\phi'' + \frac{u'}{u}\phi' + \frac{K^2\phi}{f} = 0 \tag{B-5}$$

$$u\frac{d}{dx}\left(u\frac{d\phi}{dx}\right) + \frac{K^2u^2}{f}\phi = 0 \tag{B-6}$$

Make the change of independent variable

$$u\frac{d}{dx} = \frac{d}{dt} \tag{B-7}$$

From (B-3) and (B-4) we see that

$$\frac{1}{u} = e^{+\int \frac{x}{f}} = \frac{f' + 2x}{f} \tag{B-8}$$

Integration of (B-7) yields

$$t = \int \frac{dx}{u} = \ln f + 2 \ln \frac{1}{u}$$

$$e^{-t} = \frac{u^2}{f} \tag{B-9}$$

Equation (B-6) becomes

$$\frac{d^2\phi}{dt^2} + K^2 e^{-t} \phi = 0 \tag{B-10}$$

To find an equation for x(t) we differentiate (B-7)

$$\frac{dx}{dt} = u$$

$$\frac{d^2x}{dt^2} = \frac{dx}{dt}\frac{du}{dx} = u\frac{du}{dx} = -\frac{xu^2}{f}$$

$$\frac{d^2x}{dt^2} + xe^{-t} = 0 \tag{B-11}$$

Solutions of (B-1) and (B-2) for $\phi = \phi(x)$ are given parametrically by the solutions $\phi = \phi(t)$ and $x = x(t)$ of (B-10) and (B-11) respectively. The auxiliary function f(x) is determined parametrically by (B-11) and (B-9) since:

$$f(t) = u^2 e^t = e^t \left(\frac{dx}{dt}\right)^2 \tag{B-12}$$

Finally we put (B-10) and (B-11) in more recognizable form by introducing the change of parameter

$$4e^{-t} = \tau^2 \tag{B-13}$$

$$\frac{d\phi}{dt} = \frac{d\phi}{d\tau}\frac{d\tau}{dt} = \frac{d\phi}{d\tau}\ \frac{-\tau}{2}$$

$$\frac{d^2\phi}{dt^2} = \frac{d^2\phi}{d\tau^2}\frac{\tau^2}{4} + \frac{\tau}{4}\frac{d\phi}{d\tau}$$

$$\frac{d^2\phi}{dt^2} + K^2 e^{-t}\phi = \frac{\tau^2}{4}\frac{d^2\phi}{d\tau^2} + \frac{\tau}{4}\frac{d\phi}{d\tau} + \frac{K^2\tau^2}{4}\phi = 0$$

$$\frac{d^2\phi}{d\tau^2} + \frac{1}{\tau}\frac{d\phi}{d\tau} + K^2\phi = 0 \tag{B-14}$$

Similarly (B-11) and (B-12) become

$$\frac{d^2x}{d\tau^2} + \frac{1}{\tau}\frac{dx}{d\tau} + x = 0 \tag{B-15}$$

$$f(\tau) = \left(\frac{dx}{d\tau}\right)^2 \tag{B-16}$$

General solutions of (B-14) and (B-15) are given by

$$\phi^{(\tau)} = a_1 J_o(K\tau) + b_1 Y_o(K\tau) \tag{B-17}$$

$$X^{(\tau)} = a_2 J_o(\tau) + b_2 Y_o(\tau) \tag{B-18}$$

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# IV

## THE EFFECT OF EXPERIMENTAL ERRORS IN THE NON LINEAR MODELING OF WAVE PHENOMENA

(J. E. Belyea)

It has been shown previously [1] that certain systems governed by wave equations are related in a particular non-linear manner. If $\phi$ and $\psi$ are the scalar wave functions associated with two such systems, then

$$\phi = C_3 \, {}_2F_1(A, -A; 1-B; z) + C_4 z^B \, {}_2F_1(A+B, B-A; B+1; z) \tag{1}$$

In this expression z is a linear function of $\psi$, while A, B, $C_3$ and $C_4$ are constants which depend on the nature of the systems. Relations of this type would be quite useful to experimenters since they often wish to make measurements on one system of this type and infer information about another. Before relations of this type can be used with confidence for this purpose, however, the following question must be answered: How seriously would an initial error in determining z affect the value of $\phi$ obtained from it? Absolute accuracy in any experiment cannot be hoped for, so that any method which hopelessly magnifies small experimental errors is obviously useless. In what follows we shall consider relation (1) from this viewpoint; this might be termed an initial step in the error analysis portion of the non-linear modeling study.

Suppose that the experimental error, $\delta z$, is sufficiently small that terms of order $(\delta z)^2$ are truly negligible. The resulting error in $\phi$ may then be found from the first few terms of its Taylor expansion:

$$\delta\phi = \frac{\partial\phi}{\partial z} \quad \delta z \quad . \tag{2}$$

When (1) is substituted into (2) and use is made of the identity $\frac{\partial}{\partial z} \; {}_2F_1(a, b; c; z) = \frac{ab}{c} \; {}_2F_1(a+1, b+1; c+1; z)$ [2], the error in $\phi$ is found to be

$$\delta\phi = \delta z \quad \left\{ - \frac{C_3 A^2}{(1-B)} \; {}_2F_1(A+1, -A+1; 2-B; z) + C_4 B \, z^{B-1} \, {}_2F_1(A+B, B-A; B+1; z) + \right.$$

$$\left. + C_4 \frac{(B^2 - A^2)}{B+1} \quad z^B \; {}_2F_1(A+B+1, B-A+1; B+2; z) \right\} \ldots \tag{2a}$$

Since the values of the quantities A, B, and in fact z, are, to a certain extent, subject to choice, it seems reasonable that the right side of (2a) might be made quite small. A desirable requirement would be that it be sufficiently small so that $\frac{|\delta\phi|}{|\delta z|} \sim 1$. This is equivalent to the requirement that small experimental errors are not magnified at all.

In addition to the restrictions which $\frac{|\delta\phi|}{|\delta z|} \sim 1$ places on the choice of A, B, and z, further restrictions are introduced by the character of the functions which appear in (1) and in (2a). It is known that the hypergeometric function ${}_2F_1(a, b; c; z)$ converges within and on the unit circle $|z| = 1$ if $Re(a+b-c) < 0$, so long as neither a, b, or c are negative integers. Thus from (1) come the conditions

$$|z| \leq 1, \qquad Re(B-1) < 0$$

and from (2a)

$$|z| \leq 1, \qquad Re(B) < 0^{*} . \tag{I}$$

---

* Note that these conditions are not as independent as they seem, since B and z contain the same constants of integration.

Since a term containing $z^B$ appears in expression (1), the further stipulation that Re(B) $\geqslant$ 0 must be made to insure finiteness at z = 0. This is in clear conflict with (I), and the only means by which this conflict can be satisfactorily resolved is by requiring $C_4 = 0$. As a result of this requirement (1) and (2a) become

$$\phi = C_3 \, {}_2F_1 (A, -A; 1-B; z) \quad , \tag{1'}$$

$$\delta\phi = \delta z \left( \frac{-A^2}{1-B} \, {}_2F_1 (1+A, 1-A; 2-B; z) \right) \quad , \tag{2a'}$$

in addition to which

$$|z| \leq 1 \quad , \qquad \mathrm{Re}\,(B) < 0 \ .$$

It may be easily shown that so long as $\mathrm{Re}(a+b-c) < 0$, $\mathrm{Re}(\,|a| + |b| - |c|\,) < 0$ and c is real and positive,

$$\left| {}_2F_1 (a, b; c; z) \right| \leq {}_2F_1 (\,|a|\,,\ |b|\,;\ |c|\,; 1)$$

for $|z| \leq 1$. Thus we may insure that $\dfrac{|\delta\phi|}{|\delta z|} \leq 1$ for these values of z by the requirement

$$\frac{|A^2|}{|1-B|} \; {}_2F_1 (\,|1+A|\,,\ |1-A|\,;\ |2-B|\,; 1) = 1 \ .$$

By use of the identity ${}_2F_1 (a, b; c; 1) = \dfrac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$ [2], this becomes

$$\frac{|A^2|}{|1-B|} \quad \frac{\Gamma(\,|2-B|\,)\Gamma(\,|2-B| - |1+A| - |1-A|\,)}{\Gamma(\,|2-B| - |1+A|\,)\Gamma(\,|2-B| - |1-A|\,)} = 1 \ , \tag{II}$$

with $\left[ |1+A| + |1-A| - |2-B| \right] < 0$ and $\mathrm{Im}\,(B) = 0$

It has been shown that conditions (I) and (II) constitute sufficient requirements to ensure that small errors of measurement are not magnified by the modeling technique in the case considered. The necessity of these requirements will not be investigated here.

## REFERENCES

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# V

## HIGH FREQUENCY ELECTROMAGNETIC SCATTERING

(D. M. Raybin)

In electromagnetic theory the scattering by a metallic body in the high frequency limit is generally treated by considering the body to be opaque and applying standard electromagnetic techniques to determine the cross section. In this section we shall consider the question of what happens when the frequency becomes so high as to make it possible for the incident wave to penetrate the body. To illustrate why the usual optics cross section cannot be correct, let us consider two rather similar problems.

First we consider the high frequency forward scattering by an opaque body. By usual Kirchoff theory the differential cross section* is

$$\sigma(0) = \frac{k^2 A^2}{4\pi^2} \tag{1}$$

where A is the projected area and k is the wave number. On the other hand, if we consider the quantum mechanical situation of a high frequency wave incident on a potential, we have the usual Born result [1]

---

* The radar cross section is $4\pi$ times the differential cross section.

$$\sigma(\theta) = \frac{1}{K^2} \left| \int_0^\infty r' \sin K r' \, U(r') \, d r' \right|^2 \tag{2}$$

where $K = \frac{1}{2} k \sin \frac{1}{2} \theta$, and $U(r')$ is some spherically symmetric potential. A typical result using this is the square well potential defined by

$$\begin{aligned} V(r) &= -V_o & r &< a \\ V(r) &= 0 & r &> a \end{aligned} \tag{3}$$

The differential cross section is

$$\sigma(\theta) = \left( \frac{2\mu \, V_o \, a^3}{K^2} \right)^2 g\left(2\, ka \sin \frac{1}{2} \theta\right) \tag{4}$$

where

$$g(x) = \frac{(\sin x - x \cos x)^2}{x^6}$$

In the forward direction, $\theta = 0$, and $g(x) \rightarrow 1/9$ as ka gets large, while for other directions $g(x) \rightarrow 0$. Thus we see that the electromagnetic answer diverges as k gets large, while the quantum mechanic result approaches a constant in the forward direction. This, of course, is impossible. Clearly there must be some transition between these two results.

