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EVALUATION OF THE PINCUSHION SYSTEM

by

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I. INTRODUCTION

The stated purpose of this report is an evaluation of the Angle Diversity System proposed jointly by Harry Davis and Dr. Joseph H. Vogelman (Ref. 1). This evaluation is necessarily made in the light of present theoretical and experimental knowledge. It is definitely felt that a better theoretical model and additional experimental data are necessary for a precise evaluation.

II. DISCUSSION

a. Diversity Systems

The general concept of an n^{th} order diversity system requires the combination in some fashion of n signals $S_j(t)$ which are locally coherent from the standpoint of the R.F. carrier but whose amplitudes are slowly varying functions of time which are statistically independent (Ref. 2). Thus, predetection diversity systems require circuitry for the control of RF (or IF) phase. From a practical standpoint the problem of RF (or IF) phase control can be avoided by the use of post detection combining.

The signal amplitudes encountered in practice may be correlated and have unequal means. Although the case of correlated signals has been studied by Staras (Ref. 3) and Packard (Ref. 4) and the case of unequal medians has been treated by the Bell Telephone Laboratories group (Ref. 5) and others (Ref. 6), the case of unequal medians and correlated signals has not received extensive analysis. Bolgiano, Bryant and Gordon (Ref. 7) give a treatment of correlated signals with unequal medians. Unfortunately this analysis results in an integral which must be integrated numerically. However, it is shown in this report that the method used by Packard (Ref. 4) to treat correlated signals with equal medians can be extended to the case of correlated signals with unequal medians. See Appendix.*

It is evident from these results that satisfactory diversity systems can be realized with correlated signals of unequal means provided the correlation coefficient is small enough and the ratio of the signal means is not too large. The operation of a diversity system then requires a suitable number of signal channels which provide signals which are at most partially correlated with as little difference in means as possible.

b. Angle Diversity

In an angle diversity system, offset feeds are used to obtain the necessary "independent" channels for combination. Such a system is attractive both from the standpoint of economy and space (land) since one antenna can be used to provide a number of channels. It is apparent that successful use of such a system must rely on a scattering mechanism which is volume extensive. In particular,

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*While this report was in preparation a report by the Hermes Electronic Co. (Ref. 12) came to our attention. Similar statistical methods to those given in the Appendix are used in the Hermes report. However, the results given in the Appendix are in a form which can be readily applied to the problem of combining signal channels when the mean power in each channel and the correlations between channels are known. In addition an explicit form is given for the case where the eigenvalues are not distinct.

such a system will be effective when the beamwidth of the antenna used in the system is small compared with the angle subtended by the effective scattering volume. In other words, angle diversity becomes effective when the antenna-to-medium coupling loss becomes sizable. This point is clearly recognized by the Rome group in their proposal of the pincushion system (Ref. 1).

An angle diversity system then uses a number of narrow antenna beams to illuminate and receive from a volume of space larger than the individual beamwidth. The gain of such a system depends on the variation in scattered power as a function of angle from the transmitter and/or receiver. Both the Booker-Gordon theory (Ref. 8) and the BTL layer theory (Ref. 9) may be used to predict the average scattering law. In either case, however, assumptions are made as to scale of turbulence and anisotropy or layer size and height dependence. Since there is neither strong experimental evidence nor adequate theories of turbulence to make other than an arbitrary choice, these atmospheric parameters are adjusted to produce average signal parameters which fit observed scatter propagation data.

For this reason, it is felt that a choice of theory for analysis of an angle diversity system, at this time, is mostly a matter of preference.

However, the recent publication by BTL of an extensive study of beyond-the-horizon propagation data (Ref. 5) and the fact that the BTL layer theory produces general expressions in closed form for the variation in received power as a function of antenna beamwidth and scattering angle makes this theory a more attractive choice in our opinion. For this reason, the calculations contained in this report are based on the BTL layer theory (Ref. 9).

III. PERFORMANCE OF SOME DIVERSITY SYSTEMS

a. Angle Diversity System

To determine the performance of an angle diversity system, it is necessary to determine the reduction in the mean signal level for the off-axis feeds and the correlation between adjacent beams.

If we consider the case of different frequencies for each beam, there is insufficient data to determine the correlations to be expected. For one beamwidth separation, it seems reasonable to expect the correlation to be less than 0.5. With identical frequencies, observations at Cornell (Ref. 7) have shown average correlations of 0.5 for a limited data sample taken with one beamwidth separation.

The use of different frequencies in the individual beams should provide additional decoupling resulting in a decrease in correlation between adjacent beams.

In order to determine the decrease in the mean received signal level for off-axis feeds, we consider the case of symmetrical beams with a 3 db beamwidth α separated by one beamwidth. It is assumed that the on-axis feed is aligned along the optimum great circle path. See Figure 1.

Let θ_0 be the angle between the chord, connecting the transmitter and receiver, and the tangent to the optimum great circle route. Further, we define a quantity K to be equal to the ratio of the beamwidth α to the minimum half scattering angle θ_0, i.e.

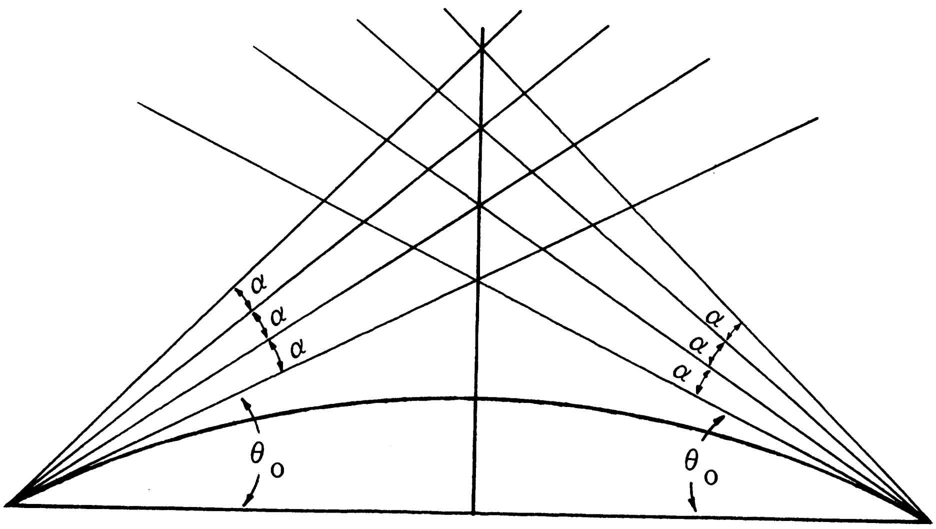


Figure 1

Scatter Path Geometry

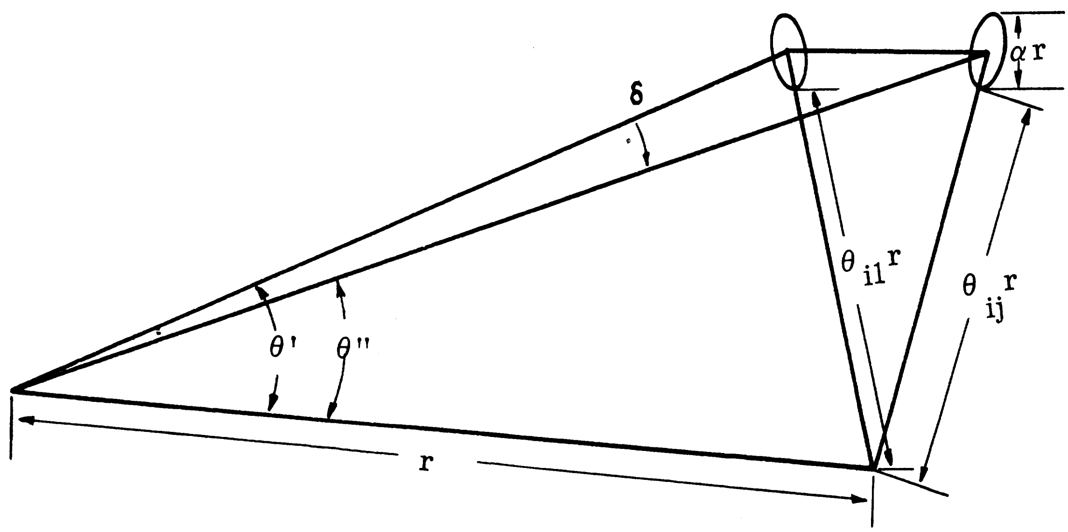


Figure 2

Geometry For θ_{ij}

$$K = \frac{\alpha}{\theta_o} .$$

We may now determine θ_{ij}, the angle between the chord and the bottom of the i, jth beam, where i is the elevation beam number and j is the azimuth beam number.

