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STUDIES IN RADAR CROSS SECTIONS XLVII -
DIFFRACTION AND SCATTERING BY REGULAR BODIES - I:
THE SPHERE

by

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PREFACE

This is the forty-seventh in a series of reports growing out of the study of radar cross sections at The Radiation Laboratory of The University of Michigan. Titles of the reports already published or presently in process of publication are listed on the preceding pages.

When the study was first begun, the primary aim was to show that radar cross sections can be determined theoretically, the results being in good agreement with experiment. It is believed that by and large this aim has been achieved.

In continuing this study, the objective is to determine means for computing the radar cross section of objects in a variety of different environments. This has led to an extension of the investigation to include not only the standard boundary-value problems, but also such topics as the emission and propagation of electromagnetic and acoustic waves, and phenomena connected with ionized media.

Associated with the theoretical work is an experimental program which embraces (a) measurement of antennas and radar scatterers in order to verify data determined theoretically; (b) investigation of antenna behavior and cross section problems not amenable to theoretical solution; (c) problems associated with the design and development of microwave absorbers; and (d) low and high density ionization phenomena.

R. E. Hiatt

I

INTRODUCTION

This is the first of a series of reports aimed at summarizing the available information about the scattering properties of selected bodies of simple shape. Perhaps the simplest of all shapes is the sphere, and it is probable that more has been written about this one body than about all other bodies put together. To detail all of the results in one report is therefore impossible, and in seeking to summarize them so as to provide an intelligible account, an author is compelled to restrict himself to those theories and those methods of solution which he feels are most significant.

In taking as the subject of this first report the diffraction of electromagnetic energy by the sphere, our object is to gather together in one place some of the more useful forms of solution, both exact and approximate, giving also a brief account of the methods of derivation. Wherever possible references are given to tabulations of the functions and series involved, particularly in connection with the standard Mie solution.

Section II is devoted to the Mie solution and since this is the starting point for most of the other approaches, a detailed description is given. Certain special applications are discussed, and references are given to computations based on the Mie series.

For sufficiently low frequencies an alternative representation of the solution is possible in which the field components are expanded in ascending positive integral powers of ka , where k is the wave number and a is the radius of the sphere. The corresponding expansion for the far field amplitude is the so-called Rayleigh series, and this is described in Section III. Two derivations are given: in the first of these the series is obtained by expanding the various terms in the Mie solution, but in the second the low frequency expansion is obtained directly without any explicit reference to the Mie result.

Section IV is concerned with the high frequency scattering behavior and the approach which is adopted is based on the Watson transform. In recent years the Watson transform technique has been generalized to an extent which permits the asymptotic solution of a large class of diffraction problems, and the general method stems from the fact that locally all convex bodies with radii of curvature much larger than the wavelength are similar to a sphere of radius equal to that of the convex body in the direction of energy flow. This local analysis led Fock [1946] to construct certain universal functions which have been computed and tabulated by Logan [1959]. Since this material does not appear in any standard reference, a relatively detailed exposition is given.

In the final section the physical optics approach is considered insofar as it applies to the sphere problem. The approximate expressions for the current

distribution and for the far field are compared with the exact expressions derived from the Mie series, and a numerical comparison for $ka = 10$ is presented.

Although the previous methods have covered the region of 'applicability' of physical optics, it was felt that this approximate but well-known technique should be included both for historical interest and because of the ease with which rough and ready answers can be obtained thereby. For a general and more critical exposition of the physical optics method, the reader is referred to Baker and Copson [1950].

II

THE EXACT SOLUTION

This section is devoted to the exact solution of the problem of scattering of a plane electromagnetic wave by a sphere. A brief account of the derivation is included since almost all subsequent computations and approximations rely to some extent on this exact result. In addition to the homogeneous sphere and the important limiting case of perfect conductivity, results for two concentric spheres are also presented. The simplifications stemming from the "far field" assumption are also discussed. A guide to computed results available in the literature is presented and some representative curves are included.

2.1 The Mie Series For the Sphere

The first exact solution for the scattering of a plane wave by a homogeneous sphere is usually attributed to Mie [1908] although much work was done before then. Thompson [1893] treated the perfectly conducting sphere with equal rigor, and in his exhaustive work on the sphere Logan [1959] gives precedence to Clebsch [1863]. Nevertheless the series solution for the sphere in terms of spherical wave functions is usually referred to as the Mie series and this general usage will be employed here. Descriptions of the solution abound in the literature, the most popular, perhaps, being that given by Stratton [1941] and it is his presentation on which the present account is based.

The problem is that of determining the electric and magnetic field vectors,

$$\underline{E} = \underline{E}^i + \underline{E}^s, \text{ and } \underline{H} = \underline{H}^i + \underline{H}^s$$

(where i and s denote incident and scattered respectively), external to a homogeneous sphere of radius a, permeability μ_1 , permittivity ϵ_1 , and conductivity s_1 in the presence of an incident or primary field given by

$$\begin{aligned} \underline{E}^i &= E_0 \hat{x} e^{-ikz}, \\ \underline{H}^i &= -H_0 \hat{y} e^{-ikz}. \end{aligned} \tag{2-1}^+$$

A rectangular Cartesian coordinate system (x, y, z) has been employed in which eqns (2-1) describe a plane wave travelling in the direction of the negative z-axis with its electric vector confined to the x direction. $H_0 = Y E_0$, where $Y = \frac{1}{Z} = \frac{k}{\omega \mu_0}$ is the intrinsic admittance of free space; k is the propagation constant of the medium in which the sphere is imbedded, which medium is assumed homogeneous, isotropic, and a perfect dielectric and is here taken as free space. In terms of the permittivity and permeability,

$$k = \omega \sqrt{\epsilon_0 \mu_0} = \frac{2\pi}{\lambda}, \tag{2-2}$$

where λ is the wavelength. M. k. s. units are employed and the harmonic time factor $e^{-i\omega t}$ has been suppressed. The restriction to free space is a trivial one because in a medium characterized by ϵ and μ different from their free space values, a propagation constant k may be defined as

⁺An underlined symbol denotes a vector and a caret denotes a unit vector.

$$k = \omega \sqrt{\epsilon \mu} \quad . \quad (2-3)$$

Similarly, if the conductivity s is non-zero, the propagation constant can be taken as

$$k = \omega \sqrt{\mu \left(\epsilon + \frac{is}{\omega} \right)} \quad . \quad (2-4)$$

It is convenient to have the center of the sphere coincide with the origin of the coordinate system. This detracts none of the generality and permits the use of spherical polar coordinates (r, θ, ϕ) where

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad \text{and} \quad z = r \cos \theta, \quad (2-5)$$

in terms of which the surface of the sphere is simply $r = a$ (see Figure 2-1).

The free space, source free, Maxwell equations are

$$\begin{aligned} \nabla \wedge \underline{E} + \mu_o \frac{\partial \underline{H}}{\partial t} &= 0 \quad , \\ \nabla \wedge \underline{H} - \epsilon_o \frac{\partial \underline{E}}{\partial t} &= 0 \quad , \\ \nabla \cdot \underline{H} = \nabla \cdot \underline{E} &= 0 \quad . \end{aligned} \quad (2-6)$$

After suppressing the harmonic time variation these equations require that all field quantities exterior to the sphere be solutions of the vector wave equation,

$$\nabla \wedge (\nabla \wedge \underline{F}) = k^2 \underline{F} \quad , \quad (2-7)$$

where \underline{F} can be \underline{E}^i , \underline{E}^s , \underline{H}^i or \underline{H}^s . Interior to the sphere, \underline{E} and \underline{H} must satisfy

$$\nabla \wedge (\nabla \wedge \underline{F}) = k_1^2 \underline{F} \quad , \quad (2-8)$$

where k_1 is the propagation constant for the material comprising the sphere.

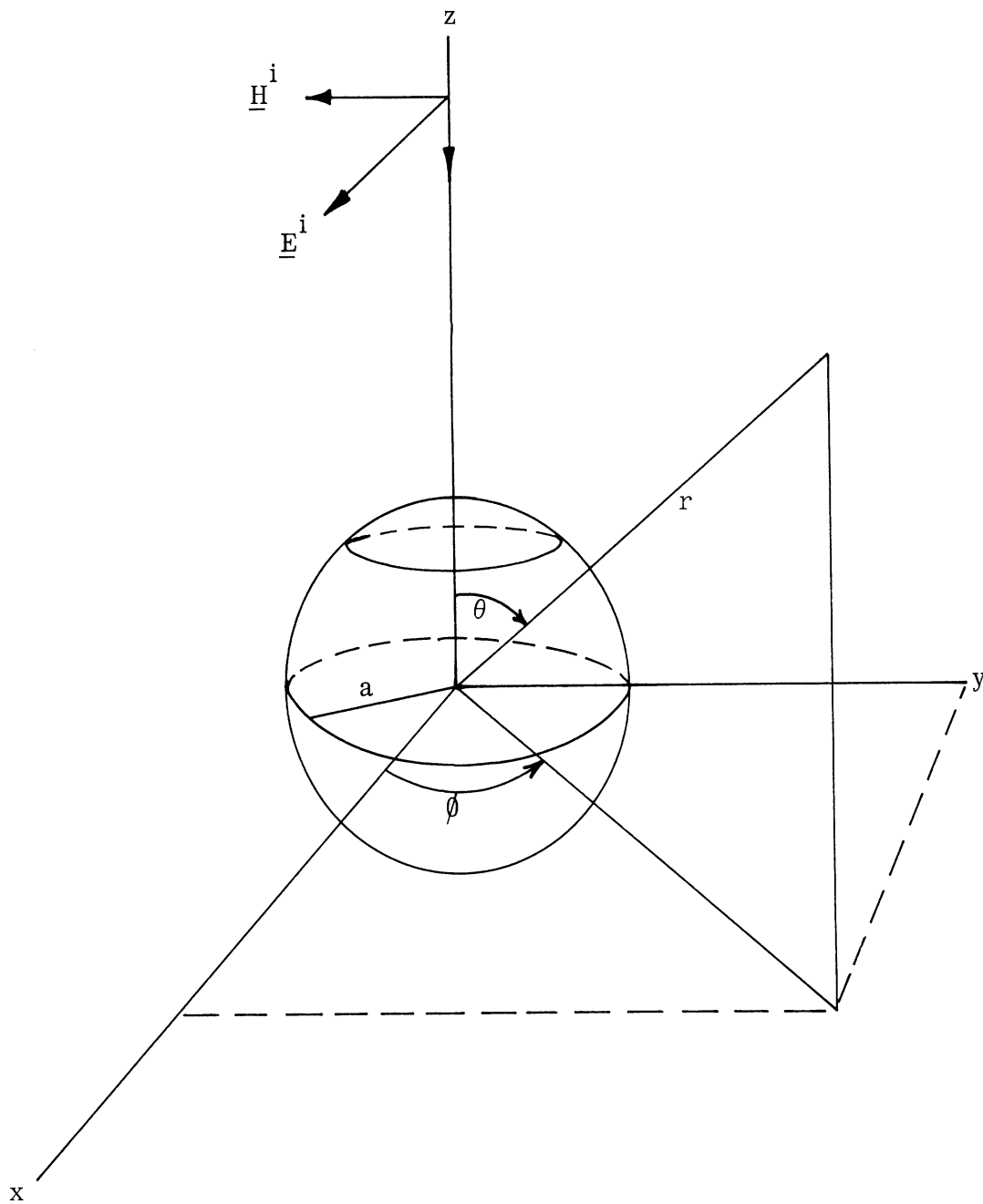


FIGURE 2-1: SPHERE GEOMETRY

The boundary conditions are really continuity conditions at the surface of the sphere, i. e. ,

$$\hat{r} \wedge (\underline{E}^i + \underline{E}^s) \Big|_{r=a^+} = \hat{r} \wedge \underline{E} \Big|_{r=a^-} , \quad (2-9)$$

$$\hat{r} \wedge (\underline{H}^i + \underline{H}^s) \Big|_{r=a^+} = \hat{r} \wedge \underline{H} \Big|_{r=a^-} .$$

General solutions of the vector wave equation can be generated by vector operations on the solutions of the scalar wave equation

$$(\nabla^2 + k^2) \psi = 0 , \quad (2-10)$$

in the following way. If ψ is a solution of eqn(2-10) then the three vectors

$$\begin{aligned} \underline{L} &= \nabla \psi \\ \underline{M} &= \nabla \wedge (\underline{r} \psi) \\ \underline{N} &= \frac{1}{k} \nabla \wedge \underline{M} , \end{aligned} \quad (2-11)$$

are orthogonal solutions of eqn (2-7). These are known as Hansen's vector wave functions, having been proposed by Hansen [1935, 1936, 1937] in his work on radiation from antennas. They are discussed more fully by Stratton [1941] and Senior [1960] .

Since field quantities are required by Maxwell's equations to be solenoidal, or divergence free, the fact that

$$\nabla \cdot \underline{L} = \nabla^2 \psi = -k^2 \psi \neq 0 , \quad (2-12)$$

shows that only the \underline{M} and \underline{N} vectors can be involved in their representation.

The appropriate scalar wave function ψ will differ depending on whether the field point lies inside or outside the sphere. The two forms are dictated by the requirements that the field remain finite at the origin and that the scattered field obey a radiation condition at infinity.

Thus, within the body,

$$\psi = j_n(k_1 r) P_n^m(\cos \theta) \frac{\cos^m \theta}{\sin^m \theta} \quad (2-13)$$

whilst for the exterior region

$$\psi = h_n(k r) P_n^m(\cos \theta) \frac{\cos^m \theta}{\sin^m \theta} \quad (2-14)$$

where P_n^m is the associated Legendre function defined in terms of the hypergeometric function as

$$P_n^m(x) = \frac{(-1)^m}{\Gamma(1-m)} \left(\frac{1+x}{1-x} \right)^{\frac{m}{2}} {}_2F_1(-n, n+1; 1-m; \frac{1-x}{2}) \quad *$$

and j_n and h_n are the spherical Bessel and Hankel functions respectively defined by

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+1/2}(x), \quad h_n(x) = \sqrt{\frac{\pi}{2x}} H_{n+1/2}^{(1)}(x) \quad (2-15)$$

The use of the Hankel function of the first kind to represent outgoing waves at infinity is necessitated by the assumed time dependence, $e^{-i\omega t}$. Since the field must be continuous and single-valued throughout the region external to the sphere, m and n can take on only integral values.

If the expression for ψ given in eqn (2-13) is now introduced into the eqns (2-11) defining the vector wave functions \underline{L} , \underline{M} , and \underline{N} , these functions take

* This is consistent with Stratton (1941) but differs by $(-1)^m$ from most standard mathematical works.

the following form:

$$\begin{aligned} \underline{L}_{e_{mn}}^{(1)} = & k j_n'(kr) P_n^m(\cos \theta) \frac{\cos m \phi}{\sin m \phi} \hat{r} + \frac{1}{r} j_n(kr) \frac{\partial}{\partial \theta} P_n^m(\cos \theta) \frac{\cos m \phi}{\sin m \phi} \hat{\theta} \\ & + \frac{m}{r \sin \theta} j_n(kr) P_n^m(\cos \theta) \frac{\sin m \phi}{\cos m \phi} \hat{\phi}, \end{aligned}$$

$$\underline{M}_{e_{mn}}^{(1)} = - \frac{m}{\sin \theta} j_n(kr) P_n^m(\cos \theta) \frac{\sin m \phi}{\cos m \phi} \hat{\theta} - j_n(kr) \frac{\partial}{\partial \theta} P_n^m(\cos \theta) \frac{\cos m \phi}{\sin m \phi} \hat{\phi},$$

$$\begin{aligned} \underline{N}_{e_{mn}}^{(1)} = & \frac{n(n+1)}{kr} j_n(kr) P_n^m(\cos \theta) \frac{\cos m \phi}{\sin m \phi} \hat{r} + \frac{1}{kr} \left[kr j_n'(kr) \right]' \frac{\partial}{\partial \theta} P_n^m(\cos \theta) \frac{\cos m \phi}{\sin m \phi} \hat{\theta} \\ & + \frac{m}{kr \sin \theta} \left[kr j_n(kr) \right]' P_n^m(\cos \theta) \frac{\sin m \phi}{\cos m \phi} \hat{\phi}, \end{aligned}$$

(2-16)

where the primes indicate differentiation with respect to kr , the subscripts e and o are short for "even" and "odd" respectively and refer to the ϕ dependence of the characteristic solution ψ , and the superscript (1) denotes the radial function used.

The superscript (2) will be used to denote the functions obtained if $j_n(kr)$ is replaced by $h_n(kr)$ and (3) will be used if $j_n(kr)$ is replaced by $j_n(k_1 r)$.

Since the field quantities are solenoidal the most general expression for the scattered electric field is

$$\underline{E}^S = E_o \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (A_{e_{mn}} \underline{M}_{e_{mn}}^{(2)} + B_{e_{mn}} \underline{N}_{e_{mn}}^{(2)}) \quad (2-17)$$

where the coefficients $A_{e_{o mn}}$ and $B_{e_{o mn}}$ involve only the propagation constants k and k_1 and the sphere radius, a . From Maxwell's equations (2-6), it is seen then that

$$\underline{H}^s = -i H_o \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (B_{e_{o mn}} \underline{M}_{e_{o mn}}^{(2)} + A_{e_{o mn}} \underline{N}_{e_{o mn}}^{(2)}) \quad , \quad (2-18)$$

where the coefficients in this equation are the same as in eqn(2-17).

Similarly, the most general expressions for the fields within the sphere ($r < a$), are

$$\underline{E} = E_o \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (C_{e_{o mn}} \underline{M}_{e_{o mn}}^{(3)} + D_{e_{o mn}} \underline{N}_{e_{o mn}}^{(3)}) \quad , \quad (2-19)$$

and

$$\underline{H} = -i \frac{E_o k_1}{\omega \mu_1} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (D_{e_{o mn}} \underline{M}_{e_{o mn}}^{(3)} + C_{e_{o mn}} \underline{N}_{e_{o mn}}^{(3)}) \quad , \quad (2-20)$$

where the constants $C_{e_{o mn}}$ and $D_{e_{o mn}}$ again involve only the propagation constants and radius.

For the incident field given by eqn (2-1), expansions in terms of vector wave functions are given in Stratton [1941] as

$$\underline{E}^i = E_o \sum_{n=1}^{\infty} (-i)^n \frac{2n+1}{n(n+1)} (M_{o1n}^{(1)} + i N_{e1n}^{(1)}) \quad (2-21)$$

$$\text{and} \quad \underline{H}^i = -i H_0 \sum_{n=1}^{\infty} (-i)^n \frac{2n+1}{n(n+1)} \left(i M_{e1n}^{(1)} + N_{o1n}^{(1)} \right). \quad (2-22)$$

No terms corresponding to $n=0$ occur because $P_0^1(\cos \theta) \equiv 0$.

Straightforward substitution of eqns (2-17) to (2-22) in the continuity relations (2-9) now leads to the following values for the unknown coefficients:

$$\begin{aligned} A_{emn} &= B_{omn} = C_{emn} = D_{omn} = 0, \text{ for all } m \text{ and } n, \\ A_{omn} &= B_{emn} = C_{omn} = D_{emn} = 0, \text{ for } m \neq 1 \text{ and all } n, \\ A_{o1n} &= (-i)^n \frac{2n+1}{n(n+1)} \frac{\mu_o j_n(ka) \left[k_1 a j_n(k_1 a) \right]' - \mu_1 j_n(k_1 a) \left[ka j_n(ka) \right]'}{\mu_1 j_n(k_1 a) \left[ka h_n(ka) \right]' - \mu_o h_n(ka) \left[k_1 a j_n(k_1 a) \right]'}, \\ B_{e1n} &= (-i)^{n+1} \frac{2n+1}{n(n+1)} \frac{\mu_o j_n(k_1 a) \left[ka j_n(ka) \right]' - \left(\frac{k}{k_1} \right)^2 \mu_1 j_n(ka) \left[k_1 a j_n(k_1 a) \right]'}{\mu_o j_n(k_1 a) \left[ka h_n(ka) \right]' - \mu_1 \left(\frac{k}{k_1} \right)^2 h_n(ka) \left[k_1 a j_n(k_1 a) \right]'}, \\ C_{o1n} &= \frac{(-i)^{n+1} (2n+1) \mu_1}{ka n(n+1) \left\{ \mu_o h_n(ka) \left[k_1 a j_n(k_1 a) \right]' - \mu_1 j_n(k_1 a) \left[ka h_n(ka) \right]' \right\}}, \\ D_{e1n} &= \frac{(-i)^n (2n+1) \mu_1 k_1 a}{n(n+1) \left\{ \mu_1 (ka)^2 h_n(ka) \left[k_1 a j_n(k_1 a) \right]' - \mu_o (k_1 a)^2 j_n(k_1 a) \left[ka h_n(ka) \right]' \right\}} \end{aligned} \quad (2-23)$$

where the prime denotes differentiation with respect to ka or $k_1 a$ as appropriate.

The situation is considerably simpler in the important case when the conductivity of the sphere becomes infinite ($\text{Im } k_1^2 \rightarrow \infty$). The continuity condition eqn (2-9) is then replaced by the boundary condition

$$\hat{r} \wedge \underline{E} \Big|_{r=a} = 0 \quad , \quad (2-24)$$

since no fields can exist within the sphere, and making use of the asymptotic forms of the spherical Bessel and Hankel functions, the coefficients in eqns (2-23) become,

$$\begin{aligned} A_{o1n} &= -(-i)^n \frac{2n+1}{n(n+1)} \frac{j_n(ka)}{h_n(ka)} \\ B_{e1n} &= (-i)^{n+1} \frac{2n+1}{n(n+1)} \frac{\left[ka j_n(ka) \right]'}{\left[ka h_n(ka) \right]'} \end{aligned} \quad (2-25)$$

$$C_{o1n} = D_{e1n} = 0 \quad .$$

2.2 The Mie Series For Two Concentric Adjoining Spheres

The more complicated problem resulting when the sphere is not homogeneous but consists of a homogeneous sphere covered with a homogeneous layer of different material, has been solved by Aden and Kerker [1951] . The geometry is essentially the same as pictured in Figure 2-1, except for the addition of a surface layer of thickness d , and this is shown in Figure 2-2 where the positive x -axis and incident \underline{E} field point out of the page.

Consistent with the notation of the previous section, the inner sphere of radius a will be characterized by $k_1, \epsilon_1, \mu_1, s_1$; the layer by $k_2, \epsilon_2, \mu_2, s_2$, and the whole spherical structure of radius $b = a + d$ will be imbedded in free space characterized by k, ϵ_0 , and μ_0 .

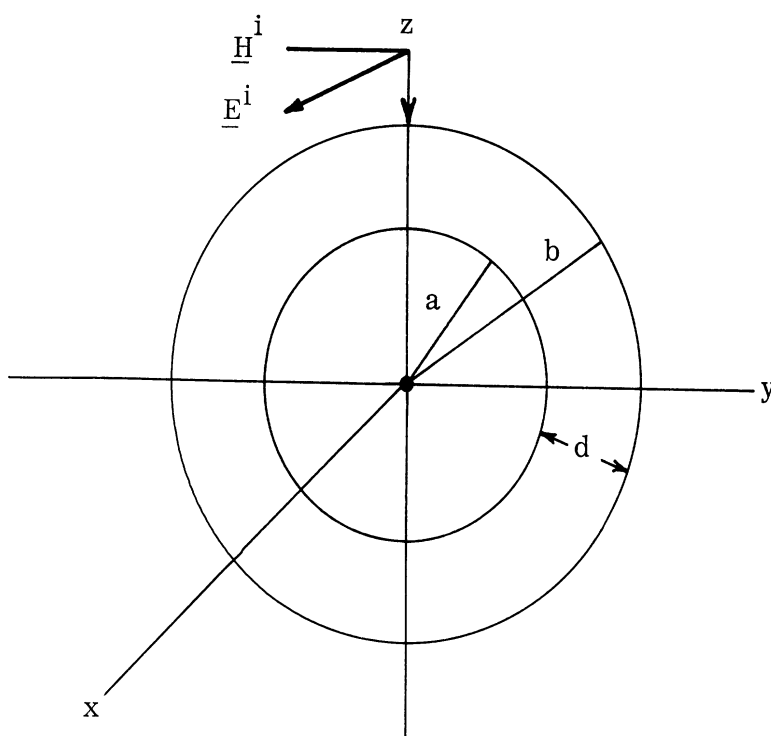


FIGURE 2-2:

In each region the representations of the field quantities are different. For the inner sphere and for free space, the representations are similar to those used in section 2.1, viz

$$r > b$$

$$\underline{E} = \underline{E}^i + \underline{E}^s, \quad \underline{H} = \underline{H}^i + \underline{H}^s$$

$$\underline{E}^i = E_0 \sum_{n=1}^{\infty} (-i)^n \frac{2n+1}{n(n+1)} (M_{o1n}^{(1)} + i N_{e1n}^{(1)}), \quad \underline{H}^i = -i H_0 \sum_{n=1}^{\infty} (-i)^n \frac{2n+1}{n(n+1)} (i M_{e1n}^{(1)} + N_{o1n}^{(1)}),$$

$$\underline{E}^s = E_0 \sum_{n=1}^{\infty} (A_n M_{n-o1n}^{(2)} + B_n N_{n-e1n}^{(2)}), \quad \underline{H}^s = -i H_0 \sum_{n=1}^{\infty} (B_n M_{n-e1n}^{(2)} + A_n N_{n-o1n}^{(2)}) \quad (2-26)^+$$

⁺For convenience the coefficients are written A_n , B_n , etc., rather than A_{o1n} , B_{e1n} , etc.

$$r < a$$

$$\underline{E} = E_o \sum_{n=1}^{\infty} (C_n \underline{M}_{o1n}^{(3)} + D_n \underline{N}_{e1n}^{(3)}), \quad \underline{H} = -i \frac{E_o k_1}{\omega \mu_1} \sum_{n=1}^{\infty} (D_n \underline{M}_{e1n}^{(3)} + C_n \underline{N}_{o1n}^{(3)}) \quad , \quad (2-27)$$

while in the layer $a < r < b$

$$\underline{E} = E_o \sum_{n=1}^{\infty} a_n \underline{M}_{o1n}^{(4)} + b_n \underline{M}_{e1n}^{(5)} + c_n \underline{N}_{e1n}^{(4)} + d_n \underline{N}_{e1n}^{(5)} \quad , \quad (2-28)$$

$$\underline{H} = -i \frac{E_o k_2}{\omega \mu_2} \sum_{n=1}^{\infty} c_n \underline{M}_{e1n}^{(4)} + d_n \underline{M}_{e1n}^{(5)} + a_n \underline{N}_{o1n}^{(4)} + b_n \underline{N}_{o1n}^{(5)} \quad ,$$

where the superscripts on the wave functions indicate the radial functions which

occur. Thus $\underline{M}^{(1)}$ and $\underline{N}^{(1)}$, defined in eqn (2-16), contain the radial function

$j_n(kr)$. If this is replaced by $h_n(kr)$, $\underline{M}^{(2)}$ and $\underline{N}^{(2)}$ result. Similarly, replacing

$j_n(kr)$ by $j_n(k_1 r)$ yields $\underline{M}^{(3)}$ and $\underline{N}^{(3)}$; replacing $j_n(kr)$ by $j_n(k_2 r)$ yields $\underline{M}^{(4)}$ and $\underline{N}^{(4)}$,

and by $h_n(k_2 r)$ gives $\underline{M}^{(5)}$ and $\underline{N}^{(5)}$. The continuity relations require $\hat{r} \cdot \underline{E}$ and

$\hat{r} \cdot \underline{H}$ to be continuous at the interfaces $r = a$ and $r = b$. This provides a sufficient

number of equations to determine the unknown coefficients, of which only A_n and

B_n , the coefficients of the scattered field, are presented here. They are

$$A_n = -(-i)^n \frac{2n+1}{n(n+1)} \frac{j_n(kb) f_1 + [kb j_n(kb)]' f_2}{h_n(kb) f_1 + [kb h_n(kb)]' f_2} \quad (2-29)$$

$$B_n = (-i)^{n+1} \frac{2n+1}{n(n+1)} \frac{j_n(kb) f_3 + [kb j_n(kb)]' f_4}{h_n(kb) f_3 + [kb h_n(kb)]' f_4} \quad ,$$

where

$$\begin{aligned}
 f_1 &= \frac{j_n(k_1 a)}{\mu_2^2} \left\{ \left[k_2 b j_n(k_2 b) \right]' \left[k_2 a h_n(k_2 a) \right]' - \left[k_2 a j_n(k_2 a) \right]' \left[k_2 b h_n(k_2 b) \right]' \right\} \\
 &\quad + \frac{\left[k_1 a j_n(k_1 a) \right]'}{\mu_1 \mu_2} \left\{ j_n(k_2 a) \left[k_2 b h_n(k_2 b) \right]' - h_n(k_2 a) \left[k_2 b j_n(k_2 b) \right]' \right\} , \\
 f_2 &= \frac{j_n(k_1 a)}{\mu_o \mu_2} \left\{ \left[k_2 a j_n(k_2 a) \right]' h_n(k_2 b) - \left[k_2 a h_n(k_2 a) \right]' j_n(k_2 b) \right\} \\
 &\quad + \frac{\left[k_1 a j_n(k_1 a) \right]'}{\mu_o \mu_1} \left\{ j_n(k_2 b) h_n(k_2 a) - j_n(k_2 a) h_n(k_2 b) \right\} , \\
 f_3 &= \frac{k^2}{\mu_o \mu_2 k_1^2} \left[k_1 a j_n(k_1 a) \right]' \left\{ j_n(k_2 a) \left[k_2 b h_n(k_2 b) \right]' - h_n(k_2 a) \left[k_2 b j_n(k_2 b) \right]' \right\} \\
 &\quad + \frac{k^2}{\mu_o \mu_1 k_2^2} j_n(k_1 a) \left\{ \left[k_2 b j_n(k_2 b) \right]' \left[k_2 a h_n(k_2 a) \right]' - \left[k_2 a j_n(k_2 a) \right]' \left[k_2 b h_n(k_2 b) \right]' \right\} , \\
 f_4 &= \frac{k_2^2}{\mu_2^2 k_1^2} \left[k_1 a j_n(k_1 a) \right]' \left\{ j_n(k_2 b) h_n(k_2 a) - j_n(k_2 a) h_n(k_2 b) \right\} \\
 &\quad + \frac{j_n(k_1 a)}{\mu_1 \mu_2} \left\{ \left[k_2 a j_n(k_2 a) \right]' h_n(k_2 b) - \left[k_2 a h_n(k_2 a) \right]' j_n(k_2 b) \right\} . \quad (2-30)
 \end{aligned}$$

Scharfman [1954] considered the limiting case when the inner sphere becomes perfectly conducting and the continuity condition at this interface is replaced by the boundary condition eqn (2-24). The expressions (2-29) for the coefficients of the scattered field are still valid but the f 's defined by (2-30) simplify as follows:

$$\begin{aligned}
 f_1 &= \frac{1}{\mu_2} \left\{ j_n(k_2 a) \left[k_2 b h_n(k_2 b) \right]' - h_n(k_2 a) \left[k_2 b j_n(k_2 b) \right]' \right\} \\
 f_2 &= \frac{1}{\mu_0} \left\{ j_n(k_2 b) h_n(k_2 a) - j_n(k_2 a) h_n(k_2 b) \right\} \\
 f_3 &= \frac{1}{\mu_0} \left(\frac{k}{k_2} \right)^2 \left\{ \left[k_2 b j_n(k_2 b) \right]' \left[k_2 a h_n(k_2 a) \right]' - \left[k_2 a j_n(k_2 a) \right]' \left[k_2 b h_n(k_2 b) \right]' \right\} \\
 f_4 &= \frac{1}{\mu_2} \left\{ h_n(k_2 b) \left[k_2 a j_n(k_2 a) \right]' - j_n(k_2 b) \left[k_2 a h_n(k_2 a) \right]' \right\} . \quad (2-31)
 \end{aligned}$$

2.3 The Mie Series For Two Concentric Disjoint Spheres

When the inner radius of the layer is larger than the radius of the inner sphere, i. e. a plane wave is incident upon a sphere with two layers of different material upon it (see Figure 2-3), the problem is even more complicated.

Plonus [1961] has treated this problem, though not in complete generality. The problem is specialized in the following ways: 1) the inner sphere of radius a is perfectly conducting; 2) the two regions $a < r < b$ and $r > c$ consist of the same material (here taken as free space and characterized by k , ϵ_0 , and μ_0), and 3) the permeability of the layer $b < r < c$ is also taken to be μ_0 although the propagation constant k_2 is different from k . The procedure is exactly the same as before. There will be three representations of the field in the three regions, two continuity conditions (at $r=b$ and $r=c$) and one boundary condition (at $r=a$).

