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ITERATIVE SOLUTIONS OF THE HELMHOLTZ EQUATION

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I
INTRODUCTION

This report presents a new approach to the solution of the problem of scalar scattering by three-dimensional finite bodies. Specifically, a method is found, of general applicability, whereby the solution of the static potential problem for a Dirichlet boundary condition on a particular surface is transformed, by successive operations, into the solution of the scalar Helmholtz equation satisfying the same boundary conditions.

Lord Rayleigh (1897) considered the relationship between potential problems (boundary value problems for the Laplace equation), and scattering problems (boundary value problems for the Helmholtz equation). In a typically virtuoso performance, Rayleigh considered two as well as three dimensional problems in the electromagnetic (vector) as well as acoustic (scalar) case. In particular he showed that the potential of an obstacle in a uniform field not only was the near field limit of the solution of the corresponding scattering problem but also could yield the first term of an expansion of the far field. He gave explicit results for a general ellipsoidal scatterer including many limiting cases of interest such as the sphere, spheroid and disc.

Since that time, considerable effort has been spent in deriving higher order terms in the expansion of these as well as other shapes. Long sought in this work is the development of a systematic procedure which will generate the solution of the Helmholtz equation, satisfying a particular boundary condition, from the solution of

Laplace's equation which satisfies the same boundary condition. It is toward the achievement of this goal that the present work is directed.

The major drawback in most of the methods proposed heretofore is their intrinsic dependence on a particular geometry. That is, the techniques result from the (often adroit) exploitation of the geometric properties of the surface on which the boundary conditions are specified. Thus, restricting attention to three dimensional scalar problems, we find a variety of methods for obtaining the low frequency expansion for a disc (and an aperture in a plane screen); see Bouwkamp (1954) and Noble (1962) for an extensive bibliography to which we may add Heins (1962), de Hoop (1954) Senior (1960) and Williams (1962a). However, success in generalizing these techniques has been limited to a class of axially symmetric problems, (Collins (1962), Heins (to be published) and Williams (1962b)), and explicit results have been obtained only for a spherical cap (Collins (1962) and Thomas (1963)). For those shapes where the Helmholtz equation is separable, of course, the low frequency expansion may always be obtained from the series solution provided sufficient knowledge of the special functions involved is available. A method for obtaining low frequency expansions for bodies which are intersections of such "separable" shapes has been proposed by Darling (1960) though as yet has been applied only to a spherically capped cone (Darling and Senior, to be published).

Most low frequency techniques, however, have as their starting point the formulation of scattering problems as integral equations using the Helmholtz representation of the solution in terms of its properties on the boundary and the free space

Green's function, e.g. Baker and Copson (1950). This formulation is also vital to the proof of the existence of solutions for a general boundary given by Weyl (1952) and Müller (1952) as well as that of Werner (1962). Noble (1962) shows how this integral formulation may be used to obtain a representation of the solution of a scattering problem for a general boundary as a perturbation of the solution of the corresponding potential problem. Each term in the low frequency expansion is the solution of an integral equation which differs only in its inhomogeneous part from term to term. However, this formulation does not yield an explicit representation for successive terms in general except as the formal inverse of an operator.

The present work describes a method whereby the solution of the general Dirichlet problem for the three dimensional Helmholtz equation is explicitly expressed in terms of the Green's function for the corresponding potential problem. A new integral equation for the scattered field is derived whose kernel is the potential Green's function for the surface instead of the free space Green's function for the Helmholtz equation. Despite the fact that the integral operates over all space, rather than just the scattering surface, and is really an integro-differential operator, it is still possible to solve the equation iteratively in a standard Neumann expansion which has a nonzero radius of convergence and may be interpreted as a partial summation of the low frequency expansion. The results are valid for complex as well as real values of wave number, k, with no restriction on the sign of the imaginary part provided k is sufficiently small in absolute value. The present work also provides a constructive proof of the existence and uniqueness of solutions of the Dirichlet problem for the Helmholtz equation based on the existence and uniqueness of the potential Green's function.

The results stem from an integral representation of functions which are regular at infinity in the sense of Kellog (1953). This representation, which is a direct consequence of Green's theorem, is derived in Section 2. Wave functions, i.e. solutions of the Helmholtz equation which satisfy a radiation condition, are not regular. However it is possible, using an expansion theorem (Wilcox, 1956b), to modify them so that the representation theorem applies. This is done in Section 3 where a new integral equation for wave functions is derived. In Section 4 this equation is solved iteratively as a Neumann series and the relation between this series and the Rayleigh expansion is given. As an illustration and a check, the method is applied to the classic problem of scattering by a sphere in Section 5. This example serves not only to corroborate the analysis but also provides further insight into the manner in which the truncated Neumann series, i.e. the Nth iterate, approximates the solution. A rigorous proof of the convergence of the Neumann series is given in the Appendix.

II
A GENERAL REPRESENTATION THEOREM

We begin with a statement of Green's theorem or Green's second identity (e.g. Stratton, 1941, p. 165) which states that if u and ω are twice differentiable functions of position everywhere in a closed region of space, V, bounded by a regular surface S then

$$\int_V \left[u\nabla^2\omega - \omega\nabla^2 u\right] dV = \int_S \left[u \frac{\partial\omega}{\partial n} - \omega \frac{\partial u}{\partial n}\right] dS , \qquad (2.1)$$

where the normal derivative $\partial/\partial n$ is directed out of the volume V. We remark that the conditions for the validity of (2.1) may be weakened but for present purposes we consider only smooth regular surfaces, deferring consideration of bodies with edges.

We choose to consider the surface S as consisting of a small sphere, S_1, with center at x_1, y_1, z_1; an arbitrary smooth surface, S_2, not necessarily connected, consisting perhaps of a finite number of closed smooth surfaces; and a large sphere, S_3, containing S_1 and S_2. Further, we erect a rectangular Cartesian coordinate system with origin in S_2, see Fig. 1.

The usual procedure in formulating an integral equation for a wave function then involves identifying one of the functions in (2.1) with the free space Green's function, $e^{\pm ikR}/R$,[+] and the other with the field scattered by S_2. The integral over S_3 is shown to vanish by virtue of the radiation condition and the integral over S_1 evaluates the scattered field. The volume integral vanishes since both functions are

[+]The sign ambiguity is removed with a particular choice of harmonic time factor.

chosen to be solutions of the homogeneous Helmholtz equation yielding the well known result

$$\omega(x_1, y_1, z_1) = \frac{1}{4\pi} \int_{S_2} \left[\frac{e^{\pm ikR}}{R} \frac{\partial \omega}{\partial n} - \omega \frac{\partial}{\partial n} \left(\frac{e^{\pm ikR}}{R} \right) \right] dS. \qquad (2.2)$$

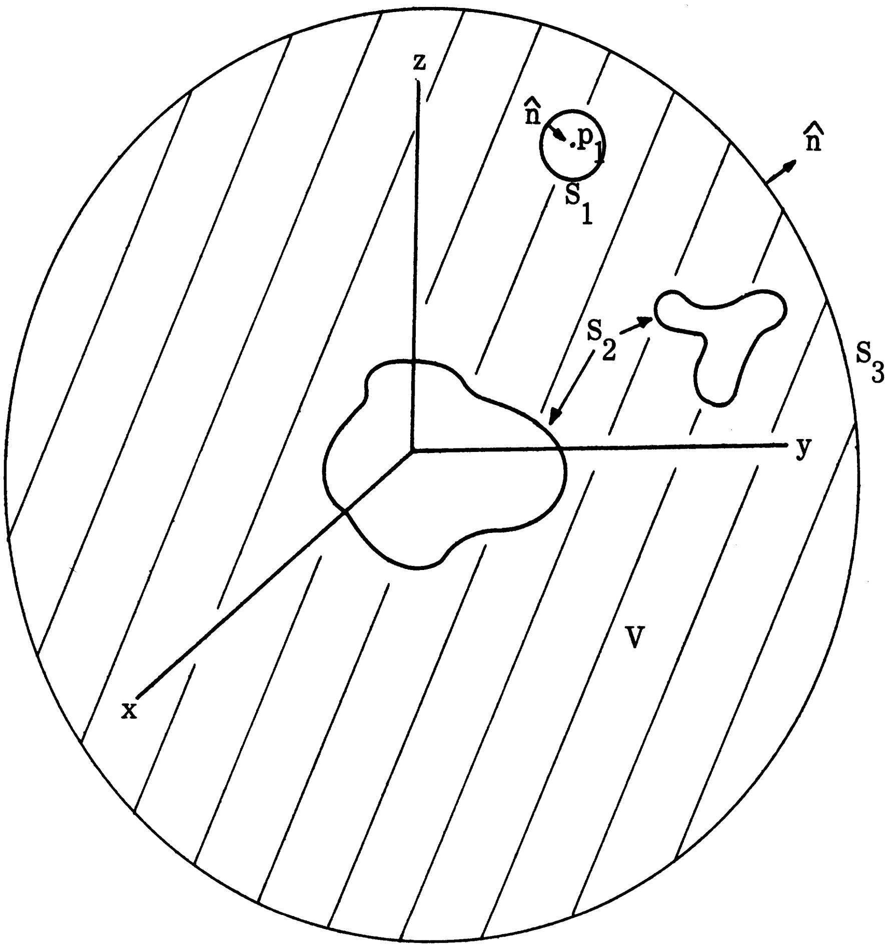

FIGURE 1

Here, however, we wish to employ, not the Green's function for the Helmholtz equation (though this is our final goal) but the Green's function for the potential

(Laplace) equation. Thus we identify u in equation (2.1) with a function of the following form

$$u = -\frac{1}{4\pi R} + u_1$$

where $R = \sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}$ and $\nabla^2 u_1 = 0$ even at $R = 0$, i.e. u_1 is regular everywhere in V in addition to the interior of S_1. Equation (2.1) then becomes

$$\int_V \left(-\frac{1}{4\pi R} + u_1\right) \nabla^2 \omega \, dV = \int_{S_1+S_2+S_3} \left[\left(-\frac{1}{4\pi R} + u_1\right) \frac{\partial \omega}{\partial n} - \omega \frac{\partial}{\partial n} \left(-\frac{1}{4\pi R} + u_1\right)\right] dS \tag{2.3}$$

If ω is assumed regular in the interior of S_1, then the integral over S_1 may be evaluated in the usual way; first integrating then letting the radius of S_1 approach zero. Clearly the regular part,

$$\int_{S_1} \left[u_1 \frac{\partial \omega}{\partial n} - \omega \frac{\partial u_1}{\partial n}\right] dS \rightarrow 0$$

and, since on S_1 $\frac{\partial}{\partial n} = -\frac{\partial}{\partial R}$,

$$\lim_{R \to 0} \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \, R^2 \sin\theta \left[\frac{1}{4\pi R} \frac{\partial}{\partial R} \omega(x_1 + R\cos\phi \sin\theta, y_1 + R\sin\phi \sin\theta, z_1 + R\cos\theta)\right.$$

$$\left. - \omega(x_1 + R\sin\theta\cos\phi, y_1 + R\sin\theta\sin\phi, z_1 + R\cos\theta) \frac{\partial}{\partial R} \left(\frac{1}{4\pi R}\right)\right]$$

$$= \omega(x_1, y_1, z_1) \tag{2.4}$$

Hence (2.3) becomes

$$\omega(x_1, y_1, z_1) = \int_V \left(-\frac{1}{4\pi R} + u_1\right) \nabla^2 \omega \, dV - \int_{S_2+S_3} \left[\left(-\frac{1}{4\pi R} + u_1\right) \frac{\partial \omega}{\partial n} - \omega \frac{\partial}{\partial n}\left(-\frac{1}{4\pi R} + u_1\right)\right] dS \qquad (2.5)$$

Now we further specify u_1 so that $\left.\left(-\frac{1}{4\pi R} + u_1\right)\right|_{S_2} = 0$, that is $-\frac{1}{4\pi R} + u_1$ is the static Green's function of the first kind (Dirichlet condition) for the surface S_2. We shall denote this function hereafter as

$$G_o(p, p_1) = -\frac{1}{4\pi R} + u_1 \qquad (2.6)$$

where the dependence on two points is indicated. The notation is a shorthand for

$$G_o(p, p_1) \equiv G_o(x, y, z, x_1, y_1, z_1) \equiv G_o\left[r, \theta, \phi, r_1 \theta_1, \phi_1\right] \qquad (2.7)$$

The volume or surface symbols V and S will be used as subscripts to indicate the variables of integration. With this in mind, equation (2.5) may now be written

$$\omega(p_1) = \int_V G_o(p_V, p_1) \nabla^2 \omega(p_V) \, dV + \int_{S_2} \omega(p_S) \frac{\partial}{\partial n} G_o(p_S, p_1) \, dS - \int_{S_3} \left[G_o(p_S, p_1) \frac{\partial \omega}{\partial n}(p_S) - \omega(p) \frac{\partial}{\partial n} G_o(p_S, p_1)\right] dS \qquad (2.8)$$

Recall that thus far, $\omega(p)$ is arbitrary except for differentiability requirements. On the other hand, $G_o(p, p_1)$ is completely specified and one of its more important properties is the fact that it is "regular at infinity" in the sense of Kellog (1953, p. 217), i.e.

$$\lim_{r \to \infty} \left| r\, G_o \right| < \infty \qquad \text{and} \qquad \lim_{r \to \infty} \left| r^2 \frac{\partial G_o}{\partial r} \right| < \infty \tag{2.9}$$

where r is the radial polar coordinate of the point p (see (2.7)).