From Figure 2 for small angles θ we find

$$\theta''^2 = \theta'^2 + \delta^2 . \qquad (1)$$

For separation of one beamwidth and $\alpha = K\,\theta_o$

$$\theta' = \theta_o + (i - 1)\,K\theta_o + \frac{K\theta_o}{2}$$

$$\theta' = \theta_o \left[1 + (i - \frac{1}{2})\,K \right]$$

and

$$\delta = (j - 1)\,K\theta_o .$$

Since

$$\theta'' = \theta_{ij} + \frac{K\theta_o}{2} ,$$

(1) becomes

$$\left[\theta_{ij} + \frac{K\theta_o}{2} \right]^2 = \theta_o^2 \left\{ \left[1 + (i - \frac{1}{2})K \right]^2 + (j - 1)^2 K^2 \right\}$$

or

$$\theta_{ij} = \theta_o \left(\left[(1 + (i - \frac{1}{2})\,K)^2 + (j - 1)^2 K^2 \right]^{1/2} - \frac{K}{2} \right) . \qquad (2)$$

The ratio of the power in the i, jth beam P_{ij} to the power in the optimum great circle beam as predicted by the BTL theory is (Ref. 9)

$$\frac{P_{11}}{P_{ij}} = \left(\frac{\theta_{ij}}{\theta_o}\right)^5 \frac{2+\frac{\alpha}{\theta_{ij}}}{2+\frac{\alpha}{\theta_o}} \frac{f\left(\frac{\alpha}{\theta_o}\right)}{f\left(\frac{\alpha}{\theta_{ij}}\right)} \qquad (3)$$

where

$$f(x) = 1 + \frac{1}{(1+x)^4} - \frac{1}{8}\left(\frac{2+x}{1+x}\right)^4 .$$

Using (2) and $\frac{\alpha}{\theta_o} = K$ we obtain

$$\frac{P_{11}}{P_{ij}} = (D_{ij\,K})^5 \frac{2+\frac{K}{D_{ij\,K}}}{2+K} \frac{f(K)}{f\left(\frac{K}{D_{ij\,K}}\right)} \qquad (4)$$

where

$$D_{ij\,K} = \left(\left[(1+(i-\tfrac{1}{2})\,K)^2 + (j-1)^2 K^2\right]^{1/2} - \frac{K}{2}\right) .$$

We have in equation (4) an expression for the ratio of the mean received power in any beam with respect to the mean received power in the optimum great circle beam which is a function of the beam number (i, j) and K only.

Figures 3, 4, and 5 show the ratios of mean received power in the various beams to the mean received power in the optimum beam for $K = 1, \frac{1}{2}$ and $\frac{1}{4}$ respectively.

From a comparison of the power ratios for the indicated values of K, the importance of the ratio of the antenna beamwidth to θ_o can be seen. The number of available channels increases rapidly as K decreases. It would appear that higher order angle diversity systems become feasible for values of K equal to or less than $\frac{1}{2}$.

i \ j	4	3	2	1	2	3	4
1	-26.0 db	-16.5 db	- 6.4 db	0 db	- 6.4 db	-16.5 db	-26.0 db
2	-30.0 db	-24.1 db	-18.8 db	-16.5 db	-18.8 db	-24.1 db	-30.0 db
3	-35.3 db	-31.2 db	-28.2 db	-26.8 db	-28.2 db	-31.2 db	-35.3 db

Figure 3

Power Ratios for K = 1

i \ j	4	3	2	1	2	3	4
1	-13.7 db	- 7.7 db	- 2.3 db	0 db	- 2.3 db	- 7.7 db	-13.7 db
2	-18.8 db	-14.7 db	-11.6 db	-10.4 db	-11.6 db	-14.7 db	-18.8 db
3	-23.6 db	-20.8 db	-18.8 db	-18.1 db	-18.8 db	-20.8 db	-23.6 db

Figure 4

Power Ratios for $K = \frac{1}{2}$

i \ j	4	3	2	1	2	3	4
1	- 5.6 db	- 2.6 db	- .7 db	0 db	- .7 db	- 2.6 db	- 5.6 db
2	-10.0 db	- 8.0 db	- 6.6 db	- 6.1 db	- 6.6 db	- 8.0 db	-10.0 db
3	-14.1 db	-12.5 db	-11.5 db	-11.2 db	-11.5 db	-12.5 db	-14.1 db

Figure 5

Power Ratios for $K = \frac{1}{4}$

If the correlations between the various beams were known, the method outlined in the Appendix could now be used to compute the reliability for different order diversity systems for specific system thresholds. Since the assumption of system threshold and correlation would be arbitrary, we simplify the calculation by assuming zero correlation and a system threshold 10 db below the mean power received in the optimum beam in order to compute reliabilities.

With these simplifying assumptions, the calculated reliabilities are as indicated in Figure 6.

K	1	.5	.25
n			
3	99.43 %	99.9 %	99.94 %
4	99.47 %	99.96 %	99.993 %
5	99.5 %	99.98 %	99.999 %
6	99.52 %	99.99 %	99.999+ %

Figure 6

Angle Diversity

Per cent Reliability for P_{mean} 10 db above System Threshold

The reliabilities for n = 4, K = 1/4 and n = 5, K = 1/2 are not optimum since a more efficient system could be achieved by straddling the optimum great circle path for these values of n and K.

b. Equivalent Space Diversity

In order to have some basis for comparison of angle diversity with space diversity, we make the assumption that the cost of an antenna is proportional to its effective area. With this assumption, the beamwidth of an equivalent n^{th} order space diversity system is found to be $\sqrt{n}\, K\theta_0$. The power ratios become

$$\frac{P_1}{P_n} = (\sqrt{n})^3 \frac{2+\sqrt{n}\,K}{2+K} \frac{f(K)}{f(\sqrt{n}\,K)} \tag{5}$$

where K and f(x) are as previously defined. The ratio of $\frac{P_1}{P_n}$ for various values of n and K are shown in Figure 7.

K	1.0	0.5	0.25
n 1	0 db	0 db	0 db
2	-4.2 db	-3.4 db	-2.7 db
3	-6.8 db	-5.6 db	-4.4 db
4	-8.7 db	-7.2 db	-5.7 db
5	-10.2 db	-8.5 db	-6.7 db
6	-11.5 db	-9.5 db	-7.6 db

Figure 7

Power Ratios

Equivalent Space Diversity

The calculated reliabilities of equivalent space diversity for various values of n and K are given in Figure 8.

n \ K	1	.5	.25
n 2	96.8 %	97.6 %	98.3 %
3	97.3 %	98.6 %	99.3 %
4	96.6 %	98.7 %	99.67 %
5	95.0 %	98.7 %	99.76 %
6	91.0 %	98.44 %	99.78 %

Figure 8

Per cent Reliabilities for P_{mean} 10 db above System Threshold

Equivalent Space Diversity

Diversity Order	Diversity Type	K		
		1.0	.5	.25
2	Equiv. Space Space	3 db 11 db	5 db 11 db	6 db 11 db
3	Angle Equiv. Space Space	11 db 4 db 23 db	19 db 7 db 23 db	21 db 10 db 23 db
4	Angle Equiv. Space Space	12 db 3 db 28 db +	23 db 8 db 28 db+	28 db+ 12 db 28 db+
5	Angle Equiv. Space Space	12 db 2 db 28 db+	26 db 8 db 28 db+	28 db+ 14 db 28 db+
6	Angle Equiv. Space Space	12 db -1 db 28 db+	28 db+ 6 db 28 db+	28 db+ 15 db 28 db+

Figure 9

Diversity Gain above a Single Channel

Note: System Reliabilities above 99.99 % are Indicated as 28 db+

c. Comparison of Angle and Space Diversity

The reliabilities calculated in the preceeding sections are based on independent signals and selection diversity. In practice, either equal gain or maximal ratio diversity are probably preferable. To some extent the two assumptions will offset one another. In any case, the increase or decrease in reliabilities will be essentially the same for the different types of systems used in this comparison.

In Figure 9, the diversity gains above a single channel are listed for the two systems previously examined. In addition, a third system is added which is an n^{th} order space diversity system utilizing n antennas with beamwidths, and therefore areas, equal to the single antenna used in the comparable n^{th} order angle diversity system.

The data in Figure 9 indicates an angle diversity system will out-perform an "equivalent" space diversity system by from 7 to 15 db depending on the value of K and the order of diversity.

In addition, it should be noted that a 3rd order angle diversity system will give equivalent performance to that of a space diversity system utilizing two antennas of the same size for $K = 1$.

d. Single-Frequency Angle Diversity

In the angle diversity calculations which have been made up to this point it has been assumed that the individual beams are fed from transmitters with different frequencies. In the course of performing such calculations one is struck with the possibility of obtaining a more efficient use of the scattering volume by utilizing the same frequency for each transmitter.

The principal advantage gained is in the vertical plane where the use of a single frequency results in a substantial increase in the scattering volume common to a given receiver beam.

