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Atmospheric Propagations From a Nuclear Explosion

by

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NOMENCLATURE

The symbols which will be used throughout Sections II, III, IV, and V are:

1. Terms describing the source.

$(R_o, 0, 0)$	=	Coordinates of the source
S_o	=	Surface surrounding the source
(r_o, θ_o, ψ_o)	=	Coordinates of a point on S_o
ϵ	=	Radius of a spherical surface concentric with $(R_o, 0, 0)$; $\epsilon = \left\| \underline{R}_o - \underline{r}_o \right\|$
V	=	Volume of gas introduced by the source
W	=	Energy introduced by the source

2. Atmospheric quantities.

p_o	=	Atmospheric pressure
ρ_o	=	Atmospheric density
T_o	=	Atmospheric temperature
c_o	=	Speed of sound in air = $(\gamma R T_o)^{1/2}$
γ	=	Ratio of specific heats
R	=	Gas constant
ℓ	=	Height of the tropopause
$\underline{v}$	=	Wind velocity

NOMENCLATURE
continued

3. Quantities detected by the observer.

p	=	Excess pressure
ρ	=	Excess density
$\underline{u}$	=	Particle velocity
ω	=	Frequency of a wave
t	=	Time
(r, θ, ψ)	=	Coordinates of the observer

4. a = Radius of the earth

5. g = Acceleration due to gravity

ABSTRACT

The pressure pulse at large distances from a nuclear explosion is investigated. A source representation is established which produces parameters characteristic of these explosions on a surface enclosing the source. An integral equation for the pressure is obtained in terms of a ring source Green's function, where the integration extends over the source. Pulse forms are obtained for explosions on the ground and in the atmosphere when various temperature models are considered. When the stratosphere is assumed to be either of the isothermal or thermospheric type, a general theory is established for determining the number of modes of propagation. In addition, a method for examining the dispersive effect of local winds is established.

This work is being continued under Contract AF 19(628)-304.

I
INTRODUCTION

We start with a qualitative description of the effect of a nuclear explosion in generating a pressure pulse in the atmosphere. The overall description is necessarily qualitative since there is no quantitative formalism which can describe the initial stages of the blast. This problem of describing the source - the initial stages - is so difficult because the tremendous release of energy sets up shock waves and turbulences for which we have no adequate mathematical description.

After propagating some distance from the blast point the pressure pulse will become so attenuated that the pressure of the pulse will be sufficiently small as compared with atmospheric pressure. At this distance from the center a linearized theory furnishes an adequate description and it is possible to handle the mathematical problems. The distance from the center at which this linear description is useful will depend upon the size and altitude of the blast. However, no precise criterion as to the amount of over pressure consistent with the linear theory can be given.

The work which has been done on this contract to determine the nature of the pressure pulse differs in many respects from that done by Scorer [1950], Hunt, Palmer and Penney [1960], Yamamoto [1957], and other earlier authors. Since the most significant contributions made here either have been or are expected to be published, we shall, in the four sections which follow, present a general treatment to emphasize the most important features. The points which are new and which bear special emphasis, are several. Perhaps the major contribution toward an interpretation of nuclear explosions is that a means has been devised whereby a pressure pulse may be determined for a source located in the atmosphere as well as on the ground. Previous authors have considered only the latter case. In addition, it has been established that as the height of burst is increased the lower frequency effects become more pronounced. Other significant contributions are as follows: (1) In Section IV an upper atmosphere with an increasing temperature profile is considered. To our knowledge, previous authors have considered only the isothermal stratosphere, which does not correspond to physical reality; (2) given a particular temperature profile, a qualitative method is established for determining the number of modes of propagation which might be expected; (3) given the wind and temperature profile in a local region, the dispersive effects of the wind may be calculated; and finally (4) a rather detailed interpretation of source phenomena is presented in Section IV.

The four sections which follow are essentially condensations of works already published. In instances where the reader wishes greater detail he may refer to the appropriate articles. Briefly, the format is as follows:

Section II: A review of the work of Weston, [1960 and 1961] in which a pressure pulse produced by a large explosion in the atmosphere is investigated. The source is characterized by a surface S_0 surrounding the explosion, in which the pressure and velocity are known quantities. With these as "source" boundary conditions an integral equation for the pressure is established where the integration extends over S_o. The gravity wave portion of the pressure pulse is then considered for three different models of the atmosphere as the height of source and distance to the observer are varied.

Section III: A review of the the work of Weston [1962a] in which the spectrum of the gravity and acoustic waves is discussed for two general models of the atmosphere. The emphasis is placed upon frequencies for which a mathematical singularity does not arise. A method is established for analyzing the discrete modes as concerns the minimum and maximum speeds of propagation and for determining the number of modes which will arise.

Section IV: A review of the work of Weston [1962b] which is essentially a continuation of the ideas and methods presented by Weston [1960 and 1961]. A general treatment of the source is presented, in that for a large explosion in the atmosphere the characterization presented by Weston [1960 and 1961] may not be sufficiently accurate. It is assumed that in the stratosphere the temperature increases linearly with altitude which has a reasonably close correspondence with the actual atmosphere. The pressure pulse produced by a large explosion is then calculated for an observer at a given distance with a particular tropospheric temperature profile. It is shown that as the height of burst is increased, the gravity wave portion of the pulse becomes increasingly dominant. For an explosion at ground level, the first three modes are evaluated, while for a burst at 76 Km only the gravity wave mode is computed.

Section V: A review of the work of Weston and van Hulsteyn [1962], which considers the effects of winds upon the gravity wave. It is shown that for a pulse traveling downwind in a uniform horizontal wind field, the effect is to increase the dispersion and the phase velocity in the gravity wave mode.

Finally, in Section VI a description of the new work (Contract AF 19(628)-304) in progress is given. This current effort is devoted to (1) an attempt to better characterize the source in terms of the behavior of the "linear" region and (2) the analysis of the linear equations for a simple model of the atmosphere in order to determine the functional behavior on the physical parameters.

II

THE PRESSURE PULSE PRODUCED BY A LARGE EXPLOSION IN THE ATMOSPHERE, PART I

The basic problem is to determine the form of the pressure pulse that an observer will detect when he is located at some large distance from a nuclear explosion. The method that will be employed here will be to set up the equations of motion in spherical coordinates and assume that winds, earth's rotation, and horizontal variations in the earth's atmosphere may be neglected. The requirement that the observer be at a great distance from the source is such that terms of the form p/p_o and ρ/ρ_o be much less than unity. On this basis, the equations of hydrodynamics may be linearized, and, after taking the harmonic time dependence and eliminating ρ and $\underline{u}$, a separable equation for the pressure is obtained (see Appendix A). The features which affect the nature of the detected pulse are the atmospheric structure and the characteristics of the explosion. The first of these may be satisfied by ascribing a certain temperature profile which, as will be shown in Section III, determines the number of modes an observer will detect and which also limits the speed of propagation. The source phenomenon may be described by locating it at a point $(R_o, 0, 0)$ and by specifying the pressure and velocity on some surface surrounding the source. The only initial restriction upon this surface, which will be denoted by S_o, is that it be large enough so that on S_o, p/p_o is much less than unity.

From appendix A the differential equation for the pressure is given by

$$(\underline{\nabla}\cdot\underline{L})\,(\rho_o^{-1/2}\,p) + \frac{q(r)}{r^2}\,(\rho_o^{-1/2}\,p) = 0\,. \tag{1}$$

The effect of vertical variations in the atmosphere is introduced through the operator $\underline{L}$ and through $q(r)$. Equation (1) is obviously separable, and, by assuming axial symmetry about the source, is independent of the azimuthal angle ψ. Writing

$$\rho_o^{-1/2}(r)\;p(r,\theta) = \phi(r)\,\chi(\theta) \tag{2}$$

one obtains, from equation (1)

$$\frac{\partial}{\partial r}\left[\frac{r^2}{h}\;\frac{\partial\phi}{\partial r}\right] + \left[q(r) + \frac{1}{4} - \mu^2\right]\;\phi(r) = 0 \tag{3}$$

and

$$\frac{1}{\sin\theta}\;\frac{\partial}{\partial\theta}\;\left[\sin\theta\;\frac{\partial\chi}{\partial\theta}\right] + \left[\mu^2 - \frac{1}{4}\right]\chi(\theta) = 0 \tag{4}$$

where $\mu^2 - \frac{1}{4}$ is the separation constant.

Using the standard boundary conditions that the kinetic energy in an infinite column is finite and that the vertical velocity is zero at the earth's surface, we have that $\rho_o^{-1/2}(r)\;p(r,\theta)$ must be squared integrable over $0 \leq r \leq \infty$, and, that at $r = a$,

$$\phi' + A\phi = 0. \tag{5}$$

It can be shown from this [Weston, 1960 and 1961] that the excess pressure may be represented as an integral over the set of continuous modes plus a sum over the set of discrete modes, but that the dominant contribution to the pressure pulse arises from the discrete set. Accordingly, for the jth mode, the eigenvalue in (3) and (4) is denoted by $\mu_j^2 - \frac{1}{4}$, while the eigenfunctions are $\phi_j(r)$ and $\chi_j(\theta)$ respectively. This feature enables us to construct a ring source Green's function which satisfies the inhomogeneous counterpart of equation (1), namely

$$(\underline{\nabla} \cdot \underline{L})\, G + \frac{q(r)}{r^2}\, G = \frac{\delta(r-r_o)\,\delta(\theta-\theta_o)}{2\pi\, r^2 \sin\theta} \qquad (6)$$

such that $G(\underline{r}, \underline{r}_o)$ is squared integrable and, that at $r = a$,

$$\frac{\partial G}{\partial r} + AG = 0 \quad . \qquad (7)$$

From (1) and (6), together with the boundary conditions, it is found that

$$\rho_o^{-1/2}\, p(r,\theta) = \int_{S_o} \underline{n} \cdot \left\{ G\, \underline{L}_o(\rho_o^{-1/2}\, p) - \rho_o^{-1/2}\, p\, \underline{L}_o\, G \right\} ds \qquad (8)$$

where $\underline{n}$ is a unit vector normal to the surface S_o and where $\underline{L}_o$ is the differential operator $\underline{L}$ which operates on the source coordinates. If $p(r,t)$ and $\underline{n} \cdot \underline{u}(r,t)$ are known on the surface S_o, the formal mathematical problem reduces to one of determining the expansion for $G(\underline{r}, \underline{r}_o)$. The eigenfunctions $\phi_j(r)$ satisfying equation (3) and the boundary conditions can be shown to be orthogonal over the range $a \leq r \leq \infty$, so that

$$G(\underline{r}, \underline{r}_o) = \sum_j \frac{\phi_j(r)\, \phi_j(r_o)\, \chi_j(\theta, \theta_o)}{\int_a^{\infty} \phi_j^2(r)\, dr} \qquad (9)$$