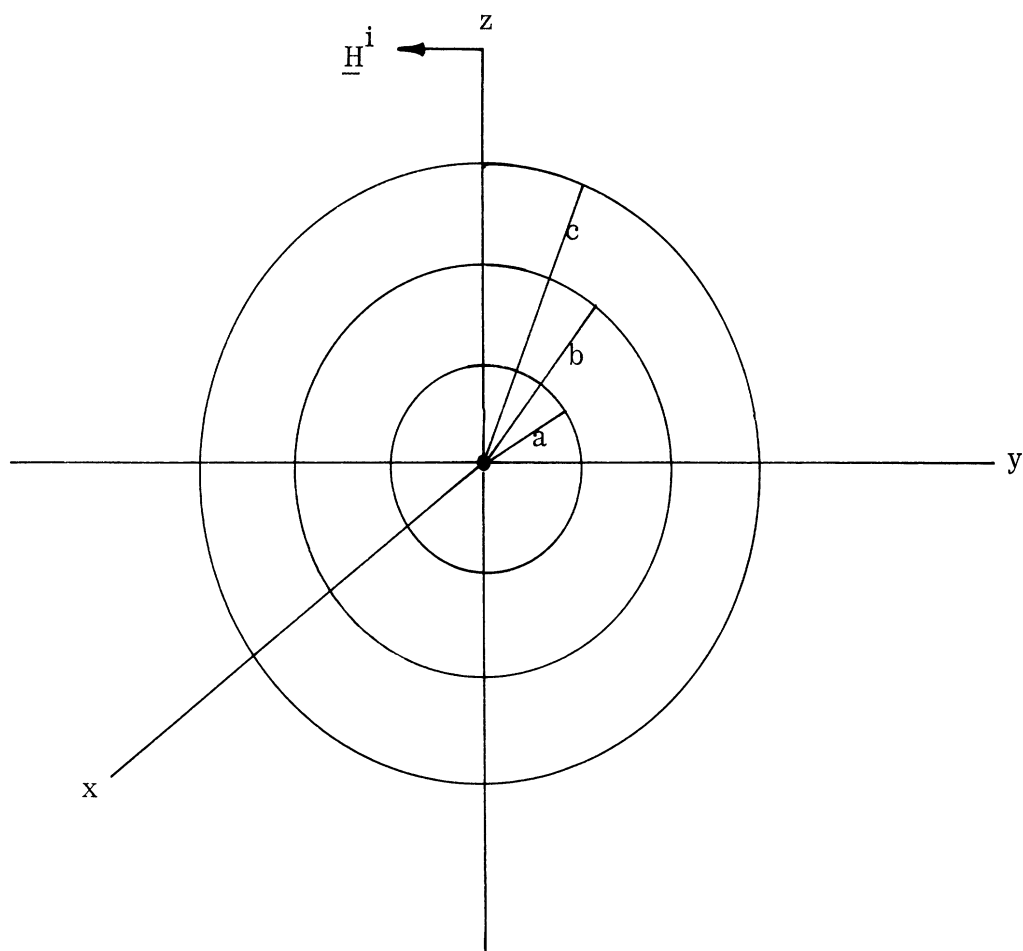


FIGURE 2-3

Thus for,

$$r > c$$

$$\underline{E} = \underline{E}^i + \underline{E}^s, \quad \text{and} \quad \underline{H} = \underline{H}^i + \underline{H}^s,$$

where the quantities are exactly as defined in eqns (2-26),

$$c > r > b$$

\underline{E} and \underline{H} are given by eqns (2-28) with $\mu_2 = \mu_0$, and for

$b > r > a$

$$\begin{aligned} \underline{E} &= E_0 \sum_{n=1}^{\infty} \alpha_n \underline{M}_{01n}^{(1)} + \beta_n \underline{M}_{01n}^{(2)} + \gamma_n \underline{N}_{e1n}^{(1)} + \delta_n \underline{N}_{e1n}^{(2)} , \\ \underline{H} &= \frac{-i E_0 k}{\omega \mu_0} \sum_{n=1}^{\infty} \gamma_n \underline{M}_{e1n}^{(1)} + \delta_n \underline{M}_{e1n}^{(2)} + \alpha_n \underline{N}_{01n}^{(1)} + \beta_n \underline{N}_{01n}^{(2)} . \end{aligned} \quad (2-32)$$

At the interfaces ($r=b$ and $r=c$), $\hat{r} \wedge \underline{E}$ and $\hat{r} \wedge \underline{H}$ must be continuous and

$\hat{r} \wedge \underline{E}|_{r=a} = 0$. These conditions provide a sufficient number of equations to determine the unknown coefficients. Again only the coefficients of the scattered field,

A_n and B_n are presented here:

$$A_n = -(-i)^n \frac{2n+1}{n(n+1)} \frac{j_n(kc)f_1 + [kc j_n(kc)]' f_2}{h_n(kc)f_1 + [kc h_n(kc)]' f_2} \quad (2-33)$$

$$B_n = +(-i)^{n+1} \frac{2n+1}{n(n+1)} \frac{j_n(kc)f_3 + [kc j_n(kc)]' f_4}{h_n(kc)f_3 + [kc h_n(kc)]' f_4} ,$$

where

$$\begin{aligned} f_1 &= [k_2 c j_n(k_2 c)]' \left\{ h_n(ka) \left(h_n(k_2 b) [kb j_n(kb)]' - j_n(kb) [k_2 b h_n(k_2 b)]' \right) \right. \\ &\quad \left. - j_n(ka) \left(h_n(k_2 b) [kb h_n(kb)]' - h_n(kb) [k_2 b h_n(k_2 b)]' \right) \right\} \\ &\quad - [k_2 c h_n(k_2 c)]' \left\{ h_n(ka) \left(j_n(k_2 b) [kb j_n(kb)]' - j_n(kb) [k_2 b j_n(k_2 b)]' \right) \right. \\ &\quad \left. - j_n(ka) \left(j_n(k_2 b) [kb h_n(kb)]' - h_n(kb) [k_2 b j_n(k_2 b)]' \right) \right\} , \end{aligned} \quad (2-34)$$

$$f_2 = j_n(ka) \left\{ j_n(k_2c) \left(h_n(k_2b) [kb h_n(kb)]' - h_n(kb) [k_2b h_n(k_2b)]' \right) \right. \\ \left. - h_n(k_2c) \left(j_n(k_2b) [kb h_n(kb)]' - h_n(kb) [k_2b j_n(k_2b)]' \right) \right\} \\ - h_n(ka) \left\{ j_n(k_2c) \left(h_n(k_2b) [kb j_n(kb)]' - j_n(kb) [k_2b h_n(k_2b)]' \right) \right. \\ \left. - h_n(k_2c) \left(j_n(k_2b) [kb j_n(kb)]' - j_n(kb) [k_2b j_n(k_2b)]' \right) \right\} ,$$

$$f_3 = [ka j_n(ka)]' \left\{ [k_2c j_n(k_2c)]' \left(\left(\frac{k}{k_2} \right)^2 h_n(kb) [k_2b h_n(k_2b)]' - h_n(k_2b) [kb h_n(kb)]' \right) \right. \\ \left. - [k_2c h_n(k_2c)]' \left(\left(\frac{k}{k_2} \right)^2 h_n(kb) [k_2b j_n(k_2b)]' - j_n(k_2b) [kb h_n(kb)]' \right) \right\} \\ - [ka h_n(ka)]' \left\{ [k_2c j_n(k_2c)]' \left(\left(\frac{k}{k_2} \right)^2 j_n(kb) [k_2b h_n(k_2b)]' - h_n(k_2b) [kb j_n(kb)]' \right) \right. \\ \left. - [k_2c h_n(k_2c)]' \left(\left(\frac{k}{k_2} \right)^2 j_n(kb) [k_2b j_n(k_2b)]' - j_n(k_2b) [kb j_n(kb)]' \right) \right\} ,$$

$$f_4 = j_n(k_2c) \left\{ [ka h_n(ka)]' \left(j_n(kb) [k_2b h_n(k_2b)]' - \left(\frac{k_2}{k} \right)^2 h_n(k_2b) [kb j_n(kb)]' \right) \right. \\ \left. - [ka j_n(ka)]' \left(h_n(kb) [k_2b h_n(k_2b)]' - \left(\frac{k_2}{k} \right)^2 h_n(k_2b) [kb h_n(kb)]' \right) \right\} \\ - h_n(k_2c) \left\{ [ka h_n(ka)]' \left(j_n(kb) [k_2b j_n(k_2b)]' - \left(\frac{k_2}{k} \right)^2 j_n(k_2b) [kb j_n(kb)]' \right) \right. \\ \left. - [ka j_n(ka)]' \left(h_n(kb) [k_2b j_n(k_2b)]' - \left(\frac{k_2}{k} \right)^2 j_n(k_2b) [kb h_n(kb)]' \right) \right\} .$$

(2-34)

When $b=a$, these expressions go over to those given in eqns (2-31) for $\mu_0 = \mu_2$.

2.4 The Far Field Amplitude

Of particular interest is the far field or far zone behavior of the scattered field. Recall that the scattered field (exterior to the sphere and any layers) is always written as

$$\begin{aligned} \underline{E}^S &= E_o \sum_{n=1}^{\infty} (A_n \underline{M}_{on}^{(2)} + B_n \underline{N}_{en}^{(2)}) , \\ \underline{H}^S &= -i H_o \sum_{n=1}^{\infty} (B_n \underline{M}_{en}^{(2)} + A_n \underline{N}_{on}^{(2)}) , \end{aligned} \quad (2-35)$$

where the A_n and B_n are given by eqns (2-23), (2-25), (2-29), or (2-33) depending on which particular sphere problem is being considered. Regardless of how these coefficients are defined the expressions for the wave functions \underline{M} and \underline{N} can be simplified in the far field of the sphere and its layers (if any). Specifically the spherical Hankel functions contained in the expressions for \underline{M} and \underline{N} can be replaced by the first terms in their asymptotic expansion for large argument and since

$$h_n(kr) \sim (-i)^{n+1} \frac{e^{ikr}}{kr} \sim -i \frac{1}{kr} \left[kr h_n(kr) \right]' , \quad (2-36)$$

the θ and ϕ components of both \underline{M} and \underline{N} are of equal order for given n . By comparison, the radial component of \underline{N} is of one higher order. Consequently, only the θ and ϕ components can appear in the far field which then has the form of a

spherically outgoing wave, and from eqns (2.16), (2.35), and (2.36)

$$\begin{aligned} \underline{E}^s \sim E_o \frac{e^{ikr}}{kr} \sum_{n=1}^{\infty} (-i)^{n+1} \left\{ \left(A_n \frac{P_n^1(\cos \theta)}{\sin \theta} + i B_n \frac{\partial}{\partial \theta} P_n^1(\cos \theta) \right) \cos \theta \hat{\theta} \right. \\ \left. - \left(A_n \frac{\partial}{\partial \theta} P_n^1(\cos \theta) + i B_n \frac{P_n^1(\cos \theta)}{\sin \theta} \right) \sin \theta \hat{\phi} \right\}, \end{aligned} \quad (2-37)$$

valid for $r \gg kc^2$, where c is the radius of the entire spherical structure with $c = a$ for a homogeneous sphere.

This result simplifies considerably for scattering in the back and forward directions. For backscattering ($\theta = 0$),

$$\left[\frac{P_n^1(\cos \theta)}{\sin \theta} \right]_{\theta=0} = \frac{n(n+1)}{2} = \frac{\partial}{\partial \theta} \left[P_n^1(\cos \theta) \right]_{\theta=0},$$

giving

$$\underline{E}^s \sim E_o \hat{x} \frac{e^{ikr}}{kr} \sum_{n=1}^{\infty} (-i)^{n+1} \frac{n(n+1)}{2} (A_n + i B_n), \quad (2-38)$$

and for forward scattering ($\theta = \pi$)

$$\left[\frac{P_n^1(\cos \theta)}{\sin \theta} \right]_{\theta=\pi} = (-1)^{n+1} \frac{n(n+1)}{2} = - \frac{\partial}{\partial \theta} \left[P_n^1(\cos \theta) \right]_{\theta=\pi},$$

so that

$$\underline{E}^s \sim E_o \hat{x} \frac{e^{ikr}}{kr} \sum_{n=1}^{\infty} i^{n-1} \frac{n(n+1)}{2} (A_n - i B_n). \quad (2-39)$$

It is now convenient to introduce the concept of a scattering function $f(\theta, \phi)$.

This will be defined by the equation

$$\underline{E}^S \sim E_o \frac{e^{ikr}}{kr} f(\theta, \phi) \hat{\tau}, \quad (2-40)$$

valid for $r \gg kc^2$, where $\hat{\tau}$ is a unit vector in the direction of \underline{E}^S , and accordingly

$f(\theta, \phi)$ can be regarded as the far field amplitude. From Maxwell's equation we

then have

$$\underline{H}^S \sim H_o \frac{e^{ikr}}{kr} f(\theta, \phi) \hat{r} \wedge \hat{\tau}, \quad (2-41)$$

and consequently the same function describes both the electric and magnetic fields.

For scattering in the backward direction, the scattering function will be written as

$f(0)$, since there is no dependence on ϕ , and eqn (2-38) then gives

$$f(0) = \sum_{n=1}^{\infty} (-i)^{n+1} \frac{n(n+1)}{2} (A_n + i B_n) \quad (2-42)$$

Similarly, for scattering in the forward direction,

$$f(\pi) = \sum_{n=1}^{\infty} i^{n-1} \frac{n(n+1)}{2} (A_n - i B_n) \quad (2-43)$$

(see eqn (2-39)).

The definition of $f(\theta, \phi)$ given in eqn (2-40) differs from that usually adopted.

The function f/k corresponds to the scattering function more commonly defined, but

this has the disadvantage of not being dimensionless. In electromagnetic theory

(and, indeed, in all branches of physical science) there seems to be every advantage attached to using a non-dimensional function, and it is for this reason that the present definition has been chosen in spite of the fact that it represents a break from conventional notation. As defined above, the function f is independent of r and can be likened to a polar diagram factor. It depends only on angular variables θ and ϕ and on the properties of the scattering body, and is sufficient to specify the far field in its entirety.

It is a simple matter to calculate the scattering cross section in terms of the function f . The differential cross section or bistatic radar cross section $\sigma(\theta, \phi)$ is defined by

$$\sigma(\theta, \phi) = \lim_{r \rightarrow \infty} 4\pi r^2 \left| \frac{E^S}{E^i} \right|^2, \quad (2-44)$$

and hence, from eqn (2-40),

$$\sigma(\theta, \phi) = \frac{4\pi}{k^2} |f(\theta, \phi)|^2. \quad (2-45)$$

An alternative expression is

$$\sigma(\theta, \phi) = \frac{\lambda^2}{\pi} |f(\theta, \phi)|^2, \quad (2-46)$$

where λ is the wavelength, and the dimensions of σ are here made explicit.

The total scattering cross section σ_T is related to $\sigma(\theta, \phi)$ by the equation

$$\sigma_T = \frac{1}{4\pi} \int \sigma(\theta, \phi) d\Omega \quad (2-47)$$

where $d\Omega$ is an element of solid angle, and by inserting eqn(2-45) we now have

$$\sigma_T = \frac{1}{k^2} \int \left| f(\theta, \phi) \right|^2 d\Omega \quad . \quad (2-48)$$

An additional relation between σ_T and f is provided by the "forward scattering theorem". This was first discovered in atomic theory and since then its electromagnetic equivalent has received a variety of independent proofs (see , for example, Schiff [1954] , Jones [1955] , and de Hoop [1959]). The theorem is merely an expression of conservation of energy and leads to the equation

$$\sigma_T = \frac{4\pi}{k^2} \text{Im. } f(\pi) \quad , \quad (2-49)$$

where Im. denotes the imaginary part. In addition to the scattering function and cross section defined above there exist many quantities in the literature with similar names but different definitions. This unfortunate situation is virtually uncorrectable at this stage and the best one can do is exercise care in checking definitions and be resigned to the fact that many existing results may require renormalization before use. Some of the more common quantities are presented here.

If the scattered far field is written

$$\underline{E}^S = E_\theta \hat{\theta} + E_\phi \hat{\phi} \quad , \quad (2-50)$$

where E_θ and E_ϕ are defined in eqn (2-37), then these components can be expressed in the form

$$E_{\theta}^S = E_o \frac{e^{ikr}}{kr} \cos \phi S_1(\theta) , \quad (2-51)$$

$$E_{\phi}^S = -E_o \frac{e^{ikr}}{kr} \sin \phi S_2(\theta) ,$$

where $S_1(\theta)$ and $S_2(\theta)$ are defined by referring to eqn (2-37) and are called the complex amplitudes of the scattered radiation for the two polarizations.

The squares of the absolute values of S_1 and S_2 are called the intensities of scattered radiation for the two polarizations.

The absorption cross section σ_a and the scattering cross section σ_s are defined as

$$\sigma_a = \frac{P_a}{P_i} , \quad (2-52)$$

$$\sigma_s = \frac{P_s}{P_i} , \quad (2-53)$$

where P_a is the power absorbed by the obstacle, P_s the power scattered, and P_i the power incident. If no power is absorbed in the obstacle and the surrounding medium is non-dissipative (e. g. free space), then σ_s is the same as σ_T defined above.

The sum $\sigma_a + \sigma_s$ is known as the extinction cross section and in cases where σ_a is non zero eqn(2-49), the forward scattering theorem, must be altered to read

$$\sigma_a + \sigma_s = \frac{4\pi}{k^2} \text{Im. } f(\pi) . \quad (2-54)$$

The various cross sections defined, i. e. differential scattering, total scattering, absorption, and extinction are referred to as efficiencies when normalized to the geometric cross section which, for a sphere of radius a , is πa^2 .

Thus

$$Q = \frac{\sigma + \sigma_s}{\pi a^2}$$

is the extinction efficiency, etc.

2.5 Computations

Kerker [1955] summarized the then available Mie theory functions and his table is reproduced here for convenience (Table II-1). To this has been added the work of Scharfman [1954] which gives the back scattering cross sections of various dielectric coated spheres. Also appended are the highly accurate tables of the back scattering function $f(0)$ for perfectly conducting spheres presented by Hey, et al [1956].

The recent work of vande Hulst [1957] is an excellent summary of work on scattering by spheres. Chapters 9-14 of this work are of particular interest to this study since they contain many tables of calculated quantities as well as a list of references containing other tabulated quantities. Table II-2 presents a brief listing of the tables given by vande Hulst. Table II-3 is a similar listing of the graphs to be found in vande Hulst's volume.

TABLE II-1: LIST OF AVAILABLE MIE THEORY FUNCTIONS

Reference	Index of Refraction ⁺	Values of ka	Quantity Calculated
Shoulejkin [1924]	1.32	1,3,∞	Scattering functions every 20°
Blumer [1925] , [1931]	1.25 1.33 1.5 ∞	0.4, 0.8, 1.6, 4, 8 1.5, 3 4 0.1, 0.5, 1, 3, 5, 10	Scattering functions every 10°
Stratton and Houghton [1931]	1.33	0-40	Scattering coefficient
Caspersson [1932] [1933]	1.63 1.56 1.50	0.71-3.16 (17 values)	Scattering functions at 0°, 45°, 90°, 135°, and 180°
Gumprich, Sung, Chin, and Sliepcevich [1952]	1.33	6, 8, 10-35(5)	Scattering functions every 10°
Gumprich and Sliepcevich [1953]	1.33 1.44 1.20	20, 30, 40, 60, 80 100, 200, 400 20, 80, 150 20, 80	Scattering coefficient
Kerker and Perlee [1953]	2.00	1.30-2.80 (12 values not in Lowan tables)	Scattering functions at 90°
Kerker and Cox [1955]	2.00	3.0-5.0 (11 values not in Lowan tables)	Scattering functions at 130°
Engelhard and Freiss [1937]	1.44	0.4, 1, 1.5, 2, 2.5, 3, 4, 6, 8	Scattering functions every 10°
Paranjpe, Naik, and Vaidya [1939]	1.33	4, 5, 6, 7, 8, 9, 10, 12, 20, 30	Scattering functions every 10°
Ruedy [1943] [1944]	1.33	1/8, 1/4, 3/8, 1/2, 3/4, 1	Scattering coefficient

(continued on next page)

$$^+ \text{ Index of refraction } m = \sqrt{\frac{\epsilon_1 \mu_1}{\epsilon_0 \mu_0} + \frac{i \sigma_1 \mu_1}{\omega \epsilon_0 \mu_0}}$$

TABLE II-1 (continued)

Reference	Index of Refraction	Values of ka	Quantity Calculated
Houghton and Chalker 1949	1.33	7-24 (33 values)	Scattering coefficient
Lowan 1948	1.33	0.5-6.0 (15 values)	Scattering functions every 10° and scattering coefficient
	1.44		
	1.50		
	2.00		
	4.21 - 2.51i	0.100-1.00(.05) 1.0-3.0(.1)	Extinction coefficient
	5.55- 2.85i	0.10-1.00(.05) 1.0-2.0(.1)	Extinction coefficient
	8.18 - 1.96i	0.100-1.00(.025)	Extinction coefficient
	3.41 - 1.94i	0.10-1.00(.05) 1.0-5.0(.1)	Extinction coefficient and A_n and B_n
	7.21 - 2.65i	0.1000-1.000(.025) 1.00-1.30(.05)	Extinction coefficient
	8.90 - .69i	0.10-0.30(.01) 0.300-0.430(.005) 0.43-0.60(.01)	Extinction coefficient and A_n and B_n
Riley 1949	1.486	0.5-3.0(.1)	Scattering functions every 10° and scattering coefficients
Aden 1950	9.01-0.43i	0.6-6.0	Scattering functions at 0°
	∞	0.6-6.0	
Gumprich and Sliepceovich 1951	1.20	1-6(1)	A_n and B_n
	1.40	8	
	1.50	10-100(5)	
	1.60	100-200(10)	
		200-400(50)	
	1.33	4, 5, 6, 8	
	1.44	10-100(5)	
		100-200(10) 200-400(50)	

(continued on next page)

TABLE II-1 (continued)

Reference	Index of Refraction	Values of ka	Quantity Calculated
Kerker, Langleben and Gunn [1951]		0.126	Backscattering by particle consisting of two concentric spheres. Inner sphere $m=1.75$, outer sphere $m=8.9-1.5i$
Scharfman [1954]		1.26	Backscattering by lossless dielectric coated perfectly conducting sphere. Outer sphere $1.6 < m < \infty$.
Hey, Stewart, Pinson and Prince [1956]	∞	0(.01)10	Backscattering function $f(0)$. ⁺

+

The actual quantity tabulated is $f/2$, not f as listed at head of each column (see Hey and Senior [1958]).

TABLE II-2: PARTIAL LISTING OF TABLES TO BE FOUND IN VAN DE HULST [1957]

Refractive Index, m	$2\pi a/\lambda$	Page No.	Quantity Calculated
∞	.1(.1)1.6 1.8-90	161	Efficiency factor
.8, .93, 1 ⁺ ₂ ϵ 1.33, 1.5, 2	-	178	Maxima and minima of the extinction curve.
m close to 1	-	180	Extinction and absorption by partially absorbing spheres.
complex	-	273-274	Complex values of m for which computations have been made.
3.41-1.94i 7.20-2.65i, ∞	1.3	277	Extinction coefficient and intensity functions.
1.50-i n' (n' small)	.5-7.0	295	Extinction by spheres.

TABLE II-3: PARTIAL LISTING OF GRAPHICAL PRESENTATIONS OF DATA IN
VAN DE HULST [1957]

Refractive			
Index, m	$2\pi a/\lambda$	Page No.	Content
2	0-4	137	Phase angle vs $2\pi a/\lambda$
2	0-12	151	Extinction curves of sphere
1.55			
1.5			
1.44			
1.33			
1.25			
2	1-6	152-153	Scattering diagrams
1.55	1-6		
1.33	1-5		
1.50	1.2-2.4		
∞	0-5	162	Efficiency factors for extinction and for radiation pressure.
∞	1/2-10	163	Scattering diagrams.
1.5	0-20	177	Extinction curves computed from Mie's formula.
1.33			
$1 \pm \epsilon$			
.93			
1.33	10	236	Scattering diagrams.
1.33	30,35,40	260	Intensity distribution.
1.27-1.37i	0-3	276	Efficiency factor for extinction, radiation pressure, absorption and scattering.
1.29(1-ik)	0-20	278	Variation of extinction curves if the imaginary part of the refractive index is varied.
8.9-.69i	0-1.5	283	Extinction curves (showing resonance peaks)
8.18-1.96i			
∞			
∞	0-3	285	Radar cross section σ computed for backscattering by water drops at $\lambda = 3\text{mm}$.
3.41-1.94i			

Some representative bistatic cross section curves for a perfectly conducting sphere are included here (Figures 2-4 through 2-20) to indicate the behavior of the sphere as a scatterer. These were computed at Air Force Cambridge Research Laboratories and appear in King and Wu [1959]. The back scattering cross section as a function of ka for the perfectly conducting sphere is also given, Figure 2-21. This was plotted from the tables of Hey et al [1956].

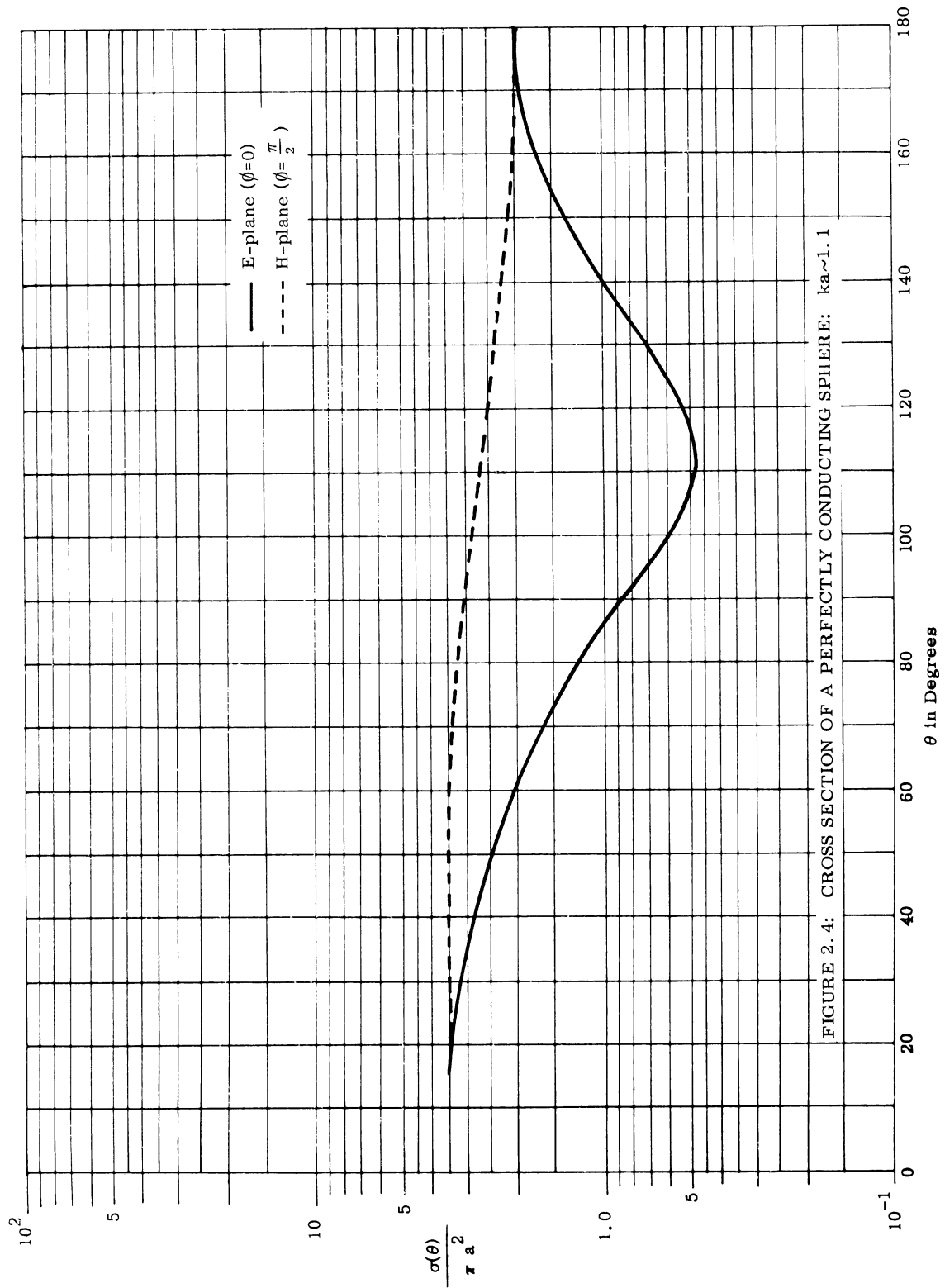


FIGURE 2.4: CROSS SECTION OF A PERFECTLY CONDUCTING SPHERE: $ka \sim 1.1$

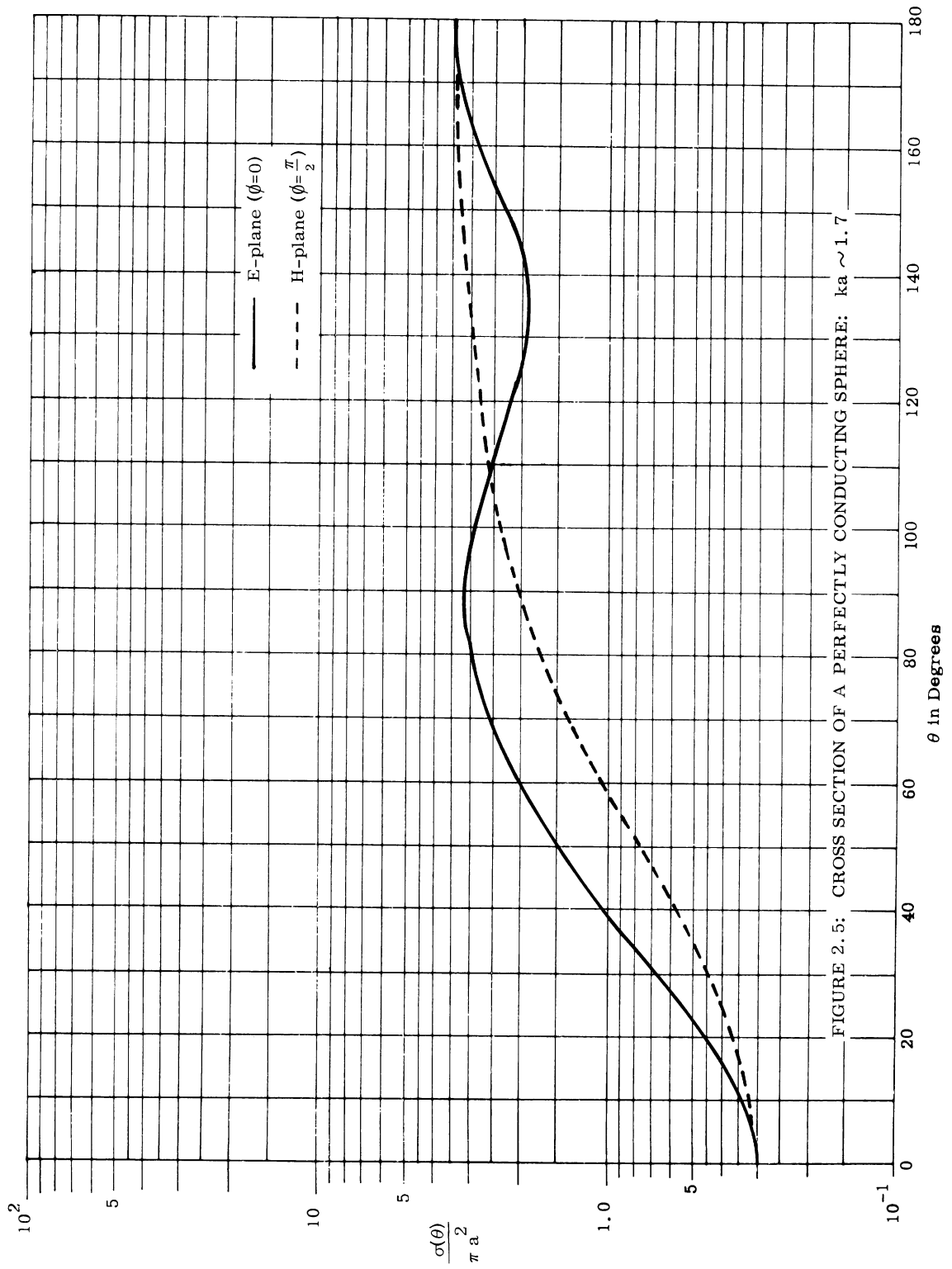
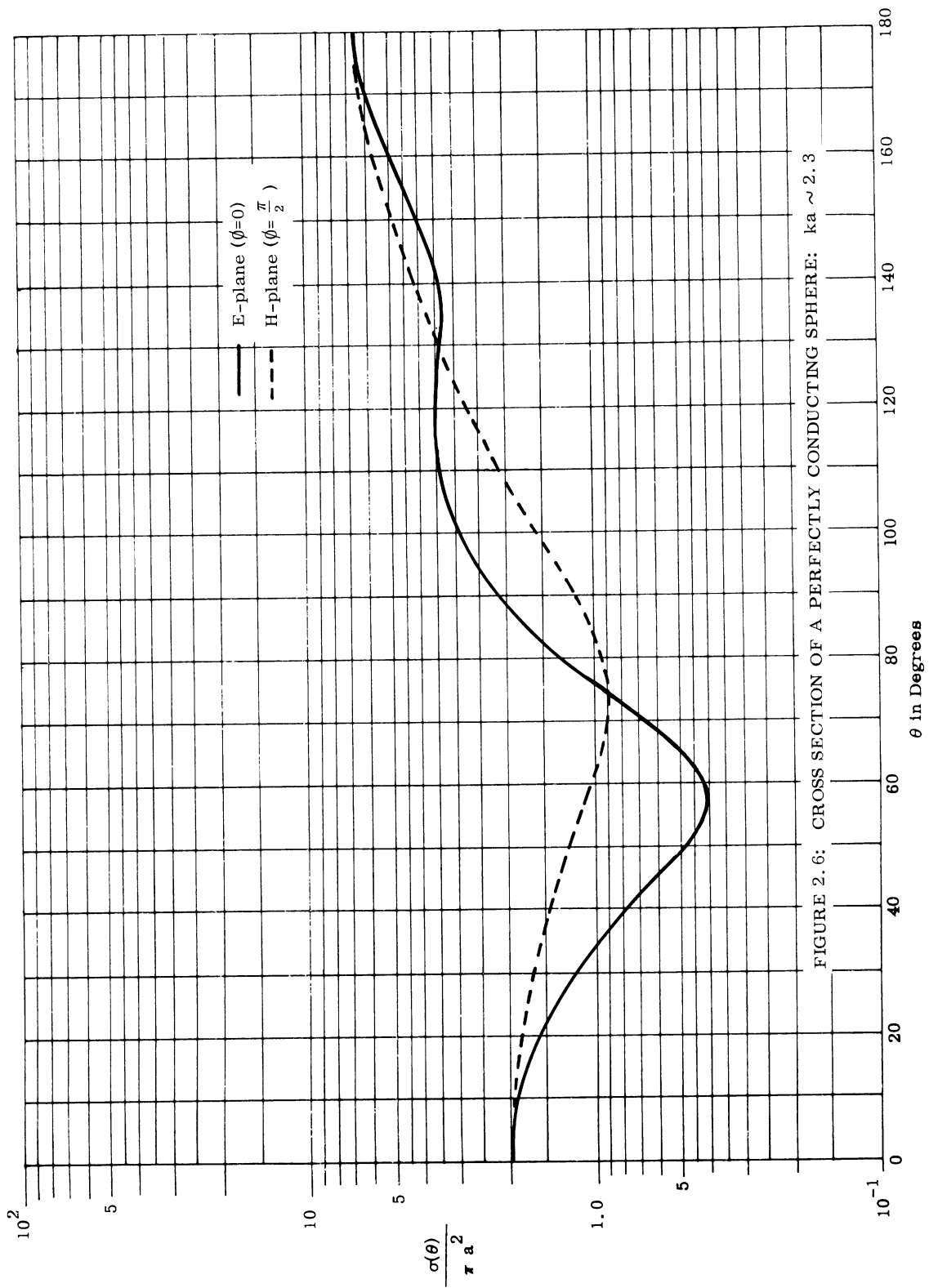
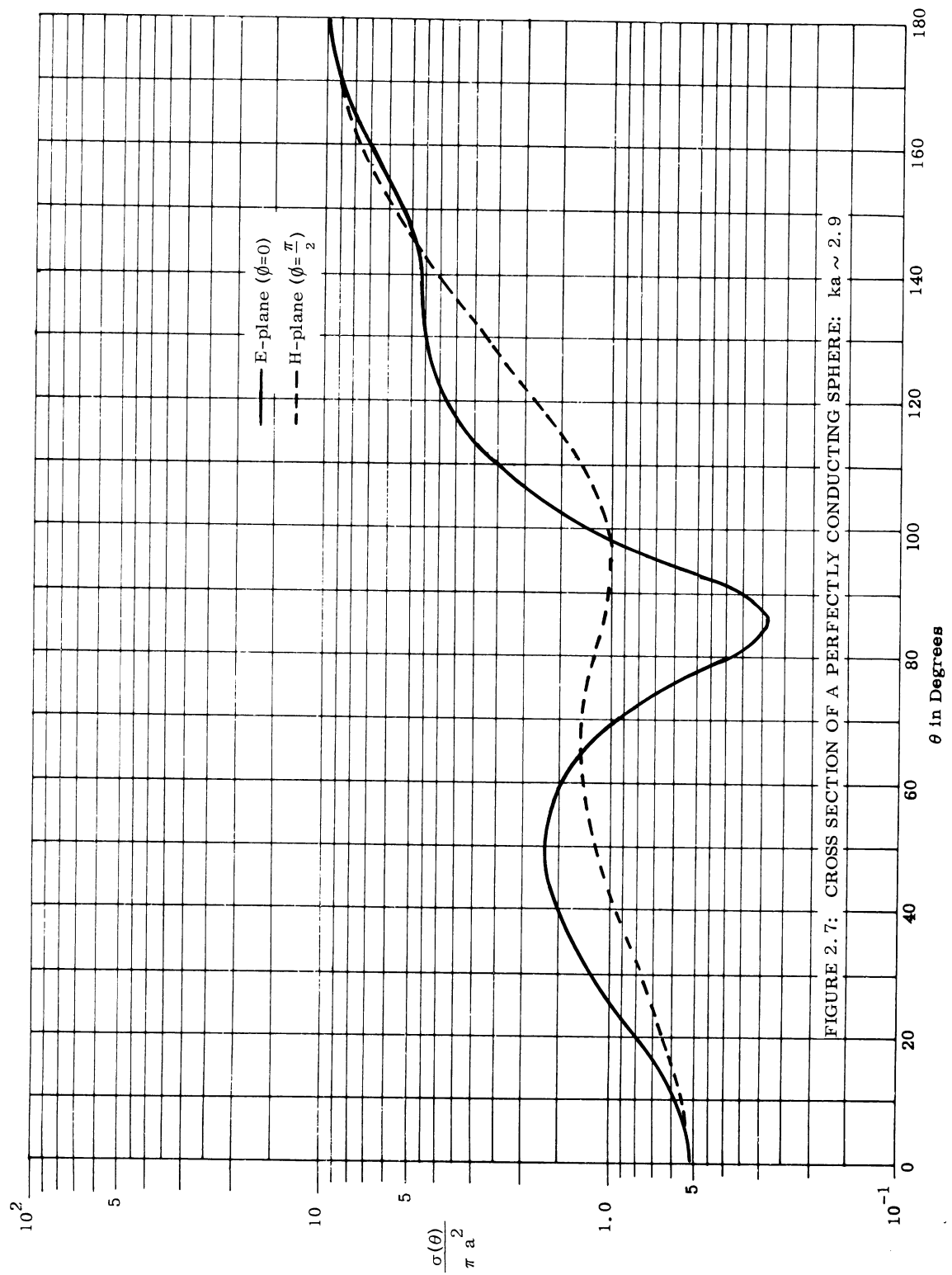