Let us now consider the integral over S_3. It is our aim to let the radius of S_3 increase so that the volume under consideration is all of 3-space exterior to the surface S_2. If G_o were a Green's function for the wave equation and ω were a wave function then the radiation condition would imply that the integral over S_3 vanish. However G_o is not a wave function and since we would still like the integral over S_3 to vanish, we must determine the requirements on ω which will accomplish this. Since S_3 is a sphere of radius r with center at the origin, $\left.\frac{\partial}{\partial n}\right|_{S_3} = \frac{\partial}{\partial r}$ and we wish to determine conditions on ω so that

$$\lim_{r \to \infty} \int_{S_3} \left[G_o(p_S, p_1) \frac{\partial}{\partial n} \omega(p_S) - \omega(p_S) \frac{\partial}{\partial n} G_o(p_S, p_1) \right] dS$$

$$= \lim_{r \to \infty} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, r^2 \sin\theta \left[G_o(p, p_1) \frac{\partial}{\partial r} \omega(p) - \omega(p) \frac{\partial}{\partial r} G_o(p, p_1) \right] = 0 \tag{2.10}$$

$$\lim_{r\to\infty}\left|\int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, r^2\sin\theta \left[G_o(p,p_1)\frac{\partial}{\partial r}\omega(p) - \omega(p)\frac{\partial}{\partial r}G_o(p,p_1)\right]\right|$$

$$\leqslant \lim_{r\to\infty}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \left\{\left|r^2 G_o(p,p_1)\frac{\partial}{\partial r}\omega(p)\right| + \left|r^2\omega(p)\frac{\partial}{\partial r}G_o(p,p_1)\right|\right\} \qquad (2.11)$$

and conditions sufficient to force this limit to vanish are

$$\lim_{r\to\infty}\left|r^2 G_o(p,p_1)\frac{\partial}{\partial r}\omega(p)\right| = 0$$

and (2.12)

$$\lim_{r\to\infty}\left|r^2\omega(p)\frac{\partial}{\partial r}G_o(p,p_1)\right| = 0$$

Rewriting these equations slightly as follows,

$$\lim_{r\to\infty}\frac{1}{r}\left|rG_o\right|\left|r^2\frac{\partial}{\partial r}\omega(p)\right| = 0 \quad \text{and} \quad \lim_{r\to\infty}\frac{1}{r}\left|r\omega(p)\right|\left|r^2\frac{\partial G_o}{\partial r}\right| = 0 \qquad (2.13)$$

clearly indicates that requiring $\omega(p)$ to satisfy the same regularity conditions at infinity as does $G_o(p,p_1)$ (i.e. eq. (2.9)) is certainly sufficient to guarantee the validity of (2.10). Thus the contribution of the integral over the large sphere S_3 is nugatory provided ω is regular at infinity. We note in passing that this condition on $\omega(p)$ may be weakened without invalidating the result but since the functions with which we shall eventually be concerned satisfy the stronger condition, we defer consideration of this refinement.

We may summarize the results obtained thus far in the following theorem:

Let V be the volume exterior to S,+ the union of a finite number of smooth, closed, bounded, disjoint surfaces, and let $G_o(p, p_1)$ be the static Green's function of the first kind for this surface (i.e.:

a) $\nabla^2 G_o(p, p_1) = \delta(|\vec{r} - \vec{r}_1|)$++

b) $G_o(p, p_1)\Big|_{p \in S} = 0$

c) $G_o(p, p_1)$ regular at infinity);

then any function $\omega(p)$ which is twice differentiable everywhere in V and regular at infinity satisfies the integral relation

$$\omega(p) = \int_V G_o(p, p_V)\, \nabla^2 \omega(p_V) dV + \int_S \omega(p_S)\, \frac{\partial}{\partial n} G_o(p, p_S)\, dS \tag{2.14}$$

where the normal is directed out of V.

+Having evaluated the integrals over S_1 and S_3 we hereafter will denote the surface as S rather than S_2.

++The δ function is normalized so that the free space static Green's function is $-\dfrac{1}{4\pi|\vec{r}-\vec{r}_1|}$.

III
APPLICATION OF THE REPRESENTATION THEOREM TO A CLASS OF SCATTERING PROBLEMS

We now consider the time harmonic scattering problem for the surface S with Dirichlet boundary conditions. This may be considered as determining the perturbation, u^s, of an incident field, u^i, due to the presence of the surface S. Specifically, for a given u^i (plane wave, point source, or superposition of such sources) we seek a function of position u(p) such that

a) $u(p) = u^i(p) + u^s(p)$

b) $(\nabla^2 + k^2) u^s(p) = 0, \quad p \epsilon V = \text{Ext } S$

c) $u^i(p) + u^s(p) \Big|_{p \epsilon S} = 0$

d) $u^s(p)$ satisfies a radiation condition, $\lim_{r \to \infty} r\left(\frac{\partial}{\partial r} - ik\right) u^s = 0.$

In this formulation a time dependence $e^{-i\omega t}$ is assumed which gives rise to the radiation condition given in (d) and implies that the free space Green's function is $-\frac{e^{ik|\vec{r}-\vec{r}_1|}}{4\pi|\vec{r}-\vec{r}_1|}$. The comparable expressions for a time dependence $e^{+i\omega t}$ are found by replacing the k by -k throughout.

It is our intent to represent the scattered field with the integral relation derived in the previous section; however, we cannot identify $u^s(p)$ with the function $\omega(p)$ since $u^s(p)$ is not regular at infinity in the required sense although it does satisfy the radiation condition. To be more specific, even though $u^s \sim \frac{e^{ikr}}{r} f(\theta, \phi)$ for large r which implies that $|r u^s|$ will be bounded, $\left| r^2 \frac{\partial}{\partial r} u^s \right|$ will not be bounded, hence u^s will not be regular at infinity. It is possible, however, to find a function closely related to the scattered field which does satisfy the regularity requirement. This is evident from the expansion theorem for "scalar radiation functions" (see Wilcox, 1956a, b; Atkinson, 1949; Sommerfeld, 1949) which may be stated for present purposes in the following form:

If S is a surface satisfying the requirements for the representation theorem, and u^s is a function satisfying (a) - (d), then the unique expansion

$$u^s = \frac{e^{ikr}}{r} \sum_{n=0}^{\infty} \frac{u_n(\theta, \phi)}{r^n} , \qquad r > c \tag{3.1}$$

is uniformly and absolutely convergent for all r, θ, ϕ provided $r > c$. It is not difficult to show that c may be chosen as the radius of the smallest sphere entirely enclosing the surface S. The question of whether this is the smallest c for which (3.1) remains valid is not without interest but is not our present concern. Now we are interested in the behavior of u^s at infinity and a glance at (3.1) reveals that while u^s is not regular in the sense of Kellog (eq. (2.9)) the function

$$e^{-ikr} u^s = \frac{1}{r} \sum_{n=0}^{\infty} \frac{u_n(\theta, \phi)}{r^n} \tag{3.2}$$

is regular in this sense. Thus we may identify the function $\omega(p)$ of the representation theorem, not with the scattered field, but with this related function, viz.,

$$\omega(p) \equiv e^{-ikr} u^s(p) . \tag{3.3}$$

With ω defined in this way it is easily seen that

$$(\nabla^2 + k^2)u^s = 0 \Rightarrow (\nabla^2 + k^2)e^{ikr}\omega(p) = 0$$

or, explicitly,

$$\left\{\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + k^2\right\} e^{ikr}\omega(p) = 0 \tag{3.4}$$

Since

$$\frac{\partial}{\partial r}\left[e^{ikr}\omega(p)\right] = ike^{ikr}\omega(p) + e^{ikr}\frac{\partial}{\partial r}\omega(p)$$

and

$$\frac{\partial^2}{\partial r^2}\left[e^{ikr}\omega(p)\right] = -k^2e^{ikr}\omega(p) + 2ike^{ikr}\frac{\partial\omega}{\partial r} + e^{ikr}\frac{\partial^2}{\partial r^2}\omega(p)$$

it follows that

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial(e^{ikr}\omega)}{\partial r}\right] = -k^2e^{ikr}\omega + 2ike^{ikr}\frac{\partial\omega}{\partial r} + e^{ikr}\frac{\partial^2}{\partial r^2}\omega(p) + \frac{2ike^{ikr}}{r}\omega + \frac{2e^{ikr}}{r}\frac{\partial\omega}{\partial r}$$

$$= -k^2e^{ikr}\omega + e^{ikr}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\omega}{\partial r}\right) + \frac{2ike^{ikr}}{r}\frac{\partial(r\omega)}{\partial r}\,. \tag{3.5}$$

Making use of (3.5) in (3.4) leads to

$$e^{ikr}\nabla^2\omega + \frac{2ike^{ikr}}{r}\frac{\partial}{\partial r}(r\omega) = 0 \tag{3.6}$$

or finally,

$$\nabla^2 \omega = \frac{-2ik}{r} \frac{\partial}{\partial r} (r\omega) . \tag{3.7}$$

Now we make use of the representation theorem of the previous section which with (3.3) and (3.7) enables us to write

$$\omega(p_1) \equiv e^{-ikr_1} u^s(p_1) = -2ik \int_V \frac{G_o(p_V, p_1)}{r_V} \frac{\partial\, r_V \omega(p_V)}{\partial r_V} dV + \int_S \omega(p_S) \frac{\partial}{\partial n} G_o(p_S, p_1) dS. \tag{3.8}$$

Furthermore the boundary condition for the scattering problem states that

$$u^s(p) = -u^i(p) , \qquad p \in S$$

which implies that on S,

$$\omega(p) = -e^{-ikr} u^i(p) . \tag{3.9}$$

Making use of this in (3.8) yields the representation

$$\omega(p_1) = -2ik \int_V \frac{G_o(p_V, p_1)}{r_V} \frac{\partial}{\partial r_V} \left[r_V \omega(p_V) \right] dV - \int_S e^{-ikr_S} u^i(p_S) \frac{\partial}{\partial n} G_o(p_S, p_1) dS \tag{3.10}$$

We may summarize the results of the present section in the following theorem:

If (1) V is the volume exterior to S, the union of a finite number of smooth, closed, bounded disjoint surfaces, (2) $G_o(p, p_1) \left(= -\frac{1}{4\pi |\vec{r}-\vec{r}_1|} + u_1(p, p_1) \right)$ is the static Green's function of the first kind for this surface $\left(G_o(p, p_1) = 0, \quad p \in S\right)$, and (3) $u(p) = u^i(p) + u^s(p)$ is the solution of the time harmonic $(e^{-i\omega t})$ Dirichlet scattering problem for this same surface; then the scattered field satisfies the following integral equation

$$u^s(p_1) = -2ik\, e^{ikr_1} \int_V \frac{G_o(p_V, p_1)}{r_V} \frac{\partial}{\partial r_V} \left[r_V e^{-ikr_V} u^s(p_V) \right] dV$$

$$- e^{ikr_1} \int_S e^{-ikr_S} u^i(p_S) \frac{\partial}{\partial n} G_o(p_S, p_1)\, dS \tag{3.11}$$

where the normal is directed out of V.

