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by

Frederick B. Sleator

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I. Introduction

The class of diffraction and radiation problems to which the spheroidal geometry is appropriate is extensive and of considerable interest from many standpoints, and considerable effort has been expended in this field over the past quarter century or more. Generally speaking, these problems yield readily to the standard methods of attack, to the extent that their solutions are expressible in terms of the appropriate sets of eigenfunctions of the separated wave equation. Unfortunately these spheroidal functions, though much more extensively tabulated now than a few years ago, still represent a serious obstacle in the attainment of quantitative results. The dependence of each function on four independent parameters or arguments indicates the immensity of the number of values which may be required, and the complete lack of recurrence relations and scarcity of other computation aids makes the calculation of these values extremely tedious for anything less than a large-scale facility. It has been the objective of this investigation, as well as an earlier one under AFOSR Grant 62-265, to develop and exploit new relations between the spheroidal functions and other better known functions which might reduce the labor and complication involved in obtaining new numerical values of the spheroidal functions and thus facilitate the quantitative solutions of new spheroidal diffraction problems or the extension of existing solutions into ranges not previously covered.

II. Discussion

In the earlier investigation it was intended to develop certain relations between spheroidal and elliptic functions with a view toward making use of the fairly comprehensive tables which exist for the latter. Although the results of this effort were largely negative, in that the relations found did not seem to be useful in general, the original ideas have yielded a relationship involving certain integrals of exponential type, which is apparently new and merits further consideration. This is derived briefly as follows.

In the standard literature on spheroidal functions[1], there appears the following expansion of the product of a radial and an angular function:

$$S_{mn}(c,\eta)\,R^{(1)}_{mn}(c,\xi) = \sum_{r=0,1}^{\infty}{}' \, i^{r+m-n}\, d^{mn}_{r}(c)\, P^{m}_{m+r}(\cos\theta)\, j_{m+r}(kr) \tag{1}$$

where the $d^{mn}_{r}(c)$ are the same coefficients which express a single spheroidal function in terms of a series of spherical ones, as in the expansion

$$S_{mn}(c,\eta) = \sum_{r=0,1}^{\infty}{}' \, d^{mn}_{r}(c)\, P^{m}_{m+r}(\eta)\ . \tag{2}$$

If eq. (2) is substituted in (1) and both sides are then multiplied by $P^{m}_{m+3}(\eta)\,d\eta$ and integrated between -1 and +1, the orthogonality of the Legendre functions leads to the form

$$\frac{2(2m+s)!}{(2m+2s+1)s!}\, d^{mn}_{s}(c)\, R^{(1)}_{mn}(c,\xi) = \sum_{r=0,1}^{\infty}{}' \, i^{r+m-n}\, d^{mn}_{r}(c) \int_{-1}^{1} P^{m}_{m+r}(\cos\theta(\eta)) \cdot$$

$$\cdot\, j_{m+r}(kr(\eta))\, P^{m}_{m+s}(\eta)\, d\eta \tag{3}$$

or, setting

$$\frac{2m+2s+1}{2}\,\frac{s!}{(2m+s)!}\cdot i^{r+m} \int_{-1}^{1} P^{m}_{m+r}(\cos\theta(\eta))\, P^{m}_{m+s}(\eta)\, j_{m+r}(kr(\eta))\, d\eta \equiv C^{m}_{rs}(c,\xi),$$

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[1] e.g. Flammer, <u>Spheroidal Wave Functions</u>, Stanford University Press, 1957.

$$i^n R^{(1)}_{mn}(c,\xi)\, d_s^{mn} = \sum_{r=0,1}^{\infty}{}' \, d_r^{mn}(c)\, C^m_{rs}(c,\xi)$$

Since the integration over η with ξ held constant is actually a line integral over an elliptic contour, the arguments of the spherical functions in the integrand are found from the geometry of the spherical and spheroidal coordinate systems to be

$$\cos\theta(\eta) = \frac{\xi\eta}{\sqrt{\xi^2+\eta^2-1}}, \qquad r(\eta) = \frac{a\sqrt{\xi^2-1}}{\sqrt{\xi^2-\eta^2}} .$$

Examination of the integrals which remain reveals that these are not elliptic but are reducible to combinations of elementary and exponential (i.e. sine or cosine) integrals. If the index s is allowed to range from 0 to ∞, there results an infinite system of linear homogeneous equations in the "unknowns" $d_r^{mn}(c)$, which can be written

$$\sum_{r=0,1}^{\infty}{}' \, d_r^{mn}(c) \left[C^m_{rs}(c,\xi) - \delta_{rs}\, i^n R^{(1)}_{mn}(c,\xi) \right] = 0, \quad s = 0, 1 \ldots \infty. \tag{4}$$