FIGURE 2.5: CROSS SECTION OF A PERFECTLY CONDUCTING SPHERE: $ka \sim 1.7$

FIGURE 2.6: CROSS SECTION OF A PERFECTLY CONDUCTING SPHERE: $ka \sim 2.3$

FIGURE 2.7: CROSS SECTION OF A PERFECTLY CONDUCTING SPHERE: $ka \sim 2.9$

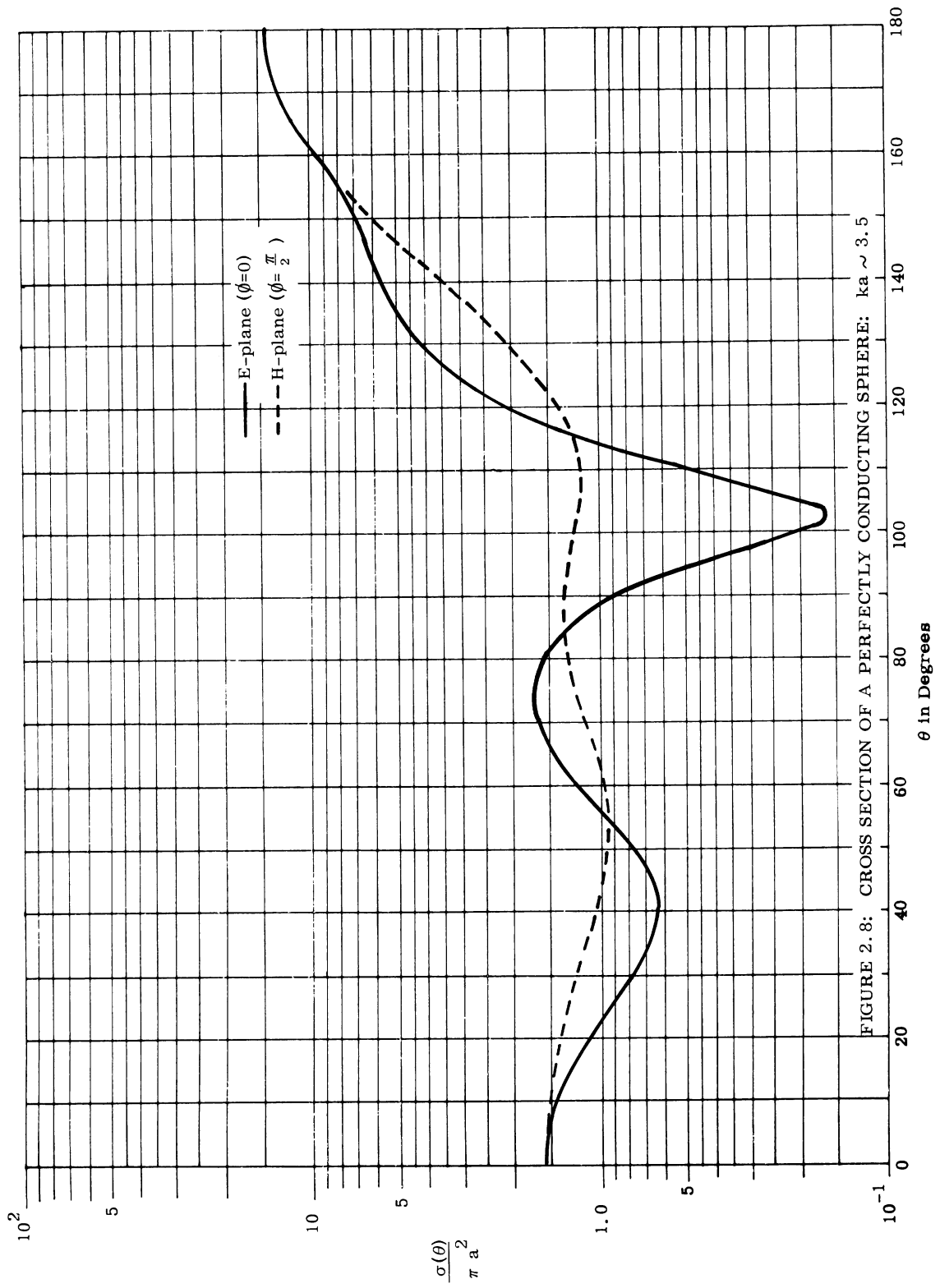
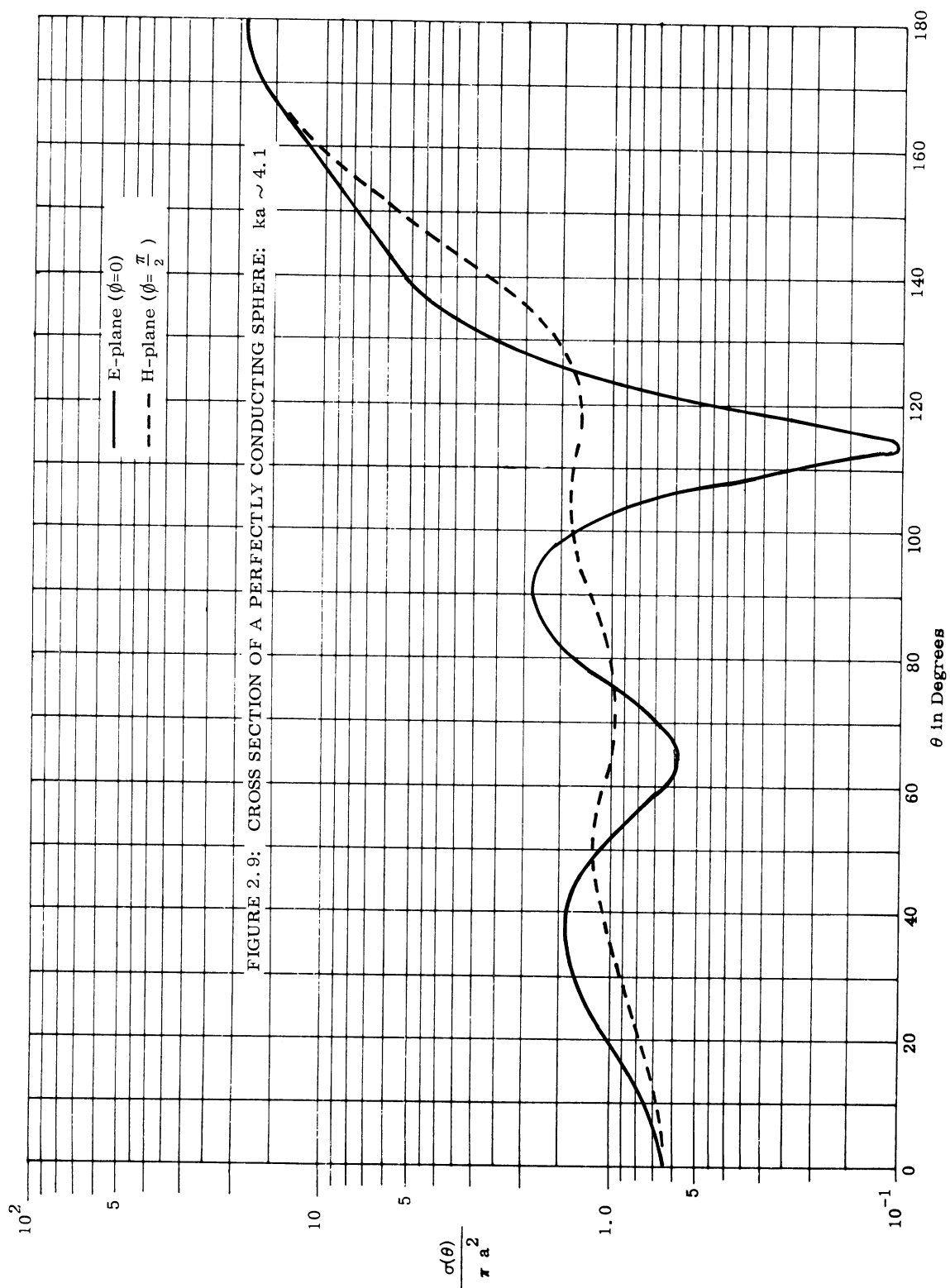
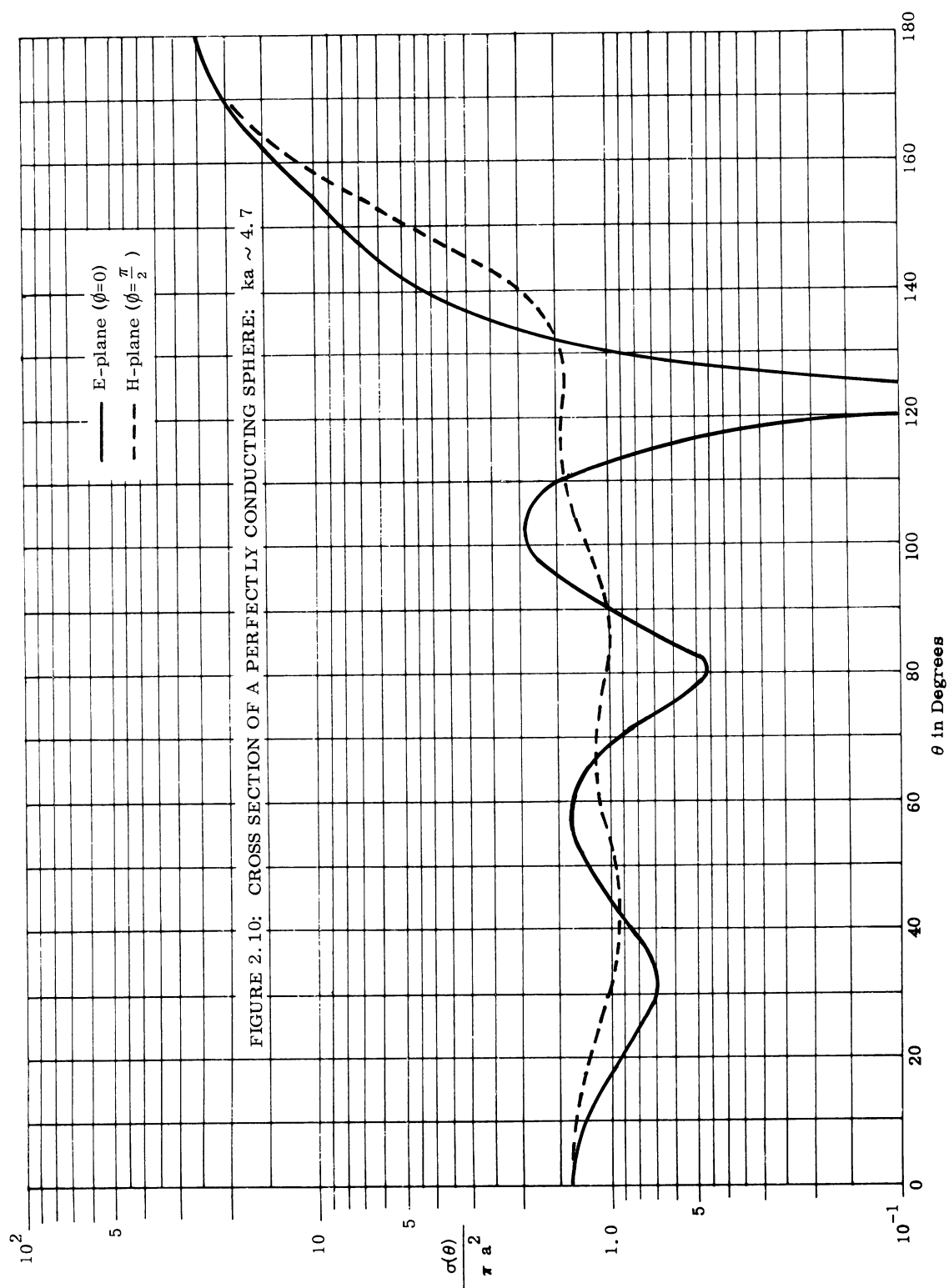
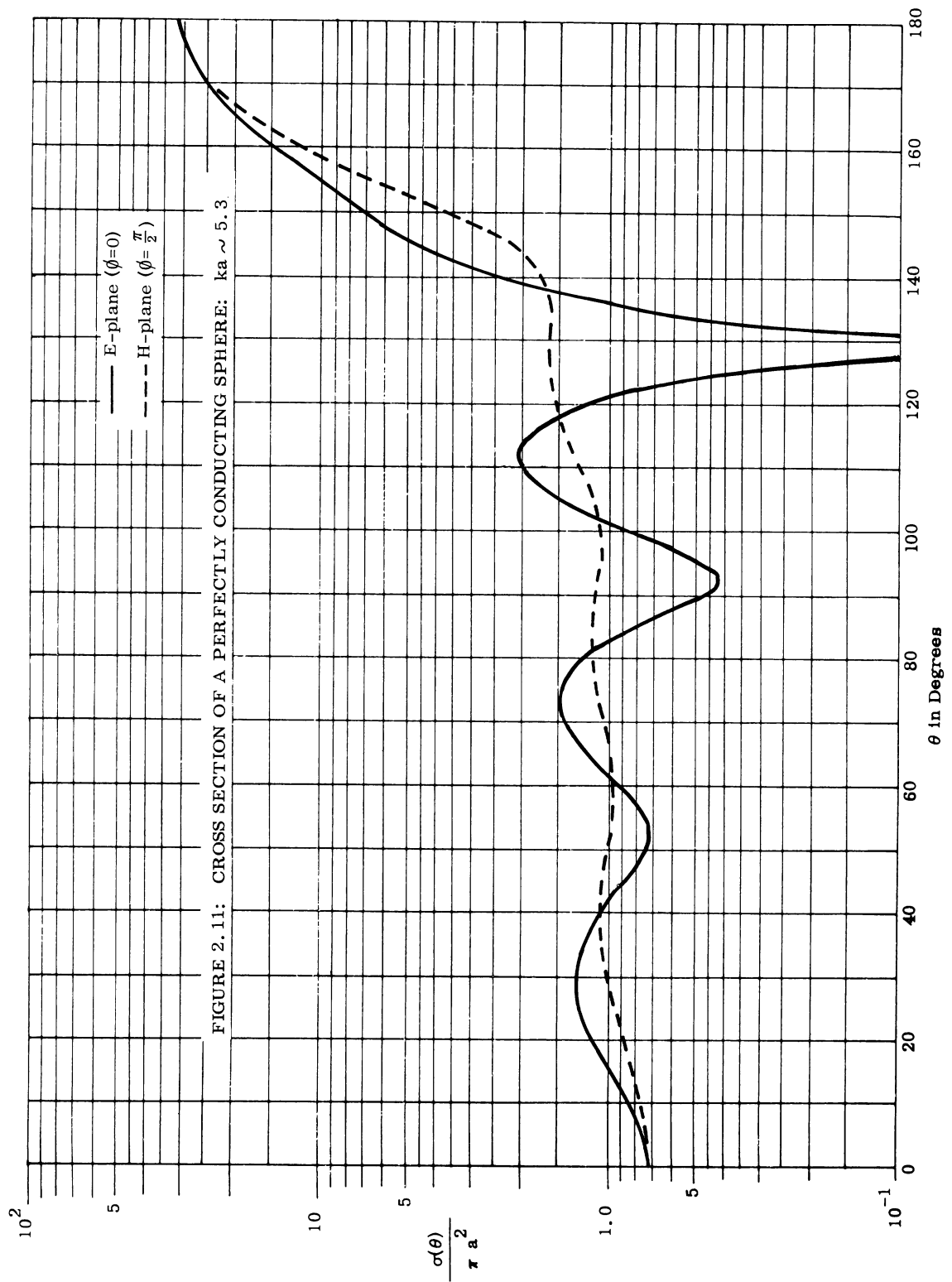
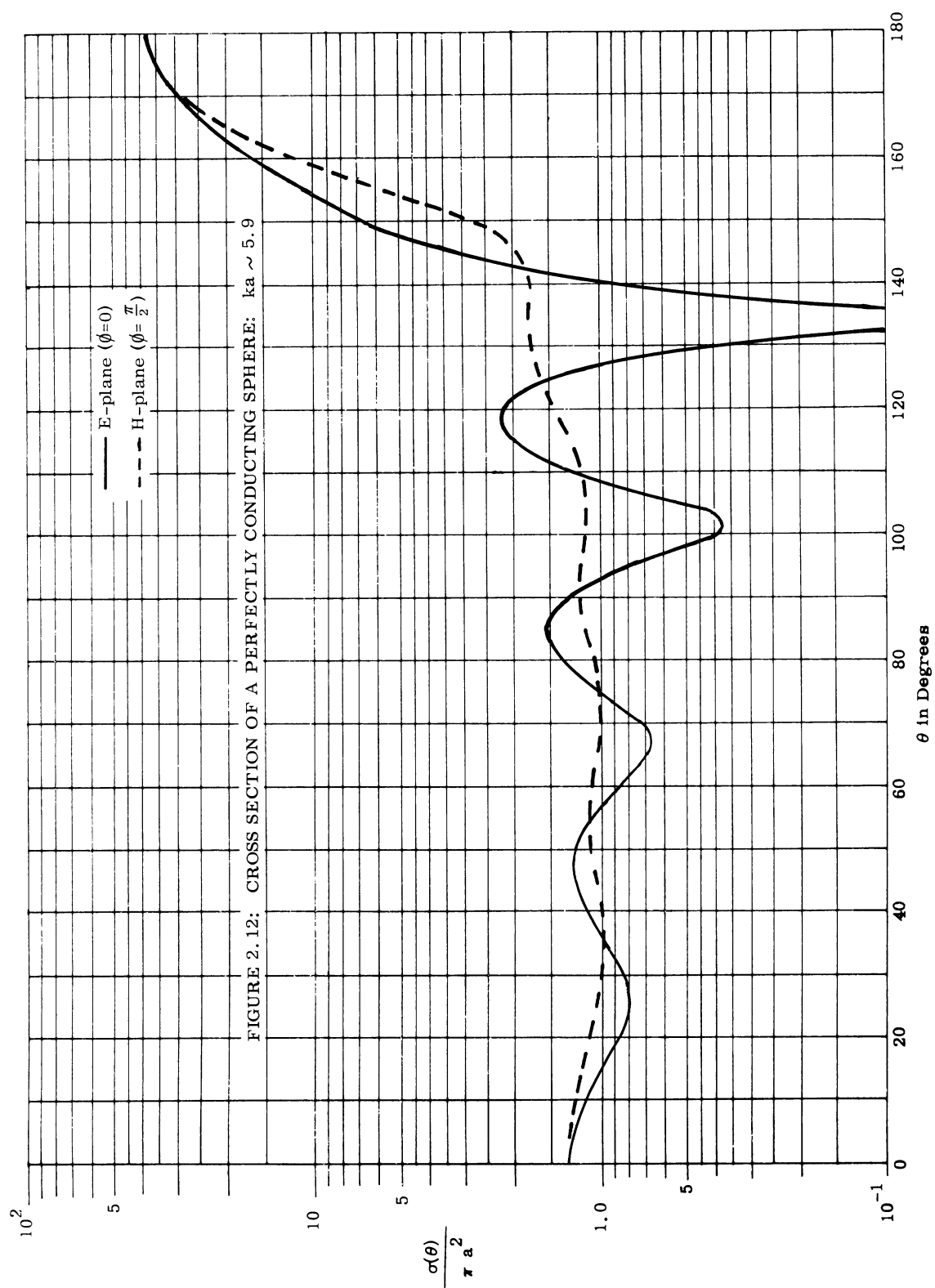


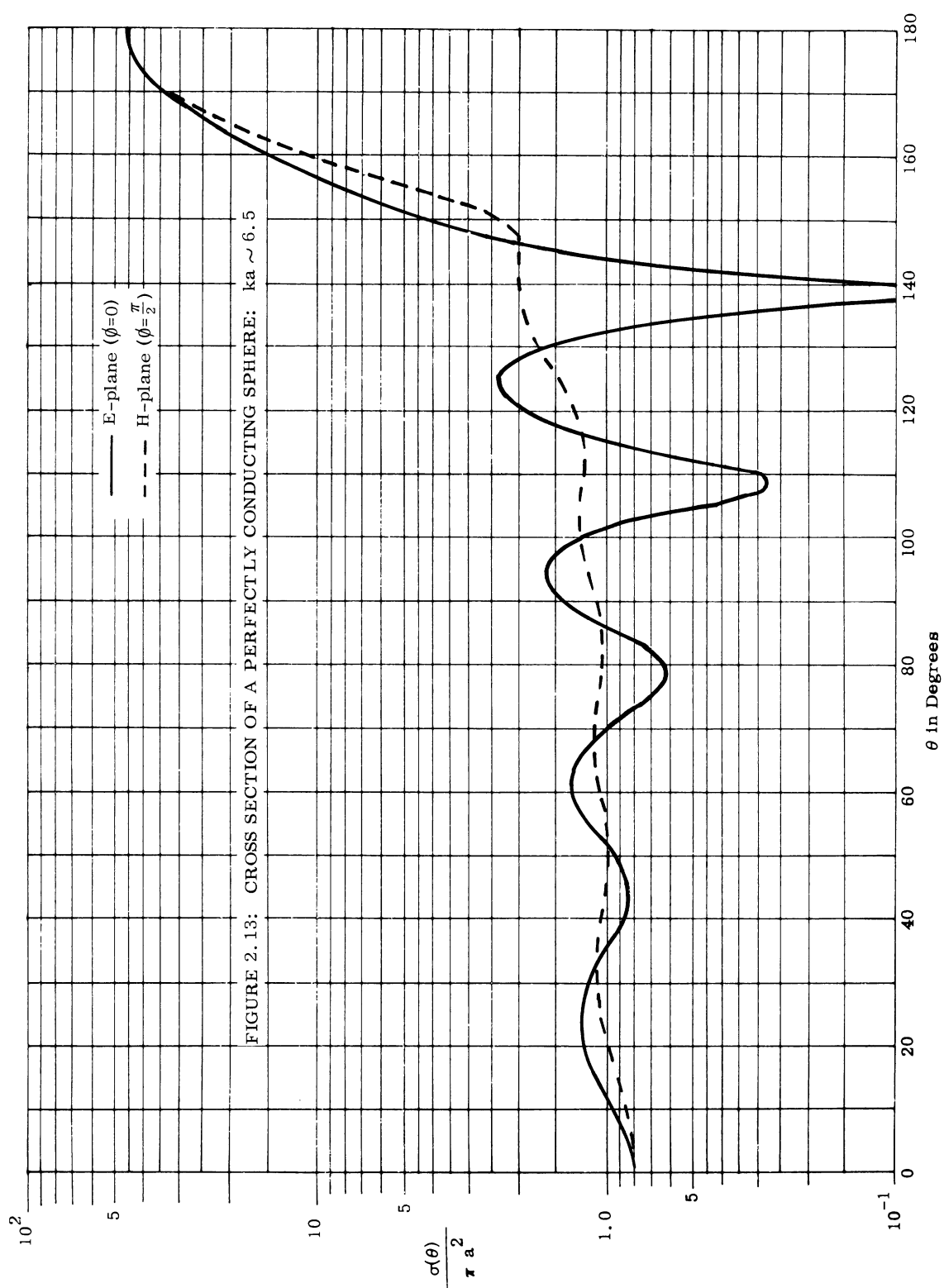
FIGURE 2.8: CROSS SECTION OF A PERFECTLY CONDUCTING SPHERE: $ka \sim 3.5$