Upon substituting this form of $G(\underline{r}, \underline{r}_o)$ into equation (6) it is found that

$$\chi_j(\theta, \theta_o) = -\frac{1}{4 \cos \pi \mu_j} \left\{ P_{-1/2+\mu_j}(\cos \theta_o) \, P_{-1/2+\mu_j}(-\cos \theta) \right\} \tag{10}$$

for $\theta_o \leq \theta \leq \pi$. The representation for $G(\underline{r}, \underline{r}_o)$ of equation (9) may then be re-expressed as

$$G(\underline{r}, \underline{r}_o) = \sum_j \frac{f_j(\underline{r}, \underline{r}_o)}{2 \cos \pi \mu_j} \tag{11}$$

which when substituted into equation (8) yields a solution of the form

$$\rho_o^{-1/2} \, p(\underline{r}) = \sum_j \frac{I_j}{2 \cos \pi \mu_j} . \tag{12}$$

In order to determine the time dependence of $p(\underline{r}, t)$, the inverse Fourier transform of (12) must be performed with the result that

$$p(\underline{r}, t) = \frac{1}{2\pi} \rho_o^{1/2}(r) \int_{-\infty}^{\infty} e^{-i\omega t} \sum_j \frac{I_j \, d\omega}{2 \cos \pi \mu_j} . \tag{13}$$

Equation (13) is the formal solution to the problem, which is dependent upon the nature of the source and the vertical structure of the atmosphere. Before specifying these, however, a quantity λ_j may be introduced which is defined by

$$\mu_j = \left[\omega^2 a^2 \lambda_j^2 + \frac{1}{4}\right]^{1/2} \simeq \omega a \lambda_j , \tag{14}$$

the approximation failing only for $\omega < 10^{-4}$ sec^{-1}. It may be shown Weston, [1961], that the portion of the pulse that arrives directly without circling the earth, is given by

$$p(\underline{r}, t) = \frac{\rho_o^{1/2}(r)}{2\pi} \int_{-\infty}^{\infty} \sum_j I_j \, e^{-i\omega\left[t - \pi a\lambda_j\right]} \, d\omega, \qquad (15)$$

where λ_j has the physical significance of being the inverse phase velocity of the j^{th} mode.

In order to determine the quantity I_j of equation (12), [Weston 1961] assumed that the surface S_o is a small sphere of radius ϵ centered at $(R_o, 0, 0)$ and that the disturbance is spherically symmetric about this point. The pressure and normal component of the velocity on S_o may be obtained from observational data with the stipulation that $p/p_o << 1$ on S_o. Since an exact description of the source is unnecessary so long as these quantities are known on S_o, we may consider them as having been produced by an equivalent "acoustic" source. In this manner the function which generates the quantities observed on S_o may be obtained. In addition, choosing an explosion of less than 1 megaton, the source is effectively an instantaneous volume source, from which it developes [Weston 1960] that

$$I_j = i\omega \, f_j(\underline{r}, \underline{R}_o) \quad V \rho_o^{1/2} (R_o)$$

where V is the total volume of gas introduced by this equivalent source.

From equations (9) - (11), together with the normalization condition for ϕ_j [Weston 1961], f_j for an observer on the ground is given by

$$f_j(a\theta,\ t) = \frac{\omega^2 \lambda_j\ \phi_j(R_o)\ P_{-\frac{1}{2}+\mu_j}(-\cos\theta)}{\left\{\dfrac{\partial}{\partial\lambda}\left[\dfrac{A\phi+\phi'}{h}\right]\right\}_{\substack{\lambda=\lambda_j\\ r=a}}} \tag{16}$$

Except for $\omega < 10^{-4}\ \sec^{-1}$, the following approximation may be made

$$P_{-\frac{1}{2}+\mu_j}(-\cos\theta) = \left[\frac{1}{2}\ \pi\ \omega\ a\lambda_j\ \sin\theta\right]^{-1/2} \cos\left\{\omega a\lambda_j(\pi-\theta) - \frac{\pi}{4}\right\} \tag{17}$$

where $0 < \theta < \pi$. Introducing the relative intensity function

$$Q_j = \frac{-\omega^2\ \phi_j(R_o)}{\left\{\dfrac{\partial}{\partial\lambda}\left[\dfrac{A\phi+\phi'}{h}\right]\right\}_{\substack{\lambda=\lambda_j\\ r=a}}} \tag{18}$$

the expression for I_j becomes

$$I_j = \frac{-i\ V\ \rho_o^{1/2}(R_o)}{\sqrt{\dfrac{\pi}{2}\ a\ \sin\theta}}\ \sqrt{\omega\lambda_j}\ \ Q_j\ \cos\left[\omega\ a\ \lambda_j(\pi-\theta) - \frac{\pi}{4}\right] \tag{19}$$

This quantity is dependent upon the size of the source through the intensity function Q_j defined in equation (18) and upon the height of burst through the term $\rho_o^{1/2}(R_o)$. Substitution of equation (19) into (15) yields after some manipulation

$$p(a\theta, t) = \left[\frac{\rho_o(R_o)\, \rho_o(a)}{2\pi\, a \sin\theta} \right]^{1/2} \frac{V}{\pi} \int_0^{\infty} \sum_j \sqrt{\omega \lambda_j}\; Q_j(\omega) \cdot$$

$$\left\{ \cos\left[\omega(t - a\,\theta\lambda_j) + \frac{\pi}{4} \right] + \cos\left[\omega\left(t - a(2\pi - \theta)\lambda_j \right) + \frac{3\pi}{4} \right] \right\} d\omega \quad . \qquad (20)$$

The second cosine term in the integrand represents the portion that reaches the observer via the antipodal route while the first represents the portion that travels directly along a great circle. The dependence of the pressure upon distance enters through the quantities $(a\theta)$ appearing in the integrand and through the $\left[a \sin\theta\right]^{-1/2}$ term in the amplitude.

The integration in equation (20) is, of course, dependent upon the dispersion relation $\lambda_j = \lambda_j(\omega)$ for a particular model of the atmosphere. The three models considered, including Scorer's model, are taken to be isothermal above a height , ℓ, and have certain temperature profiles below this height. Therefore, the eigenfunction, ϕ, of equation (3) is an exponential above $r - a = \ell$ [Weston, 1961] and must be determined as a function of r in the range $a < r < a+\ell$. The usual condition that the pressure and vertical velocity must be continuous at $r = a + \ell$ produces the desired matching conditions at this interface. The dispersion relationship, (i.e., the value of λ_j which corresponds to a given ω) is then obtained by requiring that $\phi' + A\phi = 0$ at $r = a$.

The three atmospheric models considered are described in Appendix B, together with the dispersion relationships which were obtained for the lowest or gravity wave mode. Higher modes, which were not calculated, would result in a superposition of wave trains upon the gravity wave mode, as can be seen from equation (20). Since the directly received portion of the gravity wave mode is of greatest interest, it may be written, from equation (20) as

$$p(a\theta,\ t) = \sqrt{\frac{\rho_o(R_o)\,\rho_o(a)}{2\pi\ a \sin\theta}}\ \frac{V}{\pi} \int_0^{\infty} \sqrt{\omega\lambda}\, Q(\omega) \cos\left[\omega(t - a\theta\lambda) + \frac{\pi}{4}\right] d\omega \qquad (21)$$

The second term in the integrand has been omitted, since it corresponds to the portion that followed the antipodal route. In equation (21) the subscript j has been dropped since it is understood that only the gravity wave mode is being considered. From equation (18) together with the dispersion data of Appendix B, the integral of equation (21) may be determined. For sufficiently large values of $a\theta$, the tail of the pulse may be obtained by the method of stationary phase. The head of the pulse, however, is a much more complicated problem but may be computed by asymptotic techniques [Weston, 1960 and 1961]. The pulse forms which are obtained by these methods for the various atmospheric models are shown in the figures of Appendix B.

III

GRAVITY AND ACOUSTIC WAVE MODES

Before discussing more complicated temperature structures than are presented in Appendix B, it would be worthwhile to develop some means whereby one could determine the number of modes he might expect to observe. In this section we shall indicate a method which will enable one to determine whether or not high frequency modes might arise. The analysis, however, is limited to atmospheric models which are isothermal above a height ℓ or else have temperature variations of the form $T_0 \propto z^{1+\epsilon}$ above $z = \ell$, where ϵ is a small, positive quantity. The latter model is of some interest since it has a closer correspondence with reality than does the isothermal model. The only restriction placed upon the atmospheric structure below $z = \ell$ is that the temperature does not vary radically with altitude. The majority of observed temperature profiles satisfy this requirement.

The approach is similar to that used in the preceding section, except that a flat earth approximation is used in obtaining the pressure equation. For purposes of spectrum analysis this simplification presents no inaccuracies since the interest is purely in the atmospheric structure. When the hydrodynamic equations are linearized and a time dependence of the form $\exp(i\omega t)$ is taken, the resulting equation for the pressure is

$$\nabla \cdot \underline{L}(\rho_o^{-1/2} p) + \left[\frac{\omega^2}{c_o^2} - \frac{A^2}{h} + \left(\frac{A}{h}\right)'\right](\rho_o^{-1/2} p) = 0, \tag{22}$$

in analogy with equation (1) of the last section. The functions h and A are defined in the same manner and

$$\underline{L} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{1}{h}\frac{\partial}{\partial z}\right]$$

is the Cartesian counterpart of the $\underline{L}$ mentioned in equation (1).

Equation (22) has a separable solution of the form

$$\rho_o^{-1/2}(z)\, p(\underline{r}) = \phi(z) e^{-i\omega \underline{\lambda}\cdot\underline{x}} \tag{23}$$

where, as before, $\underline{\lambda}$ is the inverse phase velocity. The equation for $\phi(z)$, then, is

$$\frac{d}{dz}\left[\frac{1}{h}\frac{d\phi}{dz}\right] + \left[\frac{\omega^2}{c_o^2} - \omega^2\lambda^2 - \frac{A^2}{h} + \left(\frac{A}{h}\right)'\right]\phi = 0 \tag{24}$$

with the boundary conditions that ϕ be squared integrable over the range (o, ∞) and that $\phi' + A\phi = 0$ at $z = 0$.