Existence of a solution to this system requires the vanishing of the determinant, i.e.

$$\det \left[C^m_{rs}(c,\xi) - \delta_{rs}\, i^n R^{(1)}_{mn}(c,\xi) \right] = 0,$$

and the quantity $i^n R^{(1)}_{mn}(c,\xi)$ is thus cast in the role of an eigenvalue of the infinite matrix $\left\{ C^m_{rs}(c,\xi) \right\}$, which could be determined by standard methods if the elements of the matrix were known. Once this eigenvalue and the requisite number of elements are known, the system (4) can presumably be solved (approximately) for some finite number of coefficients $d_r^{mn}(c)$.

Considering the intractable nature of the integrals in (3) and the difficulties inherent in the manipulation of infinite matrices, it is not clear whether or not the above development is of any immediate practical value. It appears, however, on the basis of the known properties of the spheroidal coefficients that it should be possible to construct a Hilbert space, the elements of which are vectors $\underline{d}^{mn}$ with components $d_r^{mn}(c)$, and if this is the case the considerable body of general theory on linear transformations in Hilbert space would be applicable to the matrix $\left\{C_{rs}^{m}\right\}$ and might yield results of interest.

The advantages to be gained by this line of attack appeared uncertain, and the task seemed large, and consequently it was not pursued further. Instead, the principal effort has been expended on a simpler and more direct method of obtaining approximations to the spheroidal functions, which we proceed to outline in general terms before giving the specific forms of interest.

Given a linear ordinary differential equation of the form

$$L_{\xi} U = \left[\lambda + p(\xi)\right] U \tag{5}$$

whose solutions $U(\xi, \lambda)$ are to be determined, suppose there is a simpler equation, which can be written

$$M_x V = \mu V, \tag{6}$$

whose solutions $V(x, \mu)$ are well known, and further, that we can find a function $\xi(x)$ such that

$$L_{\xi(x)} = M_x \, .$$

Now for some particular value ξ_o let $\lim_{\xi \to \xi_o} (\lambda + p(\xi)) \equiv \mu_o$. Substituting this and the function $\xi(x)$ in eq. (5) then gives an equation of the form

$$M_x U = \mu_o U + \epsilon(x) U \tag{7}$$

where, under reasonable assumptions of regularity, $\epsilon(x)$ is small in some neighborhood of $x_o = x(\xi_o)$, and in this neighborhood a solution U of eq. (7) must thus be approximated by a solution $V\big(x(\xi), \mu_o\big)$ of eq. (6) and in general an iteration scheme can be developed to extend the range or improve the approximation. The accuracy of the result of course depends intimately on the natures of the operators and functions involved and the region in which the forms are applied. Furthermore the utility is limited in that no information on the normalization is involved, i.e. the value of the unknown function is determinable only in the neighborhood of a point at which it is already known. Nonetheless the procedure seems to offer advantages in certain applications, some of which are described briefly here in terms of the spheroidal functions.

For the sake of definiteness we consider first the radial functions, for which the operator L_ξ in (5) has the form

$$L_\xi = \frac{d}{d\xi}\left[(\xi^2 - 1)\frac{d}{d\xi}\right] - \frac{m^2}{\xi^2 - 1}$$

and $p(\xi) = -c^2 \xi^2$.

If for the approximating equation (6) we take the ordinary Bessel equation, whose solutions are perhaps the most convenient set of functions we can hope to use, then the requisite transformation $\xi(x)$ is easily found to be

$$\xi = \frac{\alpha^2 + x^2}{\alpha^2 - x^2} \qquad \text{or} \qquad x = \alpha\sqrt{\frac{\xi - 1}{\xi + 1}}$$

where α is an arbitrary constant, and the transformed equation can be written explicitly as

$$\frac{d^2U_{mn}}{dx^2} + \frac{1}{x}\frac{dU_{mn}}{dx} + \left[\frac{4c^2\alpha^2(\alpha^2+x^2)^2}{(\alpha^2-x^2)^4} - \frac{m^2}{x^2} - \frac{4\alpha^2\lambda_{mn}}{(\alpha^2-x^2)^2}\right]U_{mn} = 0$$