The use of a single frequency results in common volumes which are unsymmetrical. The beam power ratios can be calculated however using the method indicated in the BTL article of September 1959 (Ref. 5). The ratio of the power in the first beam to the power in the j^{th} beam is

$$\frac{P_{11}}{P_{ij}} = \frac{\left[f\left(\frac{\alpha_T}{\theta_o}\right) + f\left(\frac{\alpha_R}{\theta_o}\right) - \left(\frac{\theta_o}{\theta_o+\alpha_R}\right)^4 f\left(\frac{\alpha_T-\alpha_R}{\theta_o+\alpha_R}\right) \right]}{\left[\left(\frac{\theta_o}{\theta_R}\right)^4 f\left(\frac{\alpha_T+\theta_o-\theta_R}{\theta_R}\right) + f\left(\frac{\alpha_R+\theta_R-\theta_o}{\theta_o}\right) - f\left(\frac{\theta_R-\theta_o}{\theta_o}\right) - \left(\frac{\theta_o}{\theta_R+\alpha_R}\right)^4 f\left(\frac{\alpha_T+\theta-\alpha_R-\theta_R}{\theta_R+\alpha_R}\right) \right]} \tag{6}$$

where

$$\alpha_T = nk\,\theta_o$$

$$\alpha_R = k\,\theta_o$$

$$\theta_R = \theta_o\left[1+(j-1)k\right]$$

$$f(x) = 1 + \frac{1}{(1+x)^4} - \left(\frac{1}{8}\right)\left(\frac{2+x}{1+x}\right)^4 .$$

In addition it is found that the ratio of the power in the on-axis beam for a single frequency operation to the power in the on-axis beam for multiple frequency operation is

$$\frac{P_{11}\ \text{(single frequency)}}{P_{11}\ \text{(multiple frequency)}} = \frac{1}{2}\ \frac{\left[f\left(\frac{\alpha_T}{\theta_o}\right) + f\left(\frac{\alpha_R}{\theta_o}\right) - \left(\frac{\theta_o}{\theta_o+\alpha_R}\right)^4 f\left(\frac{\alpha_T-\alpha_R}{\theta_o+\alpha_R}\right) \right]}{f\left(\frac{\alpha_R}{\theta_o}\right)} \qquad (7)$$

Using equation (7) the following power ratios are calculated for 2, 4 and 6 vertically stacked beams for the indicated values of K.

No. of Beams \ K	1	.5	.25
2	+ .8 db	+ 1.5 db	+ 2.0 db
4	+ 1.2 db	+ 2.2 db	+ 3.3 db
6	+ 1.4 db	+ 2.4 db	+ 3.8 db

Figure 10

$$\frac{\text{Mean Power in On-Axis Beam (Single Frequency)}}{\text{Mean Power in On-Axis Beam (Multiple Frequency)}}$$

In addition, from equation (6) the decrease in mean power for the upper beams is found to be smaller. See Figure 11.

Using the data given in Figures 3,4 and 5 for multiple-frequency operation together with the data in Figures 10 and 11 for single-frequency operation the increase in mean power for single-frequency operation can be determined. This data is shown in Figure 12.

It should be noted at this point that the power ratios given in Figure 11 are a function of the number of vertical beams used. The calculations show, however, that the change in the power ratios between the vertical beams is small

Beam Number \ K	1	.5	.25
1	0	0	0
2	- 7.2 db	- 4.6 db	-2.7 db
3	-11.3 db	- 7.8 db	-4.9 db
4	-14.2 db	-10.2 db	-6.6 db
5	-16.3 db	-12.2 db	-8.3 db
6	-18.4 db	-13.8 db	-9.4 db

Figure 11

Power Ratios for Vertical Beam Separation
(Single Frequency)

Beam Number \ K	1	.5	.25
1	+ 1.2 db	+ 2.2 db	+ 3.3 db
2	+10.5 db	+ 8.0 db	+ 6.7 db
3	+16.7 db	+12.5 db	+ 9.6 db
4	+21.0 db	+16.0 db	+12.0 db

Figure 12

Increase in Mean Power for Single-Frequency Operation as the number of beams is increased from 2 to 6, the maximum change being less than 0.2 db, which indicates as expected that a more efficient system could be realized if the transmitter power used in the vertical beams above the tangential beam were used instead in the tangential beam.

From the increase in mean power for the various beams which results from single-frequency operation it would appear that a single-frequency angle diversity system may more closely approach an optimum scatter communication system.

Dr. J.H. Vogelman has pointed out in a private communication that the problem associated with single-frequency feeding may be avoided while obtaining the full use of the scattering volume in a multiple-frequency system if the receivers associated with each feed have bandwidths which are wide enough to cover the spread in transmitter frequencies. With this type of system suitable filters would be used in the individual beam receivers to separate the different transmitter frequencies for combination in diversity combiners. Thus a system with n feeds would be a diversity system of order n^2.

IV. THE EFFECT OF CORRELATED SIGNALS

For the case of two correlated Rayleigh signals with unequal means it is shown in the Appendix that maximal-ratio combination will give a signal whose statistics are described by

$$P(r \leqslant R) = 1 - \frac{\lambda_1 e^{-\frac{R}{2\lambda_1}}}{2\lambda_1 - (\sigma_1^2 + \sigma_2^2)} - \frac{\lambda_2 e^{-\frac{R}{2\lambda_2}}}{2\lambda_2 - (\sigma_1^2 + \sigma_2^2)} \qquad (8)$$

where

$$\lambda_1 = \frac{\sigma_1^2 + \sigma_2^2 + \sqrt{(\sigma_1^2 + \sigma_2^2) - 4\sigma_1^2 \sigma_2^2 (1 - \rho_{12}^2)}}{2}$$

$$\lambda_2 = \frac{\sigma_1^2 + \sigma_2^2 - \sqrt{(\sigma_1^2 + \sigma_2^2) - 4\sigma_1^2 \sigma_2^2 (1 - \rho_{12}^2)}}{2} \quad .$$

Using equation (8) and the relation between c and ρ (see equation (7) in the Appendix), the probability of the combined signal exceeding a given level may be calculated for various ratios of σ_1 to σ_2 and correlation coefficients c.

By choosing the median of the higher channel as a reference level and assuming the local noise powers are constant and equal to 1, we obtain the graphs shown in Figures 13 through 19. The odd values of c result from assuming convenient values for ρ.

A comparison of the curves for the various values of c indicates useful diversity gain for signal correlations as high as c = .78. It can also be noted that the effect of correlation on the combined signal is essentially the same for the unequal means as it is for the equal means.

For example, in the region of interest (i.e. high reliability) the local signal-to-noise ratio for a correlation coefficient of c = .59 is approximately 2 db below that for uncorrelated signals for each of the three ratios of signal means (0 db, 6 db, 12 db).

To show the effect of correlation on the combination of more than two channels we consider the case of three channels with equal means and equal

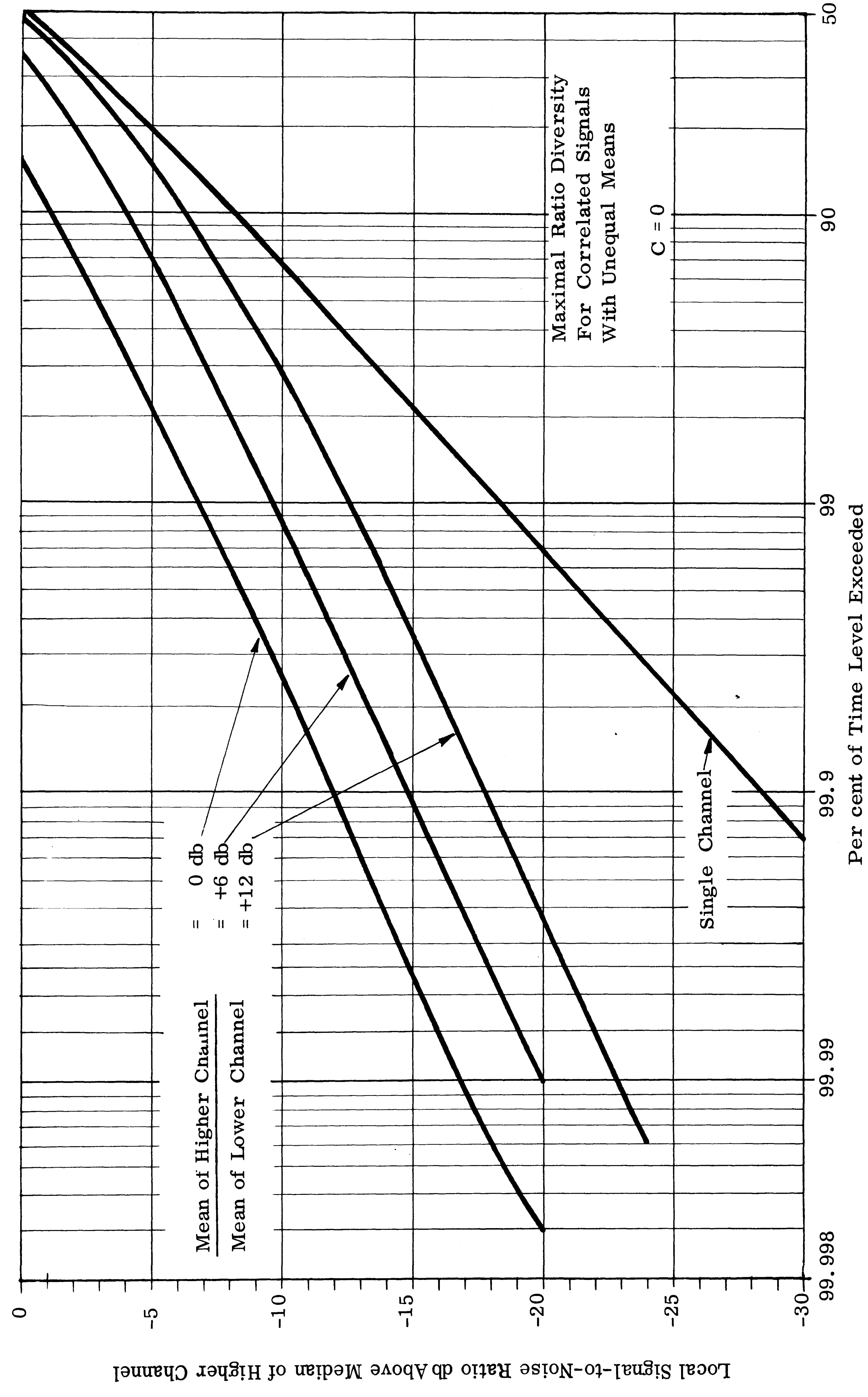

Maximal Ratio Diversity
For Correlated Signals
With Unequal Means
C = 0
Mean of Higher Channel / Mean of Lower Channel
= 0 db
= +6 db
= +12 db
Single Channel
50
90
99
99.9
99.99
99.998
0
-5
-10
-15
-20
-25
-30
Per cent of Time Level Exceeded
Local Signal-to-Noise Ratio db Above Median of Higher Channel