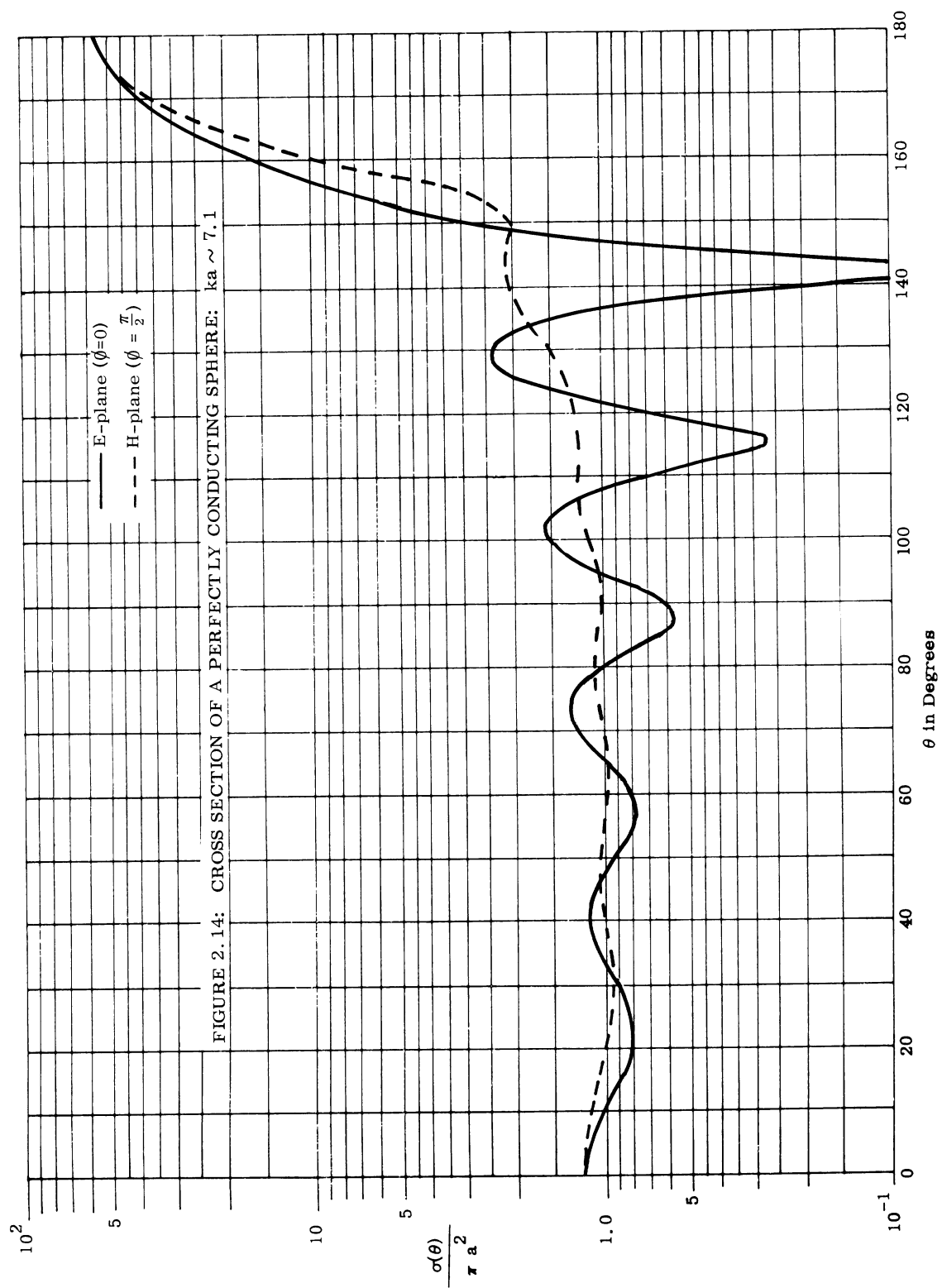


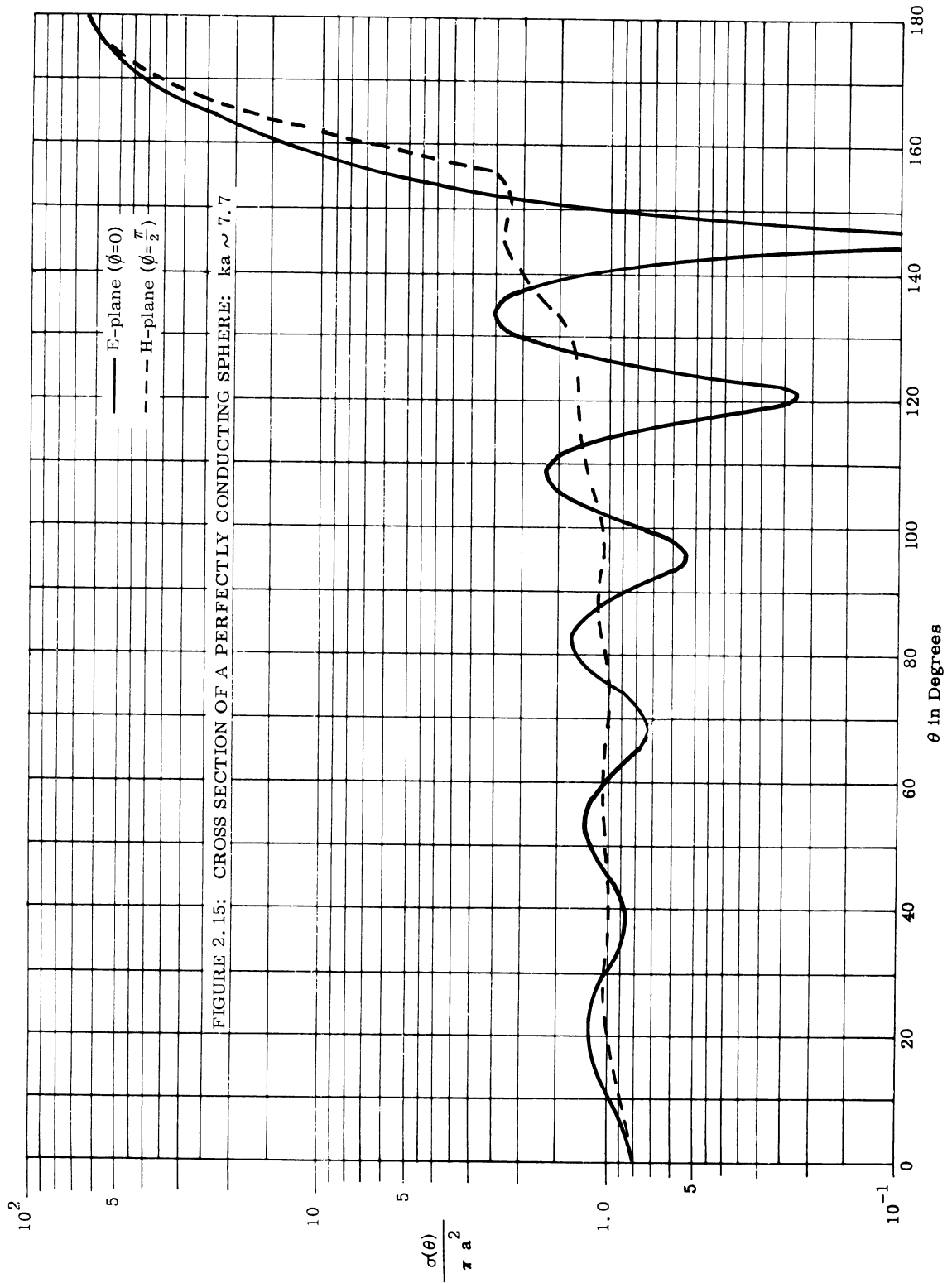


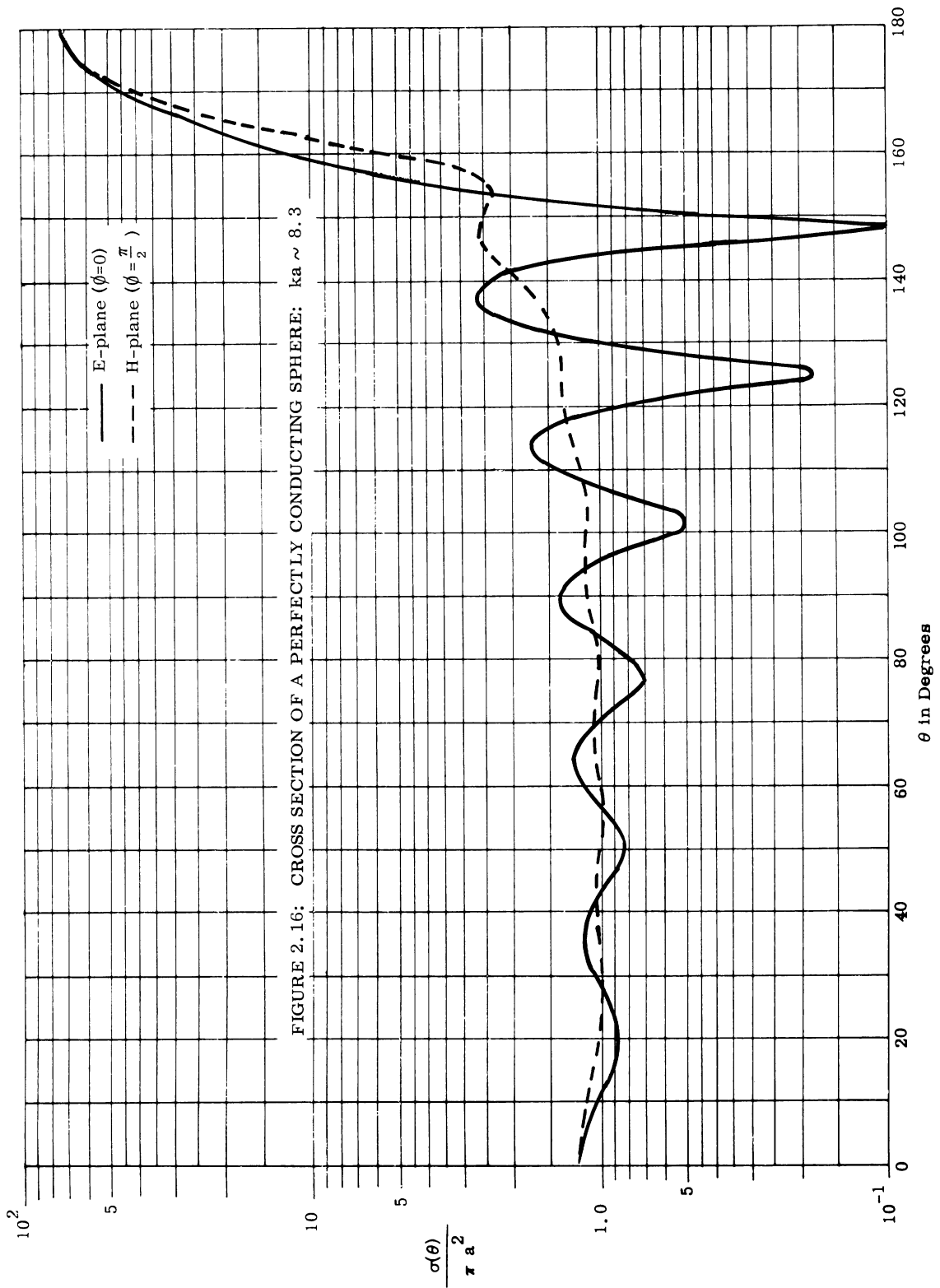


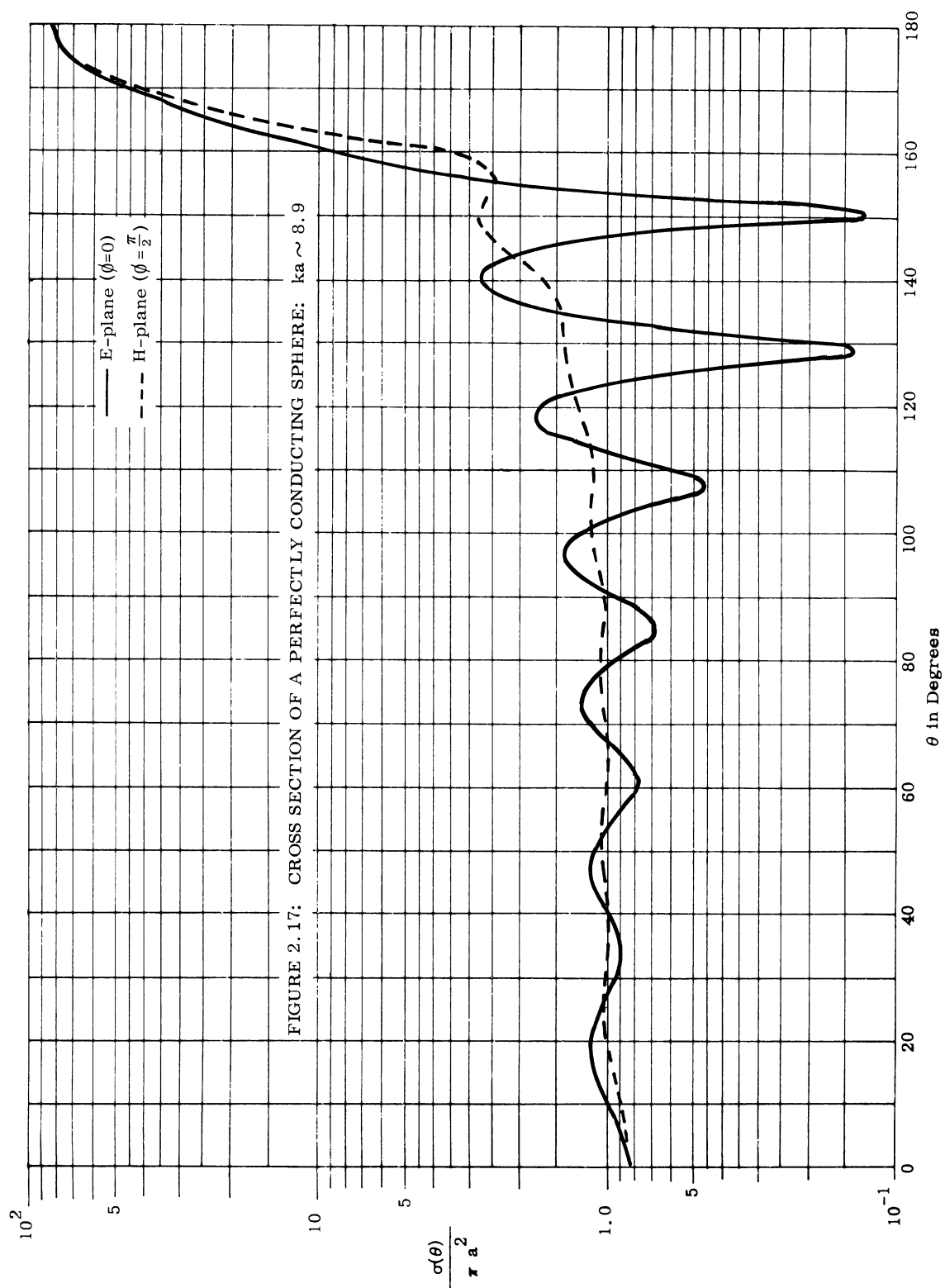


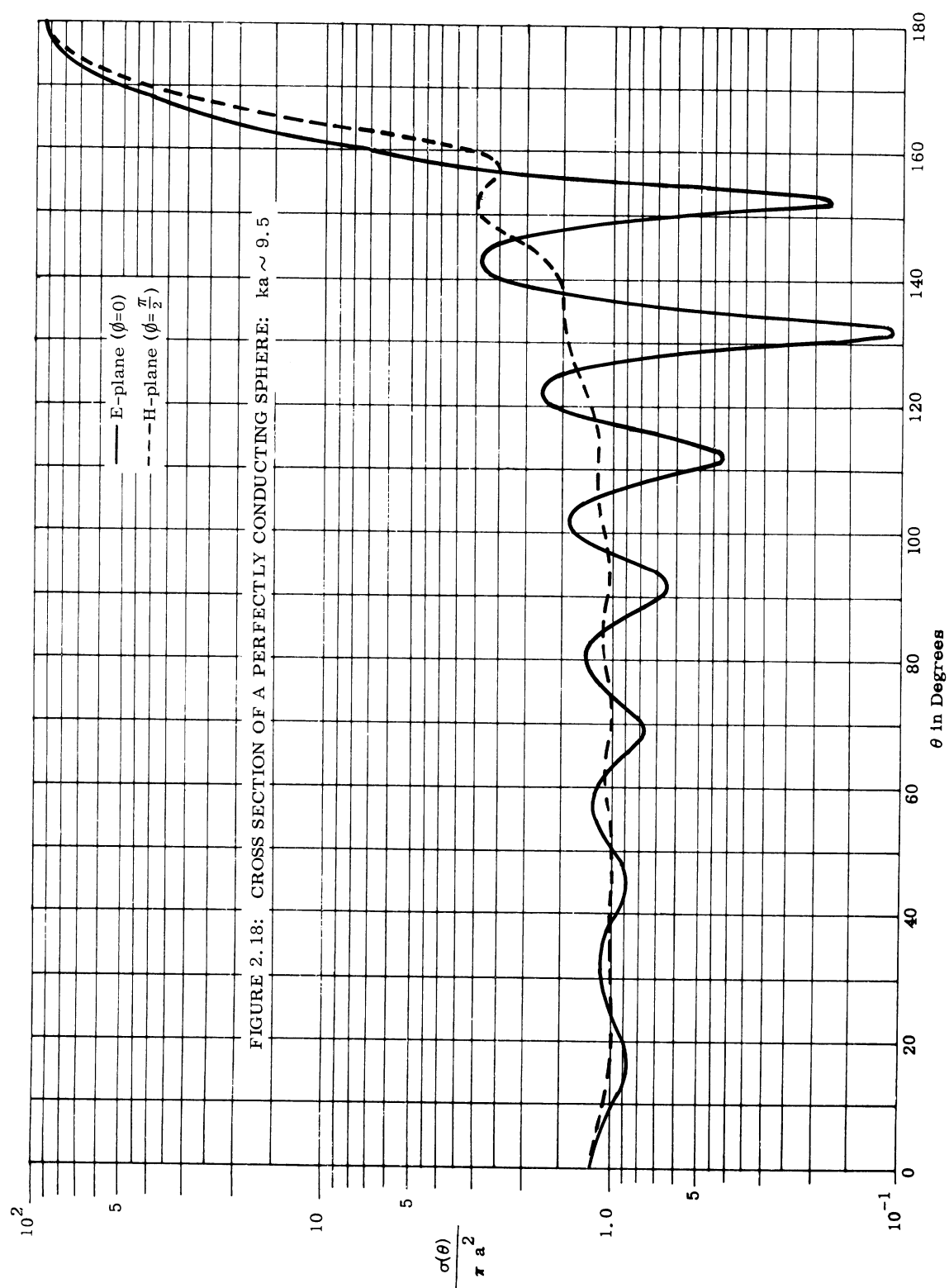


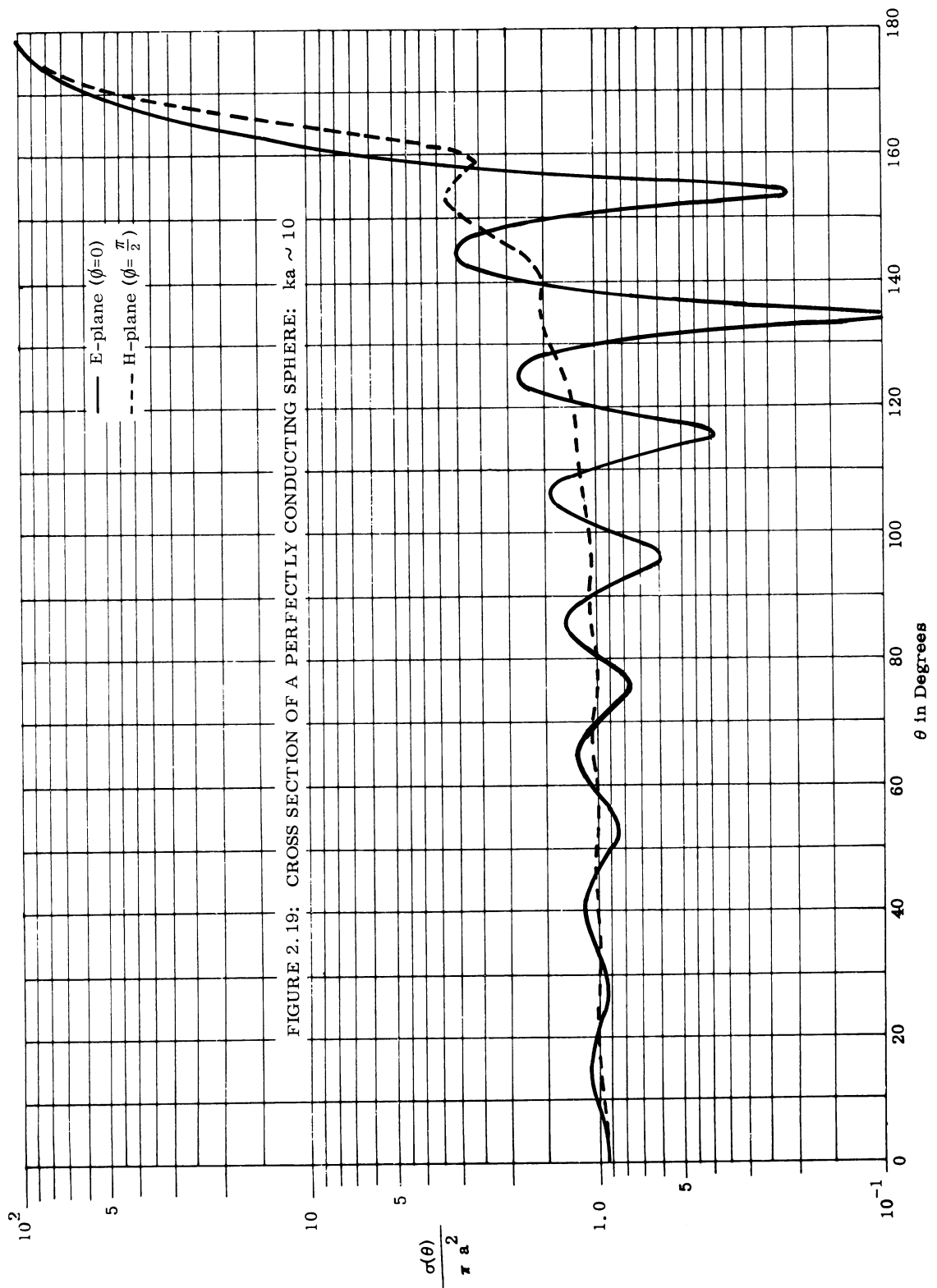


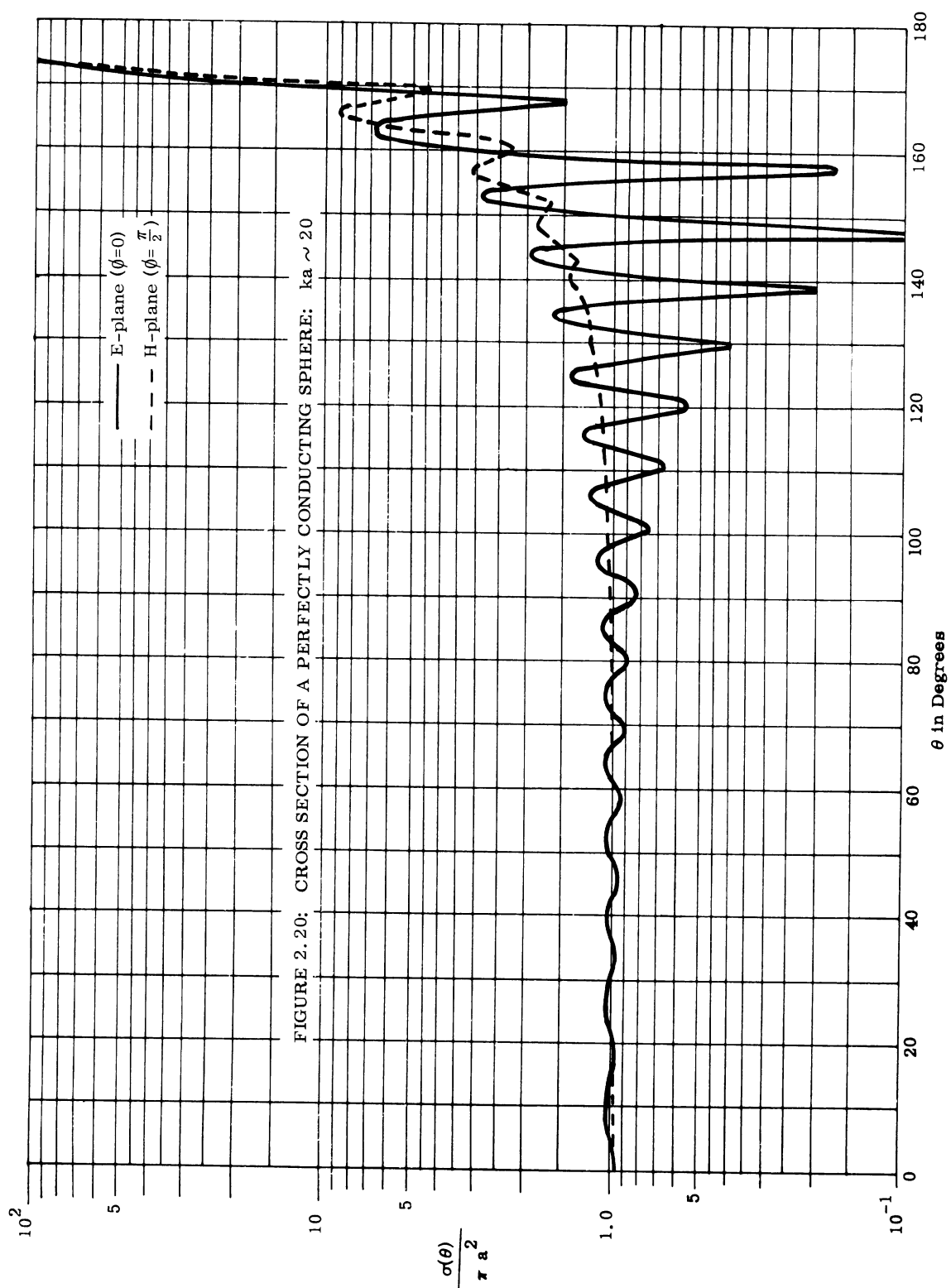












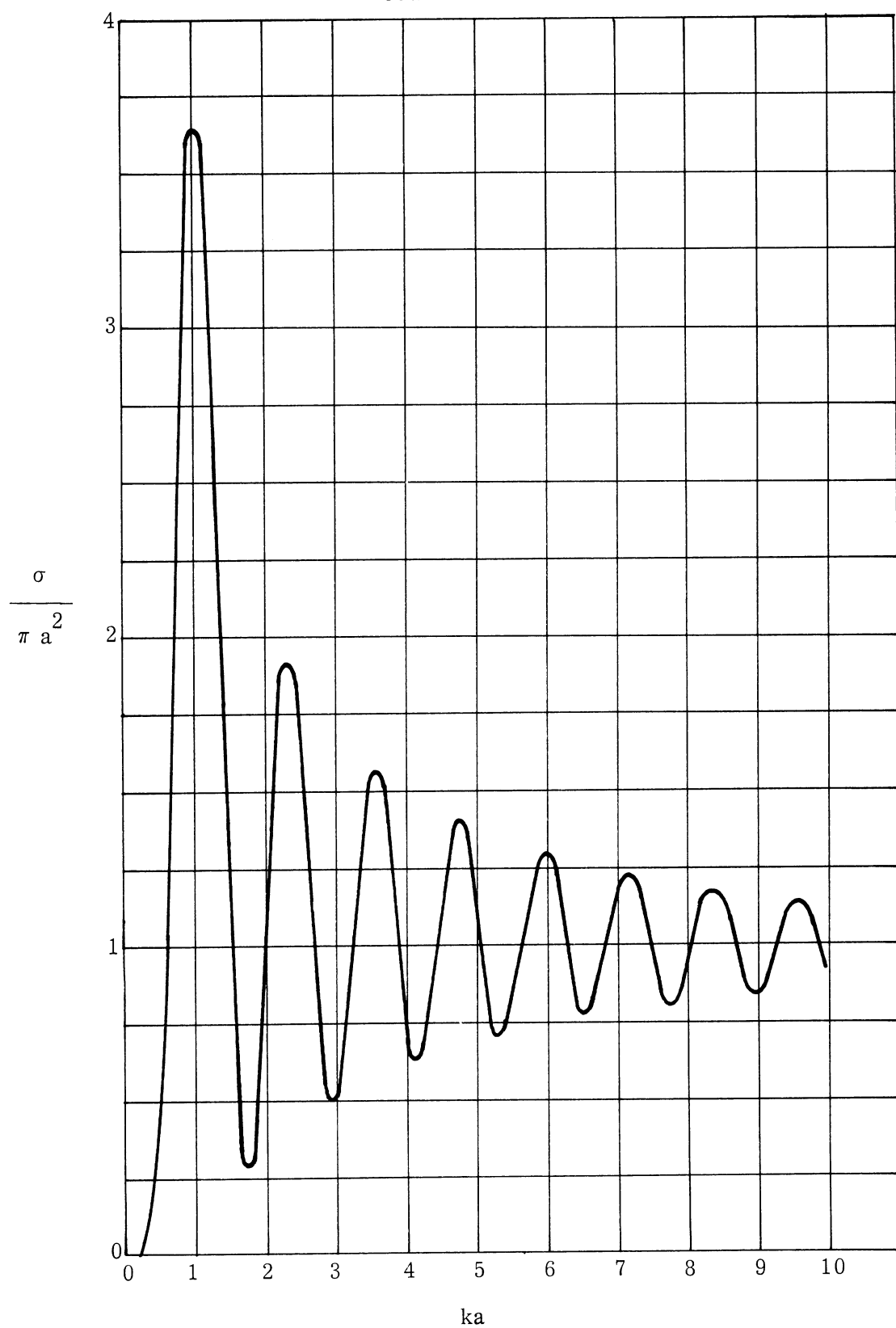


FIGURE 2-21 : BACKSCATTERING CROSS SECTION FOR PERFECTLY CONDUCTING SPHERE

III

LOW FREQUENCIES

At low frequencies the scattering function $f(\theta, \phi)$ can be expanded in series of ascending (positive) powers of k with coefficients which are functions of θ and ϕ . Associated with each power of k is the corresponding power of a parameter ℓ having the dimensions of length, and since $f(\theta, \phi)$ is independent of r , this parameter must be a characteristic of the scattering body. It is obvious that in the case of a sphere the parameter is the radius. For sufficiently small values of $k\ell$ this expansion is absolutely convergent (a fuller discussion of the convergence properties is given in section 3.4), and is generally referred to as the Rayleigh series for the body in question.

The present section is entirely concerned with this expansion, and the purpose is not only to determine the form of the series (i. e. the powers of k which it contains), but also the precise coefficients of the various powers.

In section 3.1 the series is obtained directly from the Mie solution by expanding for small argument the spherical Bessel and Hankel functions occurring in the solution. In so doing the aim was to set down explicitly a significant number of terms in the expansion, and presented here are the first five terms in the expansion for the real part of $f(\theta, \phi)$, together with the first four terms in the expansion for the imaginary part. The resulting expression for the scattering function then

includes terms in ρ^{12} , where $\rho = ka$.

In section 2.2 an alternative method is developed whereby the Rayleigh series is obtained directly without any reference to the Mie solution, and without ever having to solve a boundary value problem as such. There is no limit to the number of terms which can be calculated in this way, and while the derivation of the higher order terms can become tedious, the labour is no worse than that involved in the expansion of the Mie coefficients. In addition, the calculation is partially self-checking.

One of the main advantages of this new approach is the promise which it holds of being applicable to other (and more general) bodies for which the exact Mie-type solution is not available, but even with a spherically stratified sphere it may be quicker to use this method to obtain the first few terms in the Rayleigh series, and in section 3.3 the leading term for a dielectric coated sphere is calculated.

3.1 Derivation from the Mie Series

Since the exact solution for the sphere is known in the form of the Mie series it is only necessary to expand the radial functions for small ρ to obtain the Rayleigh series.

The coefficients of the vector wave functions for a perfectly conducting sphere are given in eqns (2-25), and using the fact that

$$j_n(\rho) = \sqrt{\pi} \frac{\rho^n}{2^{n+1}} \sum_{m=0}^{\infty} (-i)^m \frac{\left(\frac{\rho}{2}\right)^{2m}}{m! (m+n+\frac{1}{2})!} \quad (3-1)$$

$$h_n(\rho) = \sqrt{\pi} \frac{\rho^n}{2^{n+1}} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{\rho}{2}\right)^{2m}}{m! (m+n+\frac{1}{2})!} - i (-1)^n \frac{2^n}{\rho^{n+1}} \sqrt{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{\rho}{2}\right)^{2m}}{m! (m-n-\frac{1}{2})!} \quad (3-2)$$

we have immediately

$$A_{oll} = -\frac{1}{2} \rho^3 \left(1 - \frac{3}{5} \rho^2 - \frac{i}{3} \rho^3 + \frac{3}{7} \rho^4 + \frac{2i}{5} \rho^5 - \frac{11}{27} \rho^6 - \frac{71i}{175} \rho^7 \right. \\ \left. + \frac{67}{165} \rho^8 + \frac{1151i}{2335} \rho^9 \right) + O(\rho^{13})$$

$$B_{ell} = i \rho^3 \left(1 + \frac{3}{10} \rho^2 + \frac{2i}{3} \rho^3 - \frac{3}{14} \rho^4 + \frac{2i}{5} \rho^5 - \frac{17}{27} \rho^6 - \frac{79}{350} i \rho^7 \right. \\ \left. - \frac{133}{330} \rho^8 - \frac{1733i}{2335} \rho^9 \right) + O(\rho^{13})$$

$$A_{ol2} = \frac{i}{54} \rho^5 \left(1 - \frac{5}{21} \rho^2 - \frac{i}{45} \rho^3 + \frac{5}{297} \rho^4 + \frac{2}{189} i \rho^5 \right) + O(\rho^{13})$$

$$B_{el2} = \frac{1}{36} \rho^5 \left(1 - \frac{5}{42} \rho^2 + \frac{5}{108} \rho^3 + \frac{i}{30} \rho^4 - \frac{5}{264} \rho^5 - \frac{i}{126} \rho^6 \right) + O(\rho^{13})$$

$$A_{ol3} = \frac{1}{2700} \rho^7 \left(1 - \frac{7}{45} \rho^2 + \frac{7}{825} \rho^3 \right) + O(\rho^{13})$$

$$B_{e13} = - \frac{i}{2025} \rho^7 \left(1 - \frac{7}{60} \rho^2 + \frac{91}{9900} \rho^4 \right) + O(\rho^{13})$$

$$A_{o14} = - \frac{i}{20(105)^2} \rho^9 \left(1 - \frac{9}{77} \rho^2 \right) + O(\rho^{13})$$

$$B_{e14} = - \frac{1}{(420)^2} \rho^9 \left(1 - \frac{153}{1540} \rho^2 \right) + O(\rho^{13})$$

$$A_{o15} = - \frac{1}{30(945)^2} \rho^{11} + O(\rho^{13})$$

$$B_{e15} = \frac{i}{(4725)^2} \rho^{11} + O(\rho^{13})$$

For $n > 5$, A_{on} and B_{en} are $O(\rho^{13})$.

The above expansions are sufficient to specify $f(\theta, \phi)$ correct to $O(\rho^{12})$, but rather than write down the resulting series for arbitrary θ and ϕ we shall concentrate on the particular cases of back and forward scattering ($\theta = 0$ and π respectively). Substitution into eqns (2-42) and (2-43) then gives

$$\begin{aligned} f(0) = & \frac{3}{2} \rho^3 \left\{ 1 - \frac{5}{54} \rho^2 + \frac{17}{900} \rho^4 - \frac{6651923}{7938000} \rho^6 - \frac{249170261}{1875352500} \rho^8 \right\} \\ & + \frac{1}{2} i \rho^6 \left\{ 1 + \frac{6}{5} \rho^2 - \frac{1951}{2268} \rho^4 - \frac{5795}{6804} \rho^6 \right\} + O(\rho^{13}) \end{aligned} \quad (3-3)$$

$$f(\pi) = \frac{1}{2} \rho^3 \left\{ 1 + \frac{113}{90} \rho^2 - \frac{1783}{2100} \rho^4 - \frac{670057}{793800} \rho^6 - \frac{8369355737}{6876292500} \rho^8 \right\} \\ + \frac{5}{6} i \rho^6 \left\{ 1 + \frac{6}{25} \rho^2 - \frac{2137}{94500} \rho^4 - \frac{56689}{56700} \rho^6 \right\} + O(\rho^{13}) \quad (3-4)$$

3.2 The Alternative Method

In order to illustrate the term-by-term technique for deriving the low frequency expansion it is convenient to consider once again the problem of the field (eqns (2-1)) incident on a perfectly conducting sphere.

The first step is to postulate a general expression for the scattered field and the obvious form is that shown in eqn (2-35). Each of the vector wave functions \underline{M}_{oln} and \underline{N}_{eln} involves the radial distance r through the Hankel function $h_n(kr)$ and its derivatives with respect to kr , and consequently any power of r is always accompanied by a like power of k . Near to the surface of the sphere $r \sim a$, and for sufficiently small values of kr (that is, for sufficiently low frequencies)

$$h_n(kr) \sim - \frac{i}{(kr)^{n+1}} \frac{(2n)!}{2^n n!} \sim - \frac{kr}{n} \frac{1}{kr} \left[kr h_n(kr) \right]' \quad (3-5)$$

As a result, all the components of \underline{N}_{eln} are of equal order in the near field (in contrast to their behavior in the far field), while the components of \underline{M}_{oln} for the same value of n are of one higher order. If, therefore, the product $B_{ell} \underline{N}_{ell}$ is to remain finite in the near field as the frequency decreases indefinitely, it is

necessary that

$$B_{ell} \propto k^3$$

and since B_{ell} is dimensionless, k must be associated with a length parameter

which can only be the radius of the sphere. Hence $B_{ell} = O(\rho^3)$ for small ρ , and

from a consideration of the higher powers of kr in the expansion of $\frac{1}{kr} \left[kr h_n(kr) \right]'$

for small kr we are led to write

$$B_{ell} = \rho^3 (\beta_{10} + \rho\beta_{11} + \rho^2\beta_{12} + \rho^3\beta_{13} + \dots) . \quad (3-6)$$

Similarly

$$B_{el2} = \rho^5 (\beta_{20} + \rho\beta_{21} + \rho^2\beta_{22} + \rho^3\beta_{23} + \dots) \quad (3-7)$$

and so on. Any of the above coefficients may, of course, be zero.

For the product $A_{oll} \frac{M}{r_{oll}}$ a like analysis would suggest that the expansion for A_{oll} should start with a term in ρ^2 , but by choosing instead the expression for the magnetic field near to the surface (so that A_{oll} occurs in combination with $\frac{N}{r_{oll}}$) it is seen that the coefficient of ρ^2 is in fact zero. We therefore take

$$A_{oll} = \rho^3 (\alpha_{10} + \rho\alpha_{11} + \rho^2\alpha_{12} + \rho^3\alpha_{13} + \dots) \quad (3-8)$$

analogous to equation (3-6), and similarly

$$A_{ol2} = \rho^5 (\alpha_{20} + \rho\alpha_{21} + \rho^2\alpha_{22} + \rho^3\alpha_{23} + \dots), \quad (3-9)$$

etc.