IV
THE ITERATION PROCEDURE

With the theorem of the previous section established, an iteration scheme is clearly indicated. Here again it is convenient to work, not with u^s, but the related function

$$\omega(p) = e^{-ikr} u^s(p)$$

in terms of which the integral representation is given by equation (3.10).

If we rewrite (3.10) in operator form

$$\omega = kO \circ \omega + f \tag{4.1}$$

where O denotes the volume integral and f the known surface integral, the form of (4.1) suggests that a solution may be found using the Liouville-Neumann series of Fredholm theory. That is, we rewrite (4.1) as

$$\omega = \left[I - kO\right]^{-1} \circ f \tag{4.2}$$

and formally expand the inverse, obtaining

$$\omega = \sum_{n=0}^{\infty} k^n O^n \circ f . \tag{4.3}$$

Denoting by $\omega^{(N)}$ the partial sums

$$\omega^{(N)} = \sum_{n=0}^{N} k^n O^n \circ f \tag{4.4}$$

it follows that

$$\omega^{(0)} = f$$

$$\omega^{(N+1)} = kO \circ \omega^{(N)} + f\ , \qquad N \geqslant 0\ . \tag{4.5}$$

If, as is the present case, f is not independent of k but has a power series representation

$$f = \sum_{m=0}^{\infty} a_m k^m$$

then, substituting in (4.3) and formally employing Cauchy's form for the product of two series yields

$$\omega = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} k^{n+m} O^n \circ a_m \tag{4.6}$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{m} k^m O^n \circ a_{m-n}\ .$$

If we denote by ω_m the sum

$$\omega_m = \sum_{n=0}^{m} O^n \circ a_{m-n} \tag{4.7}$$

It follows that

$$\omega_o = a_o$$

$$\omega_{m+1} = \sum_{n=0}^{m+1} O^n \circ a_{m+1-n} = a_{m+1} + \sum_{n=0}^{m} O^{n+1} \circ a_{m-n}$$

$$= a_{m+1} + O \circ \omega_m\ , \qquad m \geqslant 0 \tag{4.8}$$

We have thus formally produced two representations for the function ω, namely

$$\omega = \lim_{N \to \infty} \omega^{(N)} \tag{4.9}$$

where $\omega^{(N)}$ are defined in (4.4) and (4.5) and

$$\omega = \sum_{m=0}^{\infty} \omega_m k^m \tag{4.10}$$

where ω_m are defined in (4.8).

There is of course a relation between the two. Clearly the first N+1 terms in the low frequency expansion of ω are given by

$$\sum_{m=0}^{N} \omega_m k^m$$

whereas the Nth iterate, eq. (4.4), includes these as well as terms of all order in k and may be considered as a partial summation of the low frequency series. Explicitly, $\omega^{(N)}$ may be written, (4.4) and (4.6),

$$\omega^{(N)} = \sum_{n=0}^{N} \sum_{m=0}^{\infty} k^{n+m} O^n \circ a_m ,$$

or, adding and subtracting the same quantity

$$\omega^{(N)} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} k^{n+m} O^n \circ a_m - \sum_{n=N+1}^{\infty} \sum_{m=0}^{\infty} k^{n+m} O^n \circ a_m .$$

Using Cauchy's form of the product of two series to rewrite the first sum and adjusting the index of the second enables us to write

$$\omega^{(N)} = \sum_{m=0}^{\infty} \sum_{n=0}^{m} k^m O^n \circ a_{m-n} - \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} k^{n+m+N+1} O^{n+N+1} \circ a_m .$$

Splitting the first sum and again adjusting the index of the second yield three terms

$$\omega^{(N)} = \sum_{m=0}^{N} \sum_{n=0}^{m} k^m O^n \circ a_{m-n} + \sum_{m=N+1}^{\infty} \sum_{n=0}^{m} k^m O^n \circ a_{m-n}$$

$$- \sum_{m=0}^{\infty} \sum_{n=0}^{m} k^{m+N+1} O^{n+N+1} \circ a_{m-n}$$

The first term is seen to be, with (4.7), the sum of the first N terms of the low frequency expansion while the remaining terms may be combined by further reordering to yield

$$\omega^{(N)} = \sum_{m=0}^{N} k^m \omega_m + \sum_{m=N+1}^{\infty} \sum_{n=0}^{m} k^m O^n \circ a_{m-n}$$

$$- \sum_{m=N+1}^{\infty} \sum_{n=0}^{m-N-1} k^m O^{n+N+1} \circ a_{m-N-1-n}$$

$$= \sum_{m=0}^{N} k^m \omega_m + \sum_{m=N+1}^{\infty} \left\{ \left[\sum_{n=0}^{m} k^m O^n \circ a_{m-n} - \sum_{n=N+1}^{m} k^m O^n \circ a_{m-n} \right] \right\}$$

and finally

$$\omega^{(N)} = \sum_{m=0}^{N} \omega_m k^m + \sum_{m=N+1}^{\infty} k^m \sum_{n=0}^{N} O^n \circ a_{m-n} \tag{4.11}$$

In what follows we shall assume that ω may be expanded in a power series in k satisfying sufficient convergence conditions to enable us to make explicit the formal results obtained above. However, in the Appendix we show that no assumptions on the existence of a convergent low frequency expansion are necessary and in fact these formal results are not only valid but constitute a constructive proof of the existence of the low frequency expansion.

For the present, however, let us assume a convergent expansion of $\omega(p)$ in the form

$$\omega(p) = \sum_{m=0}^{\infty} \omega_m(p)\, k^m \tag{4.12}$$

and in addition assume that the related series

$$\sum_{m=0}^{\infty} \frac{\partial}{\partial r} \left[r\omega_m(p) \right] k^m \tag{4.13}$$

converges absolutely and uniformly both for all points p in V and $0 < k < k_o$ where k_o is some finite radius of convergence. Then, since the spatial differentiability of $\omega(p)$ has been assumed, it follows that the order of summation and r-differentiation may be interchanged (Whittaker and Watson, 1952, p. 79) yielding

$$\frac{\partial}{\partial r} \sum_{m=0}^{\infty} \left[r\omega_m(p) \right] k^m = \sum_{m=0}^{\infty} \frac{\partial}{\partial r} \left[r\omega_m(p) \right] k^m . \tag{4.14}$$

Substituting (4.12) and (4.14) in (3.10) yields

$$\sum_{m=0}^{\infty} \omega_m(p_1)k^m = -2ik \int_V \frac{G_o(p_V, p_1)}{r_V} \sum_{m=0}^{\infty} \frac{\partial}{\partial r_V} \left[r_V \omega_m(p_V) \right] k^m dV$$

$$- \int_S e^{-ikr_S} u^i(p_S) \frac{\partial}{\partial n} G_o(p_S, p_1)\, dS. \tag{4.15}$$

The incident field, whether plane wave or point source, is an entire function of k as is $e^{-ikr} u^i(p)$ which may therefore be expanded in a Taylor series

$$e^{-ikr} u^i(p) = \sum_{m=0}^{\infty} A_m(p)\, k^m \tag{4.16}$$

where

$$A_m(p) = \frac{1}{m!} \frac{d^m}{dk^m} \left[e^{-ikr} u^i(p) \right] \Big|_{k=0}$$

Hence (4.15) may be written

$$\sum_{m=0}^{\infty} \omega_m(p_1)\, k^m = -2i \int_V \frac{G_o(p_V, p_1)}{r_V} \sum_{m=0}^{\infty} \frac{\partial}{\partial r_V} \left[r_V \omega_m(p_V) \right] k^{m+1}\, dV$$

$$- \int_S \sum_{m=0}^{\infty} A_m(p_S) k^m \frac{\partial}{\partial n} G_o(p_S, p_1)\, dS \tag{4.17}$$

The convergence of (4.16) is uniform and absolute for any finite k and in particular for $k < k_o$; hence,the order of summation and integration in the second (surface) integral on the right hand side of (4.17) may be interchanged. Similarly, since the sum appearing in the volume integral in (4.17) has been assumed absolutely and uniformly convergent, the order of summation and integration may again be interchanged. The justification of this interchange is slightly more involved since the volume integral extends over an infinite range whereas the surface integral is over a finite range; however, in either case the result is contained in or a minor extension of the theorems of classical analysis (e.g. Apostol, 1957, p. 451).

With this interchange of order of operation in (4.17), we equate coefficients of like powers of k obtaining

$$\omega_o(p_1) = -\int_S A_o(p_S)\frac{\partial}{\partial n} G_o(p_S, p_1)\, dS$$

$$\omega_{m+1}(p_1) = -2i \int_V \frac{G_o(p_V, p_1)}{r_V} \frac{\partial}{\partial r_V}\left[r_V \omega_m(p_V)\right] dV$$

$$- \int_S A_{m+1}(p_S)\frac{\partial}{\partial n} G_o(p_S, p_1)\, dS\ , \qquad m = 0, 1, 2\ldots \tag{4.18}$$

where

$$A_m(p) = \frac{1}{m!}\frac{d^m}{dk^m}\left[e^{-ikr} u^i(p)\right]\Bigg|_{\substack{k=0\\ p\in S}}$$

Equation (4.18) is the explicit form of (4.8), the low frequency expansion. The explicit form (4.5), the partial summation of the low frequency expansion is given by

$$\omega^{(0)}(p_1) = -\int_S e^{-ikr_S} u^i(p_S) \frac{\partial}{\partial n} G_o(p_S, p_1)\, dS$$

$$\omega^{(N+1)}(p_1) = -2ik \int_V \frac{G_o(p_V, p_1)}{r_V} \frac{\partial}{\partial r_V} \left[r_V \omega^{(N)}(p_V) \right] dV$$

$$- \int_S e^{-ikr_S} u^i(p_S) \frac{\partial}{\partial n} G_o(p_S, p_1)\, dS \quad . \qquad (4.19)$$

$$N = 0, 1, 2, \ldots$$

V
AN EXAMPLE: SCATTERING OF A PLANE WAVE BY A SPHERE

We now apply the methods derived in the previous section to the specific problem of scattering of a plane wave by a sphere and compare the results of the iteration process with the known exact result. We fix the origin of our coordinate system at the center of a sphere of radius a and consider a plane wave of unit amplitude propagating down the z-axis (see Fig. 2). Thus

$$u^i = e^{-ikz} = e^{-ikr\cos\theta} = \sum_{n=0}^{\infty} (-i)^n (2n+1)\, j_n(kr) P_n(\cos\theta) \tag{5.1}$$

and the known expression for the scattered field such that $u\big|_{r=a} = (u^i + u^s)\big|_{r=a} = 0$ is (e.g. Morse and Feshbach, 1953, p. 1483)

$$u^s = -\sum_{n=0}^{\infty} (-i)^n (2n+1) \frac{h_n(kr)}{h_n(ka)}\, j_n(ka)\, P_n(\cos\theta) \tag{5.2}$$

where P_n are Legendre polynomials and j_n and h_n are spherical Bessel functions and spherical Hankel functions of the first kind respectively.

The static Green's function for this problem is also well known (Stratton, 1941, p. 201) and is given by

$$G_o(p, p_1) = -\frac{1}{4\pi\sqrt{r^2 + r_1^2 - 2rr_1\cos\gamma}} + \frac{1}{4\pi\sqrt{a^2 + \left(\frac{rr_1}{a}\right)^2 - 2rr_1\cos\gamma}} \tag{5.3}$$

where $\cos\gamma = \cos\theta\cos\theta_1 + \sin\theta\sin\theta_1\cos(\phi - \phi_1)$.