In order to gain an understanding of this behavior, we shall consider the problem of the forward scattering by a disc of finite thickness having certain specified bulk properties. More specifically the body will be assumed to have a complex propagation vector

$$k_2 = k\,(n + i\alpha) \tag{5}$$

where

$$n^2 - \alpha^2 = 1 - 2x^2$$

$$n\alpha = x^2 a$$

where

$$x^2 = \frac{1}{2}\,\frac{\omega_p^{\,2}/\omega^2}{1 + a^2}$$

$$a = \text{resistive constant}$$

$$\omega_p^{\,2} = \frac{4\pi N e^2}{m}$$

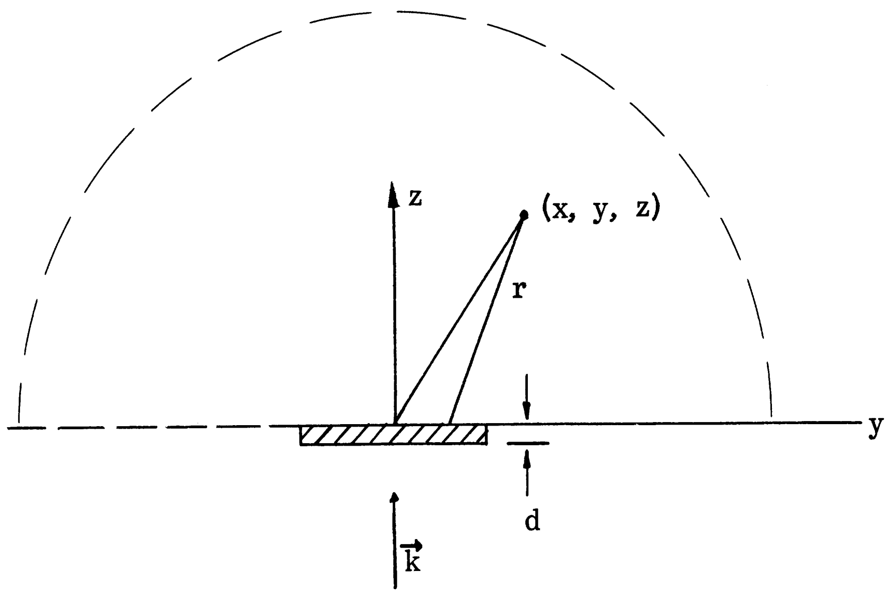
This is the simplest expression for the dielectric properties of an electron gas. In using these dielectric properties, we are in effect considering the body to be a homogeneous collection of electrons, each of which is subject to the following forces:

inertial $m\,v$

Lorentz $e\,E_{inc}$

resistive $-m\,\omega\,a\,v$

In order to relate these to an electromagnetic scattering problem, we shall consider the problem of a plane wave incident in the +z direction on a circular disc of area A, thickness d, characterized by a complex transmission coefficient $\tau$.




Since we are interested in the case where $k\,a >> 1$ we can for simplicity consider the scalar scattering given by an integral containing a Green's function, namely

$$\psi(x, y, z) = -\frac{1}{4\pi} \int_S \psi(x', y', z') \frac{\partial G}{\partial n'} \, ds' \qquad (7)$$

where the Green's function must be zero on the boundary of our surface. If we choose the surface of integration to be the $z' = 0$ plane plus a hemisphere extending to $\infty$ in the +z direction, then the Green's function is

$$G = \frac{e^{ikr_1}}{r_1} - \frac{e^{ikr_2}}{r_2} \qquad (8)$$

where

$$r_1^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$$

$$r_2^2 = (x - x')^2 + (y - y')^2 + (z + z')^2$$

and on the surface $z' = 0$,

$$\left. \frac{\partial G}{\partial n'} \right|_{z' = 0} = \frac{2z}{r} (ik) \frac{e^{ikr}}{r} \qquad (9)$$

where

$$r^2 = (x - x')^2 + (y - y')^2 + (z)^2 .$$

As long as $\psi$ satisfies the radiation condition, the integral over the hemisphere vanishes and

$$\psi(x, y, z) = -\frac{1}{i\lambda}\int \psi(x', y', z' = 0)\,\frac{e^{ikr}}{r}\left(\frac{z}{r}\right) ds' \qquad (10)$$

Since we are interested in the leading term, we can neglect edge effects and consider the field on the surface to be

$$\psi(x', y', z' = 0) = \begin{cases} 1 \text{ for } \rho > b \\ \tau \text{ for } \rho < b \end{cases} \qquad (11)$$

so the field is

$$\psi(x,y,z) = \frac{1}{i\lambda}\left\{\tau\int_0^b\int_0^{2\pi} \frac{e^{ikr}}{r}\left(\frac{z}{r}\right) ds' + \int_b^{\infty}\int_0^{2\pi} \frac{e^{ikr}}{r}\left(\frac{z}{r}\right) ds'\right\} \qquad (12)$$

Putting r in cylindrical coordinates and evaluating in the forward direction, we have

$$\psi(0,0,z) = \frac{2\pi z}{i\lambda}\left\{\tau\int_0^b \frac{e^{ik\sqrt{\rho'^2 + z^2}}}{(\rho'^2 + z^2)}\,\rho'\,d\rho' + \int_b^{\infty} \frac{e^{ik\sqrt{\rho'^2 + z^2}}}{(\rho'^2 + z^2)}\,\rho'\,d\rho'\right\} \qquad (13)$$

Now letting $x = \sqrt{\rho'^2 + z^2}$ we have

$$\psi(0,0,z) = \frac{kz}{i}\left\{ \mathcal{T} \int_{z}^{\sqrt{b^2+z^2}} \frac{e^{ikx}}{x}\, dx + \int_{\sqrt{b^2+z^2}}^{\infty} \frac{e^{ikx}}{x}\, dx \right\} \tag{14}$$

integrating by parts and retaining only the lowest order terms, gives

$$\psi(0,0,z) = \frac{kz}{i}\left\{ \frac{\mathcal{T}}{ik}\left[ \frac{e^{ik\sqrt{b^2+z^2}}}{\sqrt{b^2+z^2}} - \frac{e^{ikz}}{z} \right] - \frac{1}{ik}\, \frac{e^{ik\sqrt{b^2+z^2}}}{\sqrt{b^2+z^2}} \right\}$$

or

$$\psi(0,0,z) = -\, e^{ikz}\left\{ (\mathcal{T} - 1)\, \frac{z\, e^{ik\sqrt{b^2+z^2}}}{\sqrt{b^2+z^2}} - \mathcal{T} \right\} \tag{15}$$

and putting in the form of incident wave plus scattered wave, we have

$$\psi(0,0,z) = e^{ikz} + (1 - \mathcal{T})\, i\, \frac{kb^2}{2}\, \frac{e^{ikz}}{z} + \ldots \tag{16}$$

so that the differential cross section is

$$\sigma(0) = \left|1 - \mathcal{T}\right|^2 \frac{k^2 A^2}{4\pi^2} \tag{17}$$

where A is the area of the disc. For $\mathcal{T} = 0$, this reduces to the standard electromagnetic theory result (equation 1).

Now we are ready to evaluate the transmission coefficient $\tau$. It is at this point where we must take into account the properties of the body, or plasma region. It is important to note that in the high frequency limit where the waves penetrate the body, the forward scattering cross section is in fact very strongly dependent on these bulk properties.

From [2] the transmission coefficient is

$$\tau = \frac{4(k_2/k_1)\, e^{i(k_2 d - k_1 d)}}{(1+\frac{k_2}{k_1})^2 - (1-\frac{k_2}{k_1})^2\, e^{2ik_2 d}} \tag{18}$$

where $k_2$ is the complex propagation vector in the body, and $k_1$ is the real propagation vector outside the body. Since we are interested in the high frequency limit, we shall evaluate this for the special case of $\omega \gg \omega_p$. The index of refraction is

$$n^2 = \frac{(1-2x^2)}{2} \pm \sqrt{\frac{(1-2x^2)^2}{4} + a^2 x^4}\,. \tag{19}$$

We will use only the "+" root since n is the real part of the index. Therefore

$$n^2 = \frac{(1-2x^2)}{2}\left[1 + \sqrt{1 + \frac{4a^2 x^4}{(1-2x^2)^2}}\right] \tag{20}$$

and now evaluating for $\omega_p \ll \omega$, which implies $x^2 \ll 1$, thus

$$n = 1 - x^2 + \ldots$$

and

$$\alpha = x^2 \; a \left[ 1 + x^2 + \ldots \right] \qquad (21)$$

Now we shall evaluate the transmission coefficient $\mathcal{T}$. For simplicity we let $y = 1 - n$ so that both $y$ and $\alpha$ are of order $x^2$. In addition we require that $y\,k\,d$ and $\alpha\,k\,d$ be $\ll 1$. That this is the high frequency result follows from the fact that $y$ and $\alpha$ are proportional to $1/k^2$.

$$\mathcal{T} = \frac{4\,(n + i\alpha)\, e^{i\,(n-1)\,kd}\, e^{-\alpha kd}}{(1 + n + i\alpha)^2 - (1 - n - i\alpha)^2\, e^{2\,inkd}\, e^{-2kd\alpha}} = 1 + iykd - \alpha kd + \ldots \qquad (22)$$

so that

$$\left| 1 - \mathcal{T} \right|^2 = (kd)^2 \; (y^2 + \alpha^2)$$

or

$$\left| 1 - \mathcal{T} \right|^2 = (kd)^2 \left[ x^4 + x^4\, a^2 \right] = (kd)^2 \; x^4 \,(1 + a^2) \qquad (23)$$

so that the forward scattering cross section is

$$\sigma\,(0) = (kd)^2 \; x^4 \; (1 + a^2) \; \frac{k^2\, A^2}{4\pi^2}$$

and since $k = \omega/c$ and $x^2 = \dfrac{\frac{1}{2}\,\omega_p^{\,2}/\omega^2}{1 + a^2}$ this is

$$\sigma\,(0) = \frac{V^2}{(4\pi)^2}\,\frac{1}{(1 + a^2)}\,\left(\frac{\omega_p}{c}\right)^4 \tag{24}$$

where V is the volume of the object. In terms of the density N this is

$$\sigma\,(0) = \left[\frac{N\,V\,e^2}{m\,c^2}\right]^2 \frac{1}{1 + a^2} \tag{25}$$

or

$$\sigma\,(0) = (n\,r_o)^2\,\frac{1}{1 + a^2} \tag{26}$$

where now n = <u>total</u> number of electrons and $r_o$ is the classical electron radius, $\dfrac{e^2}{m\,c^2}$. The Thomson cross section for forward scattering of a single electron from [3] is in our notation

$$\sigma\,(0) = \frac{r_o^2}{1 + a^2} \tag{27}$$

so that our result is $n^2$ times the Thomson cross section.

The reason for getting $n^2$ rather than n, is that we are considering forward scattering and the phase shift for each electron is independent of the position of the electron. In essence we have shown that when we consider the scattering of a large body by sufficiently high frequency waves, the wave penetrates the body and we get scattering from individual electrons. It is still possible in principle at least to consider even higher frequencies in which the wave length is about the same as the electron radius. On the other hand waves of this frequency are well into the cosmic ray range.

The restrictions placed on these parameters can be expressed in many ways. First the restriction $(\omega_p/\omega)^2 \ll 1$

$$\left(\frac{\omega_p}{\omega}\right)^2 = \frac{k_p^2}{k^2} = \frac{4\pi N r_o \lambda^2}{(2\pi)^2} = \frac{N r_o \lambda^2}{\pi} \ll 1$$

or

$$N \ll \frac{\pi}{r_o \lambda^2} = \frac{10^{13}/\mathrm{cm}\cdot}{\lambda^2}$$

The restriction $y k d \ll 1$, can be written

$$y\,(kd) = x^2\, kd = \frac{1}{2} \frac{\omega_p^2/\omega^2}{1 + a^2}\, kd \ll 1$$

or

$$N \ll \frac{1 + a^2}{r_o \lambda d}$$

and the restriction $\alpha kd \ll 1$ can be written

$$\alpha kd = x^2 \; akd \ll 1$$

or

$$N \ll \frac{1 + a^2}{\lambda a d r_o}$$

Thus we see that all these restrictions essentially require that $\lambda \to 0$ or that frequency get very large.

This analysis has been restricted to the extreme high frequency limit. It has been shown by classical theory, that the usual optics cross section for an ionized gas has an upper limit of validity dependent on the collision frequency and on the density. Above this limit the body no longer can be considered as an opaque scatterer, but, in fact, must be assigned a complex index of refraction. This latter problem has, in turn, a high frequency limit where the field is the sum of the fields given by the Thomson cross sections of the electrons. This high frequency limit is now independent of frequency, except for the frequency dependence of the damping term.

## REFERENCES

## SECTION V

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# VI

## MODELING OF AN OVAL CYLINDER BY A CIRCULAR ONE

(J. E. Belyea)

In much of the previous work done on the non-linear modeling of wave scattering phenomena, the procedure used has been to concentrate on the differential equation governing such phenomena,

$$\nabla^2 \phi + k^2 \phi = 0 \tag{1}$$

and to look for certain invariant transformations associated with it. This procedure was arrived at by direct analogy with classical methods for linearly modeling (c.f. [1], p. 488), and has led to solutions in a number of interesting cases. However there are other cases, particularly those where the scattering surfaces are distorted under the modeling process, where the invariance approach becomes unmanageable.