At the surface $r = a$ the boundary conditions require the vanishing of the tangential components of the total electric field, and substituting the expressions for \underline{E}^i and \underline{E}^s , we have

$$X + \sum_{n=1}^{\infty} \left\{ \left(A_{on} h_n(\rho) \right) \frac{\partial}{\partial \theta} P_n(\cos \theta) + \left(B_{en} \frac{1}{\rho} \left[\rho h_n(\rho) \right]' \right) \frac{P_n \cos \theta}{\sin \theta} \right\} = 0, \quad (3-10)$$

and

$$X \cos \theta + \sum_{n=1}^{\infty} \left\{ \left(A_{on} h_n(\rho) \right) \frac{P_n(\cos \theta)}{\cos \theta} + \left(B_{en} \frac{1}{\rho} \left[\rho h_n(\rho) \right]' \right) \frac{\partial}{\partial \theta} P_n(\cos \theta) \right\} = 0, \quad (3-11)$$

where

$$X = e^{-i\rho \cos \theta} = \sum_{n=0}^{\infty} \frac{(-i\rho \cos \theta)^n}{n!}. \quad (3-12)$$

Since the expansions for $h_n(\rho)$ and $\frac{1}{\rho} \left[\rho h_n(\rho) \right]'$ are known, the coefficients α_{ij} and B_{ij} in the expansions for the A_{on} and B_{en} can now be determined by equating to zero the coefficients of each power of ρ in equations (3-10) and (3-11).

In both equations the lowest power of ρ is ρ^0 and the coefficient is made up of a contribution from the single vector wave function N_{-ell} and the static term in the incident field expansion. To this order in ρ the two boundary conditions reduce to

$$(1 + i\beta_{10}) = 0$$

and

$$(1 + i\beta_{10}) \cos \theta = 0$$

giving

$$\beta_{10} = i.$$

The second stage in the analysis involves the terms in ρ . Contributions from the two further wave functions \underline{M}_{oll} and \underline{N}_{el2} are now introduced, together with a contribution from \underline{N}_{ell} , and are matched to the second term in the incident field expansion. We have

$$-i \cos \theta - i\alpha_{10} \cos \theta + i\beta_{11} + 18 i\beta_{20} \cos \theta = 0$$

$$-i \cos^2 \theta - i\alpha_{10} + i\beta_{11} \cos \theta + 18 i\beta_{20} \cos 2\theta = 0$$

and by identifying coefficients of like trigonometrical functions in each equation, it is found that

$$\beta_{11} = 0, \quad \alpha_{10} = -\frac{1}{2}, \quad \beta_{20} = \frac{1}{36}.$$

Continuing in this manner the various terms in the expansions of the A_{oln} and B_{eln} can be derived, but since the analysis is so entirely straightforward there is little point in including further stages. Suffice to say that the results are in accordance with those given in section 3.1.

On the other hand, there are several features of the method which it is desirable to point out. In the first place we remark that the analysis at each stage

is to some extent self-checking, in that the n th stage provides $2n + 1$ self-consistent equations from which to calculate $2n-1$ unknowns. Moreover, the n th stage (which brings in contributions from the wave functions \underline{M}_{oln-1} and \underline{N}_{eln}) requires that

$$\beta_{n-1 \ 1}, \quad \beta_{n-3 \ 3}, \quad \beta_{n-5 \ 5}, \quad \dots$$

$$\text{and } \alpha_{n-2 \ 1}, \quad \alpha_{n-4 \ 3}, \quad \alpha_{n-6 \ 5}, \quad \dots$$

all have the value zero, and therefore introduces no new power of ρ into the expansions for the corresponding coefficients of the vector wave functions. In fact, each stage yields a correction term to the expansion for either A_{olr} or B_{elr} ($r \leq n$), but not both, and since A_{olr} and B_{elr} are of the same order in ρ , two successive stages are needed to give a new order of correction to both these coefficients.

A further point of interest concerns the real or imaginary character of the α_{ij} and the β_{ij} . At every odd stage in the analysis an even power of ρ is matched to a like power of ρ in the incident field expansion, and from eqn (3-12) it is apparent that this implies the matching of the appropriate α_{ij} and β_{ij} to a real coefficient. In contrast, the even stages produce values of α_{ij} and β_{ij} which are pure imaginary and hence

$$\alpha_{ij} \text{ is } \begin{matrix} \text{real} \\ \text{imaginary} \end{matrix} \quad \text{if } (i-j) \text{ is } \begin{matrix} \text{odd} \\ \text{even} \end{matrix}$$

whereas

$$\beta_{ij} \text{ is } \begin{matrix} \text{real} \\ \text{imaginary} \end{matrix} \quad \text{if } (i-j) \text{ is } \begin{matrix} \text{even} \\ \text{odd} \end{matrix}$$

It now follows that when n is odd all even powers of ρ in the expansions for A_{oln} and iB_{eln} have real coefficients (odd powers having pure imaginary coefficients), and the reverse situation holds where n is even. This immediately determines the power of ρ in the expression for $f(\theta, \phi)$ which have real or imaginary coefficients, and reference to eqn (2-37) shows that all even powers must have imaginary coefficients, while the odd powers have real coefficients. The fact that the first imaginary coefficient is $O(\rho^6)$ is a consequence of the vanishing of α_{11} and β_{11} , which thereby removes the ρ^4 powers. These conclusions are confirmed by eqns (3-3) and (3-4).

Our final remarks concern the initial stages in the analysis. At the first stage the coefficients of ρ^0 are matched and this requires that the incident field factor e^{-ikz} be replaced by unity, so that \underline{E}^i and \underline{H}^i are independent of one another to this approximation. Moreover, only B_{ell} contributes a term of order ρ^0 , and consequently this first approximation has produced a near-field boundary-value problem in which the electric and magnetic fields are decoupled. Although the coupling is re-introduced at the second stage, it may be of interest to consider why the initial decoupling does not affect the derivation of a complete solution.

The first stage essentially reduces the problem to a static one and gives only $\beta_{10} = i$, which corresponds to a simple electric dipole. Thus, the first stage ignores the magnetic dipole contribution to the scattered field, which contribution is of the same order as the electric one, and to obtain the electric field due to

the magnetic dipole either of two methods can be adopted. The first of these would require the corresponding first stage in the solution of the magnetic field problem and the subsequent use of the field relations to determine the contribution to \underline{E}^S . In practice, however, this is not necessary in that the second stage in the solution of the electric field problem re-introduces the coupling between the electric and magnetic fields and brings in the magnetic dipole contribution. Two stages are therefore necessary to complete the first approximation to the scattered field, and the fact that no magnetic field problem as such has to be considered is a direct consequence of the symmetry between the expressions for the scattered electric and magnetic fields in terms of the wave functions $\underline{M}_{\underline{e}mn}$ and $\underline{N}_{\underline{e}mn}$.

On the other hand, if only the first term in the expansion for $f(\theta, \phi)$ is required, it may be more convenient to replace the second stage by the first stage of the corresponding magnetic dipole analysis, since this may prove to be a somewhat easier calculation (particularly for bodies other than the simple homogeneous sphere). In this case, the whole analysis can be expressed more concisely.

Taking first the electric dipole problem, the first stage is to match $B_{\text{ell}} \underline{N}_{\text{ell}}$ to the unit vector \hat{x} at the surface $r=a$ using the boundary condition

$$\hat{n} \wedge (\underline{E}^i + \underline{E}^S) = 0,$$

and since

$$\hat{x} = \sin \theta \cos \phi \hat{r} + \cos \theta \cos \phi \hat{\theta} - \sin \phi \hat{\phi},$$

consideration of the tangential components of $\underline{N}_{\text{ell}}$ shows that

$$B_{\text{ell}} = \frac{-1}{\frac{1}{\rho} [\rho h_1(\rho)]'} \sim i\rho^3 \quad \text{for small } \rho.$$

For the magnetic dipole problem the corresponding stage is to match $i A_{\text{oll}} \underline{N}_{\text{oll}}$ to the unit vector $\hat{\underline{y}}$ using the boundary condition

$$\hat{\underline{n}} \cdot (\underline{H}^i + \underline{H}^s) = 0,$$

and in like manner this gives

$$A_{\text{oll}} = \frac{i}{\frac{2}{\rho} h_1(\rho)} \sim -\frac{1}{2}\rho^3 \quad \text{for small } \rho.$$

This completes the analysis for the two near-static problems. The electric dipole makes a direct contribution to the scattered electric field, and according to the first of eqns (2-35) we have

$$\underline{E}^s \sim i\rho^3 E_o \underline{N}_{\text{ell}}. \quad (3-13)$$

Similarly, the magnetic dipole contributes directly to the scattered magnetic field and from the second of eqns (2-35)

$$\underline{H}^s \sim -\frac{1}{2} i\rho^3 H_o \underline{N}_{\text{ell}}, \quad (3-14)$$

from which the electric field contribution can be found by using Maxwell's equations. The importance of this derivation lies in the fact that both (3-13) and

(3-14) can be obtained by appealing only to statics.

In practice however, the last step (use of Maxwell's equations) can be avoided by substituting the expressions for A_{oll} and B_{ell} directly into the first of eqn (2-35). We then have

$$\underline{E}^s \sim i\rho^3 E_o (N_{ell} + \frac{1}{2} i M_{oll}),$$

which represents the combined contribution due to the electric and magnetic dipoles, and the corresponding far field expansion is

$$\underline{E}^s \sim E_o \frac{e^{ikr}}{kr} \rho^3 \left\{ \left(\frac{1}{2} + \cos \theta \right) \cos \phi \hat{\theta} - \left(1 + \frac{1}{2} \cos \theta \right) \sin \phi \hat{\phi} \right\}. \quad (3-15)$$

3.3 A Dielectric-Coated Sphere

As an example of how the above method is used in a non-trivial problem, we shall here derive the leading term in the Rayleigh solution for a coated sphere (see section 2.2).

Consider a perfectly conducting sphere of radius a which is covered with a layer of dielectric of thickness d . The permittivity and permeability of the dielectric are ϵ and μ respectively; the conductivity, however, is zero (otherwise the Rayleigh solution is the same, to the first term, as for a perfectly conducting sphere of radius $a + d$). The whole is immersed in a homogeneous isotropic medium which, for simplicity, will be regarded as free space.

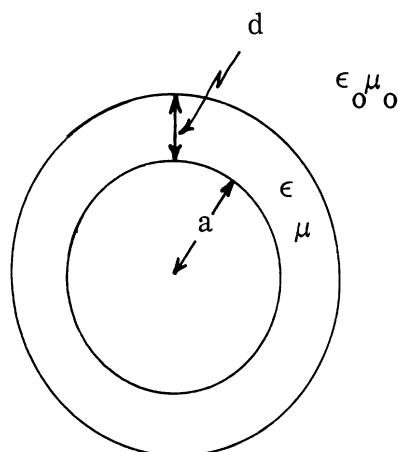


FIGURE 3-1

To determine the Rayleigh solution it is sufficient to solve two static problems and then match these results to expressions involving spherical vector wave functions. In the first static problem the incident field is merely

$$\underline{E}^i = E_o \hat{x} \quad (3-16)$$

and the task is to obtain the scattered (electrostatic) field which this excites. The second problem is analogous in that the incident field is here

$$\underline{H}^i = -H_o \hat{y}, \quad (3-17)$$

so that a magnetostatic field is now involved.

The most general solutions of Laplace's equation are of the form

$$\Phi_{e\ mn}^{(1)} = r^n P_n^m(\cos \theta) \frac{\cos m \phi}{\sin m \phi}$$

$$\Phi_{e\ mn}^{(1)} = r^{-n-1} P_n^m(\cos \theta) \frac{\cos m \phi}{\sin m \phi}$$

and if Φ is regarded as a static potential, the corresponding field can be found by taking the gradient. In the region outside the sphere the scattered field must be expressible in terms of $\Phi_{e\ mn}^{(2)}$ alone, but in the dielectric coating both types of potential will occur.

Let us take first the electrostatic problem in which the incident field is given by eqn (3-16). Since

$$\hat{x} = \nabla \Phi_{ell}^{(1)}$$

we have immediately that

$$\underline{E}^i = E_o \nabla \Phi_{ell}^{(1)} \quad (3-18)$$

If \underline{E} is the total electrostatic field in free space, so that $\underline{E} = \underline{E}^i + \underline{E}^s$, and if \underline{E}' similarly denotes the field in the layer, the boundary conditions at the dielectric interface ($r = a+d$) are

$$\hat{n} \wedge \underline{E} = \hat{n} \wedge \underline{E}'$$

and

$$\epsilon_o \hat{n} \cdot \underline{E} = \epsilon_1 \hat{n} \cdot \underline{E}'$$

At the surface ($r=a$) of the perfectly conducting sphere the only condition is

$$\hat{n} \wedge \underline{E}' = 0$$

In view of the θ and ϕ dependence implicit in the expression (eqn (3-18)) for

the incident field, it is apparent that the boundary conditions can be satisfied by choosing the following expressions for the secondary field:

$$\underline{E}^s = A \nabla \Phi_{ell}^{(2)} \quad (3-19)$$

$$\underline{E}' = B \nabla \Phi_{ell}^{(2)} + C \nabla \Phi_{ell}^{(1)} \quad (3-20)$$

where A, B and C are constants as yet undetermined. The boundary conditions now give

$$E_o + \frac{A}{(a+d)^3} = C + \frac{B}{(a+d)^3} ,$$

$$\epsilon_o \left\{ E_o - \frac{2}{(a+d)^3} A \right\} = \epsilon \left\{ C - \frac{2}{(a+d)^3} B \right\} ,$$

$$C + \frac{B}{a^3} = 0 ,$$

from which we obtain

$$A = -E_o b^3 \left\{ \frac{b^3 + 2a^3 - \frac{\epsilon_o}{\epsilon} (b^3 - a^3)}{b^3 + 2a^3 + 2 \frac{\epsilon_o}{\epsilon} (b^3 - a^3)} \right\} \quad (3-21)$$

where $b = a+d$. If $d \ll a$ so that powers of d/a higher than the first can be neglected, the expression for A becomes

$$A = -E_o (a+d)^3 \left(1 - 3 \frac{\epsilon_o}{\epsilon} \frac{d}{a} \right) , \quad (3-22)$$

which differs from the result for a perfectly conducting sphere of radius $a+d$ only in the presence of the multiplying factor $\left(1 - 3 \frac{\epsilon_o}{\epsilon} \frac{d}{a}\right)$.

From eqns (3-19) and (3-22), the field which is scattered into free space is

$$\underline{E}^s = - E_o (a+d)^3 \left(1 - 3 \frac{\epsilon_o}{\epsilon} \frac{d}{a}\right) \nabla \Phi_{ell}^{(2)},$$

and the next step is to match this to the limit of a non-static solution at low frequencies. Since

$$\nabla \Phi_{ell}^{(2)} = \frac{1}{r^3} \left\{ -2 \sin \theta \cos \phi \hat{r} + \cos \theta \cos \phi \hat{\theta} - \sin \phi \hat{\phi} \right\},$$

consideration of the vector wave functions \underline{M}_{emn} and \underline{N}_{enm} shows that for $\lambda \gg r$,

$$\nabla \Phi_{ell}^{(2)} \sim -i k^3 \underline{N}_{e11},$$

and hence, in the near-static limit

$$\underline{E}^s = i E_o k^3 (a+d)^3 \left(1 - 3 \frac{\epsilon_o}{\epsilon} \frac{d}{a}\right) \underline{N}_{e11}. \quad (3-23)$$

The far field is now obtained by inserting the first terms of the asymptotic expansions of the radial Hankel functions for large $|kr|$, and this gives,

$$\underline{E}^s \sim E_o \frac{e^{ikr}}{kr} k^3 (a+d)^3 \left(1 - 3 \frac{\epsilon_o}{\epsilon} \frac{d}{a}\right) (\cos \theta \cos \phi \hat{\theta} - \sin \phi \hat{\phi}). \quad (3-24)$$

Eqn (3-24) represents the k^3 contribution to the far field arising from the electric dipole, but it is not the only contribution of this order. There is in addition a term produced by the magnetic dipole and this is most conveniently obtained by considering a second static problem.

The incident (magnetostatic) field is now

$$\underline{H}^i = -H_o \hat{y} ,$$

which can be written as

$$\underline{H}^i = -H_o \nabla \Phi_{o11}^{(1)} , \quad (3-25)$$

and the task is to find the scattered field subject to the boundary conditions

$$\mu_o \hat{n} \cdot \underline{H} = \mu \hat{n} \cdot \underline{H}' ,$$

$$\hat{n} \wedge \underline{H} = \hat{n} \wedge \underline{H}'$$

at $r = a+d$, and

$$\hat{n} \cdot \underline{H}' = 0$$

at $r=a$. The form of eqn(3-25) leads us to adopt the following expressions for the fields:

$$\underline{H}^s = \tilde{A} \nabla \Phi_{o11}^{(2)} ,$$

$$\underline{H}' = \tilde{B} \nabla \Phi_{o11}^{(2)} + \tilde{C} \nabla \Phi_{o11}^{(1)} ,$$

where \tilde{A} , \tilde{B} , and \tilde{C} are constants as yet undetermined, and the boundary conditions

then give

$$\mu_o = \left\{ -H_o - 2 \frac{\tilde{A}}{(a+d)^3} \right\} = \mu \left\{ \tilde{C} - \frac{2\tilde{B}}{(a+d)^3} \right\} ,$$

$$-H_o + \frac{\tilde{A}}{(a+d)^3} \doteq \tilde{C} + \frac{\tilde{B}}{(a+d)^3} ,$$

$$\tilde{C} - \frac{2\tilde{B}}{a^3} = 0 .$$

Solving for \tilde{A} , we have

$$\tilde{A} = -\frac{H_o b^3}{2} \left\{ \frac{2b^3 + a^3 - 2 \frac{\mu}{\mu_o} (b^3 - a^3)}{2b^3 + a^3 + \frac{\mu}{\mu_o} (b^3 - a^3)} \right\} , \quad (3-26)$$

where $b=a+d$, and if $d \ll a$,

$$\tilde{A} = -\frac{H_o}{2} (a+d)^3 \left(1 - 3 \frac{\mu}{\mu_o} \frac{d}{a} \right) . \quad (3-27)$$

This differs from the result for a perfectly conducting sphere of radius $a+d$ only in the presence of the multiplying factor $\left(1 - 3 \frac{\mu}{\mu_o} \frac{d}{a} \right)$.

The field which is scattered into free space is now

$$\underline{H}^S = -\frac{H_o}{2} (a+d)^3 \left(1 - 3 \frac{\mu}{\mu_o} \frac{d}{a} \right) \nabla \Phi_{oll}^{(2)} ,$$

and by matching to the vector wave functions for $\lambda \gg r$, we have in the near-static limit

$$\underline{H}^S = i \frac{H_o}{2} k^3 (a+d)^3 \left(1 - 3 \frac{\mu}{\mu_o} \frac{d}{a} \right) \underline{N}_{oll} . \quad (3-28)$$

The corresponding electric vector is, by using Maxwell's equations

$$\underline{E}^S = -\frac{E_o}{2} k^3 (a+d)^3 \left(1 - 3 \frac{\mu}{\mu_o} \frac{d}{a}\right) M_{011} \quad , \quad (3-29)$$

and in the far field this becomes

$$\underline{E}^S \sim E_o \frac{e^{ikr}}{kr} \frac{k^3 (a+d)^3}{2} \left(1 - 3 \frac{\mu}{\mu_o} \frac{d}{a}\right) (\cos \phi \hat{\theta} - \cos \theta \cos \phi \hat{\phi}) \quad (3-30)$$

The complete first term in the Rayleigh expansion for the scattered electric field is obtained by adding the contributions represented by eqns (3-24) and (3-30). The required solution is therefore

$$\begin{aligned} \underline{E}^S = E_o \frac{e^{ikr}}{kr} k^3 (a+d)^3 \left\{ \left[\frac{1}{2} \left(1 - 3 \frac{\mu}{\mu_o} \frac{d}{a}\right) + \left(1 - 3 \frac{\epsilon_o}{\epsilon} \frac{d}{a}\right) \cos \theta \right] \cos \phi \hat{\theta} \right. \\ \left. - \left[\left(1 - 3 \frac{\epsilon_o}{\epsilon} \frac{d}{a}\right) + \frac{1}{2} \left(1 - 3 \frac{\mu}{\mu_o} \frac{d}{a}\right) \cos \theta \right] \sin \phi \hat{\phi} \right\} \quad , \quad (3-31) \end{aligned}$$

from which the scattering function can be determined if so desired.

3.4 Convergence

As previously remarked, the Rayleigh series is a convergent representation for sufficiently small values of $k\ell$, and in any application of the above results the actual radius of convergence is then a matter of some importance.

To see how the convergence arises, let us write the scattered electric field in the form

$$\underline{E}^S = E_o \sum_{n=1}^{\infty} \alpha_n(k\ell) \underline{f}_n(r, \theta, \phi) \quad , \quad (3-32)$$

where $\underline{f}_n(r, \theta, \phi)$ is a vector function of the coordinates. The series on the right hand side is absolutely convergent for all values of $k\ell$, the functions $\underline{f}_n(r, \theta, \phi)$ being bounded as functions of n . Each $\alpha_n(k\ell)$ can be expanded in a series of positive powers of $k\ell$ in a neighborhood of the origin of the complex $k\ell$ plane, and is therefore an analytic function of $k\ell$ within this region. By rearranging the terms in eqn (3-32) we then have a representation for \underline{E}^s as an expansion in powers of $k\ell$, which expansion converges within the least circle of convergence of the individual α_n .

If the functions α_n are now identified with the coefficients A_{on} and B_{en} in the vector wave function expansion for a perfectly conducting sphere, it is a simple task to determine the appropriate radius of convergence. From eqn (2-25) it is apparent that the only singularities of the A_{on} and B_{en} are poles at the zeros of the spherical Hankel function or its derivative, and the location of these zeros is such that the singularity nearest to the origin is provided by one of the smaller values of n . For $n = 1$ we have

$$h_1(\rho) = -\frac{e^{i\rho}}{\rho} \left(1 + \frac{i}{\rho}\right)$$

$$\frac{1}{\rho} \left[\rho h_1(\rho) \right]' = -i \frac{e^{i\rho}}{\rho} \left(1 + \frac{i}{\rho} - \frac{1}{\rho^2}\right),$$

showing that A_{o1} has a pole at $\rho = -i$ and B_{e1} has poles at $\rho = -\frac{i}{2}(1 \pm i\sqrt{3})$.

Accordingly, both A_{o1} and B_{e1} are infinite on the unit circle and since all

the higher coefficients are regular inside, the entire Rayleigh series must converge for $|\rho| < 1$. The fact that a singularity exists for which $|\rho| = 1$ implies that the series does not converge outside this region, and consequently the Rayleigh series for the perfectly conducting sphere converges only for

$$ka < 1. \quad (3-33)$$

From the above discussion it is obvious that the convergence is determined solely by the coefficients A_{o1n} and B_{e1n} , and any change in these by, for example, a modification of the boundary condition may affect the overall convergence of the Rayleigh series. To illustrate this point, let us consider the case in which the boundary condition

$$\underline{E} - (\hat{n} \cdot \underline{E}) \hat{n} = \eta Z \hat{n} \wedge \underline{H} \quad (3-34)$$

is imposed at the surface of the sphere. Here \hat{n} is a unit vector normal drawn outwards from the sphere, η is the reciprocal of the complex refractive index of the material of the sphere relative to free space, and Z is the intrinsic impedance of free space.

Eqn (3-34) is the usual impedance boundary condition and is only accurate to the first order in η . The physical situation therefore requires that η be assumed small ($\eta=0$ for infinite conductivity), though there are circumstances under which a physical significance can be attached to eqn (3-34) even when η is not small compared with unity. On the other hand, if the problem is merely regarded as a

mathematical one, it is a trivial matter to show that the boundary condition is precisely satisfied by a scattered field of the form (eqn (2-26)) with coefficients

$$A_{o1n} = \Omega_n(\rho, i\eta), \quad B_{e1n} = i \Omega_n(\rho, i/\eta) \quad (3-35)$$

where

$$\Omega_n(\rho, i\gamma) = -(-i)^n \frac{2n+1}{n(n+1)} \frac{j_n(\rho) + i\gamma \frac{1}{\rho} \left[\rho j_n(\rho) \right]'}{h_n(\rho) + i\gamma \frac{1}{\rho} \left[\rho h_n(\rho) \right]'} \quad (3-36)$$

For a fixed value of γ , $\Omega_n(\rho, i\gamma)$ is a function of ρ and can be expanded in a convergent power series within some neighborhood of the origin $\rho=0$. The circle of convergence depends on n and as in the case of a perfectly conducting sphere the least circle is provided by A_{o11} and B_{e11} , that is, by $\Omega_1(\rho, i\gamma)$. The denominator in the expression for $\Omega_1(\rho, i\gamma)$ is

$$-(1-\gamma) \frac{e^{i\rho}}{\rho^3} \left(\rho^2 + i\rho + \frac{\gamma}{1-\gamma} \right)$$

which vanishes when

$$\rho = \rho_1 = -\frac{i}{2} \left(1 + \sqrt{\frac{1+3\gamma}{1-\gamma}} \right)$$

$$\rho = \rho_2 = -\frac{i}{2} \left(1 - \sqrt{\frac{1+3\gamma}{1-\gamma}} \right) \quad .$$

These are two genuine zeros except when $\gamma=0$, in which case the second zero must be discounted. If this case is, for the moment, excluded, it follows that the Rayleigh expansion for \underline{E}^s converges only for

$$ka < \min \left(\left| \rho_1 \right|, \left| \rho_2 \right| \right) .$$

When γ is a general complex constant an explicit form for $\left| \rho_1 \right|$ or $\left| \rho_2 \right|$ is difficult to write down. If $\left| \gamma \right| \gg 1$ or $\left| \gamma \right| \ll 1$, however, the equations for ρ_1 and ρ_2 simplify considerably, leading to a more compact statement of the convergence region. Thus, for $\left| \gamma \right| \gg 1$,

$$\rho_1 \sim -\frac{i}{2} \left\{ 1 + i\sqrt{3} \left(1 + \frac{2}{3\gamma} \right) \right\}$$

$$\rho_2 \sim -\frac{i}{2} \left\{ 1 - i\sqrt{3} \left(1 + \frac{2}{3\gamma} \right) \right\} ,$$

and for $\left| \gamma \right| \ll 1$,

$$\rho_1 \sim -i (1 + \gamma)$$

$$\rho_2 \sim i\gamma .$$

Accordingly, for small η the convergence region is specified by the zero ρ_2 for the coefficient A_{011} (γ replaced by η) and is

$$ka < \left| \eta \right| \ll 1;$$

similarly, for large η the convergence is determined by the zero ρ_2 for the coefficient B_{e11} (γ replaced by $1/\eta$) and is

$$ka < \frac{1}{\left| \eta \right|} \ll 1.$$

In both cases the region of convergence is appreciably reduced in comparison with that for a perfectly conducting sphere, and can become infinitesimally small. We

observe, however, that for $\eta = 0$ the zero which is dictating the convergence disappears. The zero ρ_1 then becomes important and leads to the result given in eqn(3-33).

If η is neither large nor small compared with unity, the boundary condition is of doubtful validity, but it is still of interest to examine the convergence of the Rayleigh expansion when $\eta=0(1)$. Both ρ_1 and $\rho_2 \rightarrow \infty$ as $\gamma \rightarrow 1$ and, indeed, for $\gamma = \eta = 1$, ρ_1 and ρ_2 are infinite. This can be confirmed by looking at the expression for $\Omega_1(\rho, i)$. In this particular circumstance, however, $\Omega_2(\rho, i)$ imposes a finite radius of convergence which now becomes the important one, and from an examination of $\Omega_2(\rho, i)$ we find that the Rayleigh expansion converges only for $ka < 2$. On the other hand, note that $\eta=1$ corresponds to a sphere whose impedance is that of free space, and this is certainly a body for which the impedance boundary condition may be expected to fail. Nevertheless, the result does suggest that if the exact boundary conditions were used, the radius of convergence may be greater than unity in the case of a very diffuse sphere, and a study of the coefficients A_{0ln} and B_{e1n} in Stratton [1941, p. 565] gives additional confirmation of this.

Returning now to the previous example in which η is large or small, the fact that a marked reduction in the radius of convergence of the Rayleigh expansion accompanies the introduction of even a slight impedance into the sphere is, physically, rather surprising, and suggests that the usefulness of the Rayleigh

approximation is limited to perfectly conducting, or highly transparent, bodies.

The discontinuous change in convergence between the cases $\eta=0$ and $\eta \neq 0$ is

due to the fact that there is no expansion for A_{oln} or B_{eln} which is uniform in

η . Essentially each coefficient involves a factor of the form $\rho/\rho-\eta$, and for

$\eta \neq 0$ this can only be expanded in a series of positive powers of ρ when

$|\rho| < |\eta|$. Accordingly, any attempt to approximate the expressions for the

coefficients by neglecting terms of $O(\eta^2)$ will be limited by this same condition,

even though the final result may somewhat disguise the fact.

IV

THE WATSON TRANSFORM AND CREEPING WAVES

The problem of the diffraction of electromagnetic energy by a perfectly reflecting sphere for which ka is sizeable was made tractable by Watson [1918, 1919]. Watson found a transformation of the Mie series - the Watson transform - which resulted in a much more rapidly convergent representation of the solution. Much later Fock [1945, 1946] and Franz [1954] initiated a further analysis and generalization which indicated that the functional form of the Watson solution was applicable to problems involving other convex shapes. The mathematical counterpart of the extensions of Fock and Franz is found in the work of Langer [1932] and his followers which was essentially completed for this application in the 1930's. The more general approach has led to the presentation of the results in terms of certain "universal functions" which have been extensively computed and tabulated under the direction of N.A. Logan [1959]. In our development we follow the approach of Logan and his co-workers [1961].

4.1 The Field on the Surface

We now compute the magnetic fields induced on the surface of a perfectly conducting sphere by plane electromagnetic waves. If the incident electric field is given by

$$\begin{aligned}\underline{E}^i &= E_0 \hat{x} e^{-ikz} \\ \underline{H}^i &= -H_0 \hat{y} e^{-ikz}\end{aligned}\tag{4-1}$$

the magnetic field on the surface is

$$H_{\theta} = \frac{H_0 \sin \phi}{ka} \sum_{n=1}^{\infty} (2n+1) e^{-in\pi/2} \left[\frac{1}{\zeta_n^{(1)}(ka)} \frac{\partial P_n^{-1}(\cos \theta)}{\partial \theta} + \frac{i}{\zeta_n^{(1)'}(ka)} \frac{P_n^{-1}(\cos \theta)}{\sin \theta} \right] \quad (4-2)$$

$$H_{\phi} = i \frac{H_0 \cos \phi}{ka} \sum_{n=1}^{\infty} (2n+1) e^{-in\pi/2} \left[\frac{1}{\zeta_n^{(1)'}(ka)} \frac{\partial P_n^{-1}(\cos \theta)}{\partial \theta} - \frac{i}{\zeta_n^{(1)}(ka)} \frac{P_n^{-1}(\cos \theta)}{\sin \theta} \right]$$

where we have made use of the results of Section II, the relation

$$P_n^{-1}(\cos \theta) = -\frac{1}{n(n+1)} P_n^1(\cos \theta) \quad (4-3)$$

and the notation $\zeta_n^{(1,2)}(x) = x h_n^{(1,2)}(x)$, $\psi_n(x) = x j_n(x)$. For convenience we also make the substitution

$$\theta \rightarrow \alpha = \pi - \theta$$

which results in

$$\frac{P_n^{-1}(\cos \theta)}{\sin \theta} = (-)^{n+1} \frac{P_n^{-1}(\cos \alpha)}{\sin \alpha} \quad (4-4)$$

$$\frac{\partial P_n^{-1}(\cos \theta)}{\partial \theta} = (-)^n \frac{\partial P_n^{-1}(\cos \alpha)}{\partial \alpha}.$$

Using equation (4-4), equation (4-2) becomes

$$H_\theta = \frac{H_o \sin \phi}{ka} \sum_{n=1}^{\infty} (2n+1) e^{-in\pi/2} (-)^n \left[\frac{1}{\xi_n^{(1)}(ka)} \frac{\partial P_n^{-1}(\cos \alpha)}{\partial \alpha} - \frac{i}{\xi_n^{(1)'}(ka)} \frac{P_n^{-1}(\cos \alpha)}{\sin \alpha} \right] \quad (4-5)$$

$$H_\phi = i \frac{H_o \cos \phi}{ka} \sum_{n=1}^{\infty} (2n+1) e^{-in\pi/2} (-)^n \left[\frac{1}{\xi_n^{(1)'}(ka)} \frac{\partial P_n^{-1}(\cos \alpha)}{\partial \alpha} + \frac{i}{\xi_n^{(1)}(ka)} \frac{P_n^{-1}(\cos \alpha)}{\sin \alpha} \right].$$

We now rewrite the sums over the integers in eqn. (4-5) as sums over the odd half-integers, letting $n = \nu - \frac{1}{2}$,

$$H_{\theta} = \frac{H_0 \sin \phi}{ka} \sum_{\nu = 3/2, \dots} 2\nu e^{-i(\nu - 1/2)\pi/2} (-)^{\nu - 1/2}.$$

$$\cdot \left[\frac{1}{\xi_{\nu - 1/2}^{(1)}(ka)} \frac{\partial P_{\nu - 1/2}^{-1}(\cos \alpha)}{\partial \alpha} - \frac{i}{\xi_{\nu - 1/2}^{(1)}(ka)} \frac{P_{\nu - 1/2}^{-1}(\cos \alpha)}{\sin \alpha} \right]$$

(4-6)

$$H_{\phi} = i \frac{H_0 \cos \phi}{ka} \sum_{\nu = 3/2} 2\nu e^{-i(\nu - 1/2)\pi/2} (-)^{\nu - 1/2}$$

$$\cdot \left[\frac{1}{\xi_{\nu - 1/2}^{(1)}(ka)} \frac{\partial P_{\nu - 1/2}^{-1}(\cos \alpha)}{\partial \alpha} + \frac{i}{\xi_{\nu - 1/2}^{(1)}(ka)} \frac{P_{\nu - 1/2}^{-1}(\cos \alpha)}{\sin \alpha} \right].$$

For later use we note that since

$$P_0'(x) \equiv 0$$

the terms for $\nu = 1/2$ could have been included in these sums.