Figure 13

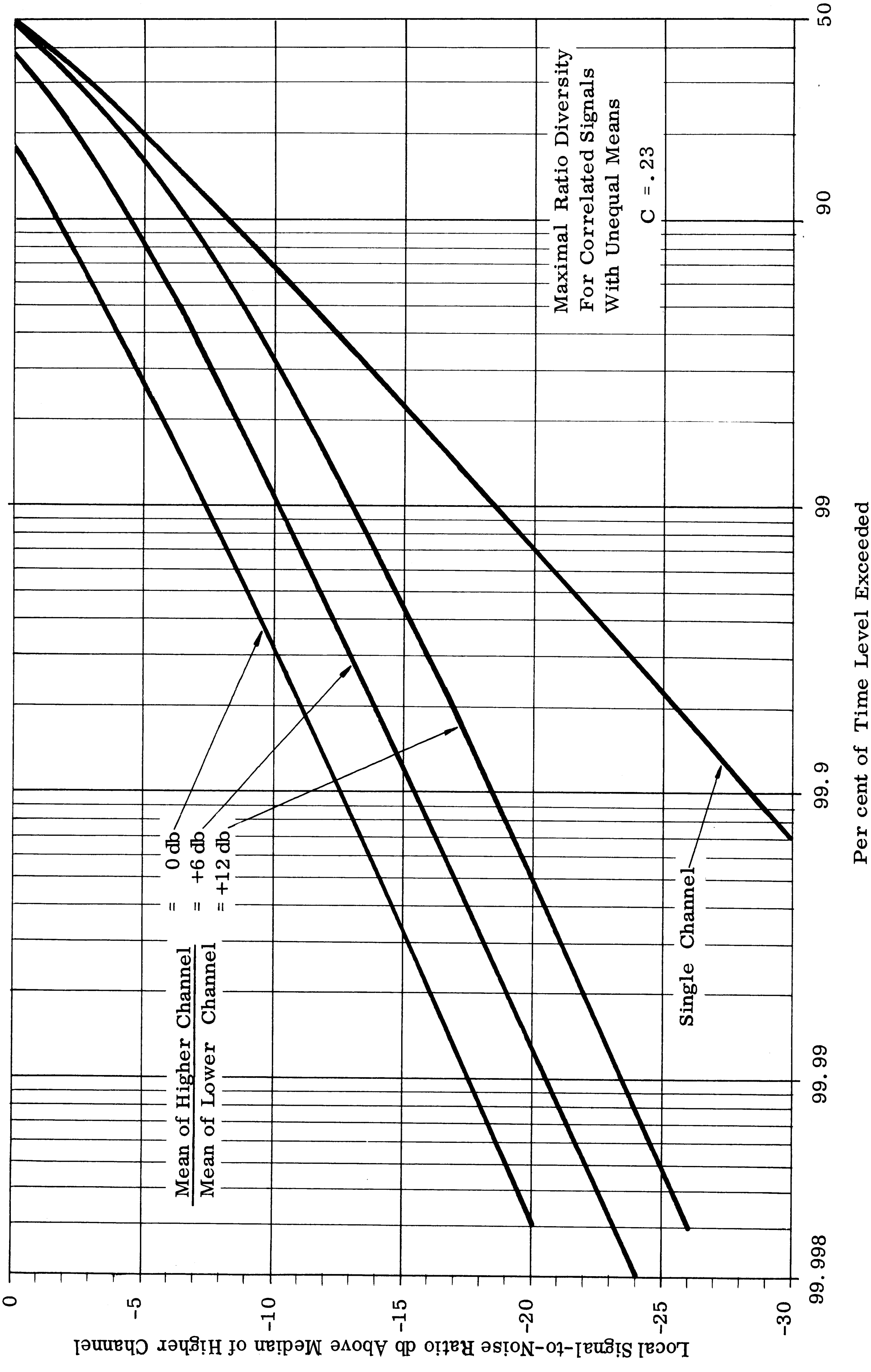

Maximal Ratio Diversity
For Correlated Signals
With Unequal Means
C = .23
Mean of Higher Channel / Mean of Lower Channel
= 0 db
= +6 db
= +12 db
Single Channel
Per cent of Time Level Exceeded
50
90
99
99.9
99.99
99.998
Local Signal-to-Noise Ratio db Above Median of Higher Channel
0
-5
-10
-15
-20
-25
-30

Figure 14

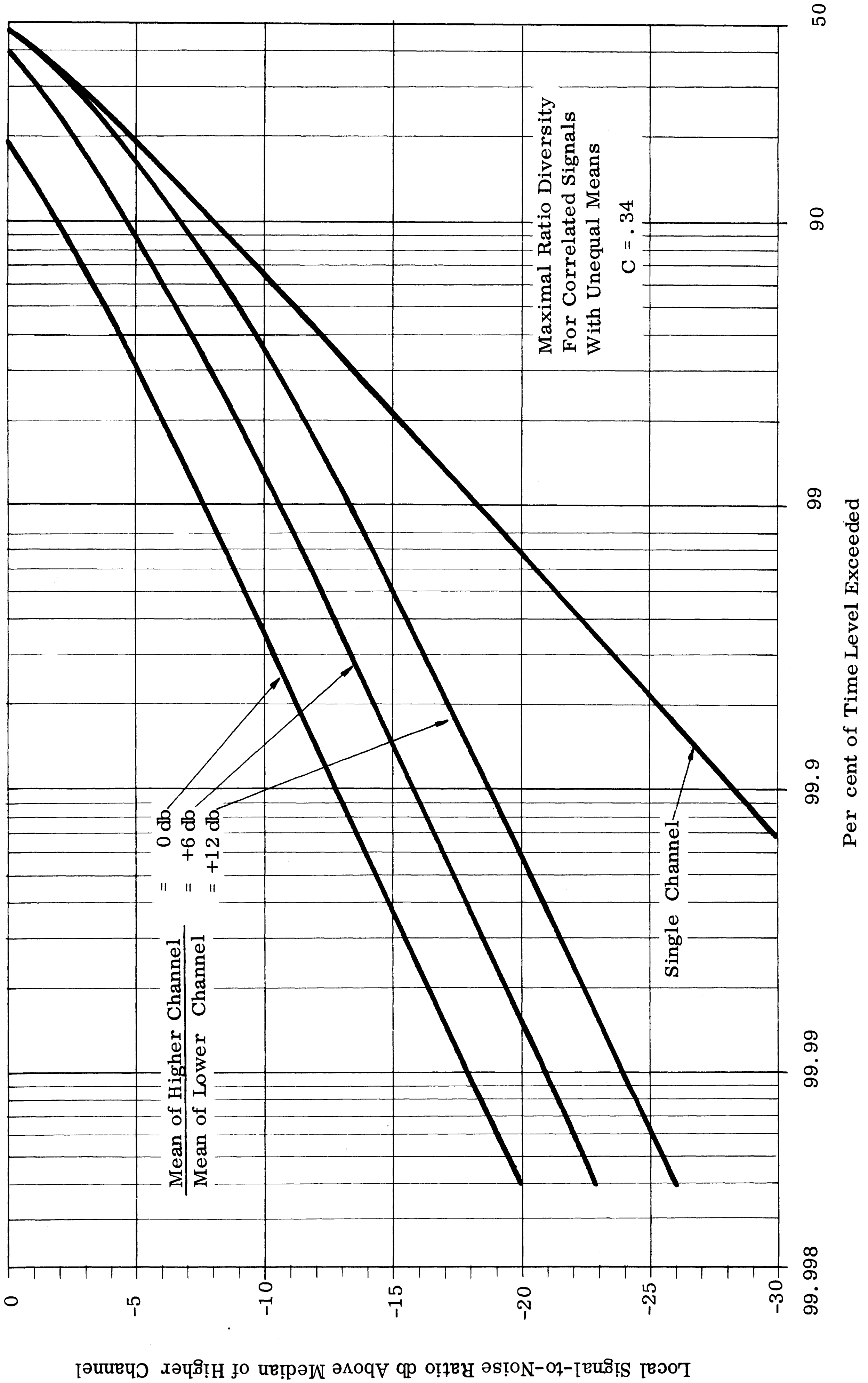

Maximal Ratio Diversity
For Correlated Signals
With Unequal Means
C = .34
Mean of Higher Channel / Mean of Lower Channel
= 0 db
= +6 db
= +12 db
Single Channel
50
90
99
99.9
99.99
99.998
Per cent of Time Level Exceeded
0
-5
-10
-15
-20
-25
-30
Local Signal-to-Noise Ratio db Above Median of Higher Channel