The function h presents some difficulties in that for certain values of ω at a given height it may have a zero. This may be seen by writing it in the form

$$h = 1 - \frac{g}{\omega^2}\left[(\gamma - 1)\frac{g}{c_o^2} + \frac{T_o'}{T_o}\right] \tag{25}$$

Because of this mathematical singularity, it is useful to introduce frequencies ω_B and ω_b such that h does not vanish anywhere for $\omega > \omega_B$ or for $\omega < \omega_b$.

For an atmosphere which is isothermal above a height $z = \ell$, equation (24) may be simplified by substituting

$$\left.\begin{aligned} x &= \int_0^z |h|^{1/2}\, dz \\ \phi &= |h|^{1/4}\, y(z) \end{aligned}\right\} \quad \text{if} \quad \omega > \omega_B \tag{26}$$

or

$$\left.\begin{aligned} x &= \omega \int_0^z |h|^{1/2}\, dz \\ \phi &= \omega^{1/2}\, |h|^{1/4}\, y(z) \end{aligned}\right\} \quad \text{if} \quad \omega < \omega_b\ . \tag{27}$$

Equation (24) then reduces to

$$\frac{d^2y}{dx^2} + y\left[\Lambda - g(x)\right] = 0 \tag{28}$$

where

$$\Lambda = \begin{cases} -\dfrac{A_\ell^2}{h_\ell} - \omega^2\lambda^2 + \dfrac{\omega^2}{c_\ell^2} & \text{if } \omega > \omega_B \\[2ex] \dfrac{A_\ell^2}{\omega^2 h_\ell} + \lambda^2 - \dfrac{1}{c_\ell^2} & \text{if } \omega < \omega_b \end{cases} \tag{29}$$

The function g(x) itself is rather complicated in the troposphere [Weston 1962a] but above $z = \ell$,

$$g(x) \equiv 0.$$

As in the preceding section, there is a continuous spectrum for $0 < \Lambda < \infty$, but this does not contribute significantly to the pulse and will thus be neglected. Since the discrete spectrum arises from Λ in $(-\infty, 0)$ [Weston 1962a] we have that

$$\lambda_i^2 \geqslant \frac{1}{c_\ell^2} - \frac{A_\ell^2}{\omega^2 h_\ell} \qquad \omega > \omega_B$$

$$\lambda_i^2 \leqslant \frac{1}{c_\ell^2} \quad \frac{A_\ell^2}{\omega^2 h_\ell} \qquad \omega < \omega_b$$

If the minimum value of c_o is c_b, it may be shown [Weston 1962a] for $\omega > \omega_B$ that there is no discrete mode corresponding to λ in the range $\lambda^2 < (c_b^2)^{-1}$. Similarly, for $\omega < \omega_b$, $\lambda^2 > (c_B^2)^{-1}$, where c_B is the maximum value of c_o. We then have the upper and lower bounds on λ^2 except for ω in the range $\omega_b < \omega < \omega_B$.

At this point the Sturmian comparison theorem may be introduced to deduce the number of discrete modes that exist. The approach is to let $u(z)$ be defined by

$$\phi(z) = u(z) \exp\left[-\int_0^z A\,d\sigma\right] \tag{30}$$

so that from equation (24), $u(z)$ satisfies

$$\frac{d}{dz}\left[\frac{\exp\left[-2\int_0^z A\,d\sigma\right]}{h} u'\right] + \omega^2\left(\frac{1}{c_o^2} - \lambda^2\right)\exp\left[-2\int_0^z A\,d\sigma\right] u = 0. \tag{31}$$

If

$$k_1 = |h|^{-1} \exp\left[-2\int_0^z A\,d\sigma\right]$$

$$k_2 = H^{-1} \exp\left[-2\int_0^z A\,d\sigma\right]$$

where H is the maximum value of $|h|$, and

$$G_1 = \omega^2(\lambda^2 - \frac{1}{c_o^2}) \exp\left[-2\int_0^z A\,d\sigma\right] \operatorname{sgn}(h)$$

$$G_2 = \frac{\exp\left[-2\int_0^z A\,d\sigma\right]}{c_o^4} \begin{cases} \omega^2(\lambda^2 c_b^2 - 1)\, c_b^2 & \omega > \omega_B \\ \omega^2(1 - \lambda^2 c_B^2)\, c_B^2 & \omega < \omega_b \end{cases}$$

then $k_1 \geqslant k_2 > 0$ and $G_1 \geqslant G_2$ in $(0, \ell)$.

Equation (31) for $u(z)$ may be rewritten as

$$\frac{d}{dz}\left[k_1 \frac{du}{dz}\right] - G_1 u = 0 \tag{32}$$

which satisfies the initial conditions

$$u(o) = \alpha_1 \qquad u'(0) = 0$$

while corresponding to k_2 and G_2 there is a function $v(z)$ defined by

$$\frac{d}{dz}\left[k_2 \frac{dv}{dz}\right] - G_2 v = 0 \tag{33}$$

satisfying the initial conditions

$$v(0) = \alpha_2 \qquad v'(0) = 0$$

The Sturmian comparison theorem states that $u(z)$ can have at most as many zeros as $v(z)$ in the interval $(0, \ell)$. The number of a mode is determined by the number of zeros it has in $(0, \ell)$, so that the gravity wave, or zeroth, mode, has no zeros, and so forth. The function $v(z)$ which satisfies (33) and the boundary condition $v'(0) = 0$, is

$$v = e^{\xi/2}\left\{\cosh\sqrt{K}\,\xi - \frac{\sinh\sqrt{K}\,\xi}{2\,K}\right\} \tag{34}$$

where

$$\xi = (2-\gamma)\, g \int_0^z \frac{d\sigma}{c_o^2}$$

and

$$K - \frac{1}{4} = \frac{H}{(2-\gamma)^2 g^2} \begin{cases} \omega^2 (\lambda^2 c_b^{\,2} - 1) c_b^{\,2} & \omega > \omega_B \\ \omega^2 (1 - \lambda^2 c_B^{\,2}) c_B^{\,2} & \omega < \omega_b \end{cases}$$

Since c_b and c_B are known for a given model of the atmosphere, the number of zeros and hence the number of modes may be determined for $\omega > \omega_B$ and $\omega < \omega_b$. From equation (34) and the Sturmian Theorem, the following may be deduced:

If ξ_ℓ is the value of ξ at $z = \ell$, then if

(1) $\xi_\ell < 2$ and $K > 0$, $u(z)$ has no zeros.

(2) $\xi_\ell < 2$ and $K < 0$, $u(z)$ has at most n zeros if $\sqrt{-K}\,\xi_\ell < n\pi$

(3) $\xi_\ell > 2$ and $K > 0$, $u(z)$ has at most one zero.

(4) $\xi_\ell > 2$ and $K < 0$, $u(z)$ has at most n zeros if $\sqrt{-K}\,\xi_\ell < (n+1)\pi$.

As an example, Scorer's model has the property that the upper isothermal atmosphere is coldest, so that $c_b = c_\ell$ and $h_\ell = H$. In addition, $\xi_\ell < 2$ and $k > 0$ so that only the gravity wave mode may exist for $\omega > \omega_B$.

The analysis in the thermosphere case where $T_o \propto z^{1+\epsilon}$ is a more complicated situation, but it may be shown by a procedure analogous to the isothermal situation that a discrete set of modes exists for $\omega > \omega_B$ which is bounded by

$$0 \leqslant \lambda_i^2 \leqslant \frac{1}{c_b^2}\,.$$

Thus, these discrete modes have a phase velocity greater than or equal to the speed of sound. There exists, however, a wide frequency range $0 \leqslant \omega \leqslant \omega_B$ which must be investigated. It can be shown that if the multiplicity of the zero of h is of order one, the effect of the vanishing of h at $z = z_o$ is to create a turning point; in effect, ϕ will be oscillitory on one side of z_o and non oscillatory on the other. The main effect of the vanishing of h is to create an additional set of discrete spectra λ_i unbounded above. These then may propagate at speeds less than the minimum speed of sound. These features will be discussed in the following section, which is similar to the first section except that the thermosphere model replaces the isothermal upper layer model.

IV

THE PRESSURE PULSE PRODUCED BY A LARGE EXPLOSION IN THE ATMOSPHERE, PART II

In the first section a specific point source representation was considered such that the time dependence of the source function made I_j proportional to V. In the more general case, if $n \cdot \underline{u}(r_0, t)$ is the normal component of velocity on S_0, and $N(r_0, \omega)$ is its Fourier transform, then it can be shown [Weston 1962 b] that

$$I_j = i\omega f_j (r, R_0) \rho_0^{\frac{1}{2}} (R_0) B \tag{35}$$

where

$$B = \lim_{\epsilon \to 0} 4\pi\epsilon^2 N(r_0, \omega) \tag{36}$$

As before ϵ represents the radius of the spherical surface S_0 described in Section I, which is allowed to shrink to zero. According to Brode [1956 and 1957] a good approximation for the positive phase of the excess pressure in S_0 prior to shrinking is given by

$$p = \frac{\mu_s}{\mu} p_1 \left[1 - \frac{\mu_s - \mu}{L}\right] \exp \left[- \frac{\mu_s - \mu}{L}\right] \tag{37}$$

where μ_s is the normalized shock radius $\left(\mu_s = \epsilon \left(\frac{p_0}{W}\right)^{\frac{1}{3}}\right)$, μ is the normalized distance from the source, W is the initial energy of the explosion, L is the normalized length of the positive phase, and p_1 is the excess pressure on S_0. If an acoustic representation is to be determined, the problem is to find a function f(t) such that

$$p = \frac{f'(t - \frac{r_1}{c})}{r_1} \tag{38}$$

and

$$\underline{n} \; \underline{u} = \frac{f'(t - \frac{r_1}{c})}{\rho_o c r_1} + \frac{f(t - \frac{r_1}{c})}{\rho_o r_1^2} \tag{39}$$

where r_1 is the radial distance from the source, and equation (38) gives the pressure profile required by equation (37). It is found that such a source function is given by

$$f(t) = (\frac{W}{p_o})^{\frac{1}{3}} \mu_s p_1 t e^{-\frac{t}{T}}, \quad t > 0 \tag{40}$$

where

$$T = \frac{L}{c_o} (\frac{W}{p_o})^{\frac{1}{3}} \tag{41}$$

and is thus related to the temporal duration of the positive phase. As the radius of S_o is shrunk to zero, the radial velocity is given by

$$u = \frac{f(t - \frac{\epsilon}{c})}{\rho_o \epsilon^2} \tag{42}$$

so that

$$B = \lim_{\epsilon \to 0} \frac{4\pi}{\rho_o(R_o)} \int_{-\infty}^{\infty} e^{i\omega t} f(t) \, dt \tag{43}$$

Performing this integration,

$$B = \frac{4\pi T^2 \mu_s p_1}{\rho_0(R_0)} \left(\frac{W}{p_0}\right)^{\frac{1}{3}} \left\{ \frac{1-(\omega T)^2 + 2i\omega T}{\left[1+(\omega T)^2\right]^2} \right\} \tag{44}$$

It is to be noted from the definition of T that it varies with the size of burst through $(W)^{\frac{1}{3}}$ and with the height of burst through $\frac{1}{c_0}\frac{1}{(p_0)^{1/3}}$. For sufficiently small values of (ωT), B is given approximately by

$$B = \frac{4\pi T^2 \mu_s p_1}{\rho_0(R_0)} \left(\frac{W}{p_0}\right)^{\frac{1}{3}} \tag{45}$$

From the definition of B in (36) however, it may be seen that for an instantaneous volume source, B = V which gives a relationship between V and W. As a particular example, if L = .35 and $\mu_s p_1 = .196\, p_0$,

$$V = \frac{4\pi W}{\gamma p_0} (.024) \tag{46}$$

From equation (41), values of T for certain heights of burst and certain values of W are calculated and given in Appendix C. Since our concern is with frequencies on the order of $5000\omega^2 = 10, (\omega T)$ is much less than unity for most explosions on the ground. For a one megaton explosion at 76 Km, however, T = 160 seconds so that the approximation of equation (45) is not valid.