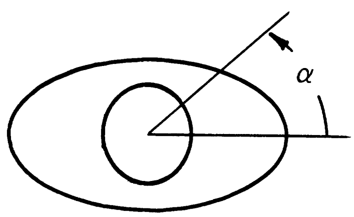
In this chapter an alternate method of attacking the problem, which concentrates wholly on boundaries and boundary values, will be initiated. This method will be presented by means of the detailed examination of a specific example, but it is capable of widespread use.

Simply stated, the procedure will be to examine the scattering of a given wave by a particular shape, on which stated boundary conditions are obeyed; then diagnose certain other boundary conditions which, when imposed at the surface of another, simpler, shape, give rise to the same far-zone field. In a formal way this diagnostic process relies heavily on the very general uniqueness proofs which exist for exterior scattering problems; its technical success or failure depends on one's ability to handle a certain integral equation which arises in the course of the analysis.

## 6.1 Statement of the Problem

The particular example to which the boundary shift method of modeling, just described, will be applied, is the scattering of a plane wave by a perfectly reflecting oval cylinder. The boundary of the model system, on which equivalent conditions will be prescribed, will be a circular cylinder.

When a time-harmonic plane acoustic wave, incident along the ray $\phi = \alpha$, encounters a perfectly reflecting oval cylinder (the equation of whose surface is $f(\rho, \phi) = \rho = \epsilon \cos^2 \phi + b)$, the boundary condition at the surface




gives that, if u is the scattered velocity potential

$$\hat{n} \cdot \nabla \quad u = -\hat{n} \cdot \nabla \left\{ e^{-ik\rho \cos(\phi - \alpha)} \right\} \tag{2}$$

on the surface. Here $\hat{n}$ is the unit vector normal to the cylinder's surface. In addition to (2), u obeys the scalar wave equation in the exterior region, and the radiation condition at $\rho = \infty$. These three conditions determine u uniquely in the exterior region since f = const. is a smooth curve.

Consider now the model system. Let v be the velocity potential scattered by a circular cylinder of (as yet) undetermined properties. That is, the explicit form of the boundary condition which v obeys on the surface is left open. Let normal boundary conditions on v, $\frac{\partial v}{\partial \rho} = \lambda(\phi)$, be prescribed at the surface of the circular cylinder, $\rho = a$. v is known to obey the scalar wave equation outside and the radiation condition at $\rho = \infty$ so it is uniquely determined. In fact, since the ooordinates are separable, it is easy to establish that

$$v = \int_0^{2\pi} \lambda(z) \sum_{m=-\infty}^{\infty} \frac{H_m^{(1)}(k\rho)\, e^{im(\phi - z)}}{2\pi\, k\, H_m^{(1)\,\prime}(ka)}\, dz\,. \tag{3}$$

Suppose that $a < b$. Then, in view of the above, a sufficient condition for u and v to be equal in that part of space where $f(\rho, \phi) > b$ is that

$$\hat{n} \cdot \nabla v = \hat{n} \cdot \nabla u = -\hat{n} \cdot \nabla \left(e^{-ik\rho \cos(\phi - \alpha)}\right) \tag{4}$$

on $f(\rho, \phi) = b$. Clearly condition (4) will not be met for just any $\lambda(\phi)$. It will be the purpose of our modeling analysis to determine $\lambda$ so that (4) is satisfied.

This may most conveniently be done by inserting (3) in the left side of (4). The result of this is that

$$-\hat{n} \cdot \nabla \left(e^{-ik\rho \cos(\phi - \alpha)}\right) = \int_0^{2\pi} \lambda(z)\, \hat{n} \cdot \nabla \sum_{m=-\infty}^{\infty} \frac{H_m^{(1)}(k\rho)\, e^{im(\phi - z)}}{2\pi k\, H_m^{(1)\,\prime}(ka)}\, dz \tag{5}$$

at $\rho = b + \epsilon \cos^2 \phi$. When the differential operations indicated in (5) are performed and the result evaluated on the proper surface, a Fredholm integral equation (of the 1st kind!) for $\lambda$ is obtained:

$$\mu(\phi;\epsilon) = \int_0^{2\pi} \lambda(z;\epsilon)\ K(\phi, z;\epsilon)\, dz \tag{6}$$

6.2 Solution for the Parameter $\epsilon$ Quite Small

If $\epsilon$ is quite small, so that powers of it higher than the first may be neglected, then it is easy to establish that

$$\hat{n} = (1,\ \frac{\epsilon}{b}\ \sin 2\phi)$$

to first order in $\epsilon$. Furthermore, on the cylinder the gradient operator has components

$$\nabla = \left(\frac{\partial}{\partial\rho}\ ,\ \frac{b - \epsilon\cos^2\phi}{b^2}\ \frac{\partial}{\partial\phi}\right)$$

to first order in $\epsilon$. Thus

$$\hat{n}\cdot\nabla = \frac{\partial}{\partial\rho} + \frac{\epsilon}{b^2}\ \sin 2\phi\ \frac{\partial}{\partial\phi}\ .$$

When the plane wave expansion $e^{-ik\rho\cos(\phi-\alpha)} = \sum_{m=-\infty}^{\infty} i^{-m} J_m(k\rho)\, e^{im(\phi-\alpha)}$ is used,

$$\mu(\phi;\epsilon) = -\hat{n}\cdot\nabla \sum_{m=-\infty}^{\infty} i^{-m} J_m(k\rho)\, e^{im(\phi-\alpha)}\Bigg|_{f=b}$$

$$= -\sum_{m=-\infty}^{\infty} i^{-m}\left(kJ_m'(k\rho) + \frac{\epsilon im}{b^2}\ \sin 2\phi\, J_m(k\rho)\right) e^{im(\phi-\alpha)}\Bigg|_{f=b}$$

When $\rho = b + \epsilon \cos^2 \phi$ is inserted in the argument of $J'_m$ and $J_m$ and the proper expansions made, it is found that

$$\mu(\phi; \epsilon) = -\sum_{m=-\infty}^{\infty} i^{-m} \left\{ k J'_m(kb) + \epsilon k^2 \cos^2\phi J''_m(kb) + \frac{\epsilon i m}{b^2} \sin 2\phi J_m(kb) \right\} e^{im(\phi - \alpha)}$$

to first order in $\epsilon$ . Similarly, to first order in $\epsilon$

$$K(\phi, z; \epsilon) = \hat{n} \cdot \nabla \sum_{m=-\infty}^{\infty} \frac{H_m^{(1)}(k\rho)\, e^{im(\phi - z)}}{2\pi k H_m^{(1)'}(ka)} \Bigg|_{f=b}$$

$$= \sum_{m=-\infty}^{\infty} \frac{e^{im(\phi - z)}}{2\pi k H_m^{(1)'}(ka)} \left[ k H_m^{(1)'}(kb) + \epsilon k^2 \cos^2\phi H_m^{(1)''}(kb) + \frac{\epsilon i m}{b^2} \sin 2\phi H_m^{(1)}(kb) \right] .$$

Thus, to first order in $\epsilon$ equation (6) of the last section becomes

$$-\sum_{m=-\infty}^{\infty} i^{-m} \left( k J'_m(kb) + \epsilon k^2 \cos^2\phi J''_m(kb) + \frac{\epsilon i m}{b^2} \sin 2\phi J_m(kb) \right) e^{im(\phi - \alpha)}$$

$$= \int_0^{2\pi} \lambda(z; \epsilon) \sum_{m=-\infty}^{\infty} \frac{e^{im(\phi - z)} dz}{2k\pi H_m^{(1)'}(ka)} \left[ k H_m^{(1)'}(kb) + \epsilon k^2 \cos^2\phi H_m^{(1)''}(kb) + \frac{\epsilon i m}{b^2} \sin 2\phi H_m^{(1)}(kb) \right] .$$

The assumption that $\lambda$ possesses a development

$$\lambda(z;\epsilon) \cong \lambda_o(z) + \epsilon\,\lambda_1(z)$$

splits (7) into 2 equations, the first of which is

$$-\sum_{m=-\infty}^{\infty} i^{-m} k J_m'(kb)\, e^{im(\phi-\alpha)} = \sum_{m=-\infty}^{\infty} \int_o^{2\pi} \frac{\lambda_o(z) H_m^{(1)'}(kb)\, e^{im(\phi-z)}}{2\pi H_m^{(1)'}(ka)}\, dz \tag{8}$$

From (8) we can conclude that if $A_i^*$ are the Fourier coefficients of $\lambda_o$, then

$$A_m = \frac{-i^{-m} k H_m^{(1)'}(ka)\, J_m'(kb)\, e^{-im\alpha}}{H_m^{(1)'}(kb)}$$

The second equation obtained from (7) is

$$-\sum_{m=-\infty}^{\infty} i^{-m}\left(k^2\cos^2\phi\, J_m''(kb) + \frac{im}{b^2}\sin 2\phi\, J_m(kb)\right) e^{im(\phi-\alpha)}$$

$$-\int_o^{2\pi} \lambda_o(z) \sum_{m=-\infty}^{\infty} \frac{\left(k^2\cos^2\phi\, H_m^{(1)''}(kb) + \frac{im}{b^2}\sin 2\phi\, H_m^{(1)}(kb)\right) e^{im(\phi-z)}}{2\pi k H_m^{(1)'}(ka)}$$

$$= \int_o^{2\pi} \lambda_1(z) \sum_{m=-\infty}^{\infty} \frac{H_m^{(1)'}(kb)}{2\pi H_m^{(1)'}(ka)}\, e^{im(\phi-z)}\, dz \tag{9}$$

From (9) we can conclude that, if $B_i$ are the Fourier coefficients of $\lambda_1(z)$,

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* i.e. $$\lambda_o = \sum_{m=\infty}^{\infty} A_n e^{inz}$$

$$\sum_{m=-\infty}^{\infty} \frac{B_m H_m^{(1)'}(kb)}{H_m^{(1)'}(ka)} e^{im\phi} =$$

$$-\sum_{m=-\infty}^{\infty} \left\{ A_m k^2 \cos^2\phi \, H_m^{(1)''}(kb) + A_m \frac{im}{b^2} \sin 2\phi \, H_m^{(1)}(kb) \right.$$

$$\left. + i^{-m} k^2 \cos^2\phi \, J_m''(kb) \, e^{-im\alpha} + \frac{i^{-m+1} m}{b^2} \sin 2\phi \, J_m(kb) \, e^{-im\alpha} \right\} e^{im\phi}$$

Multiplication by $\frac{e^{-is\phi}}{2\pi}$ , followed by integration on $\phi$ over $[0, 2\pi]$ gives, after some rearrangement

$$B_s = -\frac{H_s^{(1)'}(ka)}{H_s^{(1)'}(kb)} \left[ \frac{k^2}{2} H_s^{(1)''}(kb) A_s + \frac{k^2}{4} H_{s-2}^{(1)''}(kb) A_{s-2} \right.$$

$$+ \frac{k^2}{4} H_{s+2}^{(1)''}(kb) A_{s+2} + \frac{(s-2) H_{s-2}^{(1)}(kb)}{2b^2} A_{s-2}$$

$$- \frac{(s+2)}{2b^2} H_{s+2}^{(1)}(kb) A_{s+2} + \frac{k^2 i^{-s}}{2} J_s''(kb) \, e^{-is\alpha}$$

$$- \frac{k^2 i^{-s}}{4} J_{s-2}''(kb) \, e^{-i(s-2)\alpha} - \frac{k^2 i^{-s}}{4} J_{s+2}''(kb) \, e^{-i(s+2)\alpha}$$

$$+ \frac{(s-2) i^{-s+3}}{2b^2} J_{s-2}(kb) \, e^{-i(s-2)\alpha}$$

$$\left. - \frac{(s+2) i^{-s-1}}{2b^2} J_{s+2}(kb) \, e^{-i(s+2)\alpha} \right]$$

To sum up, it has been shown that the equivalent boundary condition, which will give rise the same scattered field, is

$$\frac{\partial v}{\partial \rho} = \lambda(\phi) = \sum_{n=-\infty}^{\infty} (A_n + \epsilon B_n) e^{in\phi}$$

where $A_n$ and $B_n$ are given above.