Figure 15

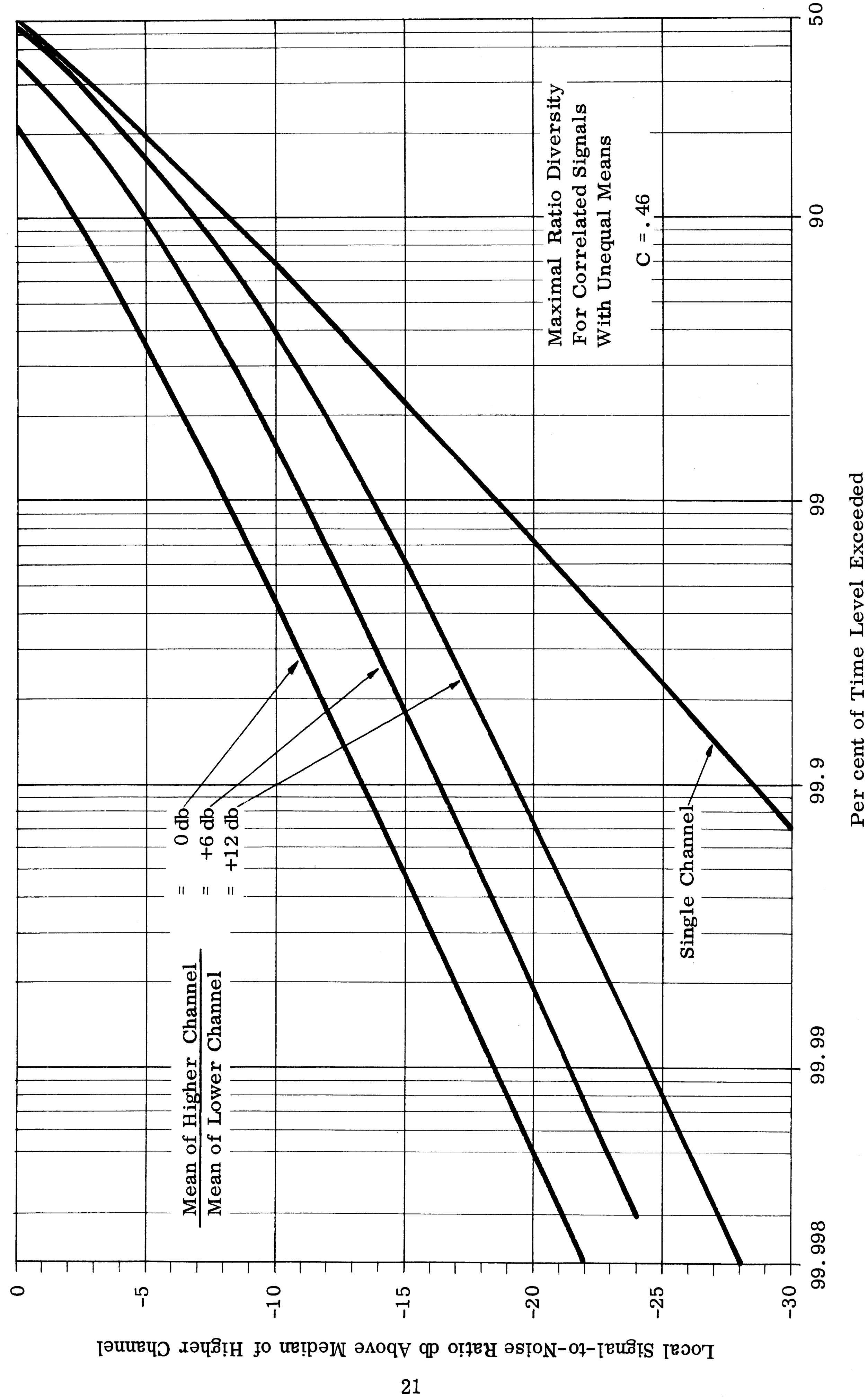
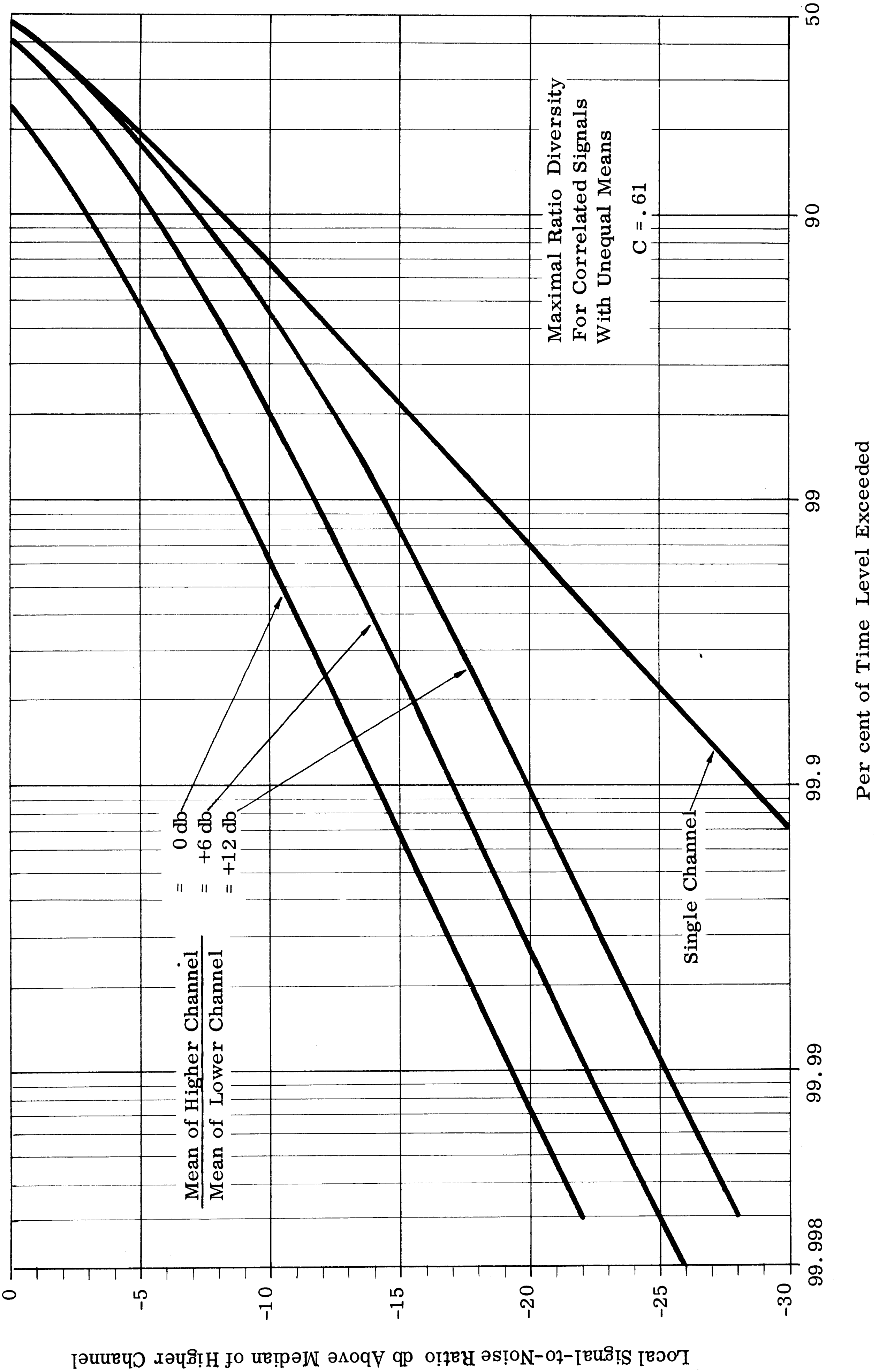

Maximal Ratio Diversity
For Correlated Signals
With Unequal Means
C = .46
Mean of Higher Channel / Mean of Lower Channel
= 0 db
= +6 db
= +12 db
Single Channel
50
90
99
99.9
99.99
99.998
Per cent of Time Level Exceeded
0
-5
-10
-15
-20
-25
-30
Local Signal-to-Noise Ratio db Above Median of Higher Channel

Figure 16


Maximal Ratio Diversity
For Correlated Signals
With Unequal Means
C = .61
Mean of Higher Channel / Mean of Lower Channel
= 0 db
= +6 db
= +12 db
Single Channel
Per cent of Time Level Exceeded
50
90
99
99.9
99.99
99.998
0
-5
-10
-15
-20
-25
-30
Local Signal-to-Noise Ratio db Above Median of Higher Channel