The summands in eqn. (4-6) are regular functions of ν in a strip along the real axis so that we can write the sum as a contour integral about the positive real

axis

$$\frac{1}{2\pi i} \int_C \frac{d\nu}{\cos \nu\pi} \pi (-)^{\nu-1/2} \quad [\text{Summand}] \quad (4-7)$$

since $\cos \nu\pi$ has simple poles at the odd half-integers. Specifically the eqns. (4-6) become

$$H_\theta = i \frac{H_o \sin \phi}{ka} \int_C \frac{\nu d\nu}{\cos \nu\pi} e^{-i(\nu-1/2)\pi/2} \Theta(\nu) \quad (4-8)$$

$$H_\phi = -\frac{H_o \cos \phi}{ka} \int_C \frac{\nu d\nu}{\cos \nu\pi} e^{-i(\nu-1/2)\pi/2} \Phi(\nu)$$

We have written the terms in square brackets in eqn. (4-6) as Θ and Φ .

We examine the terms $e^{-i\nu\pi/2} \Theta(\nu)$ and note that these are even functions of ν . Because the remainder of the integrand is odd the integrand is an odd function of ν . We consider the contour C in Figure 4-1 and note that the lower path gives

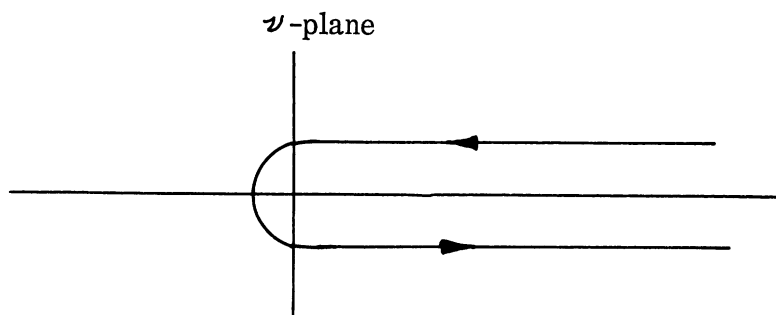


FIGURE 4-1: THE CONTOUR C

an integral of the form

$$I_e = \int_{0-i\epsilon}^{\infty-i\epsilon} d\nu \, 0(\nu) \quad (4-9)$$

where we write $0(\nu)$ for the odd functions of ν . If we reflect the contour in the origin we find

$$I_e = \int_{0+i\epsilon}^{-\infty+i\epsilon} d\nu \, 0(\nu) \quad (4-10)$$

However, interchanging the limits

$$I_e = - \int_{-\infty+i\epsilon}^{0+i\epsilon} d\nu \, 0(\nu) \quad (4-11)$$

and adding the contribution of the upper path the total integral is

$$I = - \int_{-\infty+i\epsilon}^{\infty+i\epsilon} d\nu \, 0(\nu) \quad (4-12)$$

In the sequel we will suppress the $i\epsilon$ in the limit with the understanding that the path is to run just above the real axis.

To evaluate the integrals of the form of eqn. (4-12) we need to examine the integrand in some detail. The first thing we note is that the integrands have simple poles at the zeros of $\zeta_{\nu-1/2}^{(1)}(ka)$ and $\zeta_{\nu-1/2}^{(1)'}(ka)$ and these lie in the first quadrant.

This analytic behavior suggests that if the contour $(-\infty, \infty)$ can be closed by a semi-circle in the upper half-plane we can evaluate the integral in terms of these poles.

This is indeed the case under certain circumstances as we will show below. In the contrary case we will evaluate the integral by the method of stationary phase.

As a preliminary to our examination of the integrand we define, after Logan [1961], the functions $E_m^{(1)}(\nu, \theta)$ and $E_m^{(2)}(\nu, \theta)$ by

$$(-1)^m 2P_{\nu-1/2}^{-m}(\cos \theta) \equiv E_m^{(1)}(\nu, \theta) + E_m^{(2)}(\nu, \theta) \quad (4-13)$$

where these functions have the asymptotic behavior

$$E_m^{(1,2)}(\nu, \theta) \sim \frac{1}{\nu^m} \sqrt{\frac{\theta}{\sin \theta}} H_m^{(1,2)}(\nu, \theta) \quad (4-14)$$

for $|\nu| \rightarrow \infty$ and $0 < \theta < \pi$. Explicitly

$$E_m^{(1)}(\nu, \theta) = \frac{\Gamma(\nu - m + \frac{1}{2})}{\Gamma(\nu + 1)} \sqrt{\frac{2}{\pi \sin \theta}} e^{i(\nu\theta - \frac{\pi}{4} - m\frac{\pi}{2})} \quad (4-15)$$

$$\cdot {}_2F_1\left(\frac{1}{2} + m, \frac{1}{2} - m; \nu + 1; \frac{-i e^{i\theta}}{2 \sin \theta}\right)$$

$$E_m^{(2)}(\nu, \theta) = \frac{\Gamma(\nu - m + 1/2)}{\Gamma(\nu + 1)} \sqrt{\frac{2}{\pi \sin \theta}} e^{-i(\nu\theta - \pi/4 - m\pi/2)} \cdot {}_2F_1\left(\frac{1}{2} + m, \frac{1}{2} - m; \nu + 1; \frac{ie^{-i\theta}}{2 \sin \theta}\right) \quad (4-15)$$

cont.

We have remarked that the contour runs above the real axis so that

$$\text{Im } \nu > 0;$$

hence, we can make the convergent expansion

$$\sec \nu\pi = 2e^{i\nu\pi} \sum_{\ell=0}^{\infty} (-)^{\ell} e^{2\pi i\nu\ell} \quad (4-16)$$

From the form of $E_m^{(1,2)}$ in eqn. (4-15) we make the following observations:

$$E_m^{(2)}(\nu, \alpha) = e^{-i\nu\pi} e^{+i(\pi/2 + m\pi)} E_m^{(1)}(\nu, \pi - \alpha), \quad (4-17)$$

$$e^{i\nu\ell\pi} E_m^{(1)}(\nu, \alpha) = E_m^{(1)}(\nu, \alpha + \ell\pi). \quad (4-18)$$

where eqn. (4-18) is derived from the fact that the hypergeometric functions are

periodic in α with the period π so that the continuation of the $E_m^{(1)}$ in the α -variable

to angles $\alpha > 2\pi$ is determined by the exponential alone, provided we take the radical

$(\sin \nu)^{-1/2}$ to be $|\sin \nu|^{-1/2}$. Putting these results together

$$\sec \nu\pi P_{\nu-1/2}^{-1}(\cos \alpha) = - \sum_{\ell=0}^{\infty} (-)^{\ell} \left\{ E_1^{(1)}(\nu, 2\pi(\ell+1) - \theta) - iE_1^{(1)}(\nu, 2\pi\ell + \theta) \right\} \quad (4-19)$$

where we have returned to the variable $\theta = \pi - \alpha$.

The operations $\partial/\partial\theta$ or $1/\sin\theta$ on $P_{\nu-1/2}^{-1}$ do not essentially affect the behavior of $P_{\nu-1/2}^{-1}$ as a function of ν for $|\nu| \gg 1$, $\text{Im } \nu > 0$. Hence we have from the asymptotic form of the $E_1^{(1)}(\nu, \psi)$ that the dominant term of eqn. (4-19) is of the form $e^{i\nu\theta}$ so that the dominant term of the integrand will be $e^{i\nu(\theta - \pi/2)}$. Therefore, for $\text{Im } \nu > 0$ the integrand will be a decreasing exponential in ν provided $\theta > \pi/2$.

For this case the contour can be closed, the semicircle contribution vanishes, and the integral is given by the residues of the zeros of $\xi_{\nu-1/2}^{(1)}$ and $\xi_{\nu-1/2}^{(1)'}$ provided the rest of the integrand remains bounded. This is indeed the case except on the locus of the zeros of $\xi_{\nu-1/2}^{(1)}(ka)$ and $\xi_{\nu-1/2}^{(1)'}(ka)$. It can be shown, however, that a path can be found between any two zeros on which the functions $1/\xi_{\nu-1/2}^{(1)}$ and $1/\xi_{\nu-1/2}^{(1)'}$ remain bounded.

The terms other than the dominant one in eqn. (4-19) satisfy the convergence condition on the semicircle for all values of θ . It is this behavior under the decomposition [eqn. (4-19)] that was the basis for the "creeping wave" analysis of Franz [1954].

We consider the behavior of an integral of the form

$$I = \int_{-\infty}^{\infty} \nu d\nu e^{-i\nu\pi/2} \frac{E_1^{(1)}(\nu, \psi)}{f(\nu)} \quad (4-20)$$

where we write f for either $\xi_{\nu^{-1/2}}^{(1)}$ or $\xi_{\nu^{-1/2}}^{(1)'}.$ Now from the remarks above we have for $\psi > \pi/2$ that

$$I = 2\pi i \sum_{n=0}^{\infty} \frac{e^{-i\nu_n \pi/2} E_1^{(1)}(\nu_n, \psi)}{\left. \frac{\partial f}{\partial \nu} \right|_{\nu_n}} \quad (4-21)$$

where $f(\nu_n) = 0$. Since the first zeros of f , in either case, occur for $|\nu| \sim ka$ for ka large we can use the asymptotic forms

$$\begin{aligned} \xi_{\nu^{-1/2}}^{(1)}(ka) &= -i m^{1/2} \left\{ w_1(t) - \frac{1}{60} m^{-2} t^2 w_1'(t) + \dots \right\} \\ \xi_{\nu^{-1/2}}^{(1)'}(ka) &= m^{-1/2} \left\{ w_1'(t) - \frac{1}{60} m^{-1/2} \left[(t^3 + 9) w_1(t) - 4t w_1'(t) \right] + \dots \right\} \end{aligned} \quad (4-22)$$

where we write

$$\begin{aligned} m &= (ka/2)^{1/3} \\ t &= \frac{1}{m} (\nu - ka) \end{aligned} \quad (4-23)$$

and $w_1(t)$ is the Airy function

$$w_1(t) = \sqrt{\pi} (Bi(x) + i Ai(x)) \quad (4-24)$$

Now we make a further approximation in the integrand of eqn. (4-20). We give all slowly varying functions of ν their values at $\nu = ka$ and remove them from the integral. This we do since for $\psi \geq \pi/2$, and ka and hence $|\nu_n|$ large enough

the residue series [eqn.(4-21)] converges with sufficient rapidity. Again using the asymptotic form for $E_1^{(1)}(\nu, \psi)$ under the condition $|\nu| \sin \psi \gg 1$, we have in this approximation two integrals which we write as

$$f(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{i\xi t}}{w_1(t)} dt \quad (4-25)$$

$$g(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{i\xi t}}{w_1'(t)} dt$$

where we put

$$\xi = m\left(\psi - \frac{\pi}{2}\right).$$

We now approximate the fields on the surface for $\theta > \pi/2$, $ka \sin \theta \gg 1$ by

$$H_{\theta} \approx -i H_0 \sin \phi \frac{1}{m} \frac{1}{\sqrt{\sin \theta}} \sum_{\ell=0}^{\infty} (-)^{\ell} \left\{ e^{ika\psi_{\ell}} f(\xi_{\ell}) - i e^{ika\psi'_{\ell}} f(\xi'_{\ell}) \right\}$$

$$H_{\phi} \approx -H_0 \cos \phi \frac{1}{\sqrt{\sin \theta}} \sum_{\ell=0}^{\infty} (-)^{\ell} \left\{ e^{ika\psi_{\ell}} g(\xi_{\ell}) - i e^{ika\psi'_{\ell}} g(\xi'_{\ell}) \right\} \quad (4-26)$$

where

$$\xi_{\ell} = m(2\pi \ell + \theta - \frac{\pi}{2}) = m\psi_{\ell},$$

$$\xi'_{\ell} = m(2\pi \ell + \frac{3\pi}{2} - \theta) = m\psi'_{\ell},$$

and, as above,

$$m = (ka/2)^{1/3}.$$

The next region on the surface we consider is $\theta < \pi/2$. Here we need to further decompose the region since the transition from $\theta \sim 0$ to the shadow boundary $\theta \sim \pi/2$, is accomplished by means of two different representations. First near $\theta = 0$ we evaluate the fields given by the integral representations of the form of eqn. (4-20) by a saddle point method for the first term in the expansion [eqn. (4-19)].

The result for this leading term is

$$H_{\theta}^{\text{op}} = -H_o 2 \cos \theta \sin \theta e^{-ika \cos \theta} \left\{ 1 + \frac{i \sin^2 \theta}{2ka \cos^3 \theta} + \frac{5 \sin^2 \theta - \sin^4 \theta}{2(ka)^2 \cos^6 \theta} + \dots \right\}, \quad (4-27)$$

$$H_{\phi}^{\text{op}} = -H_o 2 \cos \phi e^{-ika \cos \theta} \left\{ 1 - \frac{i \sin^2 \theta}{2ka \cos^3 \theta} - \frac{9 \sin^2 \theta - \sin^4 \theta}{2(ka)^2 \cos^6 \theta} + \dots \right\}.$$

Here we remark that the leading terms in eqn. (4-27) are just the geometric optics fields.

To bridge the gap between the shadow boundary and the optics fields in eqn. (4-27) we note that asymptotically as $\xi \rightarrow -\infty$ the functions $f(\xi)$ and $g(\xi)$ in eqn. (4-25) go to the correct leading term of the optics field in eqn. (4-27). We can then use eqn. (4-26) provided we make a substitution in the argument so that we get the correct phase. This is simply to let

$$\xi_o = m(\theta - \frac{\pi}{2}) \rightarrow \tilde{\xi}_o = m \sin(\theta - \frac{\pi}{2}) \quad (4-28)$$

in the leading terms of eqn. (4-26) for $\theta \leq \pi/2$ and $\theta > \theta_0 > 0$. Here the choice of θ_0 is somewhat arbitrary. To make the choice of θ_0 specific is not meaningful so we specify the range $[30^\circ, 60^\circ]$ so that the regions of eqns. (4-27) and (4-26) overlap.

In either of the cases we also remark that the terms in eqn. (4-26) for $\ell = 1, 2, \dots$ will also appear just as before. However the terms near the caustic in the lit region diverge as $1/\sqrt{\sin \theta}$ as $\theta \rightarrow 0$. We will find a bounded representation for these when we treat with the same behavior at the caustic in the shadow region, $\theta = \pi$.

We will now find a representation for the fields in the region near the caustic in the shadow where $ka(\pi - \theta)$ is small. After Logan [1961] we make the physical argument that the terms in eqns. (4-26) of the form $g(\xi_\ell) e^{ika\psi/\sqrt{\sin \theta}}$ describe waves which diverge from $\theta = \pi$ while terms of the form $g(\xi'_\ell) e^{ika\psi'/\sqrt{\sin \theta}}$ describe waves which converge toward $\theta = \pi$. This suggests that on the surface these waves can be represented by Hankel functions which represent converging and diverging waves in a cylindrical geometry. This behavior is also suggested by the representation of the functions $E_m^{(1,2)}(\nu, \theta)$ in eqn. (4-14).

We consider the term of eqns. (4-26)

$$\left\{ \frac{e^{ika(2\pi\ell + \theta - \pi/2)}}{\sqrt{\sin \theta}} g(\xi_\ell) - \frac{ie^{ika(2\pi\ell + \frac{3\pi}{2} - \theta)}}{\sqrt{\sin \theta}} g(\xi'_\ell) \right\} \quad (4-29)$$

This is a valid representation away from the caustic, for $ka(\pi-\theta) \gg 1$, and is suggestive of the asymptotic form of the Hankel functions. We will find such a Hankel function representation and then continue eqn. (4-29) to the caustic region.

From the asymptotic behavior of the Hankel functions

$$H_1^{(1,2)}(z) = \pm i \sqrt{\frac{2}{\pi z}} e^{\pm i(z - 3\pi/4)} \quad (4-30)$$

we have that

$$\frac{e^{ika(2\pi\ell + \theta - \pi/2)}}{\sqrt{\sin \theta}} \cong i e^{-3\pi i/4} \sqrt{\frac{ka\pi}{2}} e^{ika(2\pi\ell + \pi/2)} \cdot \sqrt{\frac{\pi-\theta}{\sin \theta}} H_1^{(2)'} [ka(\pi-\theta)] \quad (4-31)$$

and

$$\frac{e^{ika(2\pi\ell + \frac{3\pi}{2} - \theta)}}{\sqrt{\sin \theta}} \cong -i e^{3\pi i/4} \sqrt{\frac{ka\pi}{2}} e^{ika(2\pi\ell + \pi/2)} \cdot \sqrt{\frac{\pi-\theta}{\sin \theta}} H_1^{(1)'} [ka(\pi-\theta)] \quad (4-32)$$

If now we substitute eqns. (4-31) and (4-32) in (4-29) we get

$$e^{-i\pi/4} \sqrt{\frac{ka\pi}{2}} e^{ika(2\pi\ell + \pi/2)} \sqrt{\frac{\pi-\theta}{\sin \theta}} \cdot \left\{ H_1^{(2)'} [ka(\pi-\theta)] g(\xi_\ell) + H_1^{(1)'} [ka(\pi-\theta)] g(\xi'_\ell) \right\} \quad (4-33)$$

Now if we let θ approach π we have

$$g(\xi_\ell) \simeq g(\xi'_\ell) \quad (4-34)$$

and (4-33) becomes

$$e^{-i\pi/4} \sqrt{ka\pi/2} e^{ika(2\pi\ell + \pi/2)} {}_2J_1'(ka[\pi-\theta]) g(\xi_\ell) \quad (4-35)$$

which is finite.

In the above treatment we have performed the continuation into the shadow caustic using the $g(\xi)$ as an example. Of course, the same will hold using $f(\xi)$.

Using our new representations, eqn. (4-33), and the analogous one for the $f(\xi)$'s eqns., (4-26) become

$$H_\theta = H_0 \sin \theta e^{-3\pi i/4} \sqrt{m\pi} \sqrt{\frac{\pi-\theta}{\sin \theta}} \sum (-)^{\ell} e^{ika(2\pi\ell + \pi/2)} \left\{ H_1^{(2)'}(ka[\pi-\theta]) f(\xi_\ell) + H_1^{(1)'}(ka[\pi-\theta]) f(\xi'_\ell) \right\} \quad (4-36)$$

$$H_\theta = H_0 \cos \theta e^{3\pi i/4} m^{3/2} \sqrt{\pi} \sqrt{\frac{\pi-\theta}{\sin \theta}} \sum (-)^{\ell} e^{ika(2\pi\ell + \pi/2)} \left\{ H_1^{(2)'}(ka[\pi-\theta]) g(\xi_\ell) + H_1^{(1)'}(ka[\pi-\theta]) g(\xi'_\ell) \right\},$$

where as above,

$$\begin{aligned} m &= (ka/2)^{1/3}, \\ \xi_\ell &= m(2\pi\ell + \theta - \frac{\pi}{2}), \\ \xi'_\ell &= m(2\pi\ell + \frac{3\pi}{2} - \theta), \end{aligned}$$

and $\theta \geq \pi/2$.

A similar representation can be found for the higher order terms near the caustic $\theta = 0$. We give the results without repeating the analysis.

$$H_{\theta}^{cw} = H_0 \sin \phi e^{\pi i/4} \sqrt{\pi m} \sqrt{\frac{\theta}{\sin \theta}} \sum_{\ell=0}^{\infty} (-)^{\ell} e^{ika(2\pi\ell + 3\pi/2)}$$

$$\cdot \left\{ H_1^{(1)'}(ka\theta) f(\xi_{\ell+1}) + H_1^{(2)'}(ka\theta) f(\xi'_{\ell}) \right\}$$

(4-37)

$$H_{\phi}^{cw} = H_0 \cos \phi e^{-i\pi/4} m^{3/2} \sqrt{\pi} \sqrt{\frac{\theta}{\sin \theta}} \sum_{\ell=0}^{\infty} (-)^{\ell} e^{ika(2\pi\ell + 3\pi/2)}$$

$$\cdot \left\{ H_1^{(1)'}(ka\theta) g(\xi_{\ell+1}) + H_1^{(2)'}(ka\theta) g(\xi'_{\ell}) \right\}$$

where $\theta \leq \pi/2$ and $ka \sin \theta < 1$.

4.2 The Scattered Field

Again with the field

$$\underline{E}^i = E_0 \hat{x} e^{-ikz} \tag{4-38}$$

incident on a perfectly reflecting sphere of radius a , the scattered field to order $1/r$ is from eqns. (2-16), (2-17) and (2-25)

$$E_{\theta}^s = E_o i \cos \phi \frac{e^{ikr}}{kr} \sum_{n=1}^{\infty} (-)^n \frac{2n+1}{n(n+1)} \left[\frac{\psi_n'(ka)}{\xi_n^{(1)'}(ka)} \frac{\partial P_n'(\cos \theta)}{\partial \theta} - \frac{\psi_n(ka)}{\xi_n^{(1)}(ka)} \frac{P_n'(\cos \theta)}{\sin \theta} \right] \quad (4-39)$$

$$E_{\phi}^s = -E_o i \sin \phi \frac{e^{ikr}}{kr} \sum_{n=1}^{\infty} (-)^n \frac{2n+1}{n(n+1)} \left[\frac{\psi_n(ka)}{\xi_n^{(1)}(ka)} \frac{\partial P_n'(\cos \theta)}{\partial \theta} - \frac{\psi_n'(ka)}{\xi_n^{(1)'}(ka)} \frac{P_n'(\cos \theta)}{\sin \theta} \right]$$

We make the substitutions of section 4.1

$$\theta \longrightarrow \alpha = \pi - \theta$$

$$n \longrightarrow \nu = n + \frac{1}{2}$$

and use the relationships of eqn. (4-4) of section 4.1 and write

$$E_{\theta}^s = E_o i \cos \phi \frac{e^{ikr}}{kr} \sum_{\nu = \frac{3}{2}, \frac{5}{2}, \dots} 2\nu \left[\frac{\psi_{\nu-1/2}'(ka)}{\xi_{\nu-1/2}^{(1)'}(ka)} \frac{\partial P_{\nu-1/2}^{-1}(\cos \alpha)}{\partial \alpha} + \frac{\psi_{\nu-1/2}(ka)}{\xi_{\nu-1/2}^{(1)}(ka)} \frac{P_{\nu-1/2}(\cos \alpha)}{\sin \alpha} \right] \quad (4-40)$$

$$E_{\phi}^s = E_o i \sin \phi \frac{e^{ikr}}{kr} \sum_{\nu = \frac{3}{2}, \frac{5}{2}, \dots} 2\nu \left[\frac{\psi_{\nu-1/2}(ka)}{\xi_{\nu-1/2}^{(1)}(ka)} \frac{\partial P_{\nu-1/2}^{-1}(\cos \alpha)}{\partial \alpha} + \frac{\psi_{\nu-1/2}'(ka)}{\xi_{\nu-1/2}^{(1)'}(ka)} \frac{P_{\nu-1/2}(\cos \alpha)}{\sin \alpha} \right]$$

We now restrict our detailed treatment to the first of eqns. (4-40) and consider the sum

$$\Theta^S = \sum_{\nu} 2\nu \left[\frac{\psi'_{\nu-1/2}(ka)}{\xi'_{\nu-1/2}(ka)} \frac{\partial P_{\nu-1/2}^{-1}(\cos \alpha)}{\partial \alpha} + \frac{\psi_{\nu-1/2}(ka)}{\xi_{\nu-1/2}(ka)} \frac{P_{\nu-1/2}^{-1}(\cos \alpha)}{\sin \alpha} \right] \quad (4-41)$$

Eqn. (4-41) has the contour integral representation

$$\Theta^S = \int_C \frac{2\nu d\nu}{1 + e^{2i\nu\pi}} \left[\frac{\psi'_{\nu-1/2}(ka)}{\xi_{\nu-1/2}^{(1)}(ka)} \frac{\partial P_{\nu-1/2}^{-1}(\cos \alpha)}{\partial \alpha} + \frac{\psi_{\nu-1/2}(ka)}{\xi_{\nu-1/2}^{(1)}(ka)} \frac{P_{\nu-1/2}^{-1}(\cos \alpha)}{\sin \alpha} \right] \quad (4-42)$$

where the contour C encircles the positive real axis in a clockwise direction as in figure 4-2. As before we note that the contribution

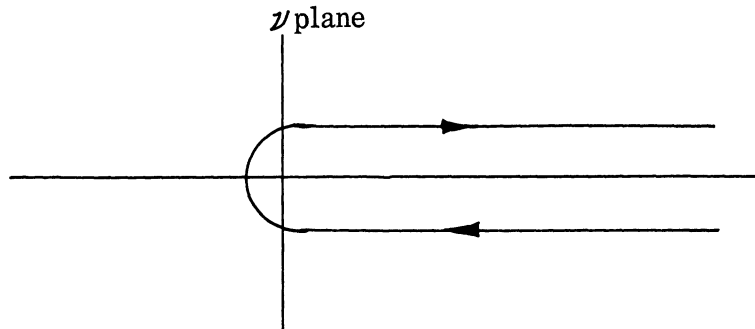


FIGURE 4-2

of the poles at $\nu = 1/2$ vanishes.

Since $2\psi_{\nu} = \xi_{\nu}^{(1)} + \xi_{\nu}^{(2)}$ eqn. (4-42) can be written as the sum of two terms.

The first we consider is

$$I_1 = \int_C \frac{\nu d\nu}{1 + e^{2i\nu\pi}} \left[\frac{\partial P_{\nu-1/2}^{-1}(\cos \alpha)}{\partial \alpha} + \frac{P_{\nu-1/2}^{-1}(\cos \alpha)}{\sin \alpha} \right] = \sum_{n=0}^{\infty} \left(n + \frac{1}{2} \right) \left[\frac{\partial P_n^{-1}(\cos \alpha)}{\partial \alpha} + \frac{P_n^{-1}(\cos \alpha)}{\sin \alpha} \right] \quad (4-43)$$

But

$$\sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right) P_n^{-1}(\cos \alpha) = \frac{1}{2} \cot \frac{\alpha}{2}, \quad (4-44)$$

hence,

$$I_1 = \frac{d}{d\alpha} \left(\frac{1}{2} \cot \frac{\alpha}{2} \right) + \frac{1}{2} \frac{\cot \frac{\alpha}{2}}{\sin \alpha} = 0 \quad (4-45)$$

From this result [eqn. (4-45)] we confine our attention to

$$\Theta^S = \int_c \frac{\nu d\nu}{1+e^{2i\nu\pi}} \left[\frac{\xi_{\nu-1/2}^{(2)'}(ka)}{\xi_{\nu-1/2}^{(1)'}(ka)} \frac{\partial P_{\nu-1/2}^{-1}(\cos \alpha)}{\partial \alpha} + \frac{\xi_{\nu-1/2}^{(2)}(ka)}{\xi_{\nu-1/2}^{(1)}(ka)} \frac{P_{\nu-1/2}^{-1}(\cos \alpha)}{\sin \alpha} \right] \quad (4-46)$$

Since the integrand in eqn. (4-46) is an odd function of ν we can reflect the lower part of c in the origin and get

$$\Theta^S = \int_{-\infty+i\epsilon}^{\infty+i\epsilon} \frac{\nu d\nu}{1+e^{2\nu\pi i}} \left[\frac{\xi_{\nu-1/2}^{(2)'}(ka)}{\xi_{\nu-1/2}^{(1)'}(ka)} \frac{\partial P_{\nu-1/2}^{-1}(\cos \alpha)}{\partial \alpha} + \frac{\xi_{\nu-1/2}^{(2)}(ka)}{\xi_{\nu-1/2}^{(1)}(ka)} \frac{P_{\nu-1/2}^{-1}(\cos \alpha)}{\sin \alpha} \right] \quad (4-47)$$

where $\epsilon > 0$ is a small parameter. In the sequel we drop the ϵ in the limits of integration with the understanding that the contour is to run just above the real axis.

As in the previous discussion of the fields on the surface of the sphere we would like to close the contour in eqn. (4-47) by a large semi-circle in the upper half plane. To this end we decompose the Legendre function as in eqn. (4-13),

$$-2P_{\nu-1/2}^{-1}(\cos \alpha) = E_1^{(1)}(\nu, \alpha) + E_1^{(2)}(\nu, \alpha) \quad (4-48)$$

and require $\alpha > 0$. Since $E_1^{(2)}(\nu, \alpha) \sim \dots e^{-i\nu\alpha}$ is the dominant term in the Legendre function in the upper-half ν -plane,

$$\frac{1}{1 + e^{2i\nu\pi}} = 1 - \frac{1}{1 + e^{-2i\nu\pi}} = 1 + O(e^{2i\nu\pi}), \quad (4-49)$$

and since in parts of this half plane

$$\frac{\xi_{\nu-1/2}^{(2)}(ka)}{\xi_{\nu-1/2}^{(1)}(ka)} = O(1) \quad (4-50)$$

we have that the terms of the integrand of eqn. (4-47) are exponentially small for

$|\nu| \gg 1$ except for

$$I^R = \int_{-\infty}^{\infty} \frac{1}{2} \nu d\nu \left[\frac{\xi_{\nu-1/2}^{(2)'}(ka)}{\xi_{\nu-1/2}^{(1)'}(ka)} \frac{\partial}{\partial \alpha} E_1^{(2)}(\nu, \alpha) + \frac{\xi_{\nu-1/2}^{(2)}(ka)}{\xi_{\nu-1/2}^{(1)}(ka)} \frac{E_1^{(2)}(\nu, \alpha)}{\sin \alpha} \right] \quad (4-51)$$

The remainder is

$$I^C = \int_{-\infty}^{\infty} \frac{1}{2} \nu d\nu \left\{ \frac{\xi_{\nu-1/2}^{(2)'}(ka)}{\xi_{\nu-1/2}^{(1)'}(ka)} \left[\frac{\partial E_1^{(1)}(\nu, \alpha)}{\partial \alpha} \frac{1}{1 + e^{2\nu\pi i}} - \frac{\partial E_1^{(2)}(\nu, \alpha)}{\partial \alpha} \frac{1}{1 + e^{-2\nu\pi i}} \right] \right. \\ \left. + \frac{\xi_{\nu-1/2}^{(2)}(ka)}{\xi_{\nu-1/2}^{(1)}(ka)} \left[\frac{E_1^{(1)}(\nu, \alpha)}{\sin \alpha} \frac{1}{1 + e^{2\nu\pi i}} - \frac{E_1^{(2)}(\nu, \alpha)}{\sin \alpha} \frac{1}{1 + e^{-2\nu\pi i}} \right] \right\} \quad (4-52)$$

Eqn. (4-52) can be determined by closing the contour and evaluating by means of the residues at the zero of $\xi_{\nu-1/2}^{(1)}(ka)$ and $\xi_{\nu-1/2}^{(1)'}(ka)$. If we designate the zeros of $\xi_{\nu-1/2}^{(1)}(ka)$ by ν_n

$$\xi_{\nu_n-1/2}^{(1)}(ka) = 0$$

and those of $\xi_{\nu-1/2}^{(1)'}(ka)$ by ν_m

$$\xi_{\nu_m-1/2}^{(1)'}(ka) = 0,$$

then

$$\begin{aligned} I^c = \pi i \sum_n \nu_n \frac{\xi_{\nu_n-1/2}^{(2)}(ka)}{\left[\frac{\partial}{\partial \nu} \xi_{\nu-1/2}^{(1)}(ka) \right]_{\nu_n}} & \left[\frac{E_1^{(1)}(\nu, \alpha)}{\sin \alpha} - \frac{1}{1+e^{2\nu_n \pi i}} \right. \\ & \left. - \frac{E_1^{(2)}(\nu_n, \alpha)}{\sin \alpha} - \frac{1}{1+e^{-2\nu_n \pi i}} \right] + \pi i \sum_m \nu_m \frac{\xi_{\nu_m-1/2}^{(2)'}(ka)}{\left[\frac{\partial}{\partial \nu} \xi_{\nu-1/2}^{(1)}(ka) \right]_{\nu_m}} \\ & \cdot \left[\frac{\partial E_1^{(1)}(\nu_m, \alpha)}{\partial \alpha} - \frac{1}{1+e^{2\nu_m \pi i}} - \frac{\partial E_1^{(2)}(\nu_m, \alpha)}{\partial \alpha} - \frac{1}{1+e^{-2\nu_m \pi i}} \right] \end{aligned} \quad (4-53)$$

Since $\xi_{\nu-1/2}^{(1)}$ and its derivative vanish at the poles we can replace $\xi_{\nu-1/2}^{(2)}$ in eqn. (4-52) by $2\psi_{\nu-1/2}$. Also, since the zeros of $\xi_{\nu-1/2}^{(1)}$ and $\xi_{\nu-1/2}^{(1)'} start at $|\nu| \sim ka$ for ka sufficiently large we can use the representations$

$$\psi_{\nu-1/2} = m^{1/2} \left\{ v(t) - \frac{1}{60} m^{-2} \left[t^2 v'(t) + 4t v(t) \right] \right\} \quad (4-54)$$

$$\xi_{\nu-1/2} = -i m^{1/2} \left\{ w_1(t) - \frac{1}{60} m^{-2} \left[t^2 w_1'(t) + 4t w_1(t) \right] \right\} + \dots$$

where

$$t = \frac{1}{m} (\nu - ka), \quad m = (ka/2)^{1/3}$$

and $v(t)$ and $w_1(t)$ are the Airy functions

$$v(t) = \sqrt{\pi} A_1(t), \quad (4-55)$$

$$w_1(t) = \sqrt{\pi} \left[B_1(t) + i A_1(t) \right].$$

Since $\text{Im } \nu > 0$ there are the convergent expansions

$$\frac{1}{1+e^{2\pi i \nu}} = \sum_{\ell=0}^{\infty} (-)^{\ell} e^{2\pi i \nu \ell} \quad (4-56)$$

$$\frac{1}{1+e^{-2\pi i \nu}} = e^{2\pi i \nu} \sum_{\ell=0}^{\infty} (-)^{\ell} e^{2\pi i \nu \ell}$$

From equations (4-17 and (4-18)

$$\frac{E_1^{(1)}(\nu, \alpha)}{1+e^{2\pi i\nu}} = \sum_{\ell=0}^{\infty} (-)^{\ell} E_1^{(1)}(\nu, \alpha + \ell\pi) \quad (4-57)$$

$$\frac{E_1^{(2)}(\nu, \alpha)}{1+e^{-2\pi i\nu}} = i \sum_{\ell=0}^{\infty} (-)^{\ell} E_1^{(1)}(\nu, 2\pi[\ell+1] - \alpha)$$

and

$$E_1^{(1)}(\nu, \alpha) \cong \frac{\Gamma(\nu - \frac{1}{2})}{\Gamma(\nu + 1)} \sqrt{\frac{2}{\pi \sin \alpha}} e^{i(\nu \alpha - 3\pi/4)} \quad (4-58)$$

So on substituting eqns. (4-54), (4-57), and (4-58) in eqn. (4-52)

$$I^c = (ka)^2 \frac{1}{m^{1/2}} e^{3\pi i/4} \sum_{\ell=0}^{\infty} (-)^{\ell} \left[q(\xi_{\ell}') e^{i\psi_{\ell}'} + iq(\xi_{\ell}) e^{i\psi_{\ell}} \right] \quad (4-59)$$

where we write, noting that this is a different notation than the previous for ξ ,

$$\begin{aligned} \xi_{\ell} &= (2\pi\ell + \pi - \theta) m = m\psi_{\ell} \\ \xi_{\ell}' &= (2\pi\ell + \pi + \theta) m = m\psi_{\ell}' \\ \hat{q} &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{i\xi t} \frac{v'(t)}{w_1'(t)} dt \end{aligned} \quad (4-60)$$

Here we use the caret notation not to designate a unit vector but to be consistent with the notation of Logan [1959].