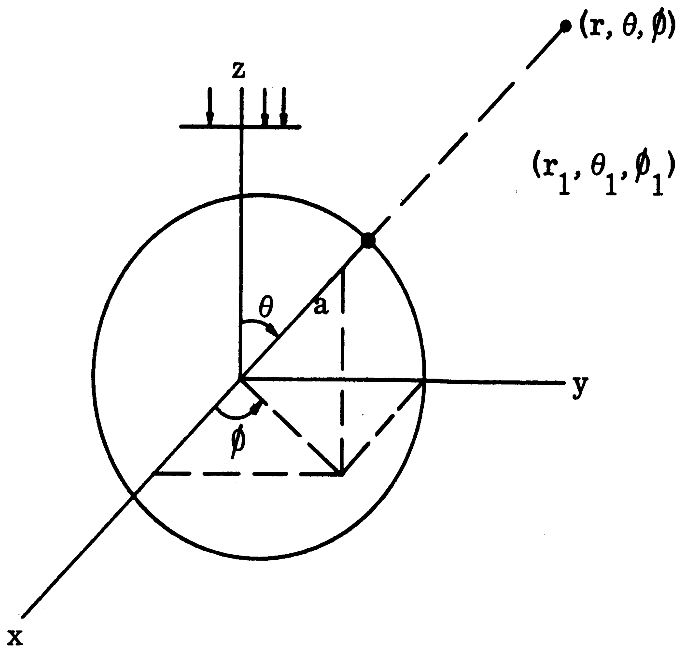

FIGURE 2

The expansion of the Green's function in spherical harmonics, which will prove useful in the subsequent analysis, is

$$G_o(p, p_1) = -\frac{1}{4\pi r_>} \sum_{n=0}^{\infty} \left(\frac{r_<}{r_>}\right)^n P_n(\cos\gamma) + \frac{1}{4\pi} \sum_{n=0}^{\infty} \frac{a^{2n+1}}{(r_1 r)^{n+1}} P_n(\cos\gamma) \qquad (5.4)$$

where $r_> = r$ if $r > r_1$ and $r_< = r$ if $r < r_1$

$= r_1$ if $r_1 > r$ and $= r_1$ if $r_1 < r$.

First we shall calculate the first few terms in the low frequency expansion of the function $\omega = e^{-ikr} u^s$ using equation (4.18). In the present case, the surface is the sphere $r = a$ on which $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$ and

$$A_m(p) = \frac{1}{m!}\frac{d^m}{dk^m}\left[e^{-ikr}u^i(p)\right]\Bigg|_{\substack{k=0\\ p\in S}}$$

$$= \frac{(-i)^m}{m!}\, a^m (1+\cos\theta)^m \ . \qquad (5.5)$$

Utilizing these facts in (4.18) allows us to write

$$\omega_o(p_1) = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, a^2 \sin\theta \frac{\partial}{\partial r} G_o(p,p_1)\Big|_{r=a}$$

and

$$\omega_{m+1}(p_1) = -2i \int_a^{\infty} dr \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, r^2 \sin\theta \frac{G_o(p,p_1)}{r}\frac{\partial}{\partial r}\left[r\omega_m(p)\right]$$

$$+ \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, a^2 \sin\theta \frac{(-ia)^{m+1}}{(m+1)!}(1+\cos\theta)^{m+1}\frac{\partial}{\partial r} G_o(p,p_1)\Big|_{r=a} \qquad (5.6)$$

$$m = 0, 1, 2, \ldots$$

where $G_o(p,p_1)$ is given in (5.3) and (5.4). Explicitly

$$\frac{\partial}{\partial r} G_o(p,p_1)\Big|_{r=a} = -\frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{a^{n-1}}{r_1^{n+1}}(2n+1)P_n(\cos\gamma) \qquad (5.7)$$

hence

$$\omega_o(p_1) = -\frac{1}{4\pi}\sum_{n=0}^{\infty}\left(\frac{a}{r_1}\right)^{n+1}(2n+1)\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta\, P_n(\cos\gamma) \ . \qquad (5.8)$$

In this section we shall assume that all interchanges of summation and integration are justifiable.

Recalling the definition of $\cos\gamma$ (eq. (5.3)) and making use of the addition theorem for Legendre polynomials (Magnus and Oberhettinger, 1949, p. 55), namely

$$P_n(\cos\gamma) = P_n\left[\cos\theta\cos\theta_1 + \sin\theta\sin\theta_1\cos(\phi-\phi_1)\right] \tag{5.9}$$

$$= P_n(\cos\theta)P_n(\cos\theta_1)+2\sum_{m=1}^{\infty}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)P_n^m(\cos\theta_1)\cos m(\phi-\phi_1),$$

it is clear that

$$\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\sin\theta P_n(\cos\gamma)P_m(\cos\theta) = 2\pi P_n(\cos\theta_1)\int_0^{\pi}d\theta\sin\theta P_n(\cos\theta)P_m(\cos\theta)$$

$$= \begin{cases} \dfrac{4\pi}{2m+1}P_m(\cos\theta_1) & n=m \\ 0 & n\neq m \end{cases} \tag{5.10}$$

Since $P_o(\cos\theta) = 1$ it follows that

$$\omega_o(p_1) = -a/r_1 . \tag{5.11}$$

Proceeding now to the next term, since

$$\frac{\partial}{\partial r}\left[r\omega_o(p)\right] = \frac{\partial}{\partial r}(-a) = 0 \tag{5.12}$$

we have

$$\omega_1(p_1) = \frac{ia}{4\pi}\sum_{n=0}^{\infty}\left(\frac{a}{r_1}\right)^{n+1}(2n+1)\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\sin\theta(1+\cos\theta)P_n(\cos\gamma) . \tag{5.13}$$

Since $1+\cos\theta = P_o(\cos\theta)+P_1(\cos\theta)$, this may be evaluated using the orthogonality of P_n, eq. (5.10), yielding

$$\omega_1(p_1) = \frac{ia^2}{r_1} + ia\left(\frac{a}{r_1}\right)^2 P_1(\cos\theta_1) \ . \qquad (5.14)$$

Proceeding to the next term, since

$$\frac{\partial}{\partial r}\left[r\,\omega_1(p)\right] = \frac{\partial}{\partial r}\left[ia^2 + \frac{ia^3}{r} P_1(\cos\theta)\right]$$

$$= -\frac{ia^3}{r^2} P_1(\cos\theta) \ , \qquad (5.15)$$

we have

$$\omega_2(p_1) = -2i\int_a^{\infty} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \, r^2 \sin\theta \, G_o(p,p_1)\left[-i\left(\frac{a}{r}\right)^3 P_1(\cos\theta)\right]$$

$$-\frac{a^4}{2}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta (1+\cos\theta)^2 \frac{\partial}{\partial r} G_o(p,p_1)\Big|_{r=a} \qquad (5.16)$$

Substituting the explicit forms for $G_o(p,p_1)$, (5.4) and (5.7), leads to

$$\omega_2(p_1) = \frac{1}{2\pi}\int_a^{r_1} dr \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, r^2 \sin\theta \left(\frac{a}{r}\right)^3 P_1(\cos\theta) \sum_{n=0}^{\infty} \frac{r^n}{r_1^{n+1}} P_n(\cos\gamma)$$

$$+\frac{1}{2\pi}\int_{r_1}^{\infty} dr \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, r^2 \sin\theta \left(\frac{a}{r}\right)^3 P_1(\cos\theta) \sum_{n=0}^{\infty} \frac{r_1^n}{r^{n+1}} P_n(\cos\gamma)$$

(cont'd)

$$-\frac{1}{2\pi}\int_a^\infty dr\int_0^{2\pi}d\phi\int_0^\pi d\theta\, r^2\sin\theta\left(\frac{a}{r}\right)^3 P_1(\cos\theta)\sum_{n=0}^{\infty}\frac{a^{2n+1}}{(rr_1)^{n+1}}P_n(\cos\gamma)$$

$$+\frac{a^2}{8\pi}\sum_{n=0}^{\infty}\left(\frac{a}{r_1}\right)^{n+1}(2n+1)\int_0^{2\pi}d\phi\int_0^\pi d\theta\sin\theta(1+\cos\theta)^2P_n(\cos\gamma)\,. \tag{5.17}$$

The angular integration is trivial using the orthogonality relation (5.10) and the fact that

$$(1+\cos\theta)^2 = \frac{2}{3}P_2(\cos\theta)+2P_1(\cos\theta)+\frac{4}{3}P_o(\cos\theta). \tag{5.18}$$

This results in

$$\omega_2(p_1) = \frac{2}{3}\int_a^{r_1}dr\,\frac{a^3}{r_1^2}P_1(\cos\theta_1)+\frac{2}{3}\int_{r_1}^{\infty}dr\,\frac{a^3r_1}{r^3}P_1(\cos\theta_1)$$

$$-\frac{2}{3}\int_a^\infty dr\left(\frac{a}{r}\right)^3\frac{a^3}{r_1^2}P_1(\cos\theta_1)+\frac{a^2}{2}\left\{\frac{2}{3}P_2(\cos\theta_1)\left(\frac{a}{r_1}\right)^3\right.$$

$$\left.+2P_1(\cos\theta_1)\left(\frac{a}{r_1}\right)^2+\frac{4}{3}\frac{a}{r_1}\right\} \tag{5.19}$$

The r integration is simply performed yielding

$$\omega_2(p_1) = \frac{2}{3}\frac{a^3}{r_1^2} P_1(\cos\theta_1)(r_1 - a) + \frac{2}{3}a^3 r_1 P_1(\cos\theta_1)\left(-\frac{1}{2r^2}\right)\Bigg|_{r=r_1}^{\infty}$$

$$- \frac{2}{3}\frac{a^6}{r_1^2} P_1(\cos\theta_1)\left(-\frac{1}{2r^2}\right)\Bigg|_{r=a}^{\infty} + \frac{a^2}{3}\left(\frac{a}{r_1}\right)^3 P_2(\cos\theta_1)$$

$$+ a^2\left(\frac{a}{r_1}\right)^2 P_1(\cos\theta_1) + a^2 \frac{2}{3}\frac{a}{r_1} \qquad (5.20)$$

or

$$\omega_2(p_1) = a^2 \left\{ \frac{2}{3} P_1(\cos\theta_1)\frac{a}{r_1} - \frac{2}{3} P_1(\cos\theta_1)\left(\frac{a}{r_1}\right)^2 + \frac{1}{3}\frac{a}{r_1} P_1(\cos\theta_1) \right.$$

$$\left. - \frac{1}{3} P_1(\cos\theta_1)\left(\frac{a}{r_1}\right)^2 + \frac{1}{3} P_2(\cos\theta_1)\left(\frac{a}{r_1}\right)^3 + \left(\frac{a}{r_1}\right)^2 P_1(\cos\theta_1) + \frac{2}{3}\frac{a}{r_1} \right\} \qquad (5.21)$$

This may be further simplified as follows

$$\omega_2(p_1) = a^2 \left\{ \frac{1}{3} P_2(\cos\theta_1)\left(\frac{a}{r_1}\right)^3 + \frac{a}{r_1} P_1(\cos\theta_1) + \frac{2}{3}\frac{a}{r_1} \right\}. \qquad (5.22)$$

Collecting our results we have, with (5.11), (5.14) and (5.22)

$$\omega(p_1) = \sum_{n=0}^{\infty} k^n \omega_n(p_1)$$

$$= \omega_o + k\omega_1 + k^2\omega_2 + O(k^3)$$

$$= -\frac{a}{r_1} + i\,ka\left[\frac{a}{r_1} + \left(\frac{a}{r_1}\right)^2 P_1(\cos\theta_1)\right] + (ka)^2 \left\{ \frac{1}{3} P_2(\cos\theta_1)\left(\frac{a}{r_1}\right)^3 \right.$$

$$\left. + \frac{a}{r_1} P_1(\cos\theta_1) + \frac{2}{3}\frac{a}{r_1} \right\} + O(k^3). \qquad (5.23)$$