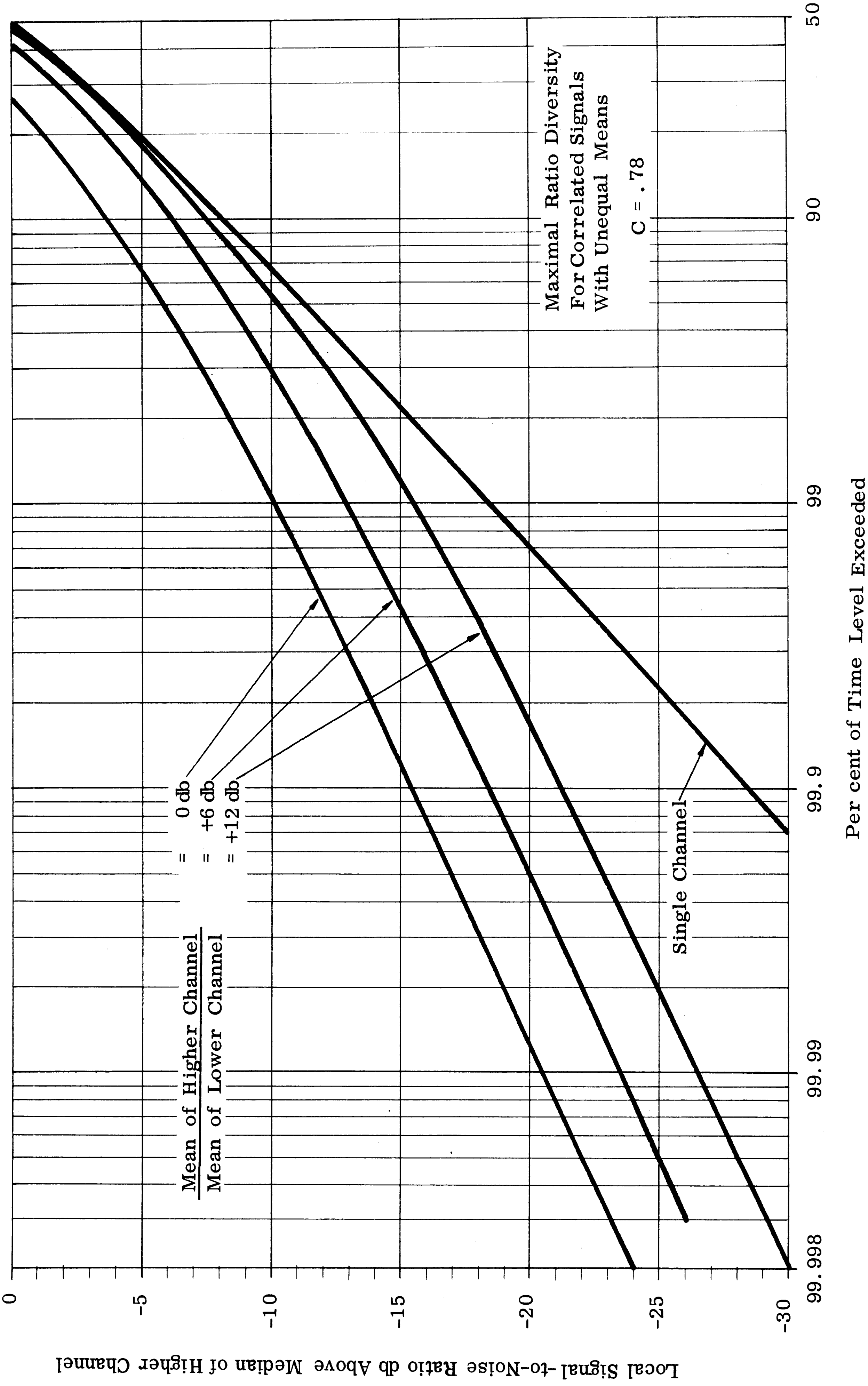
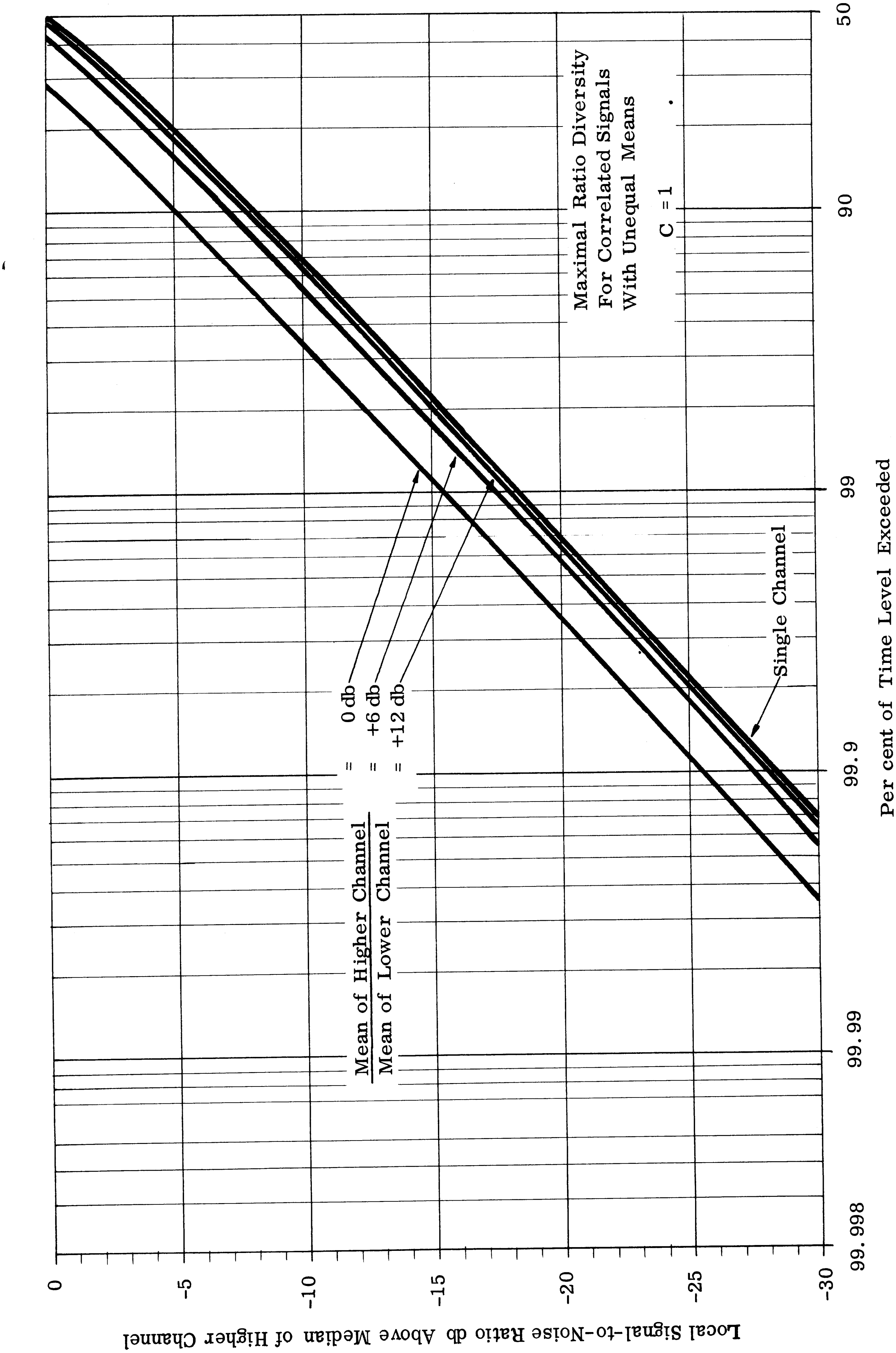

Maximal Ratio Diversity
For Correlated Signals
With Unequal Means
C = .78
Mean of Higher Channel / Mean of Lower Channel
= 0 db
= +6 db
= +12 db
Single Channel
50
90
99
99.9
99.99
99.998
0
-5
-10
-15
-20
-25
-30
Per cent of Time Level Exceeded
Local Signal-to-Noise Ratio db Above Median of Higher Channel

Figure 18


Maximal Ratio Diversity
For Correlated Signals
With Unequal Means
C = 1
Mean of Higher Channel / Mean of Lower Channel
= 0 db
= +6 db
= +12 db
Single Channel
50
90
99
99.9
99.99
99.998
Per cent of Time Level Exceeded
0
-5
-10
-15
-20
-25
-30
Local Signal-to-Noise Ratio db Above Median of Higher Channel

correlation coefficients. The correlation matrix is

$$\begin{bmatrix} \sigma^2 & \rho\,\sigma^2 & \rho\,\sigma^2 \\ \rho\,\sigma^2 & \sigma^2 & \rho\,\sigma^2 \\ \rho\,\sigma^2 & \rho\,\sigma^2 & \sigma^2 \end{bmatrix} .$$