Note that on the circular cylinder the value assumed by the normal derivative of an incident plane wave is

$$\sum_{m=-\infty}^{\infty} k i^{-m} J_m'(ka) e^{im(\phi-\alpha)} \quad .$$

This plus $\lambda$ is a non-zero quantity, and in fact it is easy to see from physical grounds that there must be a radiation from the circular cylinder's surface.

In the next section a method for obtaining the model scattered field, without using a radiating surface, will be discussed. In the following one an extension of the above perturbation analysis will be made, to the case where higher powers of $\epsilon$ are significant.

## 6.3 Conversion to a Scattering Scheme

In the preceding section it was shown that, for a plane wave incident, one may replace a rigid oval cylinder scatterer by a radiating circular cylinder without altering the field. The value which the normal derivative of the scattered field must assume on the circular cylinder was given there. In the present section the conversion of that radiation scheme to a scattering scheme will be effected.

Let a plane wave $u^{inc} = e^{-ik\rho\cos(\phi-\alpha)}$ impinge on the rigid cylinder $\rho = b + \epsilon\cos^2\phi$. Replace this by another (rigid) cylinder $\rho = a$ ($a < b$); in order that the field be unaltered by this replacement, there must be a certain flux of u through the cylinder wall. This was shown to be

$$\left.\frac{\partial u}{\partial \rho}\right|_{\rho=a} = \sum_{n=-\infty}^{\infty} (A_n + \epsilon B_n)\, e^{in\phi} - ik\cos(\phi-\alpha)\, e^{ika\cos(\phi-\alpha)}$$

If we are prepared to abandon the incident plane wave, however, and use instead an incident wave of more complicated character, it is clear that a rigid, <u>non-radiating</u> circular cylinder can be used. In order to do this we need only use an incident field $u^{inc}$ which satisfies the condition

$$\left.\frac{\partial}{\partial \rho} u^{inc}\right|_{\rho=a} + \sum_n (A_n + \epsilon B_n)\, e^{in\phi} = 0 \quad . \tag{10}$$

It is highly undesirable that this incident field satisfy the radiation condition at $\rho = \infty$, since then the resulting total field would be identically zero. Therefore we assume for it the expansion

$$u^{inc} = \sum_{n=-\infty}^{\infty} a_n J_n(k\rho)\, e^{in\phi} \tag{11}$$

where the coefficients $a_n$ are to be determined. Expression (11) satisfies the wave equation $\Delta u + k^2 u = 0$ but <u>not</u> the radiation condition.

Inserting (11) in (10) gives

$$\sum_n k\, a_n J_n'(ka)\, e^{in\phi} = \sum_n (A_n + \epsilon B_n)\, e^{in\phi} \quad . \tag{12}$$

From this we conclude that

$$a_n = \frac{A_n + \epsilon B_n}{k J_n'(ka)} \quad .$$

Thus when an incident wave of the form

$$u^{inc} = \sum_{n=-\infty}^{\infty} \frac{A_n + \epsilon B_n}{k J_n'(ka)} \; J_n(k\rho)\, e^{in\phi}$$

illuminates a rigid circular cylinder $\rho = a$, the resultant scattered field is the same as that produced by illuminating an oval cylinder $\rho = b + \epsilon \cos^2 \phi$ with a plane wave $u^{inc} = e^{-ik\rho \cos(\phi - \alpha)}$. Note that the total fields are not the same, however.

## 6.4 An Extension to Higher Order of the Perturbation Method

In section 6.1, the diffraction of a plane acoustic wave normally incident on a rigid oval cylinder was considered. The aim of the investigation was to find what boundary conditions must obtain on a smaller, circular, cylinder, in order for the same scattered field to result. This problem of prescribing equivalent boundary conditions was there phrased in terms of solving a rather formidable integral equation of the first Fredholm type:

$$\mu(\phi; \epsilon) = \int_0^{2\pi} \lambda(z; \epsilon)\, K(\phi, z; \epsilon)\, dz \tag{13}$$

In (13), $\mu$ and K are known[+] functions, while $\epsilon$ is a constant parameter, (half the difference of the axes of the oval). $\lambda$, the unknown function, is the value which the normal derivative of the scattered field must assume on the circular cylinder.

In section 6.2, $\lambda$ was obtained for the case where $\epsilon$ was an infinitesimal. In this section the iterative procedure initiated there will be extended, in order to include cases where powers of $\epsilon$ higher than the first are significant. That is, if n is the highest significant power of $\epsilon$, a method will be presented for finding all the coefficients $a_i(\phi)$ in the expansion

$$\lambda(\phi;\epsilon) \sim a_o(\phi) + a_1 \epsilon + \ldots\ldots + a_n \epsilon^n . \tag{14}$$

For our purposes it is important to remark that the equation

$$f(x) = \int_0^{2\pi} g(y)\, K(x, y; o)\, dy \quad ^{++} \tag{15}$$

is readily solvable whenever f(x) is representable by a trigonometric series on the interval $[0, 2\pi]$ . This fact is demonstrated in appendix B. Our task in this section will therefore be considered completed when we have shown that all the coefficients $a_i(\phi)$ are solutions of equations of the type (15).

The first step in the procedure is to insert the representations

$$\mu(\phi;\epsilon) = \sum_{i=0}^{\infty} \mu_i(\varphi)\, \epsilon^i \tag{16}$$

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[+] For an explicit description of these functions, see appendix A.

[++] K(x, y; o) is the kernel of (13), with $\epsilon$ set equal to zero.

and

$$K(\varphi, z; \epsilon) = \sum_{j=0}^{\infty} K_j(\phi, z)\, \epsilon^j,$$

obtained in appendix A, in (13). When, in addition, the expansion for the unknown function

$$\lambda(z; \epsilon) = \sum_{k=0}^{\infty} a_k(z)\, \epsilon^k \tag{17}$$

is inserted here, the result is

$$\sum_{i=1}^{\infty} \mu_i(\phi)\, \epsilon^i = \int_0^{2\pi} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \epsilon^{j+k}\, a_k(z)\, K_j(\phi, z)\, dz. \tag{18}$$

Here (17) may be thought of as the Maclaurin expansion of $\lambda$, considered as a function of $\epsilon$, and the unknown coefficients $a_k(z)$ may be identified with $\frac{\lambda^{(k)}(z;0)}{k!}$.

It is convenient to redefine the sums occurring on the right hand side of (18). Let $\ell = j+k$; then

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \epsilon^{j+k}\, a_k(z)\, K_j(\varphi, z) = \sum_{j=0}^{\infty} \sum_{\ell=j}^{\infty} \epsilon^{\ell}\, a_{\ell-j}(z)\, K_j(\varphi, z).$$

It is easy to see that when the order of summation in this is interchanged, the result is

$$\sum_{\ell=0}^{\infty} \sum_{j=0}^{\ell} \epsilon^{\ell}\, a_{\ell-j}(z)\, K_j(\varphi, z).$$

When this is inserted in (18), and the order of integration and summation interchanged, one obtains

$$\sum_{i=0}^{\infty} \mu_i(\phi)\,\epsilon^i = \sum_{\ell=0}^{\infty} \epsilon^{\ell} \sum_{j=0}^{\ell} \int_0^{2\pi} a_{\ell-j}(z)\,K_j(\phi, z)\,dz. \tag{19}$$

Since the powers of $\epsilon$ are linearly independent of each other, it may be concluded from (19) that a denumberable infinity of relations of the following form hold:

$$\mu_{\ell}(\phi) = \sum_{j=0}^{\ell} \int_0^{2\pi} a_{\ell-j}(z)\,K_j(\phi, z)\,dz\,, \tag{20}$$

$$\ell = 0, 1, 2, \ldots..$$

The $n\underline{th}$ equation of the set (20) may be rearranged to give

$$\mu_n(\phi) - \sum_{j=1}^{n} \int_0^{2\pi} a_{n-j}(z)\,K_j(\phi, z)\,dz = \int_0^{2\pi} a_n(z)\,K_o(\phi, z)\,dz. \tag{21}$$

Since $K_o(\phi, z) \equiv K(\phi, z; o)$, and the coefficients $a_i$, $i < n$, may be regarded as known, it is clear that the coefficients of (17), and, a fortiori, those of (14), are the solutions of equations of the type (15).

# APPENDIX A

## The Functions $\mu$ and K

Consider functions $j(\rho, \phi)$ and $h(\rho, \phi, z)$ which are defined, in sufficient detail for our purposes, by the expressions

$$j(\rho, \phi) = \sum_{n=-\infty}^{\infty} A_r J_r(k\rho)\, e^{ir\phi} \tag{1A}$$

$$h(\rho, \phi, z) = \sum_{s=-\infty}^{\infty} B_s(z)\, H_s^{(1)}(k\rho)\, e^{is\phi} \quad + \tag{2A}$$

The functions $\mu$ and K are the derivatives, of j and h respectively, along the direction normal to the cylindrical surface $\rho = b + \epsilon \cos^2\phi$. That is,

$$\mu = \hat{n} \cdot \nabla j \quad \text{and} \quad K = \hat{n} \cdot \nabla h \tag{3A}$$

where $\hat{n}$ is the unit vector normal to the cylinder's surface, and the expressions in (3A) are evaluated on that surface.

In order to obtain (3A) more explicitly, it is essential first to construct the vector $\hat{n}$. To do this, note that the field of unit vectors $\hat{N}$ which are all orthogonal to the family of surfaces $f(\rho, \phi) = \rho - \epsilon \cos^2\phi = \text{const.}$, is given by

$$\hat{N} = \nabla f / |\nabla f| \quad .$$

The gradient vector $\nabla$ in plane polar coordinates is $\left(\frac{\partial}{\partial \rho}\ ,\ \frac{1}{\rho}\ \frac{\partial}{\partial \phi}\right)$ ,

---

+ $A_r = i^{-r}$ while $B_s(z) = \dfrac{e^{-isz}}{2\pi k H_s^{(1)'}(ka)}$ .

so that

$$\nabla f = \left(1, \frac{\epsilon}{\rho} \sin 2\phi\right)$$

and

$$|\nabla f| = \left[1+\frac{\epsilon^2}{\rho^2} \sin^2 2\phi\right]^{1/2} .$$

Thus

$$\hat{N} = \frac{(\rho, \epsilon \sin 2\phi)}{\sqrt{\rho^2 + \epsilon^2 \sin^2 2\phi}} . \tag{4A}$$

The vectors $\hat{n}$ are elements of the field $\hat{N}$, and are obtained by inserting the relation f = b in (4A). When this is done, the result is

$$\hat{n} = \frac{(b+\epsilon \cos^2\phi, \ \epsilon \sin 2\phi)}{\sqrt{b^2+2b\epsilon\cos^2\phi+\epsilon^2(\cos^4\phi+\sin^2 2\phi)}}$$

In order to expand the components of $\hat{n}$ in powers of $\epsilon$, it becomes convenient to write the denominator as

$$b\left[1+2\epsilon\frac{\cos^2\phi}{b}+\epsilon^2\left(\frac{\cos^4\phi+\sin^2 2\phi}{b^2}\right)\right]^{1/2}$$

The reader will recall that $\left[1-2\mu x+x^2\right]^{-1/2}$ is the generator of the Legendre polynomials. That is,

$$\left[1-2\mu x+x^2\right]^{-1/2} = \sum_{n=0}^{\infty} P_n(\mu)\, x^n .$$

Using this fact,

$$\hat{n} = (b + \epsilon\cos^2\phi,\ \epsilon\sin 2\phi) \sum_{n=0}^{\infty} P_n \left( \frac{-\cos^2\phi}{\sqrt{\cos^4\phi + \sin^2 2\phi}} \right)$$

$$x \left( \frac{b^2}{\cos^4\phi + \sin^2 2\phi} \right)^{-n/2} \frac{\epsilon^n}{b}$$