We wish to compare this with the exact result, calculated from eq. (5.2) to the same order in k. To calculate these terms we first observe that since (Magnus and Oberhettinger, 1949, p. 22)

$$h_n(\rho) = \frac{e^{i\rho}\, i^{-n-1}}{\rho} \sum_{m=0}^{n} (-1)^m \frac{(n+\frac{1}{2}, m)}{(2i\rho)^m} ; \qquad (5.24)$$

where $$(n+\tfrac{1}{2}, 0) = 1$$

and

$$(n+\frac{1}{2}, m) = \frac{\left[(2n+1)^2 - 1\right]\left[(2n+1)^2 - 3^2\right] \ldots \left[(2n+1)^2 - (2m-1)^2\right]}{2^{2m}\, m!}$$

$$= \frac{\Gamma(n+m+1)}{\Gamma(n-m+1)m!} \qquad m = 1, 2, 3, \ldots$$

$$= \frac{(-1)^m (-n)_m (n+1)_m}{m!} , \qquad (z)_\ell = \frac{\Gamma(z+\ell)}{\Gamma(z)} ;$$

the ratio $\dfrac{h_n(kr_1)}{h_n(ka)}$ may be written

$$\frac{h_n(kr_1)}{h_n(ka)} = \frac{a\, e^{ik(r_1-a)}}{r_1} \frac{\sum_{m=0}^{n} (-1)^m \frac{(n+\frac{1}{2}, m)}{(2ikr_1)^m}}{\sum_{m=0}^{n} (-1)^m \frac{(n+\frac{1}{2}, m)}{(2ika)^m}}$$

or, reversing the order of summation and simplifying,

$$\frac{h_n(kr_1)}{h_n(ka)} = \left(\frac{a}{r_1}\right)^{n+1} e^{ik(r_1-a)} \frac{\sum_{m=0}^{n}(-1)^m(n+\frac{1}{2}, n-m)(2ikr_1)^m}{\sum_{m=0}^{n}(-1)^m(n+\frac{1}{2}, n-m)(2ika)^m} . \qquad (5.25)$$

With equations (5.2) and (5.25), the exact form of $\omega(p_1)$ may be written

$$\omega(p_1) = e^{-ikr_1} u^s(p_1)$$

$$= -e^{-ika} \sum_{n=0}^{\infty}(-i)^n(2n+1)\left(\frac{a}{r_1}\right)^{n+1} j_n(ka)\, P_n(\cos\theta_1) \frac{\sum_{m=0}^{n}(-1)^m(n+\frac{1}{2}, n-m)(2ikr_1)^m}{\sum_{m=0}^{n}(-1)^m(n+\frac{1}{2}, n-m)(2ika)^m}$$

(5.26)

Note that the quotient of two polynomials in k with non-vanishing constant term is expressible as a power series in k with non-vanishing constant term (see footnote at the end of this section). Also, from the definition of $j_n(\rho)$, namely

$$j_n(\rho) = \frac{\sqrt{\pi}\,\rho^n}{2^{n+1}} \sum_{m=0}^{\infty}(-1)^m \frac{(\rho/2)^{2m}}{m!\,\Gamma(m+n+\frac{3}{2})} \qquad (5.27)$$

we see that $j_n(ka)$ may be written as a power series in ka whose lowest order term is $(ka)^n$. Thus all terms in the series (5.26) for $n > 2$ are of order k^3 or greater, and, since

$$e^{-ika} = 1 - ka - \frac{(ka)^2}{2} + O(k^3) , \qquad (5.28)$$

we may write the exact result, (5.26), as

$$\omega(p_1) = -\left[1 - ika - \frac{(ka)^2}{2}\right]\left\{\frac{a}{r_1} j_o(ka) - 3i\left(\frac{a}{r_1}\right)^2 j_1(ka)\, P_1(\cos\theta_1)\;\cdot\right.$$

$$\left[\frac{(\frac{3}{2}, 1) - (\frac{3}{2}, 0)\, 2ikr_1}{(\frac{3}{2}, 1) - (\frac{3}{2}, 0)\, 2ika}\right] - 5\left(\frac{a}{r_1}\right)^3 j_2(ka)\, P_2(\cos\theta_1)\;\cdot$$

$$\left.\left[\frac{(\frac{5}{2}, 2) - (\frac{5}{2}, 1)(2ikr_1) + (\frac{5}{2}, 0)(2ikr_1)^2}{(\frac{5}{2}, 2) - (\frac{5}{2}, 1)(2ika) + (\frac{5}{2}, 0)(2ika)^2}\right]\right\} + O(k^3)\;. \tag{5.29}$$

Making use of the notational definition in (5.24) we have

$$(\tfrac{3}{2}, 0) = 1 \qquad (\tfrac{3}{2}, 1) = 2$$

$$(\tfrac{5}{2}, 0) = 1 \qquad (\tfrac{5}{2}, 1) = 6 \qquad (\tfrac{5}{2}, 2) = 12 \tag{5.30}$$

and from the definition of j_n, (5.27), we see that

$$j_o(ka) = 1 - \frac{(ka)^2}{6} + O(k^4)$$

$$j_1(ka) = \frac{ka}{3} + O(k^3) \tag{5.31}$$

$$j_2(ka) = \frac{(ka)^2}{15} + O(k^4)\;.$$

Substituting (5.30) and (5.31) in (5.29) we obtain

$$\omega(p_1) = -\left[1 - ika - \frac{(ka)^2}{2}\right] \frac{a}{r_1}\left[1 - \frac{(ka)^2}{6}\right]$$

$$+ 3i\left[1 - ika\right]\left(\frac{a}{r_1}\right)^2 \frac{ka}{3} P_1(\cos\theta_1) \frac{2 - 2ikr_1}{2 - 2ika}$$

$$+ 5\left(\frac{a}{r_1}\right)^3 \frac{(ka)^2}{15} P_2(\cos\theta_1) \frac{12 - 12ikr_1 - 4(kr_1)^2}{12 - 12ika - 4(ka)^2} + O(k^3) \tag{5.32}$$

This may be further simplified noting that, to the required order,

$$\frac{1 - ikr_1 - \frac{1}{3}(kr_1)^2}{1 - ika - \frac{1}{3}(ka)^2} = 1 + O(k) .$$

Hence (5.32) becomes

$$\omega(p_1) = -\frac{a}{r_1}\left[1 - ika - \frac{(ka)^2}{2}\right] + \frac{a}{r_1}\frac{(ka)^2}{6} + i\left(\frac{a}{r_1}\right)^2 ka\, P_1(\cos\theta_1)(1 - ikr_1)$$

$$+ \frac{(ka)^2}{3}\left(\frac{a}{r_1}\right)^3 P_2(\cos\theta_1) + O(k^3). \tag{5.33}$$

Collecting terms in like powers of k we find

$$\omega(p_1) = -\frac{a}{r_1} + ika\left[\frac{a}{r_1} + \left(\frac{a}{r_1}\right)^2 P_1(\cos\theta_1)\right] + (ka)^2\left[\frac{2}{3}\frac{a}{r_1} + \frac{a}{r_1} P_1(\cos\theta_1)\right.$$

$$\left. + \frac{1}{3}\left(\frac{a}{r_1}\right)^3 P_2(\cos\theta_1)\right] + O(k^3) \tag{5.34}$$

Comparison of (5.23) with (5.34) verifies the fact that to this order in k the exact result and the low frequency iteration method are in complete agreement.

Next we consider the perhaps more important iteration method given in (4.19) which represents, as was pointed out, a partial summation of the low frequency series. Specializing the surface S to be the sphere of radius a and making use of the incident field and Green's function representations given in (5.1), (5.4) and (5.7) we rewrite equation (4.19) as follows:

$$\omega^{(0)}(p_1) = -\frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \sum_{n=0}^{\infty}\left(\frac{a}{r_1}\right)^{n+1}(2n+1)\,P_n(\cos\gamma)\,e^{-ika-ika\cos\theta}$$

$$= -\frac{e^{-ika}}{4\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{a}{r_1}\right)^{n+1}(2n+1)(2m+1)(-i)^m j_m(ka)$$

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta\, P_n(\cos\gamma)\, P_m(\cos\theta) \qquad (5.35)$$

$$\omega^{(N+1)}(p_1) = +\frac{ik}{2\pi}\int_a^{r_1} dr \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, r^2\sin\theta \sum_{n=0}^{\infty}\frac{r^{n-1}}{r_1^{n+1}}\,P_n(\cos\gamma)\frac{\partial}{\partial r}\left[r\,\omega^{(N)}(p)\right]$$

$$+\frac{ik}{2\pi}\int_{r_1}^{\infty} dr \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, r^2\sin\theta \sum_{n=0}^{\infty}\frac{r_1^{n}}{r^{n+2}}\,P_n(\cos\gamma)\frac{\partial}{\partial r}\left[r\,\omega^{(N)}(p)\right]$$

$$-\frac{ik}{2\pi}\int_a^{\infty} dr \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, r^2\sin\theta \sum_{n=0}^{\infty}\frac{a^{2n+1}}{r(rr_1)^{n+1}}\,P_n(\cos\gamma)\frac{\partial}{\partial r}\left[r\,\omega^{(N)}(p)\right]$$

$$+\,\omega^{(0)}(p_1)\ .$$

With the orthogonality relation (5.10) we find that

$$\omega^{(0)}(p_1) = -e^{-ika} \sum_{n=0}^{\infty} (2n+1)(-i)^n \left(\frac{a}{r_1}\right)^{n+1} j_n(ka)\, P_n(\cos\theta_1) \tag{5.36}$$

Now calculating $\omega^{(1)}(p_1)$ using (5.35) and (5.36) we have

$$\frac{\partial}{\partial r}\left[r\,\omega^{(0)}(p)\right] = e^{-ika} \sum_{n=1}^{\infty} (2n+1)\, n(-i)^n \left(\frac{a}{r}\right)^{n+1} j_n(ka)\, P_n(\cos\theta)$$

and

$$\omega^{(1)}(p_1) = \frac{ik}{2\pi} e^{-ika} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} (2m+1)\, m(-i)^m j_m(ka) \cdot$$

$$\left\{ \int_a^{r_1} dr \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, r^2 \sin\theta\, \frac{r^{n-1}}{r_1^{n+1}} \left(\frac{a}{r}\right)^{m+1} P_n(\cos\gamma)\, P_m(\cos\theta)\right.$$

$$+ \int_{r_1}^{\infty} dr \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, r^2 \sin\theta\, \frac{r_1^n}{r^{n+2}} \left(\frac{a}{r}\right)^{m+1} P_n(\cos\gamma)\, P_m(\cos\theta)$$

$$\left. - \int_a^{\infty} dr \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, r^2 \sin\theta\, \frac{a^{2n+1}}{r_1^{n+1}\, r^{n+2}} \left(\frac{a}{r}\right)^{m+1} P_n(\cos\gamma) P_m(\cos\theta) \right\} + \omega^{(0)}(p_1). \tag{5.37}$$

Using the orthogonality relation (5.10) this becomes

$$\omega^{(1)}(p_1) = 2ike^{-ika} \sum_{m=1}^{\infty} m(-i)^m j_m(ka)P_m(\cos\theta_1) \left\{ \int_a^{r_1} dr \left(\frac{a}{r_1}\right)^{m+1} \right.$$

$$\left. + \int_{r_1}^{\infty} dr \frac{r_1^m a^{m+1}}{r^{2m+1}} - \int_a^{\infty} dr \frac{a^{3m+2}}{r_1^{m+1} r^{2m+1}} \right\} + \omega^{(0)}(p_1)$$

which, on carrying out the r-integration becomes

$$\omega^{(1)}(p_1) = 2ike^{-ika} \sum_{m=1}^{\infty} m(-i)^m j_m(ka) P_m(\cos\theta_1) \left\{ \left(\frac{a}{r_1}\right)^{m+1} (r_1 - a) \right.$$

$$\left. + \frac{a^{m+1}}{2mr_1^m} - \frac{a^{m+2}}{2mr_1^{m+1}} \right\} + \omega^{(0)}(p_1) . \quad (5.38)$$

This may also be written

$$\omega^{(1)}(p_1) = -e^{-ika} \sum_{m=1}^{\infty} (-i)^m \left(\frac{a}{r_1}\right)^{m+1} j_m(ka)P_m(\cos\theta_1)(2m+1)(-ik)(r_1 - a)$$

$$-e^{-ika} \sum_{m=0}^{\infty} (2m+1)(-i)^m \left(\frac{a}{r_1}\right)^{m+1} j_m(ka)P_m(\cos\theta_1). \quad (5.39)$$

