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THE UNIVERSITY OF MICHIGAN

COLLEGE OF ENGINEERING

DEPARTMENT OF ELECTRICAL ENGINEERING

Radiation Laboratory

FINAL REPORT - INVESTIGATION INTO THE THEORY OF SPHEROIDAL AND MATHIEU FUNCTIONS

by

Frederick B. Sleator

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FINAL REPORT ON AFOSR GRANT 62-265

The original purpose of the subject grant was to extend and edit the literature on spheroidal and Mathieu functions with a vew toward making them more convenient to use in the solution of the appropriate boundary value problems. In particular it was hoped that the inherent relations between these functions and the elliptic functions and integrals could be developed and exploited to give essentially new approximations in terms of extensively tabulated quantities. To this end several lines of investigation have been pursued, but so far with rather limited success. To begin with, the fact that the elliptic functions themselves are not eigenfunctions of any linear differential equations precludes the use of standard expansion theorems and techniques such as those provided by the Sturm-Liouville theory. However these functions are the basis of the usual representations of solutions of the Lamé equations for ellipsoidal harmonics and wave functions. Accordingly a fairly thorough investigation was made of the relations between the spheroidal differential equations and the various forms of the Lamé equations. It is known that the solutions of the former equations can be expanded in terms of those of the latter, but this does not appear attractive, since the Lamé functions are less accessible than the spherical functions, and the coefficients in these expansions may be just as difficult to obtain as those which express the spheroidal functions in terms of spherical ones. It has also been established elsewhere that the spheroidal functions are expressible, at least to a certain approximation, in terms of simpler functions whose arguments are elliptic integrals involving the original independent variables. To date the full implications of this have not been examined, but it is intended to pursue the matter further. Another incidental result of the above investigation is the observation that by proper adjustment of parameters, the spheroidal differential equation can be made to coincide with the Lamé equation out to terms of the order of the sixth power of the argument. This means that the angular function can be approximated to a fairly high degree in the region near the broadside plane by a Lamé function, which is a polynomial in an elliptic function of the independent variable. The utility of this form has not yet been determined.

In summary, then, the introduction of elliptic functions and integrals as a means of facilitating the treatment of spheroidal functions does not appear very promising at this point. There are other lines of approach to the overall objectives, however, some of which are currently being examined and appear more hopeful. One of these is the development of new explicit expressions for the expansion coefficients relating the spheroidal functions to the spherical (Bessel and Legendre) functions. These coefficients have heretofore been computed by means of a continued-fraction expression, which is laborious at best unless a large-scale machine is used, or, for small values of the parameter,

c = wave number x focal length, as power series in c which have a very limited range of convergence. A beginning in this direction is afforded by the solution of the spheroidal equation as a perturbation of the Legendre equation. The resulting expansion in Legendre functions yields explicit expressions for the desired coefficients which are in some sense equivalent to the power series expansions, but which, since they involve both separation constants explicitly, can be expected to have different convergence characteristics. The possibility that these are more favorable is also to be examined further.

Finally, since the greatest difficulties in handling the spheroidal functions occur in the range where c is large, it seems desirable to pursue the development of asymptotic forms which hold in this region. Some of these are well known, but there are still regions in the space of the several parameters which are not adequately covered by known forms, and the removal of these gaps is perhaps the most promising objective for future research in this field. The asymptotic theories of Langer and others are applicable here, and it is doubtful whether they have been fully exploited in connection with the spheroidal functions. An immediate example of this technique is afforded by the observation that a suitable transformation of the independent variable converts the spheroidal equation into Bessel's equation plus a remainder term which is inversely proportional to c^2. At a value of $c = 8.0$, the corresponding Bessel function approximation to a radial spheroidal function, when properly normalized on the axis ($\xi = 1$), agrees with the exact value out to at least four significant figures on a 10:1 prolate spheroid (ξ = 1.005), over a considerable range of the indices. By use of the standard perturbation-iteration forms, this approximation could be refined considerably. A similar treatment in terms of the Legendre equation might yield even more agreeable forms, and ultimately the requisite ones to cover the as yet unconquered regions.

The results to date on this contract have not been published as yet. Several of the developments described briefly above are currently being prepared for publication, however, and should be submitted before very long.

All the technical work under this grant has been performed by the undersigned under the direction of Dr. T. B. A. Senior and with the occasional consultation from Dr. R. E. Kleinman and others.

Respectfully submitted,

Frederick B. Sleator

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