The expression for $p(a\theta, t)$ given in equation (20) of the first section was based on the assumption that $B = V$. For large values of (ωT) however, one would have to write

$$B = |B|\, e^{i\beta} \tag{47}$$

where

$$\beta = \tan^{-1} \frac{2\omega T}{1 - (\omega T)^2} \tag{48}$$

and
$$|B| = \frac{4\pi T^2 \mu_s p_1}{\rho_0 (R_0)} \left[\frac{W}{\rho_0}\right]^{\frac{1}{3}} \frac{1}{1 + (\omega T)^2} \tag{49}$$

The real part of the expression in the integrand of equation (20) is then given for the directly received portion by

$$|B| \cos\left[\omega(t - a\theta\, \lambda_j) + \frac{\pi}{4} - \beta\right] \tag{50}$$

In the discussion of the thermosphere model that follows it will be assumed that $B(\omega) = V$ is a sufficient approximation for the energy range and heights of burst under consideration. Hence, $\beta = 0$ in equation (50).

For the particular thermosphere model chosen, $\ell = 106$ Km is the height of the tropospause, and the temperature, T_0 is given by

$$T_0 = 228 + 18(z - 106), \qquad z \geq 106 \tag{51}$$

With this temperature structure it is found that above 106 Km, the function $\phi(z)$ which satisfies equation (3) is given by

$$\phi(z) = \rho_o^{\frac{1}{2}} \left\{ (\gamma g^2 - \omega^2 c_o^2 + g c_o^2 \quad \omega) F - 2g c_o^2 \omega\lambda \; \dot{F} \right\} e^{-\frac{y}{2}} \tag{52}$$

where

$$y = 2\omega\lambda \, T_o / T_o'$$

and the dot represents differentiation with respect to y.

The function F is the particular hypergeometric function

$$F(y) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} \; {}_1F_1(a, b; y) + \frac{\Gamma(b-1)}{\Gamma(a)} \; y^{1-b} \; {}_1F_1(a-b+1, 2-b; y) \tag{53}$$

with

$$b = 1 - \frac{g}{RT_o}$$

and

$$a = \frac{b}{2} - \left|\left(\frac{\omega}{\lambda}\right)\right| (2RT_o')^{-1} - \left|\left(\frac{\lambda}{\omega}\right)\right| \; g \; \frac{\left[\gamma R T_o' + (\gamma - 1) g\right]}{2\gamma RT_o'} .$$

From the linearized equations of hydrodynamics, it may be shown (Appendix A) that the vertical velocity is given by

$$i\,\omega u_r h = \rho_o^{-\frac{1}{2}} (\phi' + A\phi) \tag{54}$$

The boundary condition that ϕ and u_r be continuous at the tropopause ($z = \ell$) requires that in the troposphere $\phi(r)$ be such that at $z = \ell$,

$$\frac{\phi' + A\phi}{\omega^2 h\phi} = \frac{\left\{\left[g(\gamma - \lambda^2 c_o^2) + c_o^2 \ \omega\lambda\right]F - 2c_o^2 \ \omega\lambda \ \dot{F}\right\}}{\left\{\left[\gamma g^2 - c_o^2 \ \omega^2 + \omega\lambda \ g c_o^2\right] F - 2g \ \omega\lambda \ c_o^2 \ \dot{F}\right\}} \tag{55}$$

Using the condition that $\phi' + A\phi = 0$ at the ground, equation (3) may be evaluated numerically in the lower atmosphere to determine the eigenvalues and eigenfunctions. The particular temperature profile considered is given in Table C-II. The values of λ corresponding to various frequencies are given for modes "0", "1" and "2" in Tables C-III, C-IV and C-V respectively. As was discussed in the second section the problem of deciding which λ belonged to which mode was determined by the number of zeros of ϕ in the interval (0, 106).

The final step is to integrate equation (20) after determining the behavior of the relative intensity function Q_j. For a burst at ground level, these are given by equation (18) of the first section, while for a burst at height z_o they are given by

$$\widetilde{Q}_j(\omega, z_o) = \left[\frac{\rho_o(z)}{\rho_o(0)}\right]^{\frac{1}{2}} Q_j(\omega) \exp \int_o^{z_o} \frac{g}{c_o^2} \, d\sigma \tag{56}$$

Omitting the details of calculation, $Q_j(\omega)$ was determined for the first three modes with a burst at ground level, and for the gravity wave mode with a burst at 76 km.

Integration of equation (20) with these results yielded expressions for $p(a\theta, t)$. These are shown in Figures C-I and C-II with $a\theta = 5500$ Km. The profile presented in Figure C-I indicates that the gravity wave arrives first with a definite compression but that the first and second modes are superimposed shortly thereafter. It would appear from these results that the high frequencies obtained by Yamomoto in his microbarograph recordings could be attributed to this complicated mixing of the modes.

These results incidentally, were obtained prior to the recent Russian test series. The microbarograph recordings obtained by Carpenter, Harwood, and Whiteside [1961] indicate that the forms obtained here are correct. The height of the firing, however, is not known to us. Since the burst recorded was in the 30 MT range this would have produced a large value of T had it been on the ground and a larger value had it been in the air. This instantaneous velocity approximation which we took would have to be replaced by source dependence given in equations (47)-(49). Since $B(\omega)$ varies as $1/1+(\omega T)^2$ the effect of increasing the height of burst would be to diminish the contribution to the pressure pulse of higher frequencies. Hence, if the location and magnitude of the burst are known, one should be able to distinguish between a surface burst and one high in the atmosphere.

V

THE EFFECT OF WINDS UPON THE GRAVITY WAVE

The final problem considered is an attempt to include the effect of winds upon the gravity wave. Microbarograph recordings have shown that winds affect not only the arrival time of the lowest mode but also tend to "stretch" the pulse. Both of these phenomena indicate that the winds appear to affect the phase velocity, or, in other words, to alter the dispersion. When one speaks of winds, however, the problem of horizontal variations in the earth's atmosphere necessarily arise. On a global scale, these would make the pressure equation unmanageable in that it would be non-separable. This difficulty is circumvented by considering the earth's surface as composed of a number of local regions such that in each the wind is uniform and horizontal. Here we shall consider the behavior in only one such region, but if one were interested in obtaining the global effect he would have to consider all such regions and match the solutions appropriately at their boundaries.

The local region considered must be small enough so that horizontal variations in the atmosphere may be neglected. If its dimensions are on the order of a few wavelengths of the gravity wave, this criterion is fulfilled. In addition, with an area this small, the flat earth approximation will be sufficient. Hence, a cartesian coordinate system may be established such that the x-axis points east, the y-axis toward the north pole and the z-axis is vertical.

The procedure is similar to that mentioned in the first section, in that an equation for the pressure must be determined. The primary difference between the linear equations involving winds and those excluding winds is that in the wind case the particle velocity is superimposed upon the wind velocity wherever a velocity term occurs. In Appendix A a derivation of the following pressure equation is presented:

$$\underline{\nabla}\cdot\left[\frac{1}{\bar{\omega}}\,\underline{L}\,\phi\right]+\frac{i}{h\,\bar{\omega}^2}\left[\frac{\partial \underline{v}}{\partial z}\cdot\underline{\nabla}\right]\left[\frac{\partial\phi}{\partial z}+A\phi\right]+\left[\frac{\bar{\omega}}{c_o^2}+\frac{\partial}{\partial z}\left(\frac{A}{h\,\bar{\omega}}\right)-\frac{A^2}{h\bar{\omega}}\right]\phi=0 \tag{57}$$

If $\underline{i}_\xi$ is a unit vector in the direction of $\underline{v}$ and if $\underline{i}_\eta$ is horizontal and normal to $\underline{i}_\xi$, we may write

$$\underline{v}=\underline{i}_\xi\, v(z)$$

With this definition, the second term in equation (57) becomes

$$\frac{i}{h\,\bar{\omega}^2}\;\frac{\partial v}{\partial z}\;\frac{\partial}{\partial\xi}\left[\frac{\partial\phi}{\partial z}+A\phi\right] \tag{58}$$

where ϕ is now a function of (ξ, η, z). If a function $\psi(\xi, \eta, z)$ defined by

$$\phi=\psi\,\exp\left\{-\int^{z} A\, d\sigma\right\} \tag{59}$$

equation (57) becomes

$$\frac{1}{h\bar{\omega}}\left\{\frac{\partial^2\psi}{\partial z^2}+\frac{\partial\psi}{\partial z}\left[-2A-\frac{1}{h\bar{\omega}}\;\frac{\partial}{\partial z}(h\bar{\omega})\right]+\frac{h\bar{\omega}^2}{c_o^2}\,\psi\right\}+$$

$$\frac{\partial}{\partial\xi}\left[\frac{1}{\bar{\omega}}\;\frac{\partial\psi}{\partial\xi}\right]+\frac{\partial}{\partial\eta}\left[\frac{1}{\bar{\omega}}\;\frac{\partial\psi}{\partial\eta}\right]+\frac{1}{h\bar{\omega}^2}\;\frac{\partial v}{\partial z}\;\frac{\partial^2\psi}{\partial\xi\,\partial z}=0 \tag{60}$$