Continuing the iteration we have

$$\frac{\partial}{\partial r}\left[r\omega^{(1)}(p)\right] = -e^{-ika} \sum_{m=1}^{\infty} (-i)^m j_m(ka)P_m(\cos\theta)(2m+1)(-ik)\, a^{m+1} \left\{ \frac{-m+1}{r^m} \right.$$

$$\left. + \frac{ma}{r^{m+1}} \right\} + \frac{\partial}{\partial r}\left[r\omega^{(0)}(p)\right] \quad (5.40)$$

and

$$\omega^{(2)}(p_1) = \frac{(ik)^2}{2\pi} e^{-ika} \sum_{m=1}^{\infty} (-i)^m j_m(ka)(2m+1)\, a^{m+1} \sum_{n=0}^{\infty}$$

$$\Bigg\{ \Bigg[\int_a^{r_1} dr \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, r^2 \sin\theta \, \frac{r^{n-1}}{r_1^{n+1}} P_n(\cos\gamma) P_m(\cos\theta) \left[\frac{-m+1}{r^m} + \frac{ma}{r^{m+1}} \right]$$

$$+ \int_{r_1}^{\infty} dr \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, r^2 \sin\theta \, \frac{r_1^n}{r^{n+2}} P_n(\cos\gamma) P_m(\cos\theta) \left[\frac{-m+1}{r^m} + \frac{ma}{r^{m+1}} \right]$$

$$- \int_a^{\infty} dr \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, r^2 \sin\theta \, \frac{a^{2n+1}}{r_1^{n+1} r^{n+2}} P_n(\cos\gamma) P_m(\cos\theta) \Bigg[\frac{-m+1}{r^m}$$

$$+ \frac{ma}{r^{m+1}} \Bigg] \Bigg\} + \omega^{(1)}(p_1) \,. \qquad (5.41)$$

Using the orthogonality relation (5.10) this becomes

$$\omega^{(2)}(p_1) = 2(ik)^2 e^{-ika} \sum_{m=1}^{\infty} (-i)^m j_m(ka) a^{m+1} P_m(\cos\theta_1) \Bigg\{ \int_a^{r_1} dr \Bigg[\frac{(-m+1)r}{r_1^{m+1}}$$

$$+ \frac{ma}{r_1^{m+1}} \Bigg] + \int_{r_1}^{\infty} da \left[\frac{(-m+1) r_1^m}{r^{2m}} + \frac{ma\, r_1^m}{r^{2m+1}} \right]$$

$$- \int_a^{\infty} dr \, \frac{a^{2m+1}}{r_1^{m+1}} \left[\frac{(-m+1)}{r^{2m}} + \frac{ma}{r^{2m+1}} \right] \Bigg\} + \omega^{(1)}(p_1)$$

(5.42)

which, on carrying out the r-integration becomes

$$\omega^{(2)}(p_1) = 2(ik)^2 e^{-ika} \sum_{m=1}^{\infty} (-i)^m j_m(ka) a^{m+1} P_m(\cos\theta_1) \left\{ \frac{(-m+1)}{r_1^{m+1}} \left(\frac{r_1^2}{2} - \frac{a^2}{2} \right) \right.$$

$$\left. + \frac{ma(r_1 - a)}{r_1^{m+1}} + \frac{(-m+1)}{(2m-1)r_1^{m-1}} + \frac{ma}{2mr_1^m} - \frac{(-m+1)a^2}{(2m-1)r_1^{m+1}} - \frac{ma^2}{2mr_1^{m+1}} \right\} + \omega^{(1)}(p_1) \tag{5.43}$$

This may also be written

$$\omega^{(2)}(p_1) = 2(ik)^2 e^{-ika} \sum_{m=1}^{\infty} (-i)^m j_m(ka) \left(\frac{a}{r_1}\right)^{m+1} (2m+1) P_m(\cos\theta_1)$$

$$\cdot \left[\frac{(-m+1)}{2(2m-1)} (r_1^2 - a^2) + \frac{a}{r}(r_1 - a) \right] + \omega^{(1)}(p_1). \tag{5.44}$$

With (5.39) we rewrite this as

$$\omega^{(2)}(p_1) = -e^{-ika} \sum_{m=1}^{\infty} (-i)^m j_m(ka) \left(\frac{a}{r_1}\right)^{m+1} (2m+1) P_m(\cos\theta_1) \left\{ \frac{(-m+1)}{2m-1} \left[(kr_1)^2 \right. \right.$$

$$\left. \left. - (ka)^2 \right] + ka(kr_1 - ka) - ikr_1 + ika + 1 \right\} - e^{-ika} \frac{a}{r_1} j_o(ka) . \tag{5.45}$$

We could of course continue iterating but it will be seen that this is not necessary in order to see the sense in which successive iterates approach the exact result. If we re-examine the known exact result for $\omega(p_1)$ given in (5.26) we see that, separating the $n = 0$ term, this may be written as follows,

$$\omega(p_1) = -e^{-ika}\frac{a}{r_1}j_o(ka) - e^{-ika}\sum_{n=1}^{\infty}(-i)^n(2n+1)\left(\frac{a}{r_1}\right)^{n+1} j_n(ka)P_n(\cos\theta_1)S_n \tag{5.46}$$

where

$$S_n = \frac{\sum_{m=0}^{n}(-1)^m(n+\frac{1}{2}, n-m)(2ikr_1)^m}{\sum_{m=0}^{n}(-1)^m(n+\frac{1}{2}, n-m)(2ika)^m}$$

and the notation $(n+\frac{1}{2}, n-m)$ is defined in (5.24).

Since $n > 0$ in (5.46), the quotient of the two polynomials, S_n, may be written

$$S_n = \frac{\sum_{m=0}^{n}(-1)^m(n+\frac{1}{2}, n-m)(2ikr_1)^m}{(n+\frac{1}{2}, n) + \sum_{m=1}^{n}(-1)^m(n+\frac{1}{2}, n-m)(2ika)^m}$$

or, dividing numerator and denominator by $(n+\frac{1}{2}, n)$

$$S_n = \frac{\sum_{m=0}^{n}(-1)^m \frac{(n+\frac{1}{2}, n-m)}{(n+\frac{1}{2}, n)}}{1+\sum_{m=1}^{n}(-1)^m \frac{(n+\frac{1}{2}, n-m)}{(n+\frac{1}{2}, n)}(2ika)^m} \tag{5.47}$$

We may always choose ka small enough so that

$$\left|\sum_{m=1}^{n}(-1)^m \frac{(n+\frac{1}{2}, n-m)}{(n+\frac{1}{2}, n)}(2ika)^m\right| < 1 \tag{5.48}$$

and, with this restriction,+ then expand the denominator in (5.47) obtaining

$$S = \sum_{m=0}^{n} (-1)^m \frac{(n+\frac{1}{2}, n-m)}{(n+\frac{1}{2}, n)} (2ikr_1)^m \sum_{\ell=0}^{\infty} (-1)^\ell \left[\sum_{m=1}^{n} (-1)^m \frac{(n+\frac{1}{2}, n-m)}{(n+\frac{1}{2}, n)} (2ika)^m \right]^\ell \tag{5.49}$$

We now calculate the first few terms of S_n expanded in powers of k. It is a convenience, in this calculation, to note that

$$\frac{(n+\frac{1}{2}, n-m)}{(n+\frac{1}{2}, n)} \equiv \frac{(-n)_m}{(-2n)_m m!} . \tag{5.50}$$

+A gross estimate of how small ka must be is found by noting that $\left| \frac{(-n)_m}{(-2n)_m} \right| \leqslant (\frac{1}{2})^m$. Using this fact we find, after substituting (5.50) in (5.58), that

$$\left| \sum_{m=1}^{n} (-1)^m \frac{(n+\frac{1}{2}, n-m)}{(n+\frac{1}{2}, n)} (2ika)^m \right| = \left| \sum_{m=1}^{n} (-1)^m \frac{(-n)_m}{(-2n)_m m!} (2ika)^m \right|$$

$$\leqslant \sum_{m=1}^{n} \frac{|ka|^m}{m!} < \sum_{m=1}^{\infty} \frac{|ka|^m}{m!} = e^{|ka|} - 1 ,$$

hence requiring $|ka| < \log 2$ is sufficient to guarantee that

$$\sum_{m=1}^{n} (-1)^m \frac{(n+\frac{1}{2}, n-m)}{(n+\frac{1}{2}, n)} (2ika)^m < 1 .$$

With (5.50) we fihd that

$$S_n = \left[1 - ikr_1 - \frac{(-n)(-n+1)}{(-2n)(-2n+1)2!} 4(kr_1)^2 + O(k^3)\right] \left[1 + \left(ika + \frac{(-n)(-n+1)}{(-2n)(-2n+1)2!} 4(ka)^2\right) - (ka)^2 + O(k^3)\right]$$

$$= 1 - ikr_1 + ika + \frac{(-n+1)}{2n-1}\left[(kr_1)^2 - (ka)^2\right] + ka(kr_1 - ka) + O(k^3) \ . \qquad (5.51)$$

Substituting (5.51) in (5.46) we find that provided ka is small enough the exact result may be written

$$\omega(p_1) = -e^{-ika}\frac{a}{r_1} j_o(ka) - e^{-ika}\sum_{n=1}^{\infty}(-i)^n(2n+1)\left(\frac{a}{r_1}\right)^{n+1} j_n(ka)\, P_n(\cos\theta_1)\left[1 - ikr_1 + ika + ka(kr_1 - ka) + \frac{(-n+1)}{(2n-1)}\left[(kr_1)^2 - (ka)^2\right] + O(k^3)\right] \ . \qquad (5.52)$$

Comparing this form of the exact result with the expression obtained by our iteration process, (5.45) shows that the two are in complete agreement to order k^3 in an expansion of the quotient S_n.

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APPENDIX: CONVERGENCE OF THE NEUMANN SERIES

This appendix is devoted to the task of showing that the unique solution of the operator equation

$$\omega = kO \circ \omega + f \tag{A.1}$$

where

$$O \circ \; = -2i \int_V dV \frac{G_o(p_V, p_1)}{r_V} \frac{\partial}{\partial r_V} \left[r_V \circ \right. \tag{A.2}$$

and

$$f = - \int_S e^{-ikr_S} u^i(p_S) \frac{\partial}{\partial n} G_o(p_S, p_1)\, dS \tag{A.3}$$

is given by the Neumann series

$$\omega = \sum_{n=0}^{\infty} k^n O^n \circ f \,. \tag{A.4}$$

Specifically we shall show that the series (A.4) converges, establish the sense in which it converges, and demonstrate that it converges to the solution of (A.1). This will be accomplished by proving that f, ω, and all the iterates $\omega^{(N)}$, i.e. first N+1 terms of (A.4) are elements of a normed vector space which is mapped into itself by the operator kO. Further we shall show that for $|k|$ sufficiently small, this operator has norm less than unity. The convergence of the Neumann series, in this norm, then follows as does the uniqueness of the solution.

First we record some properties of the spherical harmonics and known expansions of the static Green's function which will prove useful.

Denote by $Y_n(\theta,\phi)$ an nth order spherical harmonic

$$Y_n(\theta,\phi) = \sum_{m=-n}^{n} P_{mn} P_n^m(\cos\theta)\, e^{im\phi} \tag{A.5}$$

and by $Y_n(\theta,\phi;\theta_1,\phi_1)$ a symmetric nth order spherical harmonic

$$Y_n(\theta,\phi;\theta_1,\phi_1) = \sum_{m=0}^{n} A_{mn} P_n^m(\cos\theta) P_n^m(\cos\theta_1) \cos m(\phi-\phi_1). \tag{A.6}$$

These functions enjoy the orthogonality property

$$\int_0^{\pi} d\theta \int_0^{2\pi} d\phi \sin\theta\, Y_m(\theta,\phi)\, Y_n(\theta,\phi;\theta_1,\phi_1) = 0\,, \qquad m \neq n$$

$$= Y_n(\theta_1,\phi_1)\,, \quad m = n\,. \tag{A.7}$$

Here it must be kept in mind that $Y_n(\theta,\phi)$ and $Y_n(\theta_1,\phi_1)$ occurring in (A.7) are not necessarily the same function but are elements of the same equivalence class. That is, they both may be written in the form (A.5) but the constant coefficients A_{mn} may differ. In what follows, it is often unnecessary to distinguish between spherical harmonics of the same order thus we denote them all with the same symbol. This should not be overlooked in any specific calculation of the coefficients where a more precise specification is required.