Recalling that $P_n(-\mu) = (-1)^n P_n(\mu)$, and setting $\zeta(\phi) = \dfrac{\cos^2\phi}{\sqrt{\cos^4\phi + \sin^2 2\phi}}$,

we write

$$\hat{n} = (b + \epsilon\cos^2\phi,\ \epsilon\sin 2\phi) \sum_{n=0}^{\infty} \frac{(-1)^n}{b} P_n\big(\zeta(\phi)\big) \left( \frac{b\,\zeta(\phi)}{\cos^2\phi} \right)^{-n} \epsilon^n. \quad (5A).$$

As a second step toward obtaining (3A) explicitly, consider the function $F(\rho, \phi)$, which for our purposes may be either j or h, and is defined

$$F(\rho, \phi) = \sum_{r=-\infty}^{\infty} c_r Z_r(k\rho)\, e^{ir\phi}$$

The gradient of F is

$$\nabla F = \left( \frac{\partial F}{\partial \rho},\ \frac{1}{\rho}\frac{\partial F}{\partial \phi} \right)$$

$$= \left( \sum_{r=-\infty}^{\infty} C_r k Z_r'(k\rho)\, e^{ir\phi},\ \sum_{r=-\infty}^{\infty} C_r \frac{ir}{\rho} Z_r(k\rho)\, e^{ir\phi} \right).$$

Now $\mathbf{Z}_r$ is a solution of Bessel's equation, so that the relation

$$\frac{2r}{k\rho}\mathbf{Z}_r = \mathbf{Z}_{r-1} + \mathbf{Z}_{r+1}$$

holds. When this is inserted in the second component of $\nabla$ F, and the result evaluated on the surface f = b, one obtains

$$\nabla F = \left( \sum_{r=-\infty}^{\infty} k C_r \mathbf{Z}_r' (kb + k\epsilon \cos^2\phi) e^{ir\phi} \, , \right.$$

$$\left. \sum_{r=-\infty}^{\infty} \frac{ik}{2} C_r e^{ir\phi} \left[ \mathbf{Z}_{r-1}(kb + k\epsilon\cos^2\phi) + \mathbf{Z}_{r+1}(kb + k\epsilon\cos^2\phi) \right] \right) \tag{6A}$$

When the Taylor expansions

$$\mathbf{Z}_r'(kb + k\epsilon\cos^2\phi_2) = \sum_{s=0}^{\infty} \frac{\mathbf{Z}_r^{(s+1)}(kb)}{s!} (k\cos^2\phi)^s \epsilon^s$$

and

$$\mathbf{Z}_m(kb + k\epsilon\cos^2\phi) = \sum_{s=0}^{\infty} \frac{\mathbf{Z}_m^{(s)}(kb)}{s!} (k\cos^2\phi)^s \epsilon^s$$

are inserted in (6A), the result is

$$\nabla F = \left( \sum_{n=-\infty}^{\infty} \sum_{s=0}^{\infty} \frac{k^{s+1} C_r}{s!} \cos^{2s}\phi \, \mathbf{Z}_r^{(s+1)}(kb) e^{ir\phi} \epsilon^s \, , \right.$$

$$\left. \sum_{r=-\infty}^{\infty} \sum_{s=0}^{\infty} \frac{ik^{s+1} C_r}{2(s!)} \cos^{2s}\phi \, e^{ir\phi} \left[ \mathbf{Z}_{r-1}^{(s)}(kb) + \mathbf{Z}_{r+1}^{(s)}(kb) \right] \epsilon^s \right)$$

Letting $\xi_{rs}(\phi) = \dfrac{k^{s+1} C_r \cos^{2s}\phi \, e^{ir\phi}}{s}$ in this, for the sake of simplification, gives

$$\nabla F = \left( \sum_{r=-\infty}^{\infty} \sum_{s=0}^{\infty} \xi_{rs}(\varphi) \, \mathbf{Z}_r^{(s+1)}(kb) \, \epsilon^s \, , \right.$$

$$\left. \sum_{r=-\infty}^{\infty} \sum_{s=0}^{\infty} \frac{i}{2} \xi_{rs}(\varphi) \left[ \mathbf{Z}_{r-1}^{(s)}(kb) + \mathbf{Z}_{r+1}^{(s)}(kb) \right] \epsilon^s \right)$$

Thus the inner product of $\hat{n}$ with $\nabla F$ on f = b is given by

$$\hat{n} \cdot \nabla F = (b + \epsilon \cos^2\phi) \sum_{r=-\infty}^{\infty} \sum_{s=0}^{\infty} \sum_{n=0}^{\infty} \xi_{rs}(\varphi) \mathbf{Z}_r^{(s+1)}(kb) \frac{(-1)^n}{b} P_n(\zeta(\varphi)) \left[ \frac{b\zeta(\varphi)}{\cos^2\varphi} \right]^{-n} \epsilon^{n+s}$$

$$+ \epsilon \sin 2\phi \sum_{r=-\infty}^{\infty} \sum_{s=0}^{\infty} \sum_{n=0}^{\infty} \frac{i}{2} \xi_{rs}(\phi) \left[ \mathbf{Z}_{r-1}^{(s)}(kb) + \mathbf{Z}_{r+1}^{(s)}(kb) \right] \frac{(-1)^n}{b} P_n(\zeta(\varphi)) \left[ \frac{b\zeta(\varphi)}{\cos^2\varphi} \right]^{-n} \epsilon^{n+s}$$

By a redefinition of the sums over n and s which is similar to that used in Section 6.4, the details of which will not be repeated, we obtain

$$n \cdot \nabla F = (b + \epsilon \cos\phi) \sum_{r=-\infty}^{\infty} \sum_{\ell=0}^{\infty} \sum_{n=0}^{\ell} \xi_{r,\ell-n}(\phi) Q_n(\phi) \mathbf{Z}_r^{(\ell-n+1)}(kb) \epsilon^{\ell}$$

$$+ \epsilon \sin 2\phi \sum_{r=-\infty}^{\infty} \sum_{\ell=0}^{\infty} \sum_{n=0}^{\ell} \frac{i}{2} \xi_{r,\ell-n}(\phi) Q_n(\phi)$$

$$\times \left[ \mathbf{Z}_{r-1}^{(n-\ell)}(kb) + \mathbf{Z}_{r+1}^{(n-\ell)}(kb) \right] \epsilon^{\ell} \, . \qquad (7A)$$

Here the simplification

$$Q_n(\phi) = \frac{(-1)^n}{b} P_n(\xi(\phi)) \left[\frac{b\,\xi(\phi)}{\cos^2\phi}\right]^{-n}$$

was made. A trivial rearrangement of this result gives

$$\hat{n}\cdot\nabla F = \sum_{r=-\infty}^{\infty} b\,\xi_{r,0}(\phi) Q_o(\phi) Z_r^{(1)}(kb) +$$

$$\sum_{r=-\infty}^{\infty}\sum_{\ell=0}^{\infty}\sum_{n=0}\left\{ b\,\xi_{r,\ell-n+1}(\phi) Q_n(\phi) Z_r^{(\ell-n+2)}(kb)\right.$$

$$+\cos^2\phi\,\xi_{r,\ell-n}(\phi) Q_n(\phi) Z_r^{(\ell-n+2)}(kb)$$

$$\left. +\sin 2\phi\,\frac{1}{2}\,\xi_{r,\ell-n}(\phi) Q_n(\phi)\left[Z_{r-1}^{(n-\ell)}(kb) + Z_{r+1}^{(n-\ell)}(kb)\right]\right\}\epsilon^{\ell+1} \tag{8A}$$

(8A) shows that both $\mu$ and K may be written in the form

$$\mu = \sum_{\ell=0}^{\infty} \mu_\ell\,\epsilon^\ell$$

$$K = \sum_{\ell=0}^{\infty} K_\ell\,\epsilon^\ell$$

where the coefficients $\mu_\ell$ , $K_\ell$ are known. One useful point is that the kernel of equation (15) is

$$K(x,y;0) = K_o(x,y) = \sum_{r=-\infty}^{\infty} b\,\xi_{r,0}(x) Q_o(x) H_r'(kb)$$

where $\xi_{r,0}(x) = k B_r(y) e^{irx}$, $B_r(y) = \dfrac{e^{-iry}}{2\pi H_r'(ka)}$

and $Q_0(y) = 1/b$ . That is,

$$K(x, y; 0) = \sum_{n=-\infty}^{\infty} \frac{H_r'(kb) e^{ir(x-y)}}{2\pi H_r'(ka)}$$

This fact will be used in appendix B.

## APPENDIX B

### The Solution of Equation (15)

It was seen in appendix A that the kernel of (15) is

$$K(x, y; 0) = \sum_{m=-\infty}^{\infty} \frac{H_m^{(1)'}(kb)}{2\pi H_m^{(1)'}(ka)} e^{im(x-y)} . \tag{1B}$$

If f(x) possesses a Fourier representation, then, it is clear that g(y) does also, and that their Fourier coefficients are each proportional. For, insertion of (1B) in (15), multiplication by $e^{-inx}$, and integration on x over the interval $[o, 2\pi]$ gives

$$\int_o^{2\pi} f(x) e^{-inx} dx = \frac{H_n^{(1)'}(kb)}{H_n^{(1)'}(ka)} \int_o^{2\pi} g(y) e^{-iny} dy .$$

Thus

$$g(y) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{H_n^{(1)'}(ka)}{H_n^{(1)'}(kb)} e^{iny} \int_o^{2\pi} f(x) e^{-inx} dx. \tag{2B}$$

Note that if a = b, then g (y) = f (y)! Thus it would be highly advantageous, from a computational standpoint, to choose this value for a.

## REFERENCES

### SECTION VI

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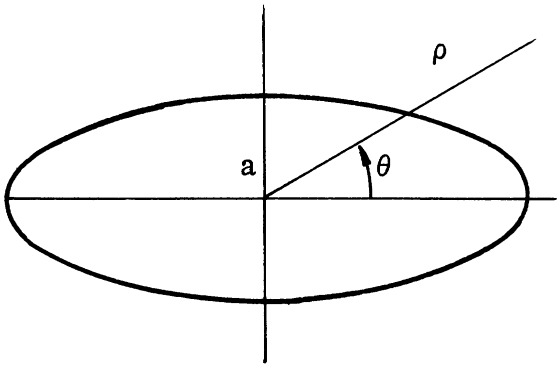
# VII

## BOUNDARY SHIFT METHOD FOR A SPHEROID

(R. D. Low)




$$\rho = \frac{a}{\sqrt{1-\epsilon\cos^2\theta}}$$

$$0 < \epsilon < 1$$

Let the plane wave $e^{ik\rho\cos\theta}$ be incident on the spheroid $\rho = a(1-\epsilon\cos^2\theta)^{-1/2}$.

Then we seek the solution $u(\rho,\theta)$ of the following problem, call it problem A:

$$\nabla^2 u + k^2 u = 0\ , \qquad \rho > a(1-\epsilon\cos^2\epsilon)^{-1/2}\ , \qquad 0 \leq \theta \leq \pi \tag{1}$$

$$u(\rho,\theta) \quad \text{or} \quad \frac{\partial u}{\partial n} = 0\ , \qquad \text{on} \quad \rho = a(1-\epsilon\cos^2\theta)^{-1/2}\ , \quad 0 \leq \theta \leq \pi \tag{2}$$

$$\rho\left(\frac{\partial u^s}{\partial \rho} - iku^s\right) \xrightarrow[\rho \to \infty]{} 0\ , \qquad \text{uniformly in } \theta \tag{3}$$

where $u^s = u - e^{ik\rho\cos\theta}$.