We return to eqn. (4-51) and make a saddle point evaluation. The stationary point occurs at

$$\nu = ka \cos \alpha/2 \quad \text{or} \quad \nu = ka \sin \theta/2 \quad (4.61)$$

So by the standard methods

$$I^R = \frac{i}{2} ka e^{-2ika \cos \theta/2} \quad (4-62)$$

Finally we have the expression for the fields

$$E_{\theta}^S \cong E_o i \cos \phi \frac{e^{ikr}}{kr} \left\{ \frac{i}{2} ka e^{-2ika \cos \theta/2} + \frac{(ka)^2}{m^{1/2}} e^{3\pi i/4} \right. \\ \left. \cdot \sum_{\ell=0}^{\infty} (-)^{\ell} \left[\hat{q}(\xi'_{\ell}) e^{i\psi'_{\ell}} + i \hat{q}(\xi_{\ell}) e^{i\psi_{\ell}} \right] \right\} \quad (4-63)$$

$$E_{\phi}^S = E_o i \sin \phi \frac{e^{ikr}}{kr} \left\{ \frac{i}{2} ka e^{-2ika \cos \theta/2} + \frac{(ka)^2}{m^{1/2}} e^{3\pi i/4} \right. \\ \left. \cdot \sum_{\ell=0}^{\infty} (-)^{\ell} \left[\hat{p}(\xi'_{\ell}) e^{i\psi'_{\ell}} + \hat{p}(\xi_{\ell}) e^{i\psi_{\ell}} \right] \right\}$$

where E_{ϕ}^S is evaluated analogously to E_{θ}^S and the function $\hat{p}(\xi)$ is given by

$$\hat{p}(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{i\xi t} \frac{v(t)}{w_1(t)} dt .$$

Near the forward direction, $\alpha \sim \pi$, it is necessary to modify our treatment because of the singular behavior of the functions $E_1^{(1,2)}(\nu, \alpha)$. We will start with the integral representation of eqn. (4-42) but retain the Legendre functions,

$$\Theta^S = \int_c \frac{2\nu d\nu}{1+e^{2i\nu\pi}} \left[\frac{\psi_{\nu-1/2}^{(1)'}(ka)}{\xi_{\nu-1/2}^{(1)'}(ka)} \frac{\partial P_{\nu-1/2}^{-1}(\cos \alpha)}{\partial \alpha} + \frac{\psi_{\nu-1/2}^{(1)}(ka)}{\xi_{\nu-1/2}^{(1)}(ka)} \frac{P_{\nu-1/2}^{-1}(\cos \alpha)}{\sin \alpha} \right]$$

where c is a clockwise contour about the positive real axis. On reflection of the odd part of the lower contour this becomes

$$\Theta^S = \int_{-\infty}^{ka} \nu d\nu \left[\frac{\xi_{\nu-1/2}^{(2)'}(ka)}{\xi_{\nu-1/2}^{(1)'}(ka)} \frac{\partial P_{\nu-1/2}^{-1}(\cos \alpha)}{\partial \alpha} + \frac{\xi_{\nu-1/2}^{(2)}(ka)}{\xi_{\nu-1/2}^{(1)}(ka)} \frac{P_{\nu-1/2}^{-1}(\cos \alpha)}{\sin \alpha} \right]$$

$$+ \int_{ka}^{\infty} 2\nu d\nu \left[\frac{\psi_{\nu-1/2}^{(1)'}(ka)}{\xi_{\nu-1/2}^{(1)'}(ka)} \frac{\partial P_{\nu-1/2}^{-1}(\cos \alpha)}{\partial \alpha} + \frac{\psi_{\nu-1/2}^{(1)}(ka)}{\xi_{\nu-1/2}^{(1)}(ka)} \frac{P_{\nu-1/2}^{-1}(\cos \alpha)}{\sin \alpha} \right]$$

$$+ \int_0^{ka} \nu d\nu \left[\frac{\partial P_{\nu-1/2}^{-1}(\cos \alpha)}{\partial \alpha} + \frac{P_{\nu-1/2}^{-1}(\cos \alpha)}{\sin \alpha} \right]$$

$$+ \int_{-\infty+i\epsilon}^{\infty+i\epsilon} \frac{-\nu d\nu}{1+e^{-2\pi i\nu}} \left[\frac{\xi_{\nu-1/2}^{(2)'}(ka)}{\xi_{\nu-1/2}^{(1)'}(ka)} \frac{\partial P_{\nu-1/2}^{-1}(\cos \alpha)}{\partial \alpha} + \frac{\xi_{\nu-1/2}^{(2)}(ka)}{\xi_{\nu-1/2}^{(1)}(ka)} \frac{P_{\nu-1/2}^{-1}(\cos \alpha)}{\sin \alpha} \right]$$

(continued)

$$\begin{aligned}
 & + \int_{\infty-i\epsilon}^0 \frac{\nu d\nu}{1+e^{2\pi i\nu}} \left[\frac{\partial P_{\nu-1/2}^{-1}(\cos \alpha)}{\partial \alpha} + \frac{P_{\nu-1/2}^{-1}(\cos \alpha)}{\sin \alpha} \right] \\
 & - \int_0^{\infty+i\epsilon} \frac{\nu d\nu}{1+e^{-2\pi i\nu}} \left[\frac{\partial P_{\nu-1/2}^{-1}(\cos \alpha)}{\partial \alpha} + \frac{P_{\nu-1/2}^{-1}(\cos \alpha)}{\sin \alpha} \right]
 \end{aligned} \tag{4-65}$$

The first two terms in eqn. (4-65) are evaluated using the Bessel function representation of the Legendre function

$$P_{\nu-1/2}^{-1}(\cos \alpha) \cong \frac{1}{\nu} J_1(\nu \alpha) \tag{4-66}$$

for $|\nu|$ large and α small. The result is for the first two terms

$$\begin{aligned}
 I_1 &= \int_{-\infty}^{ka} \frac{\xi^{(2)}_{\nu-1/2}(ka)}{\xi^{(1)}_{\nu-1/2}(ka)} \frac{P_{\nu-1/2}^{-1}(\cos \alpha)}{\sin \alpha} \nu d\nu + \int_{ka}^{\infty} \frac{2\nu_{\nu-1/2}(ka)}{\xi^{(1)}_{\nu-1/2}(ka)} \frac{P_{\nu-1/2}^{-1}(\cos \alpha)}{\sin \alpha} \nu d\nu \\
 &= -2\sqrt{\pi} \frac{J_1(\alpha ka)}{\alpha ka} m \left\{ ika p(0) + m p^{(1)}(0) + \dots \right\}
 \end{aligned} \tag{4-67}$$

where we write

$$p(\xi) = \frac{1}{2i} \frac{1}{\sqrt{\pi}} \int_{-\infty}^0 e^{i\xi t} \frac{w_2(t)}{w_1(t)} dt + \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{i\xi t} \frac{v(t)}{w_1(t)} dt \tag{4-68}$$

$$p^{(1)}(\xi) = \frac{\partial p(\xi)}{\partial \xi}$$

and $p^{(1)}(\xi) = \frac{\partial}{\partial \xi} p(\xi)$. Similarly,

$$I_2 \int_{-\infty}^{ka} \frac{\xi^{(2)'}(ka)}{\xi^{(1)'}(ka)} \frac{\partial P_{\nu-1/2}^{-1}(\cos \alpha)}{\partial \alpha} \nu d\nu + \int_{ka}^{\infty} \frac{\psi'_{\nu-1/2}(ka)}{\xi'_{\nu-1/2}(ka)} \frac{\partial P_{\nu-1/2}^{-1}(\cos \alpha)}{\partial \alpha} 2\nu d\nu$$

$$= 2\sqrt{\pi} J_1'(ka, \alpha) m \left[ika q(0) + m q^{(1)}(0) + \dots \right] \quad (4-69)$$

where the q 's are the functions

$$q(\xi) = \frac{1}{2i} \frac{1}{\sqrt{\pi}} \int_{-\infty}^0 e^{i\xi t} \frac{w_2'(t)}{w_1'(t)} dt + \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{i\xi t} \frac{v'(t)}{w_1'(t)} dt$$

$$q^{(1)}(\xi) = \frac{\partial q(\xi)}{\partial \xi} \quad (4-70)$$

The third term of eqn. (4-65) is evaluated by using the small angle expansion,

$$P_{\nu-1/2}^{-1}(\cos \alpha) = 1 - \left(\nu^2 - \frac{1}{4}\right) \sin^2 \frac{\alpha}{2} + \frac{\left(\nu^2 - \frac{1}{4}\right) \left(\nu^2 - \frac{9}{4}\right) \sin^4 \frac{\alpha}{2}}{2^2} + \dots \quad (4-71)$$

We find

$$\int_0^{ka} \nu d\nu \left[\frac{P_{\nu-1/2}^{-1}(\cos \alpha)}{\sin \alpha} + \frac{\partial P_{\nu-1/2}^{-1}(\cos \alpha)}{\partial \alpha} \right] = (ka)^2 - \left(\frac{1}{4} (ka)^4 - \frac{5}{8} (ka)^2 \right) \sin^2 \frac{\alpha}{2}$$

$$+ \left(\frac{1}{24} (ka)^6 - \frac{9}{32} (ka)^4 + \frac{3}{64} (ka)^2 \right) \sin^4 \frac{\alpha}{2} + \dots \quad (4-72)$$

The fourth term is evaluated using the Bessel function representation of the Legendre function and expanding the denominator $(1+e^{-2\nu\pi i})$. Again we remark that integrals of this form can be evaluated by residues and hence we can replace $\xi_{\nu-1/2}^{(2)}$ by 2ν so as to keep the form standard. Performing these operations and keeping the higher order terms

$$- \int_{-\infty}^{\infty} \left[\frac{\nu d\nu}{1+e^{-2\pi i\nu}} \frac{\xi_{\nu-1/2}^{(2)'}(ka)}{\xi_{\nu-1/2}^{(1)}(ka)} \frac{\partial P_{\nu-1/2}^{-1}(\cos \alpha)}{\partial \alpha} + \frac{\xi_{\nu-1/2}^{(2)}(ka)}{\xi_{\nu-1/2}^{(1)}(ka)} \frac{P_{\nu-1/2}^{-1}(\cos \alpha)}{\sin \alpha} \right]$$

$$\cong -2ka\sqrt{\pi} \sum_{\ell=0}^{\infty} (-)^{\ell} e^{ika 2\pi(\ell+1)} \left\{ J_1'(ka\alpha) \hat{q}[m(\ell+1)2\pi] + \frac{J_1(ka\alpha)}{ka\alpha} \hat{p}[m(\ell+1)2\pi] \right\}$$

where, as before

$$\hat{p}(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{i\xi t} \frac{v(t)}{w_1(t)} dt$$

(4-73)

$$\hat{q}(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{i\xi t} \frac{v'(t)}{w_1'(t)} dt$$

The last two terms of eqn. (4-65) are equal and can be evaluated using the expansion in eqn. (4-71). We get

$$\begin{aligned}
 & \left[\int_{\infty - i\epsilon}^0 \frac{1}{1 + e^{2\pi i \nu}} - \int_0^{\infty + i\epsilon} \frac{1}{1 + e^{-2\pi i \nu}} \right] \nu d\nu \left[\frac{\partial P^{-1}(\cos \alpha)}{\nu^{-1/2} \partial \alpha} + \frac{P^{-1}(\cos \alpha)}{\nu^{-1/2} \sin \alpha} \right] \\
 &= 2 \int_0^{\infty} \frac{y dy}{1 + e^{2\pi y}} \left[\frac{\partial P^{-1}(\cos \alpha)}{iy^{-1/2} \partial \alpha} + \frac{P^{-1}(\cos \alpha)}{iy^{-1/2} \sin \alpha} \right] \\
 &= \frac{1}{12} + \frac{31}{170} \sin^2 \frac{\alpha}{2} + \frac{185,767}{4,193,280} \sin^4 \frac{\alpha}{2} + \dots \quad (4-74)
 \end{aligned}$$

The fields are given by

$$\begin{aligned}
 E_{\theta}^S &= E_0 i \cos \theta \frac{e^{ikr}}{kr} \left\{ (ka)^2 - \left[\frac{1}{4} (ka)^4 - \frac{5}{8} (ka)^2 \right] \sin^2 \frac{\alpha}{2} + \dots \right. \\
 &+ 2 \sqrt{\pi} \frac{J_1(\alpha ka)}{\alpha ka} m \left[ika p(0) + m p^{(1)}(0) + \dots \right] \\
 &+ 2 \sqrt{\pi} J_1'(\alpha ka) m \left[ika q(0) + m q^{(1)}(0) + \dots \right] \\
 &- 2 ka \sqrt{\pi} m \sum_{\ell=0}^{\infty} (-)^{\ell} e^{2\pi(\ell+1)ka} \left(J_1'(ka\alpha) \hat{q} \left[m 2\pi(\ell+1) \right] \right. \\
 &\left. \left. + \frac{J_1(ka\alpha)}{ka\alpha} \hat{p} \left[m 2\pi(\ell+1) \right] \right) + \dots \right\} \quad (4-75)
 \end{aligned}$$

$$\begin{aligned}
 E_{\phi}^s = & E_o i \sin \phi \frac{e^{ikr}}{kr} \left\{ (ka)^2 - \left[\frac{1}{4} (ka)^4 - \frac{5}{8} (ka)^2 \right] \sin^2 \frac{\alpha}{2} + \dots \right. \\
 & + 2 \sqrt{\pi} \frac{J_1(\alpha ka)}{\alpha ka} m \left[ika q(0) + m q^{(1)}(0) + \dots \right] \\
 & + 2 \sqrt{\pi} J_1'(\alpha ka) m \left[ika p(0) + m p^{(1)}(0) + \dots \right] \\
 & - 2ka \sqrt{\pi} m \sum_{\ell=0}^{\infty} (-)^{\ell} e^{2\pi(\ell+1)ka} \left(J_1'(\alpha ka) \hat{p} \left[m2\pi(\ell+1) \right] \right. \\
 & \left. \left. + \frac{J_1(\alpha ka)}{\alpha ka} \hat{q} \left[m2\pi(\ell+1) \right] \right) \right\} \quad (4-76)
 \end{aligned}$$

4.3 The Formulas of Sections 4.1 and 4.2.

For a plane electromagnetic wave

$$\underline{E}^i = E_o \hat{x} e^{-ikz} \quad (4-77)$$

incident on a perfectly conducting sphere of radius a we present the formulas of the previous subsections along with a brief comment on the physical interpretation and the methods of calculation.

4.3.1 The field on the surface

In the lit region including the caustic we have found the field to consist of two

terms. The first is that due to direct illumination. This is characterized by the fact that (1) it reduces to geometric optics in the limit $ka \rightarrow \infty$ and (2) it carries the phase of the incident field. Under the somewhat arbitrary condition $0 \leq \theta \leq 60^\circ$ we have the direct field

$$H_{\theta}^{\text{op}} = -2 H_0 \cos \theta \sin \phi e^{-ika \cos \theta} \left\{ 1 + \frac{i \sin^2 \theta}{2ka \cos^3 \theta} + \frac{5 \sin^2 \theta - \sin^4 \theta}{2(ka)^2 \cos^6 \theta} + \dots \right\},$$

$$H_{\phi}^{\text{op}} = -2 H_0 \cos \phi e^{-ika \cos \theta} \left\{ 1 - \frac{i \sin^2 \theta}{2ka \cos^3 \theta} - \frac{9 \sin^2 \theta - \sin^4 \theta}{2(ka)^2 \cos^6 \theta} + \dots \right\}. \quad (4-78)$$

The second contribution in this region arises from waves that have crept around the back of the sphere and hence, is characterized by having the phases $ka(2\pi\ell + \frac{3\pi}{2})$ for $\theta = 0$. The form we give for this contribution depends upon the value of $ka \sin \theta$ although we impose the restriction $\theta \leq \pi/2$.

For $\theta \leq \pi/2$ the creeping wave contribution is for $ka \sin \theta > 1$,

$$H_{\theta}^{\text{cw}} = -i H_0 \sin \phi \frac{1}{m \sqrt{\sin \theta}} \sum_{\ell=0}^{\infty} (-)^{\ell} \left\{ -e^{ika\psi_{\ell+1}} f(\xi_{\ell+1}) - ie^{ika\psi'_{\ell}} f(\xi'_{\ell}) \right\}$$

(4-79)

$$H_{\phi}^{\text{cw}} = -H_0 \cos \phi \frac{1}{\sqrt{\sin \theta}} \sum_{\ell=0}^{\infty} (-)^{\ell} \left\{ -e^{ika\psi_{\ell+1}} g(\xi_{\ell+1}) - ie^{ika\psi'_{\ell}} g(\xi'_{\ell}) \right\}$$

and for $ka \sin \theta < 1$

$$H_{\theta}^{cw} = e^{i\pi/4} H_0 \sin \phi \sqrt{\pi m} \sqrt{\frac{\theta}{\sin \theta}} \sum_{\ell=0}^{\infty} (-)^{\ell} e^{ika(2\pi\ell + 3\pi/2)}$$

$$\cdot \left\{ H_1^{(1)'}(ka\theta) f(\xi_{\ell+1}) + H_1^{(2)'}(ka\theta) f(\xi_{\ell}') \right\}$$

(4-80)

$$H_{\phi}^{cw} = e^{-i\pi/4} H_0 \cos \phi m^{3/2} \sqrt{\pi} \sqrt{\frac{\theta}{\sin \theta}} \sum_{\ell=0}^{\infty} (-)^{\ell} e^{ika(2\pi\ell + 3\pi/2)}$$

$$\cdot \left\{ H_1^{(1)'}(ka\theta) g(\xi_{\ell+1}) + H_1^{(2)'}(ka\theta) g(\xi_{\ell}') \right\}$$

and for $\theta = 0$

$$H_{\theta}^{cw} = 2 H_0 \sin \phi e^{i\pi/4} \sqrt{\pi m} \sum_{\ell=0}^{\infty} (-)^{\ell} e^{ika(2\pi\ell + 3\pi/2)} J_1'(ka\theta) f(\xi_{\ell+1})$$

(4-81)

$$H_{\phi}^{cw} = 2 H_0 \cos \phi e^{-i\pi/4} m^{3/2} \sqrt{\pi} \sum_{\ell=0}^{\infty} (-)^{\ell} e^{ika(2\pi\ell + 3\pi/2)} J_1'(ka\theta) g(\xi_{\ell+1})$$

In eqns. (4-80) and (4-81) the function $f(\xi)$ and $g(\xi)$ appear. These have been computed and tabulated by N.A. Logan [1959].

In the transition region $30^\circ \leq \theta \leq 90^\circ$ there are again two contributions. The first is the continuation of the shadow currents, the second is the creeping wave

contribution. These are

$$H_{\theta} = -i H_0 \sin \phi \frac{1}{m} \frac{1}{\sqrt{\sin \theta}} \left[e^{-ika \cos \theta} f(-m \cos \theta) + \sum_{\ell=0}^{\infty} (-)^{\ell} \left\{ -e^{ika_{\ell+1}} f(\xi_{\ell+1}) - ie^{ika_{\ell}'} f(\xi_{\ell}') \right\} \right] \quad (4-82)$$

$$H_{\phi} = -H_0 \cos \phi \frac{1}{\sqrt{\sin \theta}} \left[e^{-ika \cos \theta} f(-m \cos \theta) + \sum_{\ell=0}^{\infty} (-)^{\ell} \left\{ -e^{ika_{\ell+1}} g(\xi_{\ell+1}) - ie^{ika_{\ell}'} g(\xi_{\ell}') \right\} \right]$$

Here we have made the substitution

$$\xi_0 = m(\theta - \frac{\pi}{2}) \rightarrow \tilde{\xi}_0 = m \sin(\theta - \frac{\pi}{2}) = -m \cos \theta$$

in the first term so that this term has the phase of the incident wave. The subsequent terms are just the creeping wave contributions recognized from their phases.

In the shadow region away from the caustic, $\theta \geq \pi/2$, $ka \sin \theta > 1$.

$$H_{\theta} = -i H_0 \sin \phi \frac{1}{m \sqrt{\sin \theta}} \sum_{\ell=0}^{\infty} (-)^{\ell} \left[e^{ika\psi_{\ell}} f(\xi_{\ell}) - i e^{ika\psi'_{\ell}} f(\xi'_{\ell}) \right] \quad (4-83)$$

$$H_{\phi} = -H_0 \cos \phi \frac{1}{\sqrt{\sin \theta}} \sum_{\ell=0}^{\infty} (-)^{\ell} \left[e^{ika\psi_{\ell}} g(\xi_{\ell}) - i e^{ika\psi'_{\ell}} g(\xi'_{\ell}) \right]$$

These are purely creeping wave terms again characterized by their phases.

Finally near the shadow caustic we have for $ka \sin \theta < 1$, $\theta > \pi/2$.

$$H_{\theta} = e^{-3\pi i/4} H_0 \sin \phi \sqrt{m\pi} \frac{\sqrt{\pi-\theta}}{\sqrt{\sin \theta}} \sum_{\ell=0}^{\infty} (-)^{\ell} e^{ika(2\pi\ell+\pi/2)} \left\{ H_1^{(2)'} [ka(\pi-\theta)] f(\xi_{\ell}) + H_1^{(1)'} [ka(\pi-\theta)] f(\xi'_{\ell}) \right\} \quad (4-84)$$

$$H_{\phi} = \cos \phi H_0 e^{3\pi i/4} m^{3/2} \sqrt{\pi} \frac{\sqrt{\pi-\theta}}{\sqrt{\sin \theta}} \sum_{\ell=0}^{\infty} (-)^{\ell} e^{ika(2\pi\ell+\pi/2)} \cdot \left\{ H_1^{(2)'} [ka(\pi-\theta)] g(\xi_{\ell}) + H_1^{(1)'} [ka(\pi-\theta)] g(\xi'_{\ell}) \right\}$$

and at $\theta = \pi$

$$H_{\theta} = 2H_0 \sin \phi e^{-3\pi i/4} m\pi \sum_{\ell=0}^{\infty} (-)^{\ell} e^{ika(2\pi\ell+\pi/2)} J_1' [ka(\pi-\theta)] f(\xi_{\ell}) \quad (4-85)$$

$$H_{\phi} = 2H_0 \cos \phi e^{3\pi i/4} m^{3/2} \sqrt{\pi} \sum_{\ell=0}^{\infty} (-)^{\ell} e^{ika(2\pi\ell+\pi/2)} J_1' [ka(\pi-\theta)] g(\xi_{\ell})$$

4.3.2 The Far Field

In the far field there are essentially just two regions. The first is the large region for which $0 \leq \theta < \pi$. The second is the forward scattered region for which $\theta \sim \pi$ and $ka \sin \theta < 1$.

For $0 \leq \theta < \pi$ the dominant contribution will arise from the terms

$$\begin{aligned} \left(E_{\theta}^s \right)_{op} \cong - E_o \cos \phi \frac{e^{ikr}}{kr} \left(\frac{ka}{2} e^{-2ika \cos \frac{\theta}{2}} \right) & \left(1 + \frac{i}{2ka \cos^3 \frac{\theta}{2}} \right. \\ & \left. - \frac{7 \sin^2 \frac{\theta}{2}}{4(ka)^2 \cos^6 \frac{\theta}{2}} + \dots \right) \end{aligned} \quad (4-86)$$

$$\begin{aligned} \left(E_{\phi}^s \right)_{op} \cong - E_o \sin \phi \frac{e^{ikr}}{kr} \left(\frac{ka}{2} e^{-2ika \cos \frac{\theta}{2}} \right) & \left(1 + i \frac{\cos \theta}{2ka \cos^3 \frac{\theta}{2}} \right. \\ & \left. - \frac{7 \sin^2 \frac{\theta}{2}}{4(ka)^2 \cos^6 \frac{\theta}{2}} + \dots \right) \end{aligned}$$

which in the limit $ka \rightarrow \infty$ reduce to the geometric optics fields. The additional terms correspond to the creeping wave contribution and are

$$\begin{aligned} \left(E_{\theta}^s \right)_{cw} = i E_o \cos \phi \frac{e^{ikr}}{kr} \frac{(ka)^2}{m^{1/2}} e^{3\pi i/4} \sum_{\ell=0}^{\infty} (-)^{\ell} & \left[\hat{q}(\xi'_{\ell}) e^{i\psi'_{\ell}} + i \hat{q}(\xi_{\ell}) e^{i\psi_{\ell}} \right] \\ \left(E_{\phi}^s \right)_{cw} = i E_o \sin \phi \frac{e^{ikr}}{kr} \frac{(ka)^2}{m^{1/2}} e^{3\pi i/4} \sum_{\ell=0}^{\infty} (-)^{\ell} & \left[\hat{p}(\xi'_{\ell}) e^{i\psi'_{\ell}} + \hat{p}(\xi_{\ell}) e^{i\psi_{\ell}} \right] \end{aligned} \quad (4-87)$$

where $m = (ka/2)^{1/3}$

$$\xi_{\ell} = (2\pi\ell + \pi - \theta) m = m\psi_{\ell}$$

$$\xi'_{\ell} = (2\pi\ell + \pi + \theta) m = m\psi'_{\ell}$$

and

$$\hat{p}(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{i\xi t} \frac{v(t)}{w_1(t)} dt$$

$$\hat{q}(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{i\xi t} \frac{v'(t)}{w'_1(t)} dt$$

are functions which have been extensively tabulated by Logan [1959]. The creeping wave contributions are a very small correction except for cases in which there is a phase correlation between the primed and unprimed terms in the brackets. This occurs for $\psi_{\ell} = \psi'_{\ell} \bmod \pi$ or $\psi_{\ell} = \psi'_{\ell} \bmod 2\pi$. In fact for $\theta = 0$, backscattering, this correlation approach can be used to predict the relative maxima and minima in the backscattering cross section as a function of ka .

In the case of forward scattering there are also two distinct contributions. The first can be recognized if we recall the cross section theorem which states that

$$\sigma_T = \frac{4\pi}{k} \operatorname{Im} f(\pi)$$

where σ_T is the total cross section and $f(\theta)$ is the complex field amplitude. Since in the limit of geometrical optics

$$\sigma_T = 4\pi a^2$$

we have in this limit that

$$\text{Im } f(\pi) = (ka)^2.$$

Recalling the form of the forward scattered field we have the optics contribution

$$\left(E_{\theta}^s \right)_{\text{op}} = i E_o \cos \phi \frac{e^{ikr}}{kr} \left\{ (ka)^2 - \left[\frac{1}{4} (ka)^4 - \frac{5}{8} (ka)^2 \right] \sin^2(\alpha/2) + \dots \right\} \quad (4-88)$$

and $\left(E_{\phi}^s \right)_{\text{op}}$ is the same expression with $\cos \phi$ replaced by $\sin \phi$

$$\left(E_{\phi}^s \right)_{\text{op}} = i E_o \sin \phi \frac{e^{ikr}}{kr} \left\{ (ka)^2 - \left[\frac{1}{4} (ka)^4 - \frac{5}{8} (ka)^2 \right] \sin^2(\alpha/2) + \dots \right\} \quad (4-89)$$

The remaining terms in the forward scattered field are those that arise

directly from the shadow boundary

$$\begin{aligned} \left(E_{\theta}^s \right)_{\text{SB}} = i E_o \cos \phi \frac{e^{ikr}}{kr} (2 \sqrt{\pi} m) & \left\{ \frac{J_1(\alpha ka)}{\alpha ka} \left[ika p(0) + mp^{(1)}(0) + \dots \right] \right. \\ & \left. + J_1'(\alpha ka) \left[ika q(0) + mq^{(1)}(0) \right] \right\} \end{aligned} \quad (4-90)$$

$$\begin{aligned} \left(E_{\phi}^s \right)_{\text{SB}} = i E_o \sin \phi \frac{e^{ikr}}{kr} (2 \sqrt{\pi} m) & \left\{ \frac{J_1(\alpha ka)}{\alpha ka} \left[ika q(0) + mq^{(1)}(0) + \dots \right] \right. \\ & \left. + J_1'(\alpha ka) \left[ika p(0) + mp^{(1)}(0) \right] \right\} \end{aligned}$$

and those which creep around the sphere one or more times

$$\begin{aligned} \left(E_{\theta}^s \right)_{cw} &= i E_o \cos \phi \frac{e^{ikr}}{kr} (-2ka \sqrt{\pi} m) \sum_{\ell=0}^{\infty} (-)^{\ell} e^{2\pi(\ell+1)ka} \\ &\cdot \left[J_1'(ka \alpha) \hat{q}(2\pi m(\ell+1)) + \frac{J_1(ka \alpha)}{ka \alpha} \hat{p}(2\pi m(\ell+1)) \right] \end{aligned}$$

(4-91)

$$\begin{aligned} \left(E_{\phi}^s \right)_{cw} &= i E_o \sin \phi \frac{e^{ikr}}{kr} (-2ka \sqrt{\pi} m) \sum_{\ell=0}^{\infty} (-)^{\ell} e^{2\pi(\ell+1)ka} \\ &\cdot \left[J_1'(ka \alpha) \hat{p}(2\pi m(\ell+1)) + \frac{J_1(ka \alpha)}{ka \alpha} \hat{q}(2\pi m(\ell+1)) \right] \end{aligned}$$

In the above

$$\begin{aligned} p(\xi) &= \frac{1}{2i} \frac{1}{\sqrt{\pi}} \int_{-\infty}^0 e^{i\xi t} \frac{w_2(t)}{w_1(t)} dt + \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{i\xi t} \frac{v(t)}{w_1(t)} dt \\ q(\xi) &= \frac{1}{2i} \frac{1}{\sqrt{\pi}} \int_{-\infty}^0 e^{i\xi t} \frac{w_2'(t)}{w_1'(t)} dt + \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{i\xi t} \frac{v'(t)}{w_1'(t)} dt. \end{aligned}$$

which have also been extensively computed and tabulated by Logan [1959].

V

THE PHYSICAL OPTICS APPROACH

Perhaps the best known technique for obtaining approximate solutions of high frequency diffraction problems is the method of physical optics. The key feature is an assumption about the current distribution on the surface of the scattering object, and in this section the method is applied to the case of a perfectly conducting sphere of radius a . The degree of approximation involved is examined by comparison with the exact Mie series, and a numerical example is treated which lends support to the use of physical optics, particular where the main purpose is to obtain general estimates of the scattering behavior.

5.1 Physical Optics for the Sphere.

The scattered magnetic field is given in terms of the current \underline{J} induced in the surface of the sphere by the equation

$$\underline{H}^s = -\frac{ik}{4\pi} \int_S \frac{e^{ikR}}{R} (\hat{\underline{R}} \wedge \underline{J}) dS \quad (5-1)$$

where $\hat{\underline{R}}$ is a unit vector from the receiver to a variable point (a, θ', ϕ') on the sphere, the distance between these points being denoted by R . If \underline{J} were accurately known, the above equation would provide an exact expression for the scattered field and the basis of the physical optics approach is an approximation to the true value of \underline{J} .