For our immediate purposes we can abandon the constant α, which seems to be immaterial, and consider the case where $\xi_o = 1$. The quantity μ_o in eq. (7) is then simply $4(c^2-\lambda_{mn})$, and for the sake of clarity in the argument which follows, we incorporate this in the independent variable, i.e. set $\sigma \equiv 2x\sqrt{c^2-\lambda_{mn}}$ and write the transformed equation finally as

$$\frac{d^2U_{mn}}{d\sigma^2} + \frac{1}{\sigma}\frac{dU_{mn}}{d\sigma} + \left[1 - \frac{m^2}{\sigma^2} + \epsilon(\sigma)\right]U_{mn} = 0$$

with

$$\epsilon(\sigma) = \frac{6(c^2-\lambda_{mn})}{\left[4(c^2-\lambda_{mn})-\sigma^2\right]^4}\left[c^2\left[4(c^2-\lambda_{mn})+\sigma^2\right]^2 - \lambda_{mn}\left[4(c^2-\lambda_{mn})-\sigma^2\right]^2\right] - 1.$$

If $\epsilon(\sigma)$ is small, the function $U_{mn}(\sigma)$ should be approximated by a cylindrical function, i.e.

$$U_{mn}(\sigma) \cong a\,Z_m(\sigma) \tag{8}$$

where a is independent of σ, but σ now depends on c and λ_{mn} as well as x. To this approximation, then, the only dependence of U_{mn} on n is through σ. The accuracy of the approximation however depends on the magnitude of $\epsilon(\sigma)$, which of course depends on n, and this dependence must be examined. In terms of the variable x, it develops that

$$\epsilon(\sigma(x)) \equiv \frac{x^2}{(1-x^2)^4}\left[\frac{4}{1-\frac{\lambda_{mn}}{c^2}} + (2-x^2)(1-x^2)\right].$$

In general, if c is fixed, the eigenvalue λ_{mn} increases with n, so that for fixed c and $x \ll 1$, there will be an infinite range of values of n over which ϵ is uniformly bounded and of order x^2. Once the constant a and the eigenvalues λ_{mn} are determined, the approximation (8) should thus be useful for all n above a certain value.

Another possible application of the above forms is in the determination of the set of eigenvalues which make the radial functions vanish on a certain spheroidal surface, as required in the Sommerfeld-type solution of a scattering problem. Suppose that for given m and some x within the range of applicability of (8) one of the requisite eigenvalues, say λ_{mo} is known. Then if $\sigma_o = 2x\sqrt{c^2 - \lambda_{mo}}$, we can write

$$U_{mo} \cong a J_m(\sigma_o) + b N_m(\sigma_o) = 0$$

in which a and b, being independent of σ, cannot depend on n. Then

$$\frac{a}{b} = -\frac{N_m(\sigma_o)}{J_m(\sigma_o)}$$

and for any other n, we can write

$$aJ_m(\sigma_n) + b N_m(\sigma_n) = \frac{b}{J_m(\sigma_o)}\Big[J_m(\sigma_o) N_m(\sigma_n) - J_m(\sigma_n) N_m(\sigma_o)\Big] \cong 0$$

or

$$J_m(\sigma_o) N_m(\sigma_n) - J_m(\sigma_n) N_m(\sigma_o) \cong 0$$

which is a familiar equation, some roots of which are tabulated in the standard literature on Bessel functions. The nth eigenvalue λ_{mn} is thus given approximately in terms of the nth root of this equation by the form

$$\lambda_{mn} \simeq c^2 - \frac{\sigma_n^2 x}{4} .$$

It appears that it should be possible to obtain analogous approximations to the ordinary eigenvalues, i.e. the values of λ_{mn} for which the angular functions are finite at $\eta = \pm 1$. To date, however, this objective has not been attained, despite a fairly intensive effort. The analysis is complicated by the facts that at the finite singular points the functions vanish to order $m/2$, and at the origin the dependence on the parameter c disappears.

Perhaps the most useful application of the above forms will prove to be in the determination of values of the functions in neighborhoods of points at which they are already known, i.e. in the development of refined methods of interpolation and extrapolation. To illustrate some of these, we now consider the angular functions, whose transformed equation is written

$$\frac{d^2 S_{mn}}{d\rho^2} + \frac{1}{\rho}\frac{dS_{mn}}{d\rho} + \left[\lambda_{mn} \cdot \frac{4}{(\rho^2+1)^2} - c^2 \cdot \frac{4(\rho^2-1)^2}{(\rho^2+1)^4} - \frac{m^2}{\rho^2}\right] S_{mn} = 0,$$

with $\rho = \sqrt{\frac{1+\eta}{1-\eta}}$. In some neighborhood of an arbitrary point η_o, the angular function $S_{mn}(c,\eta)$ can be expressed approximately by