Rather than solve A directly, we consider the following problem, call it problem B:

$$\nabla^2 v + k^2 v = 0 \quad , \quad \rho > a^* \quad , \qquad 0 \leqslant \theta \leqslant \pi \tag{4}$$

$$v(a,\theta) = v_o(\theta) \quad , \qquad 0 \leqslant \theta \leqslant \pi \tag{5}$$

$$\rho \left( \frac{\partial v^s}{\partial \rho} - ikv^s \right) \xrightarrow[\rho \to \infty]{} 0 \quad , \quad \text{uniformly in } \theta \tag{6}$$

where $v^s = v - e^{ik\rho \cos\theta}$. The idea then is to try to determine $v_o(\theta)$ so that either $v$ or $\partial v/\partial n$ will be zero on the surface $\rho = a(1-\epsilon \cos^2\theta)^{-1/2}$, where, of course, $\partial/\partial n$ denotes differentiation along the normal to the spheroid.

It is an easy job to obtain the solution of B in the form:

$$v(\rho,\theta) = \sum_{n=0}^{\infty} (2n+1)\, i^n \left[ j_n(k\rho) - \frac{j_n(ka)}{h_n(ka)}\, h_n(k\rho) \right] P_n(\cos\theta)$$

$$+ \sum_{n=0}^{\infty} \frac{2n+1}{2} \frac{h_n(k\rho)}{h_n(ka)} P_n(\cos\theta) \int_0^{\pi} v_o(\sigma)\, P_n(\cos\sigma) \sin\sigma \, d\sigma \tag{7}$$

where $j_n(ka)$ is the spherical Bessel function and $h_n(k\rho)$ is the spherical Hankel function of the first kind.

Let us now consider the case where $v(\rho,\theta) = 0$ on $\rho = a(i-\epsilon \cos^2\theta)^{-1/2}$. Hence we put $\rho = a(1-\epsilon \cos^2\theta)^{-1/2}$ in (7), interchange the order of summation and integration in the last term, and introduce the expressions

* There is no loss of generality (or practicality) in taking the sphere $\rho = a$ rather than the sphere $\rho = b$, $b < a$, as the relation between the solutions of two sphere problems is quite easy to obtain.

$$F(\theta;\epsilon) = - \sum_{n=0}^{\infty} (2n+1)\, i^n \left[ j_n(k\rho) - \frac{j_n(ka)}{h_n(ka)}\, h_n(k\rho) \right] P_n(\cos\theta) \Bigg|_{\rho = a(1-\epsilon\cos^2\theta)^{-1/2}}$$

$$K(\theta,\sigma;\epsilon) = \sum_{n=0}^{\infty} \frac{2n+1}{2}\, \frac{h_n(k\rho)}{h_n(ka)}\, P_n(\cos\theta)\, P_n(\cos\sigma)\, \sin\sigma \Bigg|_{\rho=a(1-\epsilon\cos^2\theta)^{-1/2}} .$$

Then (7) may be written as

$$\int_0^{\pi} K(\theta,\sigma;\epsilon)\, V_o(\sigma;\epsilon)\, d\sigma = F(\theta;\epsilon) \tag{8}$$

where we have written $V_o(\sigma;\epsilon)$ in place of $v_o(\sigma)$ since $v_o$, being the solution of the integral equation (8), will clearly depend on the parameter $\epsilon$. Thus the problem has been reduced to the solution of the Fredholm integral equation of the first kind (8). We proceed to solve this equation, in principle at least, by expanding each expression in (8) in a power series in $\epsilon$. Hence we have

$$K(\theta,\sigma;\epsilon) = \sum_{n=0}^{\infty} \frac{K^{(n)}(\theta,\sigma;0)}{n!}\, \epsilon^n$$

$$V_o(\sigma;\epsilon) = \sum_{s=0}^{\infty} \frac{V_o^{(s)}(\sigma;0)}{s!}\, \epsilon^s \tag{9}$$

$$F(\theta;\epsilon) = \sum_{n=0}^{\infty} \frac{F^{(n)}(\theta;0)}{n!}\, \epsilon^n$$

where $Q^{(n)}$ means $\partial^n Q/\partial\epsilon^n$ and Q stands for $K, V_o$, or F. If we insert (9) into (8) there results

$$\int_0^\pi \left( \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \frac{K^{(n)}(\theta,\sigma;0)\, V_o^{(s)}(\sigma;0)}{n!\; s!} \epsilon^{n+s} \right) d\sigma = \sum_{n=0}^{\infty} \frac{F^{(n)}(\theta;0)}{n!} \epsilon^n$$

or

$$\sum_{n=0}^{\infty} \epsilon^n \int_0^\pi \sum_{s=0}^{n} \frac{K^{(n-s)}(\theta,\sigma;0)\, V_o^{(s)}(\sigma;0)}{s!\,(n-s)!} d\sigma = \sum_{n=0}^{\infty} \frac{F^{(n)}(\theta;0)}{n!} \epsilon^n$$

which implies

$$\int_0^\pi \sum_{s=0}^{n} \binom{n}{s} K^{(n-s)}(\theta,\sigma;0)\, V_o^{(s)}(\sigma;0)\, d\sigma = F^{(n)}(\theta;0) \quad .$$

This last equation can be written in the form

$$\int_0^\pi \left[ \sum_{s=0}^{n-1} \binom{n}{s} K^{(n-s)}(\theta,\sigma;0)\, V_o^{(s)}(\sigma;0) + K^{(0)}(\theta,\sigma;0)\, V_o^{(n)}(\sigma;0) \right] d\sigma = F^{(n)}(\theta;0) \; . \tag{10}$$

Now $K^{(o)}(\theta,\sigma;0) = K(\theta,\sigma;0)$ is given by

$$K(\theta,\sigma;0) = \sum_{n=0}^{\infty} \frac{2n+1}{2} \mathbf{P}_n(\cos\theta)\, \mathbf{P}_n(\cos\sigma) \sin\sigma$$

and as a factor in an integrand this series behaves like $\delta(\theta-\sigma)$; thus (10) gives

$$V_o^{(n)}(\theta;0) = F^{(n)}(\theta;0) - \int_0^\pi \sum_{s=0}^{n-1} \binom{n}{s} K^{(n-s)}(\theta,\sigma;0)\, V_o^{(s)}(\sigma;0)\, d\sigma \; . \tag{11}$$

In addition to equation (11) we have from the equation preceeding (10) when $n = 0$,

$$\int_0^{\pi} K^{o}(\theta, \sigma; 0)\, V_o^{(o)}(\sigma; 0)\, d\sigma = F^{(o)}(\theta; 0)$$

or

$$\int_0^{\pi} K(\theta, \sigma; 0)\, V_o(\sigma; 0)\, d\sigma = F(\theta; 0)$$

or

$$\int_0^{\pi} \delta(\theta - \sigma)\, V_o(\sigma; 0)\, d\sigma = F(\theta; 0)$$

so

$$V_o(\theta; 0) = F(\theta; 0)\ .$$

Now from the definition of $F(\theta; \epsilon)$, it follows that $F(\theta;0) \equiv 0$; hence $V_o(\theta;0) \equiv 0$. Thus starting with $V_o^{(o)}(\theta;0) = 0$, we can find recursively $V_o^{(1)}(\theta;0)$, $V_o^{(2)}(\theta;0), \ldots$ from (11) and in this way determine $\dot{V}_o(\theta; \epsilon)$ from

$$V_o(\theta; \epsilon) = \sum_{n=0}^{\infty} \frac{V_o^{(n)}(\theta;0)}{n!}\, \epsilon^n\ . \tag{12}$$

The serious disadvantage of this procedure lies in the fact that for $\epsilon \approx 1$, which is the case when the spheroid is needle shaped, many terms are required and so one may ask: Why not expand the quantities appearing in (8) in a power series in $1 - \epsilon$ ? If this is done, the equation to be solved is

$$\int_0^{\pi} \sum_{s=0}^{n} \binom{n}{s} K^{(n-s)}(\theta, \sigma; 1) V_o^{(s)}(\sigma; 1) \, d\sigma = F^{(n)}(\theta; 1)$$

and in this case the equation for n = 0 becomes

$$\int_0^{\pi} K(\theta, \sigma; 1) \, V_o(\sigma; 1) \, d\sigma = F(\theta; 1) \quad . \tag{13}$$

Now one sees from their definitions that $K(\theta, \sigma; 1)$ and $F(\theta; 1)$ contain the expressions $j_n(ka \csc\theta)$ and $h_n(ka \csc\theta)$ and it does not seem possible to solve even (13) for the first term $V_o(\theta; 1)$ needed in the expansion

$$V_o(\theta; \epsilon) = \sum_{n=0}^{\infty} (-1)^n \frac{V_o^{(n)}(\theta; 1)}{n!} (1-\epsilon)^n \quad .$$

However, if this can be done one has the advantage that only a few terms will be required since now $1 - \epsilon \approx 0$. Rather than weigh the merits of either approach let us obtain at least the terms up to $\epsilon^2$ in (12). Returning to (11) we have

$$V_o^{(1)}(\theta; 0) = F^{(1)}(\theta; 0) - \int_0^{\pi} K^{(1)}(\theta, \sigma; 0) \, V_o^{(o)}(\sigma; 0) \, d\sigma$$

$$= F^{(1)}(\theta; 0)$$

since

$$V_o^{(o)}(\theta; 0) \equiv 0 \quad .$$

Then

$$V_o^{(2)}(\theta;0) = F^{(2)}(\theta;0) - \int_0^{\pi} \left[ \cancelto{0}{K^{(2)}(\theta,\sigma;0)\, V_o^{(o)}(\sigma;0)} + K^{(1)}(\theta,\sigma;0)\, V_o^{(1)}(\sigma;0) \right] d\sigma$$

$$= F^{(2)}(\theta;0) - \int_0^{\pi} K^{(1)}(\theta,\sigma;0)\, F^{(1)}(\sigma;0)\, d\sigma \quad .$$

Now

$$F^{(1)}(\theta;0) = \frac{\cos^2\theta}{2ka} \sum_{n=0}^{\infty} (2n+1)\, i^{n+1} \frac{P_n(\cos\theta)}{h_n(ka)} \quad ,$$

$$F^{(2)}(\theta;0) = \frac{\cos^4\theta}{4ka} \sum_{n=0}^{\infty} (2n+1)\, i^{n+1} \frac{P_n(\cos\theta)}{h_n(ka)} \quad ,$$

and

$$K^{(1)}(\theta,\sigma;0) = \frac{ka\cos^2\theta}{2} \sum_{n=0}^{\infty} \frac{2n+1}{2} \frac{h_n'(ka)}{h_n(ka)} P_n(\cos\theta)\, P_n(\cos\sigma) \sin\sigma \quad .$$

Hence $V_o^{(2)}(\theta;0)$ is given by

$$V_o^{(2)}(\theta;0) = \frac{\cos^4\theta}{4ka} \sum_{n=0}^{\infty} (2n+1)\cdot i^{n+1} \frac{P_n(\cos\theta)}{h_n(ka)}$$

$$- \int_0^{\pi} \left( \frac{ka\cos^2\theta}{2} \sum_{n=0}^{\infty} \frac{2n+1}{2} \frac{h_n'(ka)}{h_n(ka)} P_n(\cos\theta)\, P_n(\cos\sigma) \sin\sigma \right)$$

(eqn. is continued on next page)

$$\cdot \left( \frac{\cos^2\sigma}{2ka} \sum_{n=0}^{\infty} (2n+1)\, i^{n+1} \frac{P_n(\cos\sigma)}{h_n(ka)} \right) d\sigma$$

$$= \frac{\cos^4\theta}{4ka} \sum_{n=0}^{\infty} (2n+1)\, i^{n+1} \frac{P_n(\cos\theta)}{h_n(ka)}$$

$$- \frac{\cos^2\theta}{8} \int_0^{\pi} \left( \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} (2n+1) \frac{h_n'(ka)}{h_n(ka)} P_n(\cos\theta)\, P_n(\cos\sigma) \sin\sigma \right.$$

$$\left. \cdot \cos^2\sigma\, (2\ell+1)\, i^{\ell+1} \frac{P_\ell(\cos\sigma)}{h_\ell(ka)} \right) d\sigma$$

$$= \frac{\cos^4\theta}{4ka} \sum_{n=0}^{\infty} (2n+1)\, i^{n+1} \frac{P_n(\cos\theta)}{h_n(ka)}$$

$$- \frac{\cos^2\theta}{8} \sum_{n=0}^{\infty} \sum_{\ell=0}^{n} \left\{ (2n-2\ell+1)(2\ell+1) \frac{h_{n-\ell}'(ka)\, i^{\ell+1}}{h_{n-\ell}(ka) h_\ell(ka)} P_{n-\ell}(\cos\theta) \right\}$$

$$\cdot \left\{ \int_0^{\pi} P_{n-\ell}(\cos\sigma)\, P_\ell(\cos\sigma) \cos^2\sigma \sin\sigma\, d\sigma \right\} .$$

The integral here can be reduced to two integrals containing products of three Legendre polynomials and can thus be evaluated. Hence $V_o^{(2)}(\theta;0)$ is "determined". Then to the second order in $\epsilon$, we have

$$V_o(\theta;\epsilon) = V_o(\theta;0) + V_o^{(1)}(\theta;0)\,\epsilon + V_o^{(2)}(\theta;0)\,\frac{\epsilon^2}{2}$$

$$= V_o^{(1)}(\theta;0)\,\epsilon + V_o^{(2)}(\theta;0)\,\frac{\epsilon^2}{2}$$

since $V_o(\theta;0) \equiv 0$.