According to ray theory there exists a sharply-defined shadow region behind the sphere in which the total field is zero, and since

$$\underline{J} = \hat{n} \wedge \underline{H} ,$$

where \hat{n} is a unit vector normal, the current distribution over the shadow area must be identically zero. For the illuminated portion of the sphere \underline{J} is obtained on the assumption that the field is reflected at every point as though an infinite plane wave were incident on an infinite tangent plane, and this gives

$$\underline{J} = 2\hat{n} \wedge \underline{H}^i \quad (5-2)$$

that is, twice the tangential component of the incident magnetic field. The current distribution is now completely specified by choosing the incident field as that given in eqn. (2-1), and hence, at a point (a, θ', ϕ') on the illuminated side of the sphere,

$$\underline{J} = (\cos \theta' \hat{x} - \sin \theta' \cos \phi' \hat{z}) 2 H_0 e^{-ika \cos \theta'} . \quad (5-3)$$

Moreover, the fact that r is large compared with the radius of the sphere means that \hat{R} is effectively directed toward the origin and

$$R \sim r - a \cos \theta \cos \theta' - a \sin \theta \sin \theta' \cos (\phi - \phi')$$

from which it follows that

$$H_r^s = 0 ,$$

$$H_\theta^s = \frac{ik}{2\pi} H_o a^2 \frac{e^{ikr}}{r} \sin \phi \int_0^{\pi/2} \int_0^{2\pi} e^{-ika\beta} \sin \theta' \cos \theta' d\theta' d\phi' ,$$

$$H_\phi^s = -\frac{ik}{2\pi} H_o a^2 \frac{e^{ikr}}{r} \sin \theta \int_0^{\pi/2} \int_0^{2\pi} e^{-ika\beta} \sin^2 \theta' \cos \phi' d\theta' d\phi'$$

$$-\frac{ik}{2\pi} H_o a^2 \frac{e^{ikr}}{r} \cos \theta \cos \phi \int_0^{\pi/2} \int_0^{2\pi} e^{-ika\beta} \sin \theta' \cos \theta' d\theta' d\phi' ,$$

where

$$\beta = (\cos \theta + 1) \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi') .$$

Of more direct interest, however, are the components of the electric vector in the scattered field and by using the equation

$$\underline{E}^s = \frac{iZ}{k} \nabla \wedge \underline{H}^s$$

we have

$$E_\theta^s = \frac{ika^2}{2\pi} E_o \frac{e^{ikr}}{r} \left\{ I_1 \sin \theta + I_2 \cos \theta \cos \phi \right\} \quad (5-4)$$

$$E_\phi^s = -\frac{ika^2}{2\pi} E_o \frac{e^{ikr}}{r} I_2 \sin \phi \quad (5-5)$$

with

$$I_1 = \int_0^{\pi/2} \int_0^{2\pi} e^{-ika\beta} \sin^2 \theta' \cos \phi' d\phi' d\theta'$$

and

$$I_2 = \int_0^{\pi/2} \int_0^{2\pi} e^{-ika\beta} \sin \theta' \cos \theta' d\phi' d\theta'.$$

The component E_r^s is of the order $1/r^2$ and therefore negligible by comparison.

The above integrals can only be evaluated exactly in certain special cases and for arbitrary values of θ and ϕ it is necessary to rely upon approximate techniques based upon the (assumed) large value of ka . These techniques can be illustrated by reference to I_1 . Here the ϕ -integral is

$$\begin{aligned} & \int_0^{2\pi} e^{-ika \sin \theta \sin \theta' \cos(\phi - \phi')} \cos \phi' d\phi' \\ &= \int_{-\phi}^{2\pi - \phi} e^{-ika \sin \theta \sin \theta' \cos \phi'} (\cos \phi \cos \phi' - \sin \phi \sin \phi') d\phi'. \end{aligned}$$

and since the term involving $\sin \phi'$ contributes nothing, the integration being over a complete period, we are left with

$$\cos \phi \int_{-\phi}^{2\pi - \phi} e^{-ika \sin \theta \sin \theta' \cos \phi'} \cos \phi' d\phi' = -2\pi i \cos \phi J_1(ka \sin \theta \sin \theta')$$

and hence

$$I_1 = -2\pi i \cos \phi \int_0^{\pi/2} e^{-ika(\cos \theta + 1)\cos \theta'} J_1(ka \sin \theta \sin \theta') \sin^2 \theta' d\theta'.$$

If it is now assumed that $ka \sin \theta \sin \theta'$ is large compared with unity* the Bessel function can be replaced by its asymptotic expansion to give

$$I_1 \sim -2i \sqrt{\frac{2\pi}{ka \sin \theta}} \cos \phi \int_0^{\pi/2} e^{-ika(\cos \theta + 1)\cos \theta'} \cos(ka \sin \theta \sin \theta' - \frac{3\pi}{4}) \sin^{\frac{3}{2}} \theta' d\theta'$$

and by writing $\cos(ka \sin \theta \sin \theta' - \frac{3\pi}{4})$ in exponential form it can be verified that the only saddle point for which $0 \leq \theta \leq \pi$ and $0 \leq \theta' \leq \frac{\pi}{2}$ is $\theta' = \theta/2$. This is equivalent to substituting

$$\frac{1}{2} \exp i \left[\frac{3\pi}{4} - ka \sin \theta \sin \theta' \right]$$

for the cosine factor and gives

$$I_1 \sim e^{i\pi/4} \sqrt{\frac{2\pi}{ka \sin \theta}} \cos \phi \int_{-\theta/2}^{\frac{\pi}{2} - \frac{\theta}{2}} e^{-2ika \cos \frac{\theta}{2} \cos \theta'} \sin^{3/2} \left(\frac{\theta}{2} + \theta' \right) d\theta'$$

$$\sim e^{i\pi/4} \frac{\sqrt{\pi}}{ka} \tan \frac{\theta}{2} \cos \phi e^{-2ika \cos \frac{\theta}{2}} \int_{-\sqrt{ka \cos \frac{\theta}{2} \frac{\theta}{2}}}^{\sqrt{ka \cos \frac{\theta}{2} \left(\frac{\pi}{2} - \frac{\theta}{2} \right)}} e^{it^2} dt.$$

* The failure of this condition at the lower limit of integration clearly indicates that even if the subsequent evaluation of I_1 and I_2 were performed exactly the results would at best be approximate.

Providing $\theta \neq 0, \pi$ (cases already excluded by the requirement that $ka \sin \theta \sin \theta'$ be large), the limits of integration can be replaced by $\pm \infty$ and since

$$\int_{-\infty}^{\infty} e^{it^2} dt = \sqrt{\pi} e^{i\pi/4}$$

we finally obtain

$$I_1 = \frac{i\pi}{ka} \tan \frac{\theta}{2} \cos \phi e^{-2ika \cos \frac{\theta}{2}}.$$

An analogous treatment applied to the integral I_2 leads to the result

$$I_2 = \frac{i\pi}{ka} e^{-2ika \cos \frac{\theta}{2}}$$

and if these expressions for I_1 and I_2 are inserted into eqns. (5-4) and (5-5), the scattered field takes the form

$$E_{\theta}^s = -\frac{a}{2} E_0 \frac{e^{ikr}}{r} \cos \phi e^{-2ika \cos \frac{\theta}{2}}, \quad (5-6)$$

$$E_{\phi}^s = \frac{a}{2} E_0 \frac{e^{ikr}}{r} \sin \phi e^{-2ika \cos \frac{\theta}{2}}. \quad (5-7)$$

The corresponding scattering cross sections are

$$\sigma_{\theta} = \pi a^2 \cos^2 \phi \quad (5-8)$$

and

$$\sigma_{\phi} = \pi a^2 \sin^2 \phi. \quad (5-9)$$

The above results have been obtained by an approximate evaluation of I_1 and I_2 , the basis of which is the assumption that $ka \sin \theta$ is large compared with unity. The smaller the value of $\sin \theta$, the larger must ka be in order to fulfill this requirement (see footnote p. 119) and in the limiting cases for which $\sin \theta = 0$ the method is not longer valid. It is fortunate that these cases are the very ones for which I_1 and I_2 can be treated exactly.

$\theta = \pi$ corresponds to forward scatter and since β is then zero

$$I_1 = 0$$

and

$$I_2 = \pi$$

giving

$$E_{\theta}^s = -\frac{ika^2}{2} E_o \frac{e^{ikr}}{r} \cos \phi$$

and

$$E_{\phi}^s = -\frac{ika^2}{2} E_o \frac{e^{ikr}}{r} \sin \phi.$$

The polarization is therefore identical to that of the incident field and the scattering cross sections are

$$\sigma_{\theta} = \pi k^2 a^4 \cos^2 \phi$$

and

$$\sigma_{\phi} = \pi k^2 a^4 \sin^2 \phi,$$

implying

$$\sigma = \sigma_{\theta} + \sigma_{\phi} = \frac{4\pi A^2}{\lambda^2}$$

where $A = \pi a^2$ is the geometric cross sectional area of the sphere.

The other limiting case is $\theta = 0$ and corresponds to back scatter. Since $\beta = 2 \cos \theta'$,

$$I_1 = 0$$

and

$$I_2 = 2\pi \int_0^{\pi/2} e^{-2ika \cos \theta'} \sin \theta' \cos \theta' d\theta' = -\frac{\pi}{ika} \left\{ e^{-2ika} + \frac{1}{2ika} (e^{-2ika} - 1) \right\}$$

$$\sim -\frac{\pi}{ika} e^{-2ika}$$

if ka is large. Hence

$$E_{\theta}^s = -\frac{a}{2} E_o \frac{e^{ikr}}{r} \cos \theta e^{-2ika}$$

$$E_{\phi}^s = \frac{a}{2} E_o \frac{e^{ikr}}{r} \sin \theta e^{-2ika}$$

and these are in complete agreement with equations (5-6) and (5-7) notwithstanding the fact that the approximate method of evaluating I_1 and I_2 breaks down when $\theta=0$.

In view of this continuity as θ approaches 0 it is reasonable to put forward the scattering cross sections given by equations (5-8) and (5-9) as valid for all θ not near to π and the implications of such a statement will now be considered.

5.2 A Comparison of Formulae

According to physical optics the scattered electric field at any point consists

of two in-phase rectangular components in the aperture plane of a receiver directed towards the center of the sphere. This is certainly in partial agreement with the conclusions of the exact analysis.

As regards the nature of the scattered field and its dependence on θ , the predictions of the physical optics method are correct, as may be seen from a comparison of equations (5-6) and (5-7) with (2-37). But the approximate treatment has produced components which are in phase for all θ and moreover, has destroyed most of the dependence on θ . Indeed if we exclude for the moment the case of θ near to π use of the physical optics method is equivalent to replacing

$$S_1(\theta) = \sum_{n=1}^{\infty} (-i)^{n+1} A_n \left\{ \frac{P_n^1(\cos \theta)}{\sin \theta} + i B_n \frac{\partial P_n^1(\cos \theta)}{\partial \theta} \right\} \quad (5-10)$$

and

$$S_2(\theta) = \sum_{n=1}^{\infty} (-i)^{n+1} A_n \left\{ \frac{\partial P_n^1(\cos \theta)}{\partial \theta} + i B_n \frac{P_n^1(\cos \theta)}{\sin \theta} \right\} \quad (5-11)$$

by $\mp \frac{ka}{2} e^{-2ika \cos \frac{\theta}{2}}$ respectively, where A_n and B_n are defined in eqn. (2-25).

In particular, for backscatter it replaces

$$\sum_{n=1}^{\infty} (-i)^{n+1} n \left(\frac{n+1}{2} \right) (A_n + i B_n) \quad \text{by} \quad -\frac{ka}{2} e^{-2ika}$$

and for the exceptional case of forward scatter

$$\sum_{n=1}^{\infty} i^{n-1} n \left(\frac{n+1}{2} \right) (A_n - i B_n) \quad \text{by} \quad \frac{(ka)^2}{2}.$$

The degree of approximation which these imply has been examined for a selected value of ka and the results are given in section 5.3.

Such a comparison is not, however, a fair test of the physical optics method in that additional approximations were made in order to evaluate the integrals I_1 and I_2 . The basic assumptions as to the form of the currents can only be tested by a study of the currents themselves and this will now be done.

At any point (a, θ, ϕ) on the surface of the sphere the physical optics approximations to the current can be obtained from eqn. (5-3) as

$$J_r = 0 \quad (5-12)$$

$$J_\theta = 2 H_0 \cos \phi e^{-ika \cos \theta} \quad (5-13)$$

and

$$J_\phi = -2 H_0 \sin \phi \cos \theta e^{-ika \cos \theta} \quad (5-14)$$

for $0 \leq \theta \leq \pi/2$ (illuminated portion of sphere), with $J_r = J_\theta = J_\phi = 0$ otherwise.

In contrast the exact current distribution is

$$\underline{J} = \hat{n} \wedge (\underline{H}^i + \underline{H}^s) \quad (5-15)$$

where \underline{H}^i and \underline{H}^s are given by eqns. (2-18), (2-22) and (2-25). When substituted into eqn. (5-15) these expressions, together with the Wronskian relations

$$\frac{j_n(ka) - h_n(ka)}{\left[ka h_n(ka) \right]'} = \frac{i/ka}{\left[ka h_n(ka) \right]}$$

and

$$\left[ka j_n(ka)\right]' - \frac{j_n(ka)}{h_n(ka)} \left[ka h_n(ka)\right]' = - \frac{i/ka}{h_n(ka)},$$

lead to the formulae

$$J_\theta = \frac{H_o}{ka} \cos \phi \sum_{n=1}^{\infty} (-i)^n \frac{2n+1}{n(n+1)} \left\{ \frac{P_n^1(\cos \theta)}{ka h_n(ka) \sin \theta} + i \frac{\frac{\partial P_n^1(\cos \theta)}{\partial \theta}}{\left[ka h_n(ka)\right]'} \right\} \quad (5-16)$$

and

$$J_\phi = \frac{H_o}{ka} \sin \phi \sum_{n=1}^{\infty} (-i)^{n+1} \frac{2n+1}{n(n+1)} \left\{ \frac{P_n^1(\cos \theta)}{\left[ka h_n(ka)\right]' \sin \theta} - i \frac{\frac{\partial P_n^1(\cos \theta)}{\partial \theta}}{ka h_n(ka)} \right\}. \quad (5-17)$$

A comparison of equations (5-13) and (5-14) with (5-16) and (5-17) now shows that the physical optics approximation to the current replaces

$$T_1 = \frac{1}{ka} \sum_{n=1}^{\infty} (-i)^n \frac{2n+1}{n(n+1)} \left\{ \frac{P_n^1(\cos \theta)}{ka h_n(ka) \sin \theta} + i \frac{\frac{\partial P_n^1(\cos \theta)}{\partial \theta}}{\left[ka h_n(ka)\right]'} \right\} \quad (5-18)$$

and

$$T_2 = \frac{1}{ka} \sum_{n=1}^{\infty} (-i)^{n+1} \frac{2n+1}{n(n+1)} \left\{ \frac{P_n^1(\cos \theta)}{\left[ka h_n(ka)\right]' \sin \theta} - i \frac{\frac{\partial P_n^1(\cos \theta)}{\partial \theta}}{ka h_n(ka)} \right\} \quad (5-19)$$

by

$$2e^{-ika \cos \theta} \quad \text{and} \quad -2 \cos \theta e^{-ika \cos \theta}$$

respectively for $0 \leq \theta \leq \pi/2$ and by zero for other values of θ .

It must be emphasized, however, that the usefulness of the physical optics approach in scattering problems does not depend entirely upon the accuracy of the current distribution which it predicts. The parameters of most practical importance are the far field amplitudes and the fact that these can be expressed as stationary forms involving the currents (as in the variational formulation) suggests that slight errors in these currents do not necessarily reveal themselves as errors in the far field amplitudes. Ideally it would be desirable to carry out a direct comparison of eqns. (5-4) and (5-7) with (2-37) with no approximations made to I_1 and I_2 , but the labor involved in a numerical integration of these integrals prohibits such an undertaking. The current distribution is the only alternative basis of comparison not involving approximations additional to those of the physical optics method itself.

5.3 A Particular Case

A significant test of the predictions of physical optics can be achieved by confining attention to a single, judiciously-chosen value of ka . The case where $ka = 10$ is convenient for computational purposes and, in addition, leads to a back-scattering cross section whose exact value (see, for example, Hey, Stewart, Pinson and Prince, [1956]) differs from that of physical optics by (about) the local mean of

these deviations as a function of ka . To this extent it is representative. Moreover, it corresponds to a sphere of sufficient size for the results to be of practical use, but small enough ($a \lesssim 1.6\lambda$) to give a stringent test of the physical optics method.

The basic assumption of physical optics is an approximation to the current distribution which would seem to be justified if all dimensions of the body (including the radii of curvature) are large compared with a wavelength. Nevertheless, the method is known to give good results for a wide variety of bodies not excluding those having point singularities or sharply curved surfaces, and indeed, a sphere of radius $5\lambda/\pi$ falls into the latter category.

A comparison of the postulated currents with their exact counterparts for such a sphere (see figures 5-1 and 5-2) reveals a remarkable amount of agreement over the entire illuminated surface, the only real discrepancy being near to the shadow edge in that current which is assumed to be discontinuous there. Over the shadow area the currents are not zero, contrary to assumption, but the amplitudes are appreciably less than for the other hemisphere, particularly in the case of the 'continuous' current J_ϕ . At $\theta = \pi$ the currents are identical, with their amplitudes showing a marked increase as this point is approached.

The failure of the physical optics approximation to the current in the shadow is not surprising since the currents here have to 'fit in' with the unnatural form forced upon them in the other region. Moreover, the discrepancies are unlikely to

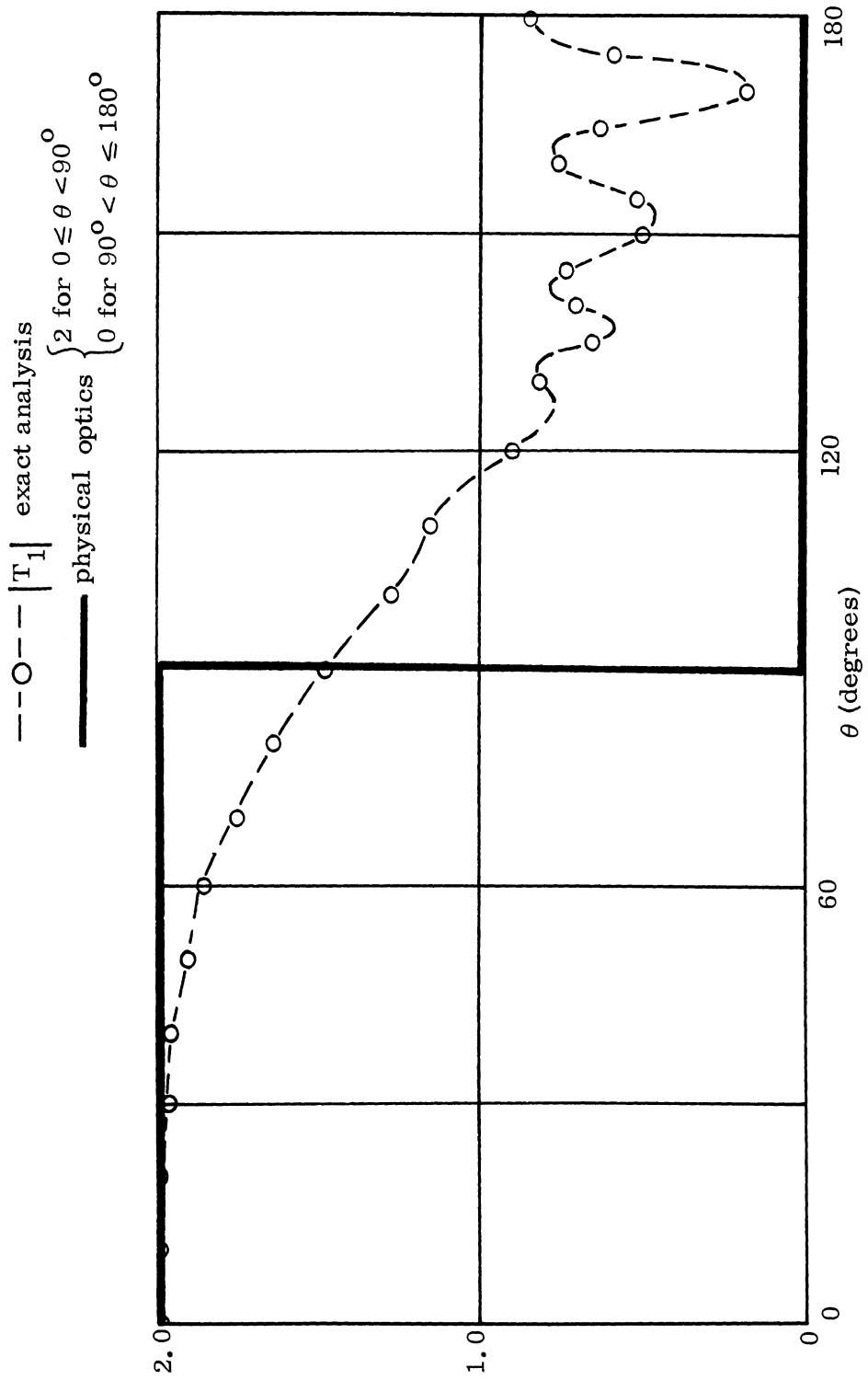


FIGURE 5-1: AMPLITUDE OF CURRENT J_θ ($ka = 10$)

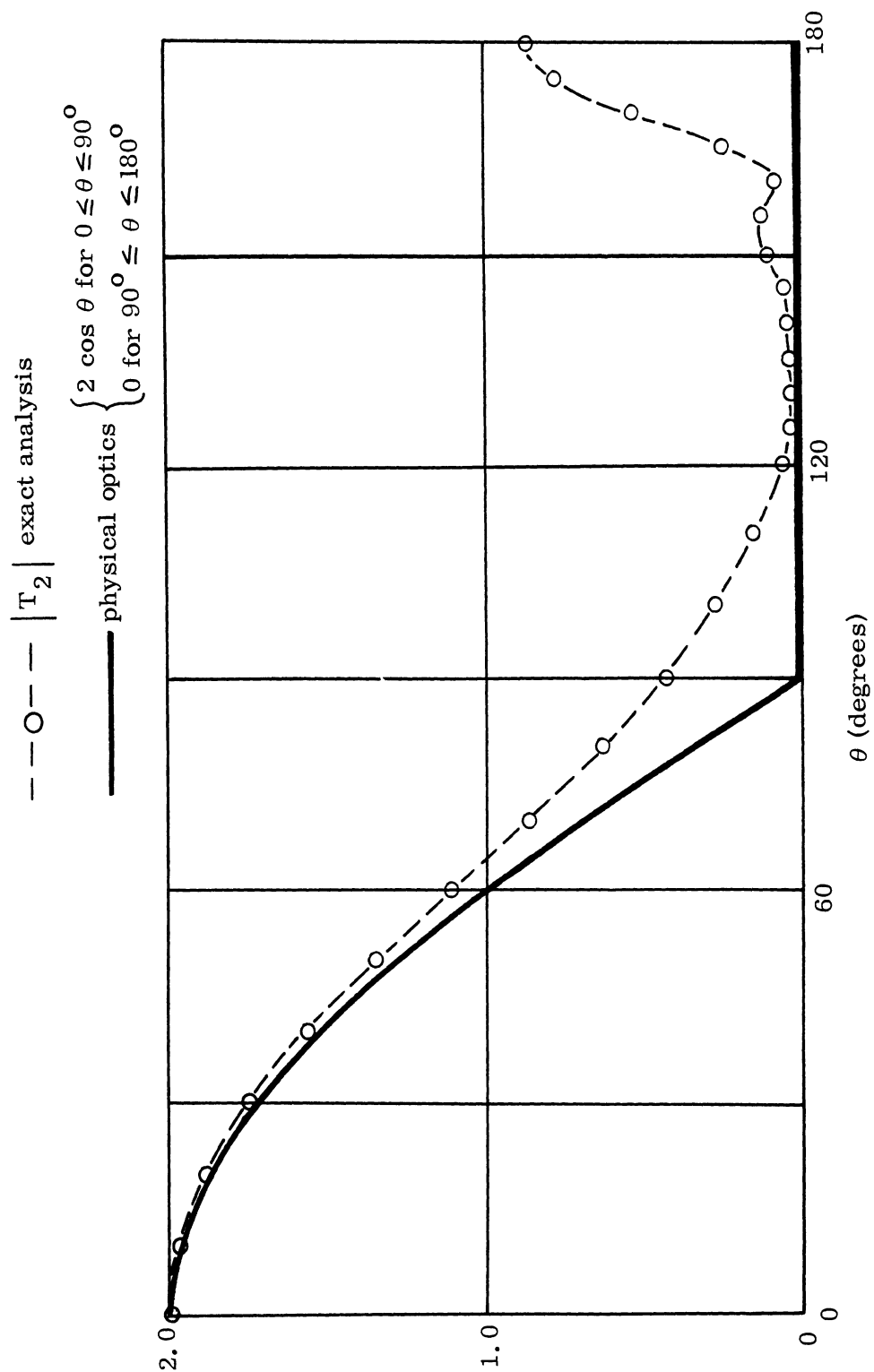


FIGURE 5-2: AMPLITUDE OF CURRENT J_ϕ ($ka = 10$)

have much effect on near-backscattering in view of the shielding action of the illuminated hemisphere.

The fact that the overall agreement between the current distributions is greater than had been supposed does suggest that the discrepancies which have been found in physical optics values of scattering cross sections are not necessarily attributable to errors in the currents themselves. It may well be that for bodies having no surface singularities, and with a receiver in the illuminated half-space, the major inaccuracies in the calculated scattering behavior are produced in the (approximate) evaluation of the physical optics integrals. In the present case, however, this approximate evaluation yields results which are quite acceptable for many purposes. The qualitative agreement between the component echoing areas is good (see figure 5-3) and indeed, the approximate values are in error by no more than 10 per cent for a sphere of radius $5\lambda/\pi$ providing the receiver lies in the half-space containing the incident field. Even if the bistatic angle exceeds 90° the errors in using the optics formulae remain small, and for $a = 5\lambda/\pi$ a permitted error of 16 per cent would extend their validity to cases of scattering through angles as large as 120° .

in view of this agreement there seems every reason for putting forward the physical optics scattering cross section for use in practical calculations involving spheres of radius greater than $3\lambda/2$. Providing the receiver is directed at the

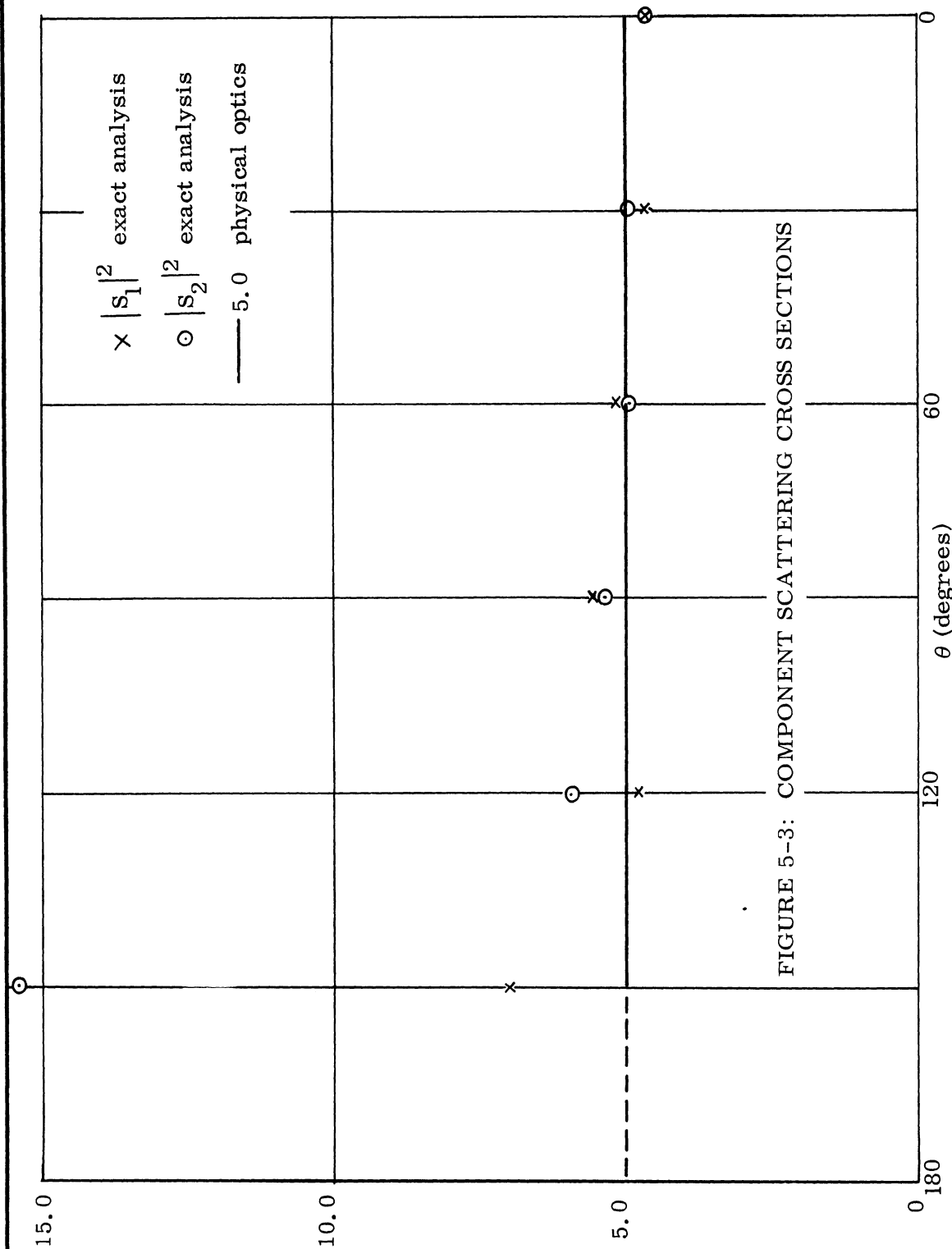


FIGURE 5-3: COMPONENT SCATTERING CROSS SECTIONS

illuminated portion of the sphere, the component scattering cross sections for a linearly polarized incident plane wave are

$$\sigma_{\theta} = \pi a^2 \cos^2 \phi$$

and

$$\sigma_{\phi} = \pi a^2 \sin^2 \phi$$

where θ and ϕ are polar coordinates defined with reference to the directions of incidence and of the incident magnetic vector. These results are sufficient to define the apparent cross section applicable to any receiving system and for any type of incident polarization.

REFERENCES

- Aden, A. 1950. Harvard Cruft Laboratory Report 106.
- Aden, A. L. and Kerker, M. 1951. J. Appl. Phys., 22, No.10, 1242.
- Baker, B. B. and Copson, E. T. 1950. The Mathematical Theory of Huygen's Principle, Oxford Clarendon Press.
- Blumer, H. 1925. Z. Physik, 32, 119.
- Blumer, H. 1931. Z. Physik, 38, 159.
- Caspersson, T. 1932. Kolloid-Z., 60, 151.
- Caspersson, T. 1933. Kolloid-Z., 65, 162.
- Clebsch, A. 1863. Crelle's J., 61.
- Engelhard, H. and Freiss, H. 1937. Kolloid-Z., 81, 129.
- Fock, V. A. 1945. J. Phys., 9, 255.
- Fock, V. A. 1946. J. Phys., 10, 399.
- Franz, W. 1954. Z. Natur., 9a, 705.
- Gumprich, R. O. and Sliepcevich, C. M. 1951. Tables of Light Scattering Functions for Spherical Particles, University of Michigan, Ann Arbor.
- Gumprich, R. O. and Sliepcevich, C. M. 1953. J. Phys. Chem. 57, 90.
- Gumprich, R. O., Sung, Chin and Sliepcevich, C. M. 1952. J. Opt. Soc. Am. 42, 226.
- Hansen, W. W. 1935. Phys. Rev. 47, 139.
- Hansen, W. W. 1936. Physics 7, 460.
- Hansen, W. W. 1937. J. Appl. Phys. 8, 284.
- Hey, J. S. and Senior, T. B. A. 1958. Proc. Phys. Soc. LXXII, 981.

- Hey, Stewart, Pinson and Prince. 1956. Proc. Phys. Soc. B, LXIX, 1036.
- deHoop, A. T. 1959. Appl. Sci. Res. B, 7, 463.
- Houghton, H. and Chalker, W. 1949. J. Opt. Soc. Am., 39, 955.
- van de Hulst, H. C. 1957. Light Scattering by Small Particles, John Wiley and Sons.
- Jones, D. S. 1955. Phil. Mag., 46, 957.
- Kerker, M. 1955. J. Opt. Soc. Am., 45, 12.
- Kerker, M. and Cox, A. L. 1955. J. Opt. Soc. Am. 45, 1080.
- Kerker, Langleben and Gunn. 1951. J. Meteorology, 8, 424.
- Kerker, M. and Perlee, H. 1953. J. Opt. Soc. Am. 43, 49.
- King, R. W. P. and Wu, T. T. 1959. The Scattering and Diffraction of Waves, Harvard University Press.
- Langer, R. E. 1932. Trans. Am. Math. Soc., 34, 447.
- Logan, N. A. 1959. Lockheed Missile Systems Division Report LMSD 288087.
- Logan, N. A. 1961. Private communication.
- Lowan, A. 1948. Tables of Scattering Functions for Spherical Particles, NBS Appl. Math. Series 4, Washington 25, D. C.
- Mie, G. 1908. Ann. Phys. 25, 377.
- Paranjpe, Naik, and Vaidya. 1939. Proc. Indian Acad. Sci. A9, 333, 352.
- Plonus, M. A. 1961. Trans. IRE, PGAP, 9, No. 6, 573.
- Riley, J. D. 1949. U. S. Naval Research Laboratory ORB Information Bulletin 9.
- Ruedy, R. 1943. Can. J. Res. A21, 79, 99.
- Ruedy, R. 1944. Can. J. Res. A22, 53.

- Scharfman. 1954. J. Appl. Phys. 25, 11, 1352.
- Schiff, L. I. 1954. Prog. Theor. Phys., 11, 288.
- Senior, T. B. A. 1960. Can. J. Phys., 38, 1702.
- Shoulejkin, W. 1924. Phil. Mag. 48, 307.
- Stratton, J. A. 1941. Electromagnetic Theory, McGraw Hill.
- Stratton, J. and Houghton, H. 1931. Phys. Rev., 38, 159.
- Thompson, J. J. 1893. Recent Researches in Electricity and Magnetism, Oxford.
- Watson, G. N. 1918. Proc. Roy. Soc. London, 95.
- Watson, G. N. 1919. Proc. Roy. Soc. London, A.