Since the winds are uniform and horizontal the coefficients in this equation are functions of z alone so that a separable solution

$$\psi(\xi, \eta, z) = \chi(z) \exp\left[-i\omega(\xi\bar{\lambda}_\xi + \eta\bar{\lambda}_\eta)\right] \tag{61}$$

exists. With this form, the equation which $\chi(z)$ satisfies is

$$\frac{d^2\chi}{dz^2} + \left[-2A - \frac{1}{h\bar{\omega}}\frac{d(h\bar{\omega})}{dz} + \frac{\omega}{\bar{\omega}}\frac{dv}{dz}\bar{\lambda}_\xi\right]\frac{d\chi}{dz} - \left[h\omega^2\bar{\lambda}^2 - \frac{h\bar{\omega}^2}{c_o^2}\right]\chi = 0 \tag{62}$$

which, since

$$\frac{1}{\bar{\omega}}\frac{d\bar{\omega}}{dz} = -\frac{\omega}{\bar{\omega}}\frac{dv}{dz}\bar{\lambda}_\xi$$

reduces to

$$\frac{d^2\chi}{dz^2} - \left[2A + \frac{1}{h\bar{\omega}^2}\frac{d(h\bar{\omega}^2)}{dz}\right]\frac{d\chi}{dz} + \left[\frac{h\bar{\omega}^2}{c_o^2} - h\omega^2\bar{\lambda}^2\right]\chi = 0 \tag{63}$$

as the wind speed approaches zero, $\bar{\omega} \to \omega$ and this equation becomes identical to its non-wind counterpart. The dispersive effects of the wind are introduced through the separable form of ψ in equation (61) where the eigenvalue is $\bar{\lambda}$ instead of λ and through the definition $\bar{\omega} = \omega\left[1 - \underline{v} \cdot \bar{\lambda}\right]$. Accordingly, for a given atmospheric model equation (63) may be solved with appropriate boundary conditions, thereby obtaining the relationship between $\bar{\lambda}$ and ω.

The temperature and wind models which were taken are presented in Appendix D, together with a graph indicating the manner in which wind affects the dispersion. The effect of winds on the horizontal propagation is to speed the arrival time significantly at large distances. In addition, the shape of the pulse is altered, or "stretched" in that low frequencies are most affected.

VI

CURRENT STUDIES

The aspects of the problem which are currently under investigation may be divided into two groups.

The first group concerns the region near the source of the pressure pulse where the excess pressure can not be regarded as a small perturbation of the undisturbed pressure. The equations which must be solved in this region are non-linear and a numerical solution using the method of characteristics is the best that can be expected. The particular approach to this rather general problem that is being used is to start at a certain distance from the source with an assumed time-distribution of pressure, which is similar to the observed pulse and which differs from the undisturbed pressure by less than 10 per cent. The distribution of radial velocity which is consistent with this assumed pressure and with the acoustic solution can be determined and these values are used as initial values for a backward space integration of the equations, i.e., to positions nearer the source. The solution will provide evidence of the nature of the modifications produced in the form of the pressure pulse as it traverses the non-linear region and, in particular, an estimate of the accuracy of the use of the acoustic approximation in this region. A knowledge of what happens in the non-linear region is necessary if an attempt is made to deduce the nature of the source from observations of the pressure pulse produced at great distances from the source.

The second group of topics under investigation concerns the difficulties introduced by the use of a mathematical model of the atmosphere extending to infinity vertically and by the use of the finite energy condition as the boundary condition at infinity. Physically, the use of an infinite atmosphere is not very realistic but it may be justified by the idea that the pressure pulse is not much affected by the nature of the atmosphere at great heights. Unfortunately the mathematical solution varies considerably with the boundary condition that is assumed to hold at infinity and it is difficult to find an acceptable model of a finite atmosphere without enclosing it with a rigid boundary. If the finite energy condition is used with the infinite atmosphere (which is the usual practice and is obviously a necessary condition), the solution involves velocities which increase exponentially with height, which is unrealistic and also violates the linearization process by which the equations were derived. If, however, the condition is imposed that the velocities must remain finite everywhere, no solutions are possible. One way to overcome this difficulty is to introduce the effect of viscosity on the wave. Viscosity is of negligible importance near the ground, but when the density decreases, the kinematic viscosity increases and so it is important at great heights. In the absence of viscosity the choice in the upper isothermal layer is between two solutions, both giving increasing velocities but only one having finite energy in a vertical column. If it can be shown that one of these solutions gives finite velocities when the effect of viscosity is introduced, the solution of the problem can proceed as before and if the same solution is involved as was chosen by the finite energy condition, the use of this condition is justified. It may, however, appear that the other solution is required, possibly for a limited range of frequencies, in which case all previous solutions will have to be modified. Another possibility is to use a finite atmosphere with a top layer in which the temperature decreases linearly so that the pressure decreases to zero at a finite height and it is not necessary to put a rigid boundary on the top. This model admittedly bears no resemblance to the actual atmosphere but neither does the infinite isothermal layer usually assumed and it does have the advantage of finiteness.

Another topic under investigation is an examination of the simplest model of the atmosphere, that of two isothermal layers, which is treated incorrectly by Hunt, Palmer and Penney [1960]. The correct form of the frequency equation is being examined analytically to see if any deductions can be made of the effect of varying the relevant parameters.

APPENDIX A

DERIVATION OF PRESSURE EQUATION

The equation for pressure will be derived in cartesian coordinates in the presence of winds. Modifications to the spherical case in the absence of winds will be indicated at the end of this appendix.

The equations of motion with wind included are:

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla)\, \underline{u} + (\underline{u} \cdot \nabla)\, \underline{v} + (\underline{v} \cdot \nabla)\, \underline{u} = \qquad \text{Momentum}$$

$$- \frac{1}{\rho_o + \rho}\, \underline{\nabla}\, (p_o + p) - \underline{i}_z g \qquad (A.1)$$

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho_o \underline{u}) + \underline{\nabla} \cdot (\rho_o \underline{v}) + \underline{\nabla} \cdot (\rho \underline{u}) + \underline{\nabla} \cdot (\rho \underline{v}) = 0 \qquad \text{Continuity} \quad (A.2)$$

$$\frac{\partial p}{\partial t} + (\underline{u} \cdot \underline{\nabla}) p_o + (\underline{v} \cdot \underline{\nabla}) p_o + (\underline{u} \cdot \underline{\nabla}) p + (\underline{v} \cdot \underline{\nabla}) p = \qquad \text{Energy}$$

$$(c_o^2 + c^2) \left\{ \frac{\partial \rho}{\partial t} + (\underline{u} \cdot \underline{\nabla}) \rho_o + (\underline{v} \cdot \underline{\nabla}) \rho_o + (\underline{u} \cdot \underline{\nabla}) \rho + (\underline{v} \cdot \underline{\nabla}) \rho \right\} \qquad (A.3)$$

Second order terms in the perturbed variables will be neglected. In addition, since p_o, ρ_o, and c_o^2 are functions of z alone and since the wind velocity ($\underline{v}$) is uniform and horizontal, several terms in (A.1 - A.3) are identically zero. The harmonic time dependent forms of the equations then become, respectively

$$i\omega \underline{u} + (\underline{v} \cdot \underline{\nabla})\, \underline{u} + u_z \frac{\partial \underline{v}}{\partial z} = - \frac{1}{\rho_o} \underline{\nabla} p - \underline{i}_z \frac{g\rho}{\rho_o} \tag{A.4}$$

$$i\omega \rho + (\underline{v} \cdot \underline{\nabla})\rho + \underline{\nabla} \cdot (\rho_o \underline{u}) = 0 \tag{A.5}$$

$$i\omega p + (\underline{v} \cdot \underline{\nabla}) p - c_o^2 \left[i\omega\rho + \underline{v} \cdot \underline{\nabla}\rho \right] +$$

$$\underline{u} \cdot \left[\underline{\nabla}\, p_o - c_o^2\, \underline{\nabla}\, \rho_o \right] = 0 \tag{A.6}$$

If it is assumed that p, ρ, and $\underline{u}$ have separable solutions whose horizontal variations are of the form $\exp\left[-i\omega(\bar{\lambda}_x x + \bar{\lambda}_y y)\right]$, then, since $(\underline{v} \cdot \underline{\nabla})$ operates only on the horizontal variation,

$$(\underline{v} \cdot \underline{\nabla})\, \chi = - i\omega\, \underline{v} \cdot \underline{\tilde{\lambda}}\, \chi \tag{A.7}$$

(where χ represents p, ρ, or one of the components of $\underline{u}$). For simplification we define $\bar{\omega}$ such that

$$\bar{\omega} = \omega \left[1 - \underline{v} \cdot \underline{\tilde{\lambda}} \right] \tag{A.8}$$

Introducing the quantities

$$\phi = \rho_o^{-\frac{1}{2}}\, p$$

$$R = \rho_o^{-\frac{1}{2}}\, \rho$$

$$\underline{U} = \rho_o^{+\frac{1}{2}}\, \underline{u}$$

and using the definition of $\bar{\omega}$ in (A. 8), equations (A. 4 - A. 6) become

$$i\bar{\omega}\,\underline{U} + U_z \frac{\partial \underline{v}}{\partial z} = -\underline{\nabla}\phi - \frac{1}{2}\,\underline{i}_z \frac{\rho_o'}{\rho_o}\phi - \underline{i}_z\, gR \tag{A.9}$$

$$i\bar{\omega}R + (\underline{\nabla}\cdot\underline{U}) + \frac{1}{2}\frac{\rho_o'}{\rho_o} U_z = 0 \tag{A.10}$$

$$i\bar{\omega}\phi - c_o^2\, i\bar{\omega}R + \underline{U}\cdot\left[\frac{\underline{\nabla}p_o}{\rho_o} - c_o^2\frac{\underline{\nabla}\rho_o}{\rho_o}\right] = 0 \tag{A.11}$$

where the prime denotes differentiation with respect to z.