It is well known (e.g. Kellog, 1953, p. 143) that potential functions may be expanded in spherical harmonics. In particular the static Green's function for the surface S may be written

$$G_o(p, p_1) = -\frac{1}{4\pi R(p, p_1)} + \sum_{n=0}^{\infty} \frac{Y_n(\theta, \phi)}{r^{n+1}} , \qquad r \geqslant a \tag{A.8}$$

$$= -\frac{1}{4\pi R(p, p_1)} + \sum_{n=0}^{\infty} \frac{Y_n(\theta_1, \phi_1)}{r_1^{n+1}} , \qquad r_1 \geqslant a \tag{A.9}$$

$$= -\frac{1}{4\pi R(p, p_1)} + \sum_{n=0}^{\infty} \frac{Y_n(\theta, \phi; \theta_1, \phi_1)}{(rr_1)^{n+1}} , \qquad rr_1 \geqslant a \tag{A.10}$$

where the series are uniformly and absolutely convergent and may be differentiated or integrated any number of times with respect to r, θ, or ϕ; $a = c + \epsilon$, $\epsilon > 0$; and c is the radius of the sphere enclosing S. The reciprocity relation is explicitly exhibited. It is useful to note that the source term may also be expanded in spherical harmonics,

$$\frac{1}{R(p, p_1)} = \sum_{n=0}^{\infty} \frac{r_<^n}{r_>^{n+1}} P_n \left[\cos\theta \cos\theta_1 + \sin\theta \sin\theta_1 \cos(\phi - \phi_1)\right] \tag{A.11}$$

where $r_> = \max(r, r_1)$, $r_< = \min(r, r_1)$. Note that the expansion has the same convergence properties as the series in (A.8) - (A.10) provided $r \neq r_1$.

In addition to the orthogonality of spherical harmonics, it will be useful to define a related property.

Definition. A function $f_n(\theta, \phi)$ will be called a "pseudo-spherical harmonic of order n" if

$$\int_0^{\pi} d\theta \int_0^{2\pi} d\phi \sin\theta f_n(\theta,\phi) Y_m(\theta,\phi;\theta_1,\phi_1) = 0 \quad , \qquad m < n$$

$$= Y_m(\theta_1,\phi_1) \, , \qquad m \geqslant n \qquad \text{(A.12)}$$

With the understanding that zero may be considered a spherical harmonic of any order (all coefficients in (A.5) are zero) if follows that any spherical harmonic of order n is also a pseudo-spherical harmonic of order n.

Now we are in a position to define a particular function space in which we will establish the convergence of the iterations. Recalling that V is the volume exterior to the surface S and a is the radius of a sphere entirely containing S in its interior we define W as follows:

$W = \left\{ w \;\middle|\; \right.$

a) $w \in c^2(V)$

b) $w = \frac{1}{r} \sum_{n=0}^{\infty} \frac{f_n(\theta,\phi)}{r^n}$, $r \geqslant a$ and the series is uniformly and absolutely convergent, term by term differentiable, with respect to r, θ, or ϕ and the resulting series are uniformly and absolutely convergent.

c) $f_n(\theta,\phi)$ are pseudo-spherical harmonics, i.e., satisfy (A.12) $\left. \right\}$

(A.13)

Further we specify the following norm, implied both by the pointwise convergence of the series, (A.13b), and the fact that elements of W are twice differentiable everywhere in V,

$$\|w\| = \max_{p \in V} |w(p)| \ . \tag{A.14}$$

It is clear that much more could be said of W than that it is a linear normed vector space; however, rather than investigate this space in general, we confine our attention to those properties necessary for our present purpose. These are established in the following lemmas which are then used to prove the main result of the report.

Lemma 1: $f \in W$

Proof: We complete the definition of f given in (A.3) by restricting u^i to be either a plane wave or a point source not on the surface S. Then, since the surface is finite and $G_o(p_S, p_1)$ is infinitely differentiable with respect to coordinates of the point p_1, as long as this point does not lie on S, it follows that the order of integration and differentiation may be interchanged and $f \in c^2(V)$. In fact $f \in c^\infty(V)$. Actually Kellog (1953, p. 172) established that the potential due to a double layer, with twice differentiable moment, of which f is an example, is also continuously differentiable for p on S, i.e. $f \in c^1(\bar{V})$. Furthermore when $r \geqslant a$ we utilize (A.8) and (A.11) to obtain

$$G_o(p, p_S) = -\frac{1}{4\pi} \sum_{n=0}^{\infty} \frac{r_S^n}{r^{n+1}} P_n \left[\cos\theta\cos\theta_S + \sin\theta\sin\theta_S\cos(\phi-\phi_S)\right] + \sum_{n=0}^{\infty} \frac{Y_n(\theta, \phi)}{r^{n+1}} \tag{A.15}$$

or since P_n is an nth order spherical harmonic

$$G_o(p, p_S) = \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \sum_{m=-n}^{n} A_{nm}(p_S) P_n^m(\cos\theta) e^{im\phi} . \qquad (A.16)$$

This series converges uniformly as does the derived series; therefore, we may rewrite (A.3) as

$$f = -\sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \sum_{m=-n}^{n} P_n^m(\cos\theta) e^{im\phi} \int_S e^{-ikr_S} u^i(p_S) \frac{\partial}{\partial n} A_{nm}(p_S) dS \qquad (A.17)$$

which is again of the form

$$f = \sum_{n=0}^{\infty} \frac{Y_n(\theta, \phi)}{r^{n+1}} \qquad (A.18)$$

Hence conditions (A.13b, c) are satisfied as well as (A.13a) and the lemma is proven.

Lemma 2: If $w \in W$ then $O \circ w \in W$.

Proof: With the definition of O, eq. (A.2) we write

$$O \circ w = -2i \int_V dV \frac{G_o(p, p_V)}{r_V} \frac{\partial}{\partial r_V} r_V w(p_V) \qquad (A.19)$$

We separate the volume over which the integration is performed into an infinite volume, V_e, where $r_V \geq a$ and the expansion theorem (3.1) holds, and a finite volume, V_i where it does not. V_i thus is the volume interior to the sphere of a radius a and exterior to the surface S. Thus we define two functions

$$w_{\substack{e\\i}}(p) = -2i \int_{V_{\substack{e\\i}}} dV \frac{G_o(p, p_V)}{r_V} \frac{\partial}{\partial r_V} \left[r_V w(p_V) \right] . \qquad \text{(A.20)}$$

Clearly $w_e(p) + w_i(p) = O \circ w$ and if we can demonstrate that w_e and w_i are elements of W then, since the space is linear, it follows that $O \circ w$ is also in W.

Consider first the finite volume. $w_i(p)$ is the potential of a volume distribution which Kellog (1953) has shown to be twice differentiable, for finite volume, provided the density is piecewise continuous (p. 156). This is certainly satisfied in the present case since $w \in W$ which implies that the density $\frac{1}{r_V} \frac{\partial}{\partial r_V} \left[r_V w(p_1) \right]$ is continuously differentiable. Therefore, $w_i(p) \in c^2(V)$. When $r \geqslant a$ the expansion of the Green's function (A.16) is valid, with p_V replacing p_S. The uniform convergence of the expansion and the fact that the integration is carried out over finite limits permits interchange of order yielding

$$w_i(p) = \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \sum_{m=-n}^{n} P_n^m(\cos\theta)\, e^{im\phi} \int_{V_i} dV(-2i) \frac{A_{mn}(p_V)}{r_V} \frac{\partial}{\partial r_V} \left[r_V w(p_V) \right] \qquad r \geqslant a \qquad \text{(A.21)}$$

which is of the form

$$w_i(p) = \sum_{n=0}^{\infty} \frac{Y_n(\theta, \phi)}{r^{n+1}} , \qquad r \geqslant a \qquad \text{(A.22)}$$

hence

$$w_i(p) \in W . \qquad \text{(A.23)}$$

Turning now to $w_e(p)$ we see that if V_e is replaced by any large but finite volume then the fact that $w_e \in c^2(V)$ again follows from Kellog's work. It is only necessary to show that w_e remains well defined when V_e becomes infinite. Explicitly

$$w_e(p) = \lim_{\rho \to \infty} -2i \int_a^\rho dr_V \int_0^{2\pi} d\phi_V \int_0^\pi d\theta_V \, r_V^2 \sin\theta_V \, \frac{G_o(p, p_V)}{r_V} \frac{\partial}{\partial r_V} \left[r_V w(p_V) \right] \tag{A.24}$$

and it is sufficient to show that the integrand is $O(1/r_V^2)$ for large r_V. Since $w(p_V) \in W$ it follows that

$$w(p_V) = \sum_{n=0}^{\infty} \frac{f_n(\theta_V, \phi_V)}{r_V^{n+1}}, \qquad r_V \geqslant a \tag{A.25}$$

and therefore that

$$\frac{\partial}{\partial r_V} \left[r_V w(p_V) \right] = - \sum_{n=1}^{\infty} \frac{n f_n(\theta_V, \phi_V)}{r_V^{n+1}} . \tag{A.26}$$

Thus for large r_V, $\frac{\partial}{\partial r_V} \left[r_V w(p_V) \right] = O(1/r_V^2)$. Furthermore, the expansions of $G_o(p, p_1)$ given in (A.9) and (A.10) show that for r_V sufficiently large, $\frac{G_o(p, p_V)}{r_V} = O(1/r_V^2)$. Hence, despite the factor r_V^2 in the volume element, the integrand is indeed $O(1/r_V^2)$ and it makes sense to let the r_V integration extend to ∞. This calculation may be pursued more carefully to show that in addition, $w_e(p)$ satisfies the expansion properties required of elements of W. Thus we rewrite (A.24) for $r, r_V \geqslant a$ as

$$w_e(p) = \int_a^\infty dr_V \int_0^{2\pi} d\phi_V \int_0^\pi d\theta_V r_V \sin\theta_V \left\{ -\frac{1}{4\pi R(p, p_V)} + \sum_{m=0}^\infty \frac{Y_m(\theta,\phi;\theta_V,\phi_V)}{(rr_V)^{m+1}} \right\} \sum_{n=1}^\infty \frac{f_n(\theta_V,\phi_V)}{r_V^{n+1}} \tag{A.27}$$

where we have absorbed the factor 2in in the functions $f_n(\theta_V, \phi_V)$. Now consider separately the integrals involving the regular and singular parts of the static Green's function, treating the regular part, w_e^{reg}, first. In this case both series are uniformly convergent and the integral has been shown to exist, thus we may interchange order of integration and summation and perform the integration using the pseudo-orthogonality condition (A.12) to obtain

$$w_e^{reg}(p) = \int_a^\infty dr_V \int_0^{2\pi} d\phi_V \int_0^\pi d\theta_V \sin\theta_V \sum_{m=0}^\infty \sum_{n=1}^\infty \frac{Y_m(\theta,\phi;\theta_V,\phi_V) f_n(\theta_V,\phi_V)}{r^{m+1} r_V^{m+n+1}}$$

$$= \sum_{n=1}^\infty \sum_{m=n}^\infty \frac{Y_m(\theta,\phi)}{(m+n)a^{m+n} r^{m+1}} \, . \tag{A.28}$$

Absorbing the constant factors in the spherical harmonics and renaming the second summation index yields

$$w_e^{reg}(p) = \sum_{n=1}^\infty \sum_{m=0}^\infty \frac{Y_{m+n}(\theta,\phi)}{r^{m+n+1}} \, . \tag{A.29}$$

Using Cauchy's formula to rearrange terms, which is allowed since the convergence is absolute, we obtain

$$w_e^{reg}(p) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{Y_{n+1}(\theta, \phi)}{r^{n+2}} . \tag{A.30}$$

While the coefficients in $Y_{n+1}(\theta\ \phi)$ may depend on m, the summation over m is still a spherical harmonic of order n+1 hence (A.30) is of the form

$$w_e^{reg}(p) = \sum_{n=0}^{\infty} \frac{Y_{n+1}(\theta, \phi)}{r^{n+2}} . \tag{A.31}$$

The analysis involving the singular part of the Green's function is slightly more involved since the expansion of 1/R, (A.11), is not uniformly convergent at $r = r_1$. From (A.27) we see that

$$w_e^{sing}(p) = -\frac{1}{4\pi} \int_a^{\infty} dr_V \int_0^{2\pi} d\phi_V \int_0^{\pi} d\theta_V \frac{\sin\theta_V}{R(p, p_V)} \sum_{n=1}^{\infty} \frac{f_n(\theta_V, \phi_V)}{r_V^n} \tag{A.32}$$