The roots of the characteristic equation can easily be shown to be

$$\lambda_1 = \sigma^2 (1 + 2\rho)$$

$$\lambda_2 = \lambda_3 = \sigma^2 (1 - \rho) \quad .$$

Using the results from the Appendix for the case of non-distinct roots we find

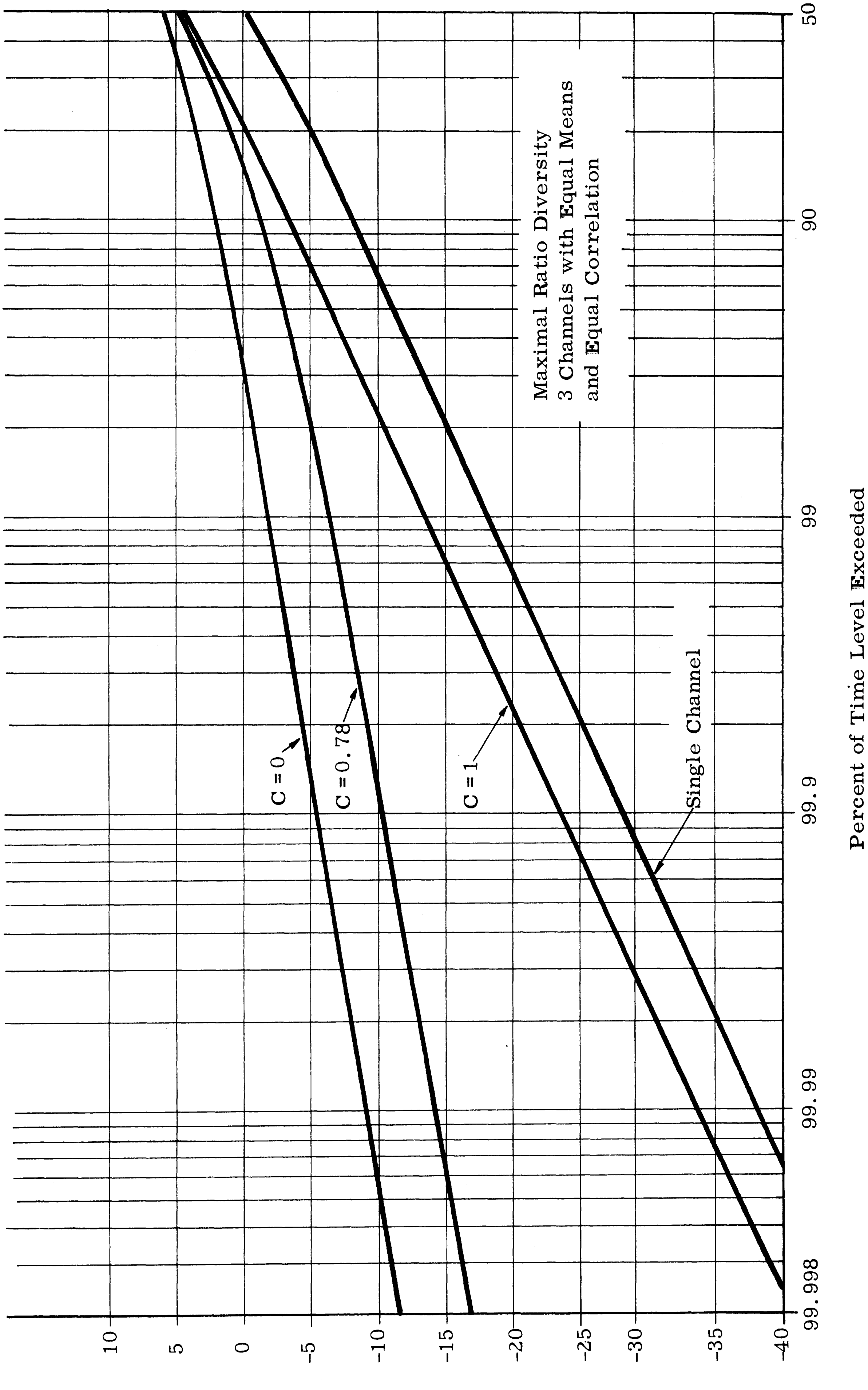
$$P(r \leqslant R) = 1 - \frac{(1 + 2\rho)^2}{9\rho^2} e^{-\frac{R}{2\sigma^2 (1+2\rho)}} + e^{-\frac{R}{2\sigma^2(1-\rho)}} \left[\frac{R}{6\sigma^2\rho} + \frac{(1+5\rho)(1-\rho)}{9\rho^2} \right] .$$

This result is plotted in Figure 20 for c = 0, .78 and 1. An examination of the curves indicates substantial diversity gain at the higher reliability levels for correlation coefficients as high as c = .78.

V. SUMMARY AND CONCLUSIONS

An analysis of the performance of the pincushion system using the BTL layer theory indicates reasonable diversity gains over an "equivalent" space diversity system. Although the Booker-Gordon theory would produce different numbers the conclusions can be expected to be the same. Both theories predict average performance and from this standpoint suffer from the same limitation.

The analysis shows that the pincushion system as originally proposed does not efficiently utilize the effective scattering volume. It can be concluded


Maximal Ratio Diversity
3 Channels with Equal Means
and Equal Correlation
C = 0
C = 0.78
C = 1
Single Channel
Percent of Time Level Exceeded
Local Signal-to-Noise Ratio db Above Median of One Channel
50
90
99
99.9
99.99
99.998
10
5
0
-5
-10
-15
-20
-25
-30
-35
-40

that a more efficient system can be obtained either by use of a single frequency or by utilizing the method proposed by Dr. Vogelman.

In the analysis of the effects of correlated signals a general method is derived for computing the combined reliability of n correlated signals with unequal means. Curves have been plotted for the special cases of two correlated channels with unequal means and three correlated channels with equal means. These curves indicate useful diversity gains at the higher reliability levels for signals with correlation as high as $c = .78$.

It is felt that the pincushion system when used with a single frequency or with wide-band receivers as proposed by Dr. Vogelman will more closely approach an optimum scatter communication system.

Appendix

The effect of correlation on maximal-ratio combiner diversity for signals of unequal mean may be computed in the following manner.

Consider n Rayleigh distributed signals

$$p_1(z_i) = \frac{z_i}{\sigma_i^2} e^{-\frac{z_i^2}{2\sigma_i^2}} \tag{1}$$

with correlation c_{ij} between z_i and z_j

$$c_{ij} = \frac{\langle (z_i - \langle z_i \rangle)(z_j - \langle z_j \rangle) \rangle}{\sqrt{\langle (z_i - \langle z_i \rangle)^2 \rangle \langle (z_j - \langle z_j \rangle)^2 \rangle}} . \tag{2}$$

Each of variables z_i may be resolved into two uncorrelated variables x_i and y_i which are normally distributed with zero means and variance σ_i^2,

$$z_i^2 = x_i^2 + y_i^2 . \tag{3}$$

The correlation between the x's and y's is such that

$$\langle x_i x_j \rangle = \rho_{ij} \sigma_i \sigma_j \qquad i \neq j \tag{4}$$

$$\langle y_i y_j \rangle = \rho_{ij} \sigma_i \sigma_j \qquad i \neq j \tag{5}$$

$$\langle x_i y_j \rangle = 0 \quad . \tag{6}$$

The correlations between the z's and the x's and y's are related by the equation (Ref. 7)

$$c_{ij} = \frac{\pi}{4-\pi} \sum_{n=1}^{\infty} \left[\frac{(2n!)\,\rho_{ij}^{n}}{2^{2n}(n!)^2(2n-1)} \right]^2 \cong .915\,\rho_{ij}^{2} + .0572\,\rho_{ij}^{4}. \tag{7}$$

Consider the linear change of variables

$$u] = Ax] \tag{8}$$

$$v] = Ay] \tag{9}$$

where the symbol] indicates a column matrix (Ref. 10).

In addition we require the following conditions

$$\langle u_i\, u_j \rangle = 0 \qquad i \neq j \tag{10}$$

$$\langle v_i\, v_j \rangle = 0 \qquad i \neq j \tag{11}$$

$$\sum u_i^2 = \sum x_i^2, \qquad \sum v_i^2 = \sum y_i^2 \tag{12}$$

$$\langle u_i^2 \rangle = \lambda_i \tag{13}$$

$$\langle v_i^2 \rangle = \lambda_i\ . \tag{14}$$

Since the u_i and v_i are related to x_i and y_i by the same transformation matrix A, it is only necessary to discuss one of the transformations.

From condition (12) we obtain

$$\lfloor u \rfloor\ u] = \lfloor x \rfloor\ A^T A\, x] = \lfloor x \rfloor\ x] \tag{15}$$

thus $A^T A = U$ and A is an orthogonal matrix.

Forming the product $u]\ \lfloor u \rfloor$ and taking the average we obtain

$$\begin{bmatrix} \langle u_1^2\rangle & \langle u_1u_2\rangle & \langle u_1u_3\rangle & \cdots & \langle u_1u_n\rangle \\ \langle u_1u_2\rangle & \langle u_2^2\rangle & \langle u_1u_3\rangle & \cdots & \langle u_2u_n\rangle \\ \vdots & & & & \\ \langle u_1u_n\rangle & \langle u_2u_n\rangle & \langle u_3u_n\rangle & \cdots & \langle u_n^2\rangle \end{bmatrix} = A \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \cdots & \rho_{1n}\sigma_1\sigma_n \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 & \cdots & \rho_{2n}\sigma_2\sigma_n \\ \vdots & & & \\ \rho_{1n}\sigma_1\sigma_n & \cdots & \cdots & \sigma_n^2 \end{bmatrix} A \tag{16}$$

Using equations (10), (11), (13) and (14) this reduces to

$$\begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} = A \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \cdots & \rho_{1n}\sigma_1\sigma_n \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 & \cdots & \rho_{2n}\sigma_2\sigma_n \\ \vdots & & & \\ \rho_{1n}\sigma_1\sigma_n & \cdots & \cdots & \sigma_n^2 \end{bmatrix} A^T . \tag{17}$$

Since the u_i's and v_i's are linear combinations of normally distributed variables, they are also normally distributed and each u_i and v_i is completely determined by its variance λ_i. Furthermore, by virtue of the condition imposed by equations (10) and (11), the new variables are uncorrelated.