The second case, namely the determination of $v_o(\theta)$ so that on $\rho = a(1-\epsilon\cos^2\theta)^{-1/2}$, $\partial v/\partial n = 0$, can also be reduced to an integral equation like (8). In fact the result is

$$\int_0^{\pi} L(\theta, \sigma; \epsilon)\, V_o(\sigma;\ )\, d\sigma = G(\theta, \epsilon) \tag{14}$$

where

$$L(\theta,\sigma;\epsilon) = \sum_{n=0}^{\infty} \frac{2n+1}{2}\,\frac{P_n(\cos\sigma)}{h_n(ka)}\,\sin\sigma\,\frac{\partial}{\partial n}\Big[h_n(k\rho)\,P_n(\cos\theta)\Big]_{\rho=a(1-\epsilon\cos^2\theta)^{-1/2}}$$

$$G(\theta;\epsilon) = -\sum_{n=0}^{\infty}(2n+1)\,i^n\,\frac{\partial}{\partial n}\left[\left\{j_n(k\rho) - \frac{j_n(ka)}{h_n(ka)}\,h_n(k\rho)\right\}P_n(\cos\theta)\right]_{\rho=a(1-\epsilon\cos^2\theta)^{-1/}}$$

and

$$\frac{\partial}{\partial n} = \left[1 + \frac{\epsilon^2\sin^2\theta\cos^2\theta}{(1-\epsilon\cos^2\theta)^2}\right]^{-1/2}\left(\frac{\partial}{\partial\rho} - \frac{\epsilon\sin\theta\cos\theta}{a\sqrt{1-\epsilon\cos^2\theta}}\,\frac{\partial}{\partial\theta}\right).$$

Since the operator $\partial/\partial n$ appears both in $L(\theta,\sigma;\epsilon)$ and $G(\theta;\epsilon)$ one may disregard the factor $\left[1+\frac{\epsilon^2\sin^2\theta\cos^2\theta}{(1-\epsilon\cos^2\theta)^2}\right]^{-1/2}$. However, if the boundary condition were $\partial v/\partial n = g(\theta)$ or $\partial v/\partial n + hv = 0$ then of course this factor would remain. In principle, at least, one may proceed to solve (14) in exactly the same way as (8) was treated. However, the details are quite messy, the main reason being that now $\epsilon$ enters because of its presence in the operator $\partial/\partial n$ as well as in the equation, $\rho = a(1-\epsilon\cos^2\theta)^{-1/2}$ of the spheroid. Let us illustrate the result at least for the first integral equation, namely:

$$\int_0^{\pi} L^{(o)}(\theta,\sigma;0)\, V_o^{(o)}(\sigma;0)\, d\sigma = G^{(o)}(\theta;0) \quad .$$

From their definition one finds that

$$L^{(0)}(\theta,\sigma;0) = K \sum_{n=0}^{\infty} \frac{2n+1}{2} \frac{h_n'(ka)}{h_n(ka)} P_n(\cos\theta)\, P_n(\cos\sigma) \sin\sigma$$

$$G^{(o)}(\theta;0) = \frac{1}{ka^2} \sum_{n=0}^{\infty} (2n+1)\, i^{n+1} \frac{P_n(\cos\theta)}{h_n(ka)} \quad .$$

If these are inserted into the above integral equation one finds after a little rearrangement that

$$\sum_{n=0}^{\infty} \frac{h_n'(ka)}{h_n(ka)} P_n(\cos\theta) \frac{2n+1}{2} \int_0^{\pi} V_o^{(o)}(\sigma;0)\, P_n(\cos\sigma) \sin\sigma\, d\sigma$$

$$= \frac{1}{k^2a^2} \sum_{n=0}^{\infty} (2n+1)\, i^{n+1} \frac{P_n(\cos\theta)}{h_n(ka)}$$

which implies that

$$V_o^{(o)}(\sigma;0) = \frac{1}{k^2a^2} \sum_{n=0}^{\infty} (2n+1)\, i^{n+1} \frac{P_n(\cos\sigma)}{h_n'(ka)} \quad .$$

It is not too difficult to show that

$$L^{(1)}(\theta,\sigma;0) = \sum_{n=0}^{\infty} \frac{2n+1}{2} \left[ \frac{k^2a\cos^2\theta}{2} \frac{h_n''(ka)}{h_n(ka)} P_n(\cos\theta) + \frac{\sin^2\theta\cos\theta}{a} P_n'(\cos\theta) \right]$$

$$\cdot\ P_n(\cos\sigma)\sin\sigma \quad .$$

Then the next integral equation to be solved would be

$$\int_0^{\pi} \left[ L^{(1)}(\theta,\sigma;0)\, V_o^{(o)}(\sigma;0) + L^{(o)}(\theta,\sigma;0)\, V_o^{(1)}(\sigma;0) \right] d\sigma = G^{(1)}(\theta,0)$$

where $G^{(1)}(\theta;0)$ is given by

$$G^{(1)}(\theta;0) = -\frac{\cos^2\theta}{ka^2} \sum_{n=0}^{\infty} (2n+1)\, i^{n+1} \frac{P_n(\cos\theta)}{h_n(ka)} \quad .$$

Thus we must solve the integral equation

$$\int_0^{\pi} L^{(o)}(\theta,\sigma;0)\, V_o^{(1)}(\sigma;0)\, d\sigma = -\frac{\cos^2\theta}{ka^2} \sum_{n=0}^{\infty} (2n+1)\, i^{n+1} \frac{P_n(\cos\theta)}{h_n(ka)}$$

$$- \int_0^{\pi} L^{(1)}(\theta,\sigma;0)\, V_o^{(o)}(\sigma;0)\, d\sigma$$

where, of course, $V_o^{(o)}(\sigma;0)$ is now known. Whether or not this equation can be solved in any reasonable manner (I have not tried to do so), it seems clear that one can carry out the same type of iterative solution as was indicated in the case in which $v = 0$ on the spheroid. In this case, too, it goes without saying that the number of terms required will be very large in the event that $\epsilon \approx 1$. Thus at best the above outlined procedures seem to be impractical when $\epsilon \approx 1$. The other alternative of expanding in power series in $(1 - \epsilon)$ has the disadvantage that one cannot apparently solve easily even the first integral equation.

# VIII

## MODELING OF A SPHEROID BY A SPHERE

## (THE SCALAR PROBLEM WITH ARBITRARY ECCENTRICITY)

(F. B. Sleator)

The continuous dependence of a solution of the exterior scattering problem on the boundary values indicates that it should be possible to duplicate the far-zone field of a given body by substituting a different (but topologically equivalent) body with different boundary conditions. If the scattering properties of the given body are unknown, or unmeasurable, it may be possible to determine them by calculating or measuring the field of the substituted body, providing the proper boundary conditions can be determined or produced. The determination of the boundary conditions on a sphere which would produce the same scattered field as a hard or soft spheroid of arbitrary eccentricity is the present concern.

A convenient tool for this job is the Helmholtz formula

$$V(P) = \frac{1}{4\pi} \int_{S'} \left[ V(P') \frac{\partial}{\partial n'} G(P, P') - \frac{\partial}{\partial n'} V(P') G(P, P') \right] dS' \qquad (1)$$

which relates the potential $V(P)$ at the point $P$ in space to its value at $P'$ on the scattering surface $S'$. Here $G(P, P')$ is the Green's function of free space, i.e. $e^{ik\rho}/\rho$, where $\rho$ is the distance from $P'$ to $P$, and $\partial/\partial n'$ is the derivative in the direction normal to the surface $S'$. If a term of the form $e^{ikz}$, where $z$ is a space coordinate, is added to the r.h.s. of the equation, then $V(P)$ represents the total field produced by a plane wave incident on the body in the z-direction. If the quantity V(P) is assumed known, equation (1) is an integral equation in the two unknowns V(P') and $\partial V(P')/\partial n'$, which can be attacked by means of the usual expansion procedures.

Accordingly, for a given scattering surface S', which we will assume to be spherical, we introduce the following expansions:

$$V(P) = V(r,\theta) = \sum_{\mu} a_{\mu} h_{\mu}(kr)\, P_{\mu}(\cos\theta)$$

$$V(P') = V(\theta') = \sum_{\mu} b_{\mu} P_{\mu}(\cos\theta')$$

$$\frac{\partial V(P')}{\partial n'} = \frac{\partial V(\theta')}{\partial n'} = \sum_{\mu} c_{\mu} P_{\mu}(\cos\theta')$$

$$e^{ikz} = e^{ikr\cos\theta} = \sum_{\mu} i^{\mu}(2\mu+1)\, j_{\mu}(kr)\, P_{\mu}(\cos\theta) \tag{2}$$

$$G(P,P') = ik \sum_{m}\sum_{n} \epsilon_{m}(2n+1)\frac{(n-m)!}{(n+m)!}\cos\left[m(\phi'-\phi)\right] \cdot$$

$$\cdot\, P_n^m(\cos\theta)\, P_n^m(\cos\theta')\, j_n(kr')\, h_n(kr) \qquad (r' \leq r)$$

where the $j_{\mu}$ and $h_{\mu}$ are spherical Bessel and Hankel functions, the $P_{\nu}^{\mu}$ are Legendre functions ($P_{\nu} \equiv P_{\nu}^{o}$), and $a_{\mu}, b_{\mu}, c_{\mu}$ are undetermined coefficients depending on the geometry of the system, which is assumed axially symmetric, so that the fields are independent of $\phi$. Substitution of these in the Helmholtz formula with included plane wave yields

$$\sum_{\mu} a_\mu h_\mu(kr)\, P_\mu(\cos\theta) = \sum_{\mu} i^\mu (2\mu+1)\, j_\mu(kr)\, P_\mu(\cos\theta) + \frac{ik}{4\pi} \int_{S'} \Bigg[ \sum_{\mu} b_\mu P_\mu(\cos\theta') \cdot$$

$$\cdot\; k \sum_{m}\sum_{n} \epsilon_m (2n+1) \frac{(n-m)!}{(n+m)!} \cos\left[m(\phi'-\phi)\right] P_n^m(\cos\theta')\, P_n^m(\cos\theta) \cdot$$

$$\cdot\; \cos\left[m(\phi'-\phi)\right] P_n^m(\cos\theta')\, P_n^m(\cos\theta)\, j_n(kr')\, h_n(kr) \Bigg] dS'$$

$$= \sum_{\mu} i^\mu (2\mu+1)\, j_\mu(kr)\, P_\mu(\cos\theta) + \frac{ik}{4\pi} \sum_{m}\sum_{n}\sum_{\mu} \epsilon_m (2n+1) \frac{(n-m)!}{(n+m)!} \cdot$$

$$h_n(kr)\, P_n^m(\cos\theta) \int_0^\pi \int_0^{2\pi} \cos\left[m(\phi'-\phi)\right] P_n^m(\cos\theta')\, P_\mu(\cos\theta') \cdot$$

$$\cdot \left[k b_\mu j_n'(kr') - c_\mu j_n(kr')\right] r'^2 \sin\theta'\, d\phi'\, d\theta' \quad .$$