Defining $B = \dfrac{\rho_o'}{\rho_o} + \dfrac{g}{c_o^2}$, (A. 11) becomes

$$i\bar{\omega}\phi - c_o^2\, i\bar{\omega}R - c_o^2\, B\, U_z = 0 \tag{A.12}$$

which when substituted into (A. 9) and (A. 10) respectively, yields:

$$-\bar{\omega}^2\,\underline{U} + i\bar{\omega}U_z\frac{\partial \underline{v}}{\partial z} = -i\bar{\omega}\,\underline{\nabla}\phi - \frac{1}{2}\,\underline{i}_z\frac{\rho_o'}{\rho_o}\, i\bar{\omega}\phi - \underline{i}_z\, g\left[\frac{1}{c_o^2}\, i\bar{\omega}\phi - B\,U_z\right] \tag{A.13}$$

and

$$\frac{1}{c_o^2}\, i\bar{\omega}\phi - B\,U_z + \underline{\nabla}\cdot\underline{U} + \frac{1}{2}\frac{\rho_o'}{\rho_o}U_z = 0 \tag{A.14}$$

Upon introducing the quantities

$$A = \frac{1}{2}\frac{\rho_o'}{\rho_o} + \frac{g}{c_o^2}$$

and

$$h = \frac{g}{\bar{\omega}^2}\left[\frac{\rho_o'}{\rho_o} + \frac{g}{c_o^2}\right]$$

equation (A.13) in matrix form becomes

$$\begin{bmatrix} -\bar{\omega}^2 & 0 & i\bar{\omega}\dfrac{\partial v_x}{\partial z} \\ 0 & -\bar{\omega}^2 & i\bar{\omega}\dfrac{\partial v_y}{\partial z} \\ 0 & 0 & -h\bar{\omega}^2 \end{bmatrix}\begin{bmatrix} U_x \\ U_y \\ U_z \end{bmatrix} = -i\bar{\omega}\begin{bmatrix} \dfrac{\partial \phi}{\partial x} \\ \dfrac{\partial \phi}{\partial y} \\ \dfrac{\partial \phi}{\partial z} + A\phi \end{bmatrix} \tag{A.15}$$

while (A.14) may be rewritten as

$$i\bar{\omega}\phi + c_o^2\, \underline{\nabla}\cdot \underline{U} - c_o^2 A U_z = 0 \tag{A.16}$$

When $h \neq 0$, the solution of (A.15) is

$$\begin{bmatrix} U_x \\ U_y \\ U_z \end{bmatrix} = \frac{i}{h\bar{\omega}^3}\begin{bmatrix} h\bar{\omega}^2 & 0 & i\bar{\omega}\dfrac{\partial v_x}{\partial z} \\ 0 & h\bar{\omega}^2 & i\bar{\omega}\dfrac{\partial v_y}{\partial z} \\ 0 & 0 & \bar{\omega}^2 \end{bmatrix}\begin{bmatrix} \dfrac{\partial \phi}{\partial x} \\ \dfrac{\partial \phi}{\partial y} \\ \dfrac{\partial \phi}{\partial z} + A\phi \end{bmatrix} \tag{A.17}$$

Substitution of (A.17) into (A.16) yields, after some simplification

$$\underline{\nabla}\cdot\left[\frac{1}{\bar{\omega}}\,\underline{L}\phi\right] + \frac{i}{h\bar{\omega}^2}\left(\frac{\partial \underline{v}}{\partial z}\cdot\underline{\nabla}\right)\left(\frac{\partial \phi}{\partial z} + A\phi\right) + \left[\frac{\bar{\omega}}{c_o^2} + \left(\frac{A}{h\bar{\omega}}\right)' - \frac{A^2}{h\bar{\omega}}\right]\phi = 0 \tag{A.18}$$

where

$$\underline{L} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{1}{h}\frac{\partial}{\partial z}\right]$$

When no winds are present, $\underline{v} = 0$ and $\overline{\omega} = \omega$, so that (A.18) reduces to

$$(\underline{\nabla} \cdot \underline{L})\, \phi + \left[\frac{\omega^2}{c_o^2} + \left(\frac{A}{h}\right)' - \frac{A^2}{h}\right] \phi = 0 \qquad \text{(A.19)}$$

In spherical coordinates, when winds are neglected, the derivation proceeds in a similar manner, except that the radial dependence of the divergence of $\underline{U}$ is given by $\frac{2}{r} U_r + \frac{\partial U_r}{\partial r}$. This, in effect introduces an extra term, so that, in spherical coordinates

$$(\underline{\nabla} \cdot \underline{L})\, \phi + \frac{q(r)}{r^2}\, \phi = 0 \qquad \text{(A.20)}$$

where

$$\underline{L} = \left[\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{1}{r \sin\theta}\frac{\partial}{\partial \psi}\right]$$

and

$$q(r) = \left[\frac{\omega^2 r^2}{c_o^2} + \frac{r^2}{h}\left(-A^2 + A' - \frac{Ah'}{h} + \frac{2A}{r}\right)\right]$$

From the definition of ϕ given above, we finally arrive at equation (1) of Section II, namely

$$(\underline{\nabla} \cdot \underline{L})\left(\rho_o^{-\frac{1}{2}} p\right) + \frac{q(r)}{r^2}\left(\rho_o^{-\frac{1}{2}} p\right) = 0 \qquad \text{(A.21)}$$

APPENDIX B

TEMPERATURE MODELS CONSIDERED IN SECTION II AND NUMERICAL RESULTS OBTAINED

Two of the three temperature models considered in Section II are as follows:

TABLE B-I

Atmosphere I		Atmosphere II	
Altitude (gkm)	Temperature (^oC)	Altitude (gkm)	Temperature (^oC)
0	29	0	17
1.5	18	1.5	8
3.1	10	3.0	- 3
5.9	- 5	5.6	- 19
9.7	-31	9.2	- 46
12.4	-53	11.9	- 49
14.2	-67	13.8	- 47
16.6	-81	16.4	- 47
20	-66	20	- 48
25	-65	25	- 46
30	-45	30	- 42
35	-34	35	- 34
40	-20	40	- 18
45	- 5	45	- 2
50	7	50	4

where altitude above sea-level is measured in geopotential kilometers and temperature in degrees centigrade. Above 50 gkm, these models are isothermal at 7^oC and 4^oC respectively. The third, which is Scorer's model (1950), was considered for the sake of comparison. For this case the lower atmosphere has a constant temperature gradient with temperatures 286.91^oK and 229.53^oK at sea-level and

9.6137 km respectively, while the upper atmosphere is isothermal at 229.53°K.

The calculated values of the inverse phase velocities for the gravity wave mode in each of these atmospheres is presented in the following table.

TABLE B-II

	Inverse Phase Velocities λ(km^{-1} sec)		
5000 ω^2	Atmosphere I	Atmosphere II	Scorer's Model
0	3.15283	3.16379	3.18947
1	3.15819	3.16773	3.19148
2	3.16277	3.17072	3.19355
3	3.16695	3.17319	3.19570
4	3.17091	3.17534	3.19793
5	3.17472	3.17728	3.20025
6	3.17844	3.17907	3.20266
7	3.18209	3.18074	3.20517
8	3.18569	3.18233	3.20778
9	3.18923	3.18384	3.21052
10	3.19274	3.18530	3.21338
11	3.19620	3.18672	3.21638
12	3.19962	3.18809	3.21953
13	3.20301	3.18943	3.22286
14	3.20635	3.19073	3.22637
15	3.20965	3.19201	3.23006
16	3.21290	3.19327	---
17	3.21611	3.19450	---
18	3.21928	3.19571	---
19	3.22239	3.19690	---
20	3.22545	3.19807	---
21	3.22846	3.19922	---
22	3.23142	3.20035	---
23	3.23433	3.20147	---
24	3.23718	3.20257	---
25	3.23998	3.20365	---

The pressure pulse forms for Scorer's model of the atmosphere for explosions both on the ground and at a height of 9.6137 km are plotted in Figure B-1. In addition, pulse forms for atmospheres I and II are given in Figures B-2 and B-3, for a range of 7000 miles.

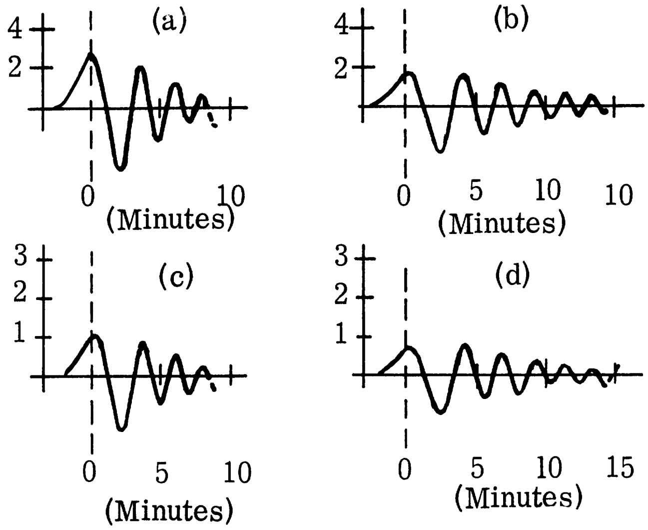

FIGURE B-1: THE PRESSURE PULSE AT THE GROUND AT A DISTANCE OF 3600 KM (a) AND (c), AND 6000 KM (b) AND (d) FOR SCORER'S ATMOSPHERE, FOR AN EXPLOSION ON THE GROUND (a) AND (b), AND AT A HEIGHT OF 9.6137 KM (c) AND (d). ONE UNIT OF AMPLITUDE CORRESPONDS TO 0.614 μBAR PER KM^3 OF GAS RELEASED.

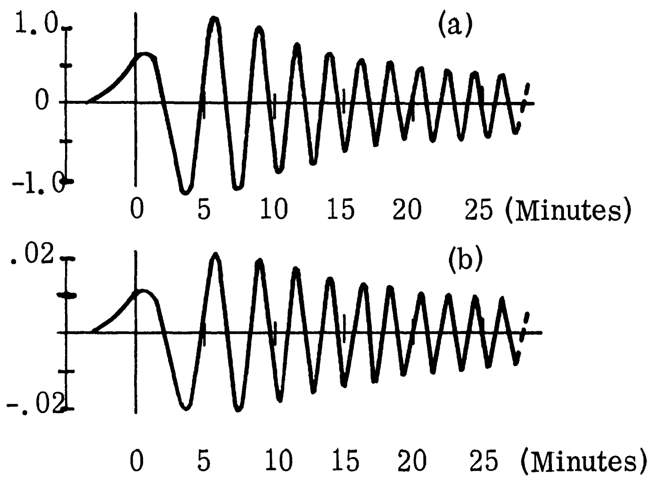

FIGURE B-2: THE HEAD OF THE PRESSURE PULSE AT GROUND LEVEL AT A DISTANCE OF 7000 KM FROM AN EXPLOSION ON THE GROUND (a), AND AT A HEIGHT OF 39 KM (b), FOR ATMOSPHERE MODEL I. ONE UNIT OF AMPLITUDE CORRESPONDS TO 1 μ BAR PER 1 KM^3 OF GAS RELEASED. (THE TAIL OF THE PULSE WHICH EXTENDS FOR A VERY LONG TIME INTERVAL BEYOND THE BROKEN LINES IS NOT GIVEN.