It can now be seen that it is not necessary to determine the matrix A explicitly for the λ's may be determined directly from the solution of the characteristic equation (Ref. 11),

$$\begin{vmatrix} \sigma_1^2 - \lambda & \rho_{12}\sigma_1\sigma_2 & \cdots & \rho_{1n}\sigma_1\sigma_n \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 - \lambda & \cdots & \rho_{2n}\sigma_2\sigma_n \\ & & & \\ \rho_{1n}\sigma_1\sigma_n & \rho_{2n}\sigma_2\sigma_n & \cdots & \sigma_n^2 - \lambda \end{vmatrix} = 0 = f(\lambda)\,(-1)^n . \tag{18}$$

where $f(\lambda)$ is a polynomial of degree n in λ expressible as

$$f(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ldots . a_n \quad . \tag{19}$$

It is now possible to determine the distribution which results from the maximal-ratio combination of the n Rayleigh variables z_i. The variable of the combined signals is

$$z^2 = \sum_{i=1}^{n} z_i^2 = \sum_{i=1}^{n} (x_i^2 + y_i^2) \quad . \tag{20}$$

From equation (12) we have

$$z^2 = \sum_{i=1}^{n} (x_i^2 + y_i^2) = \sum_{i=1}^{n} (u_i^2 + v_i^2). \tag{21}$$

Let $r_i = u_i^2 + v_i^2$ and $\sum_i r_i = r$. Because u_i and v_i are normally distributed variables with zero means and variances λ_i, the probability density function of r_i is

$$p_{r_i}(r_i) = \frac{1}{2\lambda_i} e^{-\frac{r_i}{2\lambda_i}} \qquad 0 \leqslant r_i < \infty \quad . \tag{22}$$

The characteristic function of (22) is

$$y_i(\omega) = \int_0^{\infty} e^{j\omega r_i} p_{r_i}(r_i)\, d_{r_i} = \frac{1}{2\lambda_i} \left(\frac{1}{\frac{1}{2\lambda_i} - j\omega} \right) \quad . \tag{23}$$

Since the r_i's are uncorrelated the probability density function of r (or z^2) is

$$p(r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-j\omega r} \left\{ \prod_{i=1}^{n} y_i(\omega) \right\} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-j\omega r}}{2^n} \left\{ \prod_{i=1}^{n} \frac{1}{\lambda_i \left(\frac{1}{2\lambda_i} - j\omega\right)} \right\} d\omega \quad . \tag{24}$$

The integral in equation (24) can be evaluated easily since the integrand is analytic everywhere except at the poles $\omega = \frac{1}{2j\lambda_i}$.

If the λ's are all distinct p(r) becomes

$$p(r) = \frac{1}{2} \sum_{i=1}^{n} \lambda_i^{n-2} e^{-\frac{r}{2\lambda_i}} \left\{ \prod_{\substack{k=1 \\ k \neq i}}^{n} \frac{1}{\lambda_i - \lambda_k} \right\} . \qquad (25)$$

Equation (25) can be expressed in terms of the characteristic equation (18) as

$$p(r) = \frac{1}{2} \sum_{i=1}^{n} \frac{\lambda_i^{n-2} e^{-\frac{r}{2\lambda_i}}}{\left. \frac{d f(\lambda)}{d\lambda} \right|_{\lambda = \lambda_i}} .$$

The probability that $r \leqslant R$ is by definition

$$P(r \leqslant R) = \int_0^R p(r)\, dr = \sum_{i=1}^{n} \frac{\lambda_i^{n-1}}{\left. \frac{d f(\lambda)}{d\lambda} \right|_{\lambda = \lambda_i}} - \sum_{i=1}^{n} \frac{\lambda^{n-1} e^{-\frac{R}{2\lambda_i}}}{\left. \frac{d f(\lambda)}{d\lambda} \right|_{\lambda = \lambda_i}} . \qquad (26)$$

The first term in equation (26) can be evaluated as follows.

From equation (18) $f(\lambda)$ is a polynomial of degree n

$$f(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \ldots + a_n \qquad (27)$$

choose

$$\phi(\lambda) = a_1 \lambda^{n-1} + \ldots + a_n . \qquad (28)$$

The rational function $\frac{\phi(\lambda)}{f(\lambda)}$ can be expanded in partial fractions as

$$\frac{\phi(\lambda)}{f(\lambda)} = \sum_{i=1}^{n} \frac{\phi(\lambda_i)}{\left.\frac{d f(\lambda_i)}{d\lambda}\right|_{\lambda=\lambda_i}} \; \frac{1}{\lambda - \lambda_i} \quad . \tag{29}$$

From (27) and (28), $\frac{\phi(0)}{f(0)} = 1$ and $\phi(\lambda_i) = -\lambda_i^n$. Therefore (29) becomes

$$1 = \sum_{i=1}^{n} \frac{\lambda^{n-1}}{\left.\frac{d f(\lambda)}{d\lambda}\right|_{\lambda=\lambda_i}} \quad .$$

The desired result is then obtained from equation (26) as

$$P(z^2 \leqslant R) = P(r \leqslant R) = 1 - \sum_{i=1}^{n} \frac{\lambda_i^{n-1} \, e^{-\frac{R}{2\lambda_i}}}{\left.\frac{d f(\lambda)}{d\lambda}\right|_{\lambda=\lambda_i}} \quad . \tag{30}$$

The extension of the above formulation to the case of multiple roots is straight-forward. The result can be generalized to the following:

If $f(\lambda)$ has roots λ_i of order p_i, then

$$P(z^2 \leqslant R) = P(r \leqslant R) = 1 - \sum_{i=1}^{n} \left[p_i \, \frac{\partial^{p_i - 1}}{\partial \lambda_i^{p_i - 1}} \left(\frac{\lambda_i^{n-1} \, e^{-\frac{R}{2\lambda_i}}}{f^{(p_i)}(\lambda_i)} \right) \right] \tag{31}$$

where $f^{(p_i)}(\lambda_i)$ denotes the p_i th derivative of $f(\lambda)$ evaluated at $\lambda = \lambda_i$.

Specifically, for a double pole, the term corresponding to λ_i in the summation is

$$e^{-\frac{R}{2\lambda_i}} \left[\frac{(n-1)\lambda_i^{n-2} + \lambda_i^{n-3} \left(\frac{R}{2}\right)}{\frac{1}{2!} f^{(2)}(\lambda_i)} - \frac{\lambda_i^{n-1} \frac{1}{3!} f^{(3)}(\lambda_i)}{\left[\frac{1}{2} f^{(2)}(\lambda_i)\right]^2} \right] . \tag{32}$$

Similarly, expressions can be written for the contributions due to higher order poles.

For the special case n = 2, equation (30) becomes

$$P(z^2 \leqslant R) = P(r \leqslant R) = 1 - \frac{\lambda_1 e^{-\frac{R}{2\lambda_1}}}{2\lambda_1 - (\sigma_1^2 + \sigma_2^2)} - \frac{\lambda_2 e^{-\frac{R}{2\lambda_2}}}{2\lambda_2 - (\sigma_1^2 + \sigma_2^2)} \tag{33}$$

where

$$\lambda_1 = \frac{(\sigma_1^2 + \sigma_2^2) + \sqrt{(\sigma_1^2 + \sigma_2^2)^2 - 4\sigma_1^2 \sigma_2^2 (1 - \rho_{12}^2)}}{2}$$

$$\lambda_2 = \frac{(\sigma_1^2 + \sigma_2^2) - \sqrt{(\sigma_1^2 + \sigma_2^2)^2 - 4\sigma_1^2 \sigma_2^2 (1 - \rho_{12}^2)}}{2} .$$

In the special case $\sigma_1^2 = \sigma_2^2$

$$\lambda_1 = \sigma_1^2 \left[1 + \rho_{12}\right]$$

$$\lambda_2 = \sigma_1^2 \left[1 - \rho_{12}\right] ,$$

(30) becomes

$$P(r = R)\ 1 + \frac{1-\rho}{2\rho}\, e^{-\frac{R}{2\sigma^2(1-\rho)}} - \frac{(1+\rho)\, e^{-\frac{R}{2\sigma^2(1+\rho)}}}{2\rho} \tag{34}$$

which is in agreement with Packard (Ref. 4), when the Gaussian variance used above is replaced by the variance of a Rayleigh distribution; i.e., $2\sigma^2 = \sigma_R^2$.

By a series expansion it may be shown that equation (34), in the special case $\rho = 0$, reduces to

$$P(z^2 \leqslant R) = P(r \leqslant R) = 1 - e^{-\frac{R}{2\sigma^2}} \left[1 + \frac{R}{2\sigma^2}\right] .$$

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