Orthogonality properties of the angular functions of $\theta'$ and $\phi'$ simplify the right side of this equation to give

$$\sum_{\mu} a_\mu h_\mu(kr)\, P_\mu(\cos\theta) = \sum_{\mu} i^\mu (2\mu+1)\, j_\mu(kr)\, P_\mu(\cos\theta) + \qquad (3)$$

$$+\; ikr'^2 \sum_{\mu} h_\mu(kr)\, P_\mu(\cos\theta) \left[k b_\mu j_\mu'(kr') - c_\mu j_\mu(kr')\right]$$

Although there are three sets of coefficients appearing in this equation, it is clear that the $b_\mu$ and $c_\mu$ are not independent, since either set, along with the radiation condition, determines the solution uniquely, and the other should therefore be expressible in terms of the first. The explicit relation between $b_\mu$ and $c_\mu$ for any $\mu$ can be obtained easily for the case when S' is spherical by letting the point P approach this surface. In this case $r \to r'$ and $a_\mu \longrightarrow \frac{b_\mu}{h_\mu(kr')}$. The coefficients of $P_\mu(\cos\theta')$ can then be equated to give

$$b_\mu = i^\mu (2\mu+1) j_\mu(kr') + ikr'^2 h_\mu(kr') \left[ k b_\mu j'_\mu(kr') - c_\mu j_\mu(kr') \right] .$$

Rearrangement and substitution of the Wronskian for spherical Bessel functions leaves this in the simpler form

$$-b_\mu \cdot k h'_\mu(kr') + c_\mu h_\mu(kr') = \frac{i^{\mu-1}(2\mu+1)}{kr'^2} . \tag{4}$$

This can be inserted in (3) to yield, after some manipulation

$$\sum_\mu \left( a_\mu - \frac{b_\mu}{h_\mu(kr')} \right) P_\mu(\cos\theta)\, h_\mu(kr) = \sum_\mu i^{\mu+1} (2\mu+1) \frac{P_\mu(\cos\theta)}{h_\mu(kr')}$$

$$\cdot \left[ j_\mu(kr)\, n_\mu(kr') - j_\mu(kr')\, n_\mu(kr) \right] \tag{5}$$

where the $n_\mu$ are spherical Neumann functions.

Up to now the point $P(r, \theta)$ has been restricted only to the exterior of the sphere S'. If we now require it to lie on another spherical surface, i.e. fix $r$ at some constant value greater than $r'$, then the coefficients of $P_\mu(\cos\theta)$

in (5) can be matched and the resulting relation between $a_\mu$ and $b_\mu$ is

$$a_\mu = \frac{b_\mu}{h_\mu(kr')} + \frac{i^{\mu+1}(2\mu+1)}{h_\mu(kr)h_\mu(kr')}\left[j_\mu(kr)\,n_\mu(kr') - j_\mu(kr')\,n_\mu(kr)\right] . \quad (6)$$

If the exterior (hypothetical) surface is to be spheroidal, however, then r depends on $\theta$, and it is no longer possible to match the coefficients directly. We can expect only to get a matrix relation which permits the determination of the difference $a_\mu - \frac{b_\mu}{h_\mu(kr')}$ as the solution of an infinite linear algebraic system. One such system is obtained by multiplying both sides of equation (5) by $P_\nu(\cos\theta)\,j_\nu(kr)\,\phi(\theta)\,d\theta$ and integrating from 0 to $\pi$, where r is now $r(\theta) = a\sqrt{\frac{\xi^2-1}{\xi^2\cos^2\theta}}$, a is the semi-major axis of the spheroid, $\xi$ is the inverse of the eccentricity, $\phi(\theta)$ is some arbitrary weight function, and $\nu = 0, 1, 2, \ldots \infty$. With the definition

$$\lambda_{\mu\nu} \equiv \int_0^\pi P_\mu(\cos\theta)\,h_\mu\big(kr(\theta)\big)\,P_\nu(\cos\theta)\,j_\nu\big(kr(\theta)\big)\,\phi(\theta)\,d\theta$$

the linear system is written as

$$\sum_\mu \left(a_\mu - \frac{b_\mu}{h_\mu(kr')}\right)\lambda_{\mu\nu} = \sum_\mu \frac{i^{\mu+1}(2\mu+1)}{h_\mu(kr')}\left[n_\mu(kr')\,\mathrm{Re}\,\lambda_{\mu\nu} - j_\mu(kr')\,\mathrm{Im}\,\lambda_{\mu\nu}\right] . \quad (7)$$

The first task one faces in the solution of this system is the evaluation of the integral $\lambda_{\mu\nu}$ for arbitrary $\mu$ and $\nu$. Since both the Bessel and Legendre functions are expressible in finite series, it is clear that for any finite indices

$\mu$ and $\nu$ the integral should be expressible as a finite combination of integrals of some elliptic type; however, the labor involved in deriving a sufficient number of cases appears to be prohibitive. The best alternative found as yet is an expansion in spheroidal functions which is obtained as follows:

Consider the quantity

$$\Gamma_{\mu\nu} \equiv \lim_{\xi'\to\xi} \int_{\xi'}\int_{\xi} G(\mathbf{P},\mathbf{P}')\, \mathbf{P}_{\mu}(\eta)\, \mathbf{P}_{\nu}(\eta')\, dS\, dS' \tag{8}$$

where $G(\mathbf{P},\mathbf{P}')$ is the Green's function of the points $\mathbf{P}(\xi, \eta, \phi)$ and $\mathbf{P}'(\xi', \eta', \phi')$ in spheroidal coordinates, $\mathbf{P}_{\mu}$ and $\mathbf{P}_{\nu}$ are Legendre polynomials, and the integrations cover the two spheroids given by the coordinates $\xi$ and $\xi'$. If the Green's function is represented as a Fourier integral [1]

$$G(\mathbf{P},\mathbf{P}') = \frac{1}{2\pi^2} \iiint_{-\infty}^{\infty} \frac{e^{i\vec{K}\cdot\vec{\rho}}}{\vec{K}^2 - k^2}\, d\vec{K} \tag{9}$$

where $\vec{K}$ is the vector $(K_x, K_y, K_z)$, $d\vec{K} = dK_x dK_y dK_z$, and $\vec{\rho} \equiv (x-x', y-y', z-z')$ then six of the seven integrations can be carried out in the manner described in [2], and in the limit as $\xi' \longrightarrow \xi$ the result is

$$\Gamma_{\mu\nu} = \frac{-16\pi\, i^{\mu+\nu+1}\, k}{ab^2 F^2} \int_0^{\pi} P_{\mu}(\cos\psi)\, P_{\nu}(\cos\psi)\, h_{\mu}(k\rho)\, j_{\nu}(k\rho)\, \rho^3 \sin\psi\, d\psi \tag{10}$$

where a, b and F are respectively half of the major axis, minor axis and focal length of the ellipse specified by the relations

$$\rho = a\sqrt{\frac{\xi^2 - 1}{\xi^2 - \cos^2\psi}}$$

$$\xi = \frac{a}{\sqrt{a^2-b^2}} = \frac{a}{F} \quad .$$

Alternatively we can expand the Green's function in (6) in series of spheroidal functions [3]:

$$G(P,P') = 2ik \sum_m \sum_n \frac{1}{\Lambda_{mn}} S_{mn}(\eta) S_{mn}(\eta) j_{e_{mn}}(\xi) h_{e_{mn}}(\xi') e^{im(\phi-\phi')} .$$

Here $\Lambda_{mn}$ is the normalization constant for the angular spheroidal functions $S_{mn}$, and $j_{e_{mn}}$ and $h_{e_{mn}}$ are radial functions of the first and third kinds.

When this expression is inserted in (6), the integrations over $\phi$ and $\phi'$ yield immediately the factor $4\pi^2 \delta_{om}$, and those over $\eta$ and $\eta'$ have the form

$$\int_{-1}^{1} S_{on}(\eta) P_\mu(\eta) d\eta = \frac{2i^{\mu-n}}{2\mu+1} \cdot d^{on}_{\frac{\mu-n}{2}}$$

where $d^{mn}_k$ is a spheroidal coefficient. Thus the resulting expression for $\Gamma_{\mu\nu}$ is

$$\Gamma_{\mu\nu} = \frac{32i^{\mu+\nu+1} k\pi^2}{(2\mu+1)(2\nu+1)} \sum_n \frac{(-1)^n}{\Lambda_{on}} j_{e_{on}}(\xi) h_{e_{on}}(\xi) d^{on}_{\frac{\mu-n}{2}} d^{on}_{\frac{\nu-n}{2}} . \tag{11}$$

Equating the two expressions (10) and (11) for $\Gamma_{\mu\nu}$ gives at once the aforementioned expansion of the integral $\lambda_{\mu\nu}$ with the particular weight function $\phi(\theta) = \rho^3 \sin\theta$,

$$\lambda_{\mu\nu} \equiv \int_0^{\pi} P_\mu(\cos\theta)\, P_\nu(\cos\theta)\, h_\mu(k\rho)\, j_\nu(k\rho)\, \rho^3 \sin\theta \, d\theta$$

$$= \frac{-2\pi F^5 \xi(\xi^2-1)}{(2\mu+1)(2\nu+1)} \sum_n \frac{(-1)^n}{\Lambda_{on}} \; j_{e_{on}}(\xi)\, h_{e_{on}}(\xi)\, d^{on}_{\frac{\mu-n}{2}}\, d^{on}_{\frac{\nu-n}{2}} \; . \tag{12}$$

Existence of a solution to the linear system (7) which can be approximated by the solution of the corresponding truncated (finite) system remains to be demonstrated. This requires an examination of the series in equation (12) based on known properties of the spheroidal functions and coefficients. The constant terms of the linear system, given by the series on the right in equation (7) must also be evaluated. One characteristic of the system appears immediately on examination of (12). The parity about the point $\theta = \pi/2$ of the functions appearing in the integral is such that this vanishes if $\mu + \nu$ is odd, with the result that the odd and even indices in the linear system can be treated separately, and the labor involved in the solution is considerably reduced.

Once the solution of (7) is obtained, the field and its normal derivative on the sphere are known for any arbitrary distribution of the field on the hypothetical spheroid as specified by the coefficients $a_\mu$ in the expansion assumed. In particular, of course, the solution for the Dirichlet condition $V(P) = 0$ on the spheroid comes out at once. However the corresponding solution for the case where the normal derivative of the field on the spheroid is specified is not obtainable from the above expressions, and the derivation of analogous ones which will furnish it presents considerable difficulty. The principal problem involved is that the differentiation in a direction normal to the spheroid expressed in spherical coordinates produces integrands which are not easily amenable to the treatments described above. If a spheroidal coordinate system is employed, an expression analogous to (4) is easily derived, giving each coefficient of the field on the spheroid in terms of the corresponding one for the normal derivative. However the reconciliation of the two expansions, one in spherical coordinates and the other in spheroidal ones, is by no means trivial.

Various possible means of resolving these difficulties are currently being investigated. It is also projected to investigate the linear system (7) in more detail in the light of certain methods which appear in the literature [4] for the simplification of such systems. Finally, although it appears highly improbable that any diagonal or essentially diagonal system relating the required coefficients could be obtained, it does seem possible that various forms could be derived, and an investigation of these to determine which is optimum seems in order.

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