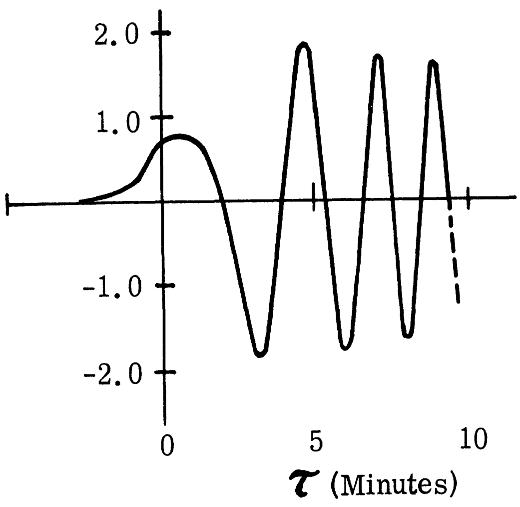

FIGURE B-3: THE HEAD OF THE PRESSURE PULSE AT GROUND LEVEL AT A DISTANCE OF 7000 KM FROM AN EXPLOSION ON THE GROUND FOR ATMOSPHERE MODEL II. ONE UNIT OF AMPLITUDE CORRESPONDS TO 1 μBAR PER 1 KM^3 OF GAS RELEASED.

APPENDIX C

C-I: VARIATION WITH ENERGY AND ALTITUDE OF CHARACTERISTIC PERIOD

TABLE C-I

ESTIMATES OF THE CHARACTERISTIC PERIOD T OF THE EXPLOSION FOR VARIOUS ENERGIES AND ALTITUDES

Altitude \ Energy W	1 Kiloton	10 Kilotons	100 Kilotons	1 Megaton
0 Km	.356	.766	1.65	3.56
25 Km	1.41	3.03	6.55	14.1
50 Km	3.77	8.1	17.5	37.7
76 Km	16.1	34.6	74.8	161.

Period T in seconds

APPENDIX C

C-II: TEMPERATURE MODELS CONSIDERED IN SECTION IV AND NUMERICAL RESULTS OBTAINED

The temperature model considered in Section III was of the thermosphere variety, where, for $z \gtrsim 106$ Km,

$$T_o = 228 + 18(z-106).$$

For $z \lesssim 106$ Km the following temperature profile was used.

TABLE C-II

TEMPERATURE PROFILE

Height (Km)	Temperature (^{o}K)
0	290
1.47	281
2.94	270
5.49	254
9.01	227
11.65	224
13.51	226
16.05	226
19.59	225
24.5	227
26.4	231
34.3	239
39.7	255
44.1	271
49.	277
54.	277
80	165
91	165
106	228

The calculated values of the inverse phase velocities for modes "0", "1", and "2" for this temperature model are presented in Tables C-III, C-IV and C-V respectively

TABLE C-III

INVERSE PHASE VELOCITY λ FOR MODE "0"

$5000\ \omega^2$	λ
.05	3.18077
.2	3.18093
.5	3.18149
.9	3.18195
1.8	3.18388
2.33	3.18519
2.8	3.18663
3.2318	3.18841
3.5069	3.18983
3.7457	3.19143
3.9712	3.19340
4.2	3.19674
4.35	3.19979
4.5	3.20427
4.6	3.20859
4.7	3.21463
4.8	3.22330
4.9	3.23574
4.95	3.2436

TABLE C-IV

INVERSE PHASE VELOCITY λ FOR MODE "1"

$5000\,\omega^2$	λ	$5000\,\omega^2$	λ
3	1.5025	6.5	3.17991
3.2318	2.05585	6.75	3.18120
3.5069	2.37990	7	3.18228
3.7457	2.57611	8	3.18557
3.9712	2.71701	8.5	3.18699
4.2	2.85164	9	3.18841
4.5	2.98294	9.5	3.18995
4.8	3.08486	10	3.19178
5	3.12938	10.5	3.19422
5.25	3.15641	11	3.19793
5.5	3.16743	11.5	3.20451
5.75	3.17287	11.8	3.21084
6	3.17610	12	3.21624
6.25	3.17828	12.2	3.2202

TABLE C-V

INVERSE PHASE VELOCITY λ FOR MODE "2"

$5000\,\omega^2$	λ	$5000\,\omega^2$	λ
5.5	2.37693	15	3.18958
6	2.53209	15.5	3.19057
7	2.74838	16	3.19146
8	2.89013	17	3.19304
9	2.99269	18	3.19445
10	3.07155	19	3.19576
11	3.13242	20	3.19699
11.5	3.15506	21	3.19818
11.75	3.16365	22	3.19933
12	3.17021	23	3.20044
12.5	3.17869	24	3.20153
13	3.18265	25	3.20261
13.5	3.18525	26	3.20366
14	3.18705	27	3.20470
14.5	3.18844	28	3.20573

In the following figures, the head of the pressure pulse at ground level at a distance of 5500 Km from the source is shown. The solid, dashed, and dotted curves of Figure C-1 correspond to the contributions of modes "0", "1" and "2" respectively for an explosion on the ground, while in Figure C-2 only the gravity wave mode for an explosion at 76 Km is shown.

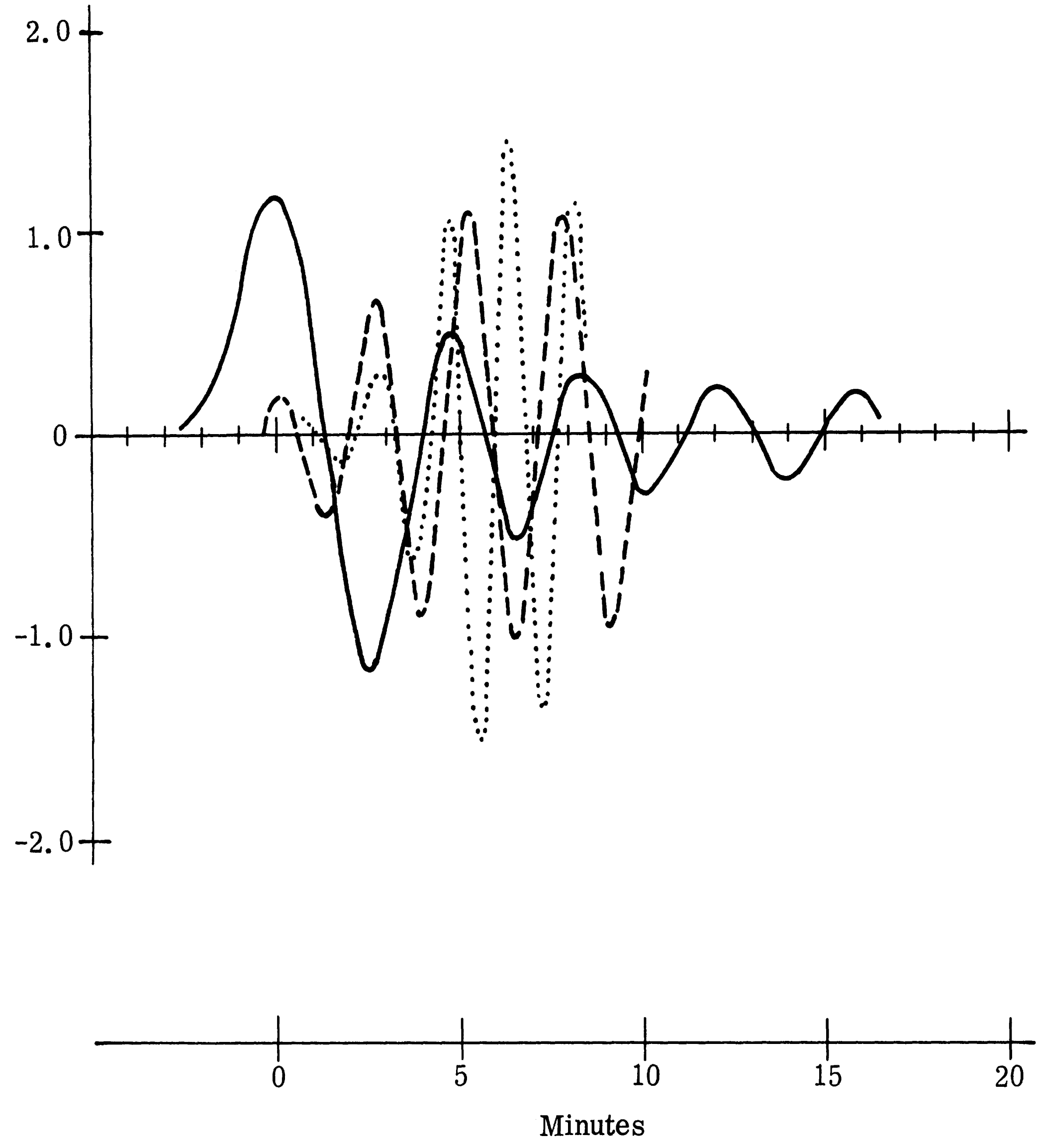

FIGURE C-1: THE HEAD OF THE PRESSURE PULSE AT GROUND LEVEL AT A DISTANCE OF 5500 KM FROM AN EXPLOSION ON THE GROUND. THE SOLID, DASHED AND DOTTED CURVES CORRESPOND TO THE CONTRIBUTIONS OF MODES "0", "1" and "2" RESPECTIVELY. THE AMPLITUDES ARE GIVEN IN μ BARS PER 1 KM^3 OF GAS RELEASED.