Since the series occurring in (A.32) is uniformly convergent and the infinite integral has been shown to exist, we may interchange order of summation and integration, and absorb the factor $(-1/4\pi)$ into f_n, obtaining

$$w_e^{sing}(p) = \sum_{n=1}^{\infty} \int_a^{\infty} dr_V \int_0^{2\pi} d\phi_V \int_0^{\pi} d\theta_V \frac{\sin\theta_V}{R(p, p_V)} \frac{f_n(\theta_V, \phi_V)}{r_V^n} \tag{A.33}$$

Now we employ the expansion (A.11) to obtain

$$w_e^{sing}(p) = \sum_{n=1}^{\infty} \left\{ \int_a^r dr_V \int_0^{2\pi} d\phi_V \int_0^{\pi} d\theta_V \frac{\sin\theta_V}{r_V^n} f_n(\theta_V, \phi_V) \sum_{m=0}^{\infty} \frac{r_V^m}{r^{m+1}} Y_m(\theta, \phi, \theta_V, \phi_V) \right.$$

$$\left. + \int_r^{\infty} dr_V \int_0^{2\pi} d\phi_V \int_0^{\pi} d\theta_V \frac{\sin\theta_V}{r_V^n} f_n(\theta_V, \phi_V) \sum_{m=0}^{\infty} \frac{r^m}{r_V^{m+1}} Y_m(\theta, \phi, \theta_V, \phi_V) \right\} . \tag{A.34}$$

Although the inner summation is singular at $r = r_V$, $\theta = \theta_V$, $\phi .= \theta_V$, it is a straightforward matter to exclude a small neighborhood of (r, θ, ϕ) from the integral (in which case the interchange of summation and integration is legitimate)and then show that the integral over the excluded neighborhood may be made as small as we wish by making the neighborhood sufficiently small. Thus we find, again using the pseudo-orthogonality property (A.12)[+]

$$w_e^{sing}(p) = \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \left\{ \frac{Y_m(\theta, \phi)(r^{m-n+1} - a^{m-n+1})}{r^{m+1}(m - n + 1)} + \frac{Y_m(\theta, \phi)}{(n+m)\, r^n} \right\} \tag{A.35}$$

Again absorbing the constants in the spherical harmonics we obtain

$$w_e^{sing}(p) = \sum_{n=1}^{\infty} \frac{1}{r^n} \sum_{m=n}^{\infty} Y_m(\theta, \phi) + \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \frac{Y_m(\theta, \phi)}{r^{m+1}} \tag{A.36}$$

The second sum in (A.36) is of precisely the same form as (A.29) hence the same argument allows us to write

[+] The justification for requiring this apparently artificial restriction on the space W is found here since without this property, terms involving $\log r$ would occur.

$$w_e^{sing}(p) = \sum_{n=1}^{\infty} \frac{1}{r^n} \sum_{m=n}^{\infty} Y_m(\theta, \phi) + \sum_{n=0}^{\infty} \frac{Y_{n+1}(\theta, \phi)}{r^{n+2}} \tag{A.37}$$

and, with (A.31), we find that $w_e(p)$ is also of this form, i.e.,

$$w_e(p) = w_e^{reg}(p) + w_e^{sing}(p) = \sum_{n=1}^{\infty} \frac{1}{r^n} \sum_{m=n}^{\infty} Y_m(\theta, \phi) + \sum_{n=0}^{\infty} \frac{Y_{n+1}(\theta, \phi)}{r^{n+2}} \tag{A.38}$$

or by a trivial change of notation

$$w_e(p) = \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \sum_{m=n+1}^{\infty} Y_m(\theta, \phi) + \sum_{n=1}^{\infty} \frac{Y_n(\theta, \phi)}{r^{n+1}} \tag{A.39}$$

This is precisely the form required for $w_e(p)$ to be in W, i.e.,

$$w_e(p) = \sum_{n=0}^{\infty} \frac{f_n(\theta, \phi)}{r^{n+1}} , \qquad r \geqslant a \tag{A.40}$$

where

$$\left.\begin{aligned} f_o &= \sum_{m=1}^{\infty} Y_m(\theta, \phi) \\ f_n &= \sum_{m=n}^{\infty} Y_m(\theta, \phi), \qquad n \geqslant 1 \end{aligned}\right\} \tag{A.41}$$

Having shown that $w_e(p)$ satisfies (A.13b), it remains only to demonstrate that $f_n(\theta, \phi)$ defined in (A.41) are pseudo-spherical harmonics. This follows immediately from the orthogonality of spherical harmonics, (A.7), since

$$\int_0^\pi d\theta \int_0^{2\pi} d\phi\, f_n(\theta,\phi) Y_\ell(\theta,\phi;\theta_1,\phi_1)\sin\theta$$

$$= \int_0^\pi d\theta \int_0^{2\pi} d\phi \sum_{m=n}^{\infty} Y_m(\theta,\phi)\, Y_\ell(\theta,\phi;\theta_1,\phi_1)\sin\theta = 0, \qquad \ell < n \tag{A.42}$$

$$= Y_\ell(\theta_1,\phi_1), \qquad \ell \geqslant n$$

Therefore we may conclude that

$$w_e(p) \in W \tag{A.43}$$

which, with (A.23) proves the lemma.

Lemma 3: O is bounded, i.e., $\exists M^{real} < \infty \ni \|O\| \leqslant M$.

Proof: If $w \in W$ then $O \circ w \in W$ by lemma 2. Therefore $O \circ w$ is twice differentiable everywhere in V which guarantees that $O \circ w$ has no singularities in V. Thus $\exists M_w \ni$

$$|O \circ w| \leqslant M_w, \qquad p \in V \tag{A.44}$$

and, recalling the definition of the norm (A.14),

$$\|O \circ w\| \leqslant M_w . \tag{A.45}$$

Let $M = \sup_{w \in W,\ \|w\| = 1} M_w$. This must exist or else for some $w \in W$, $|O \circ w|$ would be unbounded which would contradict lemma 2, the fact that $O \circ w \in W$. Thus

$$\sup_{w \in W,\ \|w\| = 1} \|O \circ w\| \leqslant M. \tag{A.46}$$

The lemma then follows since, by definition,

$$\|O\| = \sup_{w \in W,\ \|w\| = 1} \|O \circ w\| \ . \tag{A.47}$$

Lemma 4: $\exists k_o \ni \|kO\| < 1$ if $|k| < |k_o|$.

Proof: Since O is linear it follows from the definition of norm that

$$\|kO\| = |k| \ \|O\| \ . \tag{A.48}$$

With lemma 3 we obtain

$$\|kO\| \leqslant |k| M \tag{A.49}$$

therefore by choosing $|k| < 1/M$ or equivalently, letting $|k_o| = 1/M$, the lemma follows.

Lemma 5: $\omega \in W$.

Proof: The definition of ω in terms of u^s, $\omega = e^{-ikr} u^s$, together with the fact that $u^s \in c^2(V)$ imply that $\omega \in c^2(V)$. Furthermore, the expansion theorem (3.1) guarantees that we may write

$$\omega(p) = \sum_{n=0}^{\infty} \frac{f_n(\theta, \phi)}{r^{n+1}} \ , \qquad r \geqslant a. \tag{A.50}$$

It remains only to demonstrate that these $f_n(\theta, \phi)$ are "pseudo-spherical harmonics". To accomplish this we employ the well known expansion of wave functions in spherical harmonics, e.g., Sommerfeld (1949, p. 143),

$$u^s(p) = \sum_{n=0}^{\infty} h_n(kr) \, Y_n(\theta, \phi) \qquad r \geqslant a \tag{A.51}$$

where $h_n(kr)$ are spherical Hankel functions of the first kind,

$$h_n(kr) = \frac{e^{ikr}\, i^{-n-1}}{r} \sum_{m=0}^{n} \frac{(n+m)!}{(n-m)!\, m!} \left(\frac{-1}{2ikr}\right)^m . \tag{A.52}$$

With this expression together with (A.51) we find that

$$\omega(p) = e^{-ikr} u^s(p) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{Y_n(\theta,\phi)\, i^{-n-1}}{r^{m+1}} \frac{(n+m)!}{(n-m)!\, m!} \frac{(-1)^m}{(2ik)^m} \tag{A.53}$$

or, upon rearranging terms and absorbing the multiplicative factors in the spherical harmonics,

$$\omega(p) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{Y_{n+m}(\theta,\phi)}{r^{m+1}} \tag{A.54}$$

This can be rewritten, with the obvious changes in notation so as to correspond to (A.50)

$$\omega(p) = \sum_{\ell=0}^{\infty} \frac{f_n(\theta,\phi)}{r^{n+1}}$$

where

$$f_n(\theta,\phi) = \sum_{\ell=0}^{\infty} Y_{n+\ell}(\theta,\phi) . \tag{A.55}$$

The functions $f_n(\theta,\phi)$ thus obviously satisfy the pseudo-orthogonality condition, (A.12) and the lemma is proven. Note that this proof essentially duplicates Sommerfeld's derivation of the expansion theorem but that, as is clear from the above, his statement (Sommerfeld, 1949, p. 191) that the $f_n(\theta,\phi)$ are <u>finite</u> sums of spherical harmonics is in error.

We now at last are in a position to prove that the Neumann series (A.4) converges to the solution we seek, and that this solution is unique.

First we show that the series converges to the solution, that is, for any $\epsilon > 0$, $\exists N_0 \ni \|\omega - \omega^{(N)}\| < \epsilon$ if $N > N_0$.

Lemmas 1, 2, and 5 establish that ω, f, and all the iterates $\omega^{(N)}$ are in the space W, hence it makes sense to write $\|\omega - \omega^{(N)}\|$ for any N. With (A.1) and the definition of the iterates it follows that

$$\omega - f = kO \circ \omega$$

$$\omega - \omega^{(1)} = \omega - kO \circ f - f = kO \circ (\omega - f) = k^2 O^2 \circ \omega$$

$$\omega - \omega^{(2)} = \omega - kO \circ \omega^{(1)} - f = kO \circ (\omega - \omega^{(1)}) = k^3 O^3 \circ \omega$$

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$$\omega - \omega^{(N)} = k^{N+1} O^{N+1} \circ \omega \tag{A.56}$$

hence

$$\|\omega - \omega^{(N)}\| < \|kO\|^{N+1} \|\omega\| . \tag{A.57}$$

But lemma 4 states that $\|kO\| < 1$ if $|k| < |k_0|$ and $\|\omega\|$ is bounded since $\omega \in W$ (lemma 5) hence it is always possible to find N large enough so that

$$\|kO\|^{N+1} \|\omega\| < \epsilon . \tag{A.58}$$

Specifically, since $\log \|kO\| < 0$, we find that (A.58) is satisfied if

$$N+1 > \frac{\log \frac{\epsilon}{\|\omega\|}}{\log \|kO\|} \quad . \tag{A.59}$$

We have thus established that for any $\epsilon > 0$, $\|\omega - \omega^{(N)}\| < \epsilon$ if $N > N_o$ and $|k| < |k_o|$ where N_o is the greatest integer in $\left[\frac{\log \frac{\epsilon}{\|\omega\|}}{\log \|kO\|} - 1\right]$ and k_o exists by lemma 4.

To prove uniqueness we assume that existence of two solutions of (A.1), ω_1 and ω_2 such that

$$\omega_{\substack{1\\2}} = kO \circ \omega_{\substack{1\\2}} + f \tag{A.60}$$

and

$$\|\omega_1 - \omega_2\| \neq 0 \; . \tag{A.61}$$

Then

$$\omega_1 - \omega_2 = kO \circ \omega_1 - kO \circ \omega_2 \tag{A.62}$$

and taking norms we obtain

$$\|\omega_1 - \omega_2\| \leqslant \|kO\| \; \|\omega_1 - \omega_2\| \quad . \tag{A.63}$$

By assumption $\|\omega_1 - \omega_2\| \neq 0$, hence we may divide obtaining

$$1 \leqslant \|kO\| \tag{A.64}$$

which violates lemma 4.