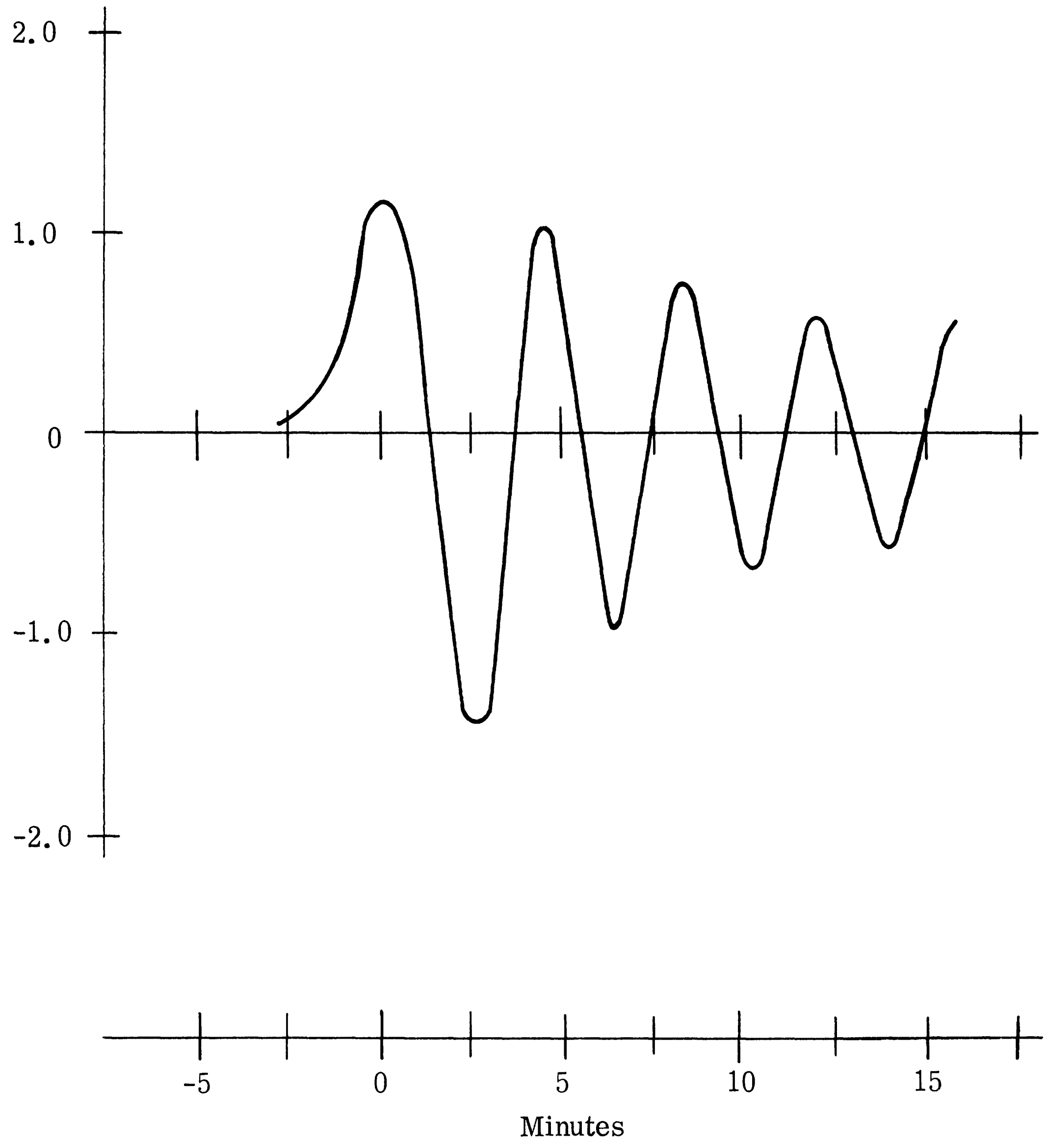

FIGURE C-2: THE HEAD OF THE PRESSURE PULSE AT GROUND LEVEL CORRESPONDING TO MODE "0", AT A DISTANCE OF 5500 KM FROM AN EXPLOSION AT AN ALTITUDE OF 76 KM. THE AMPLITUDE IS EXPRESSED IN UNITS OF 2.9655 x 10^{-4} μ BARS PER 1 KM^3 OF GAS RELEASED.

APPENDIX D

TEMPERATURE AND WINDS PROFILE CONSIDERED IN SECTION V, DETERMINATION OF DISPERSIVE RESULTS FOR THIS CASE

The particular model of the atmosphere to be considered has the following temperature and wind profiles:

TABLE D-I

TEMPERATURE PROFILE

Alt. (gkm)	Temp (^{o}C)
0	18
1.5	8
3.1	2
5.7	-12
9.5	-35
12.2	-51
14.0	-60
16.5	-66
20.0	-67
25.0	-56
30.0	-48
35.0	-38
40.0	-26
45.0	-11
50.0	6

TABLE D-II

WIND PROFILE

Alt. (gkm)	Wind Vel. (M sec^{-1})
0	0
12.2	-9.5
16.5	-8.5
20.0	7.5
25.0	13.0
30.0	15.5
40.0	29.0
61.2	38.0
69.0	30.0
76.5	0

where the altitude is measured in geopotential kilometers (equivalent of .98 dynamic kilometers).

Between the surface of the earth and 12.2 gkm the wind velocity is assumed to decrease at a constant rate. Above an altitude of 50 gkm, the atmosphere is isothermal at 6^{o}C, while above 76.5 gkm the wind velocity is zero. It should be mentioned at this point that in the numerical calculations that were performed the wind velocity enters only through the term $\underline{v} \cdot \underline{\lambda}$. The results which were obtained are based on the assumption that the wind velocities given in Table D-II are the components of the wind in the direction of $\bar{\underline{\lambda}}$.

The method of solving equation (63) simply involves the application of fairly standard boundary conditions which are as follows:

a) Above 76.5 gkm (75 Km), the atmosphere is isothermal as has been stated, so that the solution of equation (63) is an exponential. The condition that the kinetic energy be finite in a vertical column above this altitude requires that the solution be

$$\chi = \exp\left[-\beta z + A_s z\right] \tag{D.1}$$

where

$$\beta^2 = A_s^2 - h_s \omega^2 \left[\frac{1}{c_s^2} - \bar{\lambda}^2\right] \tag{D.2}$$

and

$$A_s = (2-\gamma/2)\frac{g}{c_s^2} \tag{D.3}$$

Below 75 Km, equation (63) was solved by numerical methods, and the imposed boundary conditions were:

b) At 75 Km, the pressure and the vertical component of the velocity are continuous. This implies that ψ and ψ'/h must be continuous here. Normalizing ψ so that $\psi = 1$ at this altitude, the boundary condition from (D.1) is that $\psi'(75) = (-\beta + A_s)$.

c) At 49 Km where there is a discontinuity in the temperature gradient, pressure and the vertical component of the velocity must be continuous. Therefore, we require that at this interface, ψ and $\psi'/\bar{\omega}^2 h$ be continuous.

d) On the surface of the earth, the vertical component of the velocity is zero, so that $\psi'(0) = 0$. This final condition makes it possible to obtain $\bar{\lambda}$.

The results obtained for this particular wind profile are plotted in Figure D-1 (solid curve). The dotted line, on the other hand, represents the results which were obtained when the wind was neglected.

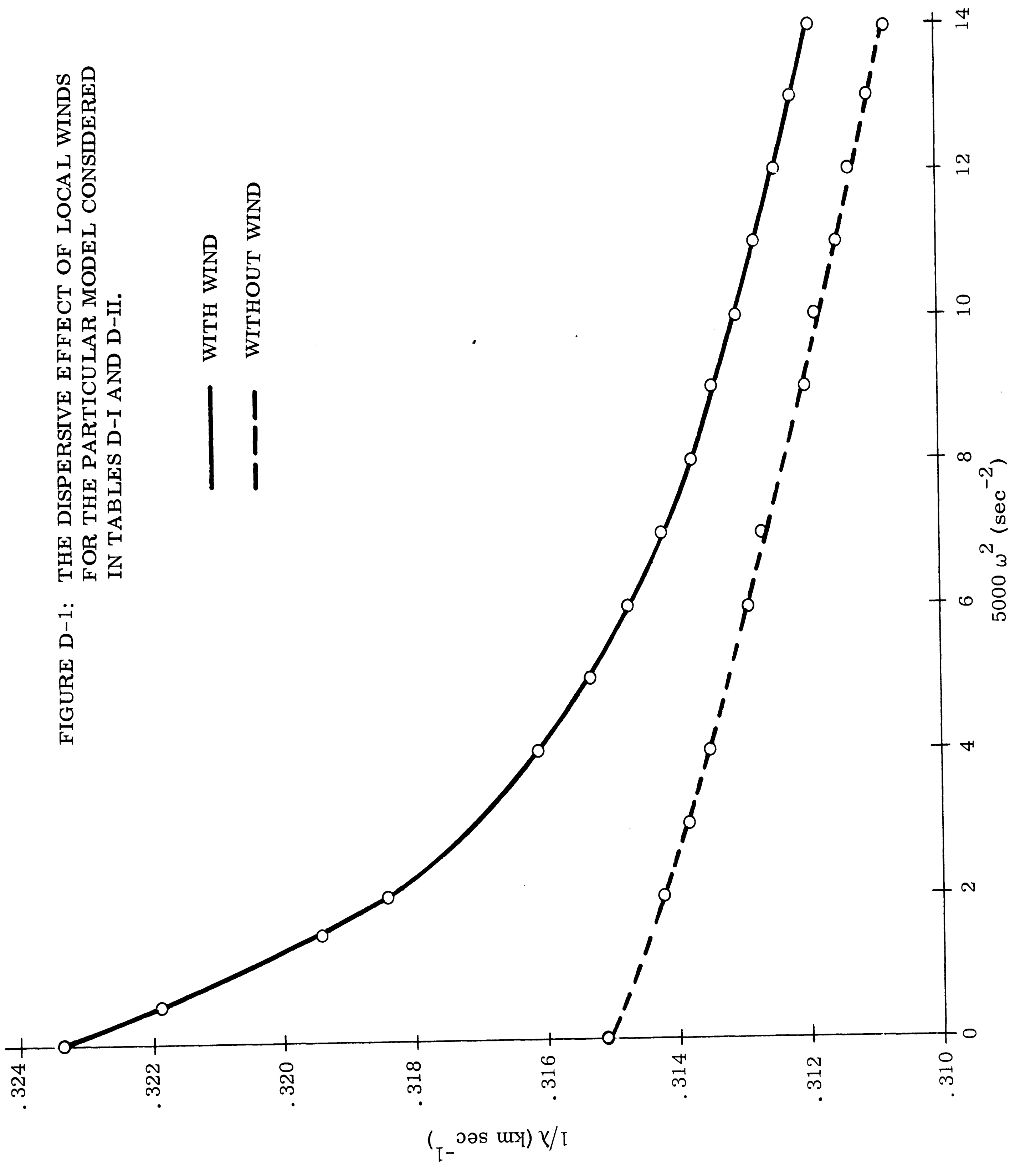

FIGURE D-1: THE DISPERSIVE EFFECT OF LOCAL WINDS FOR THE PARTICULAR MODEL CONSIDERED IN TABLES D-I AND D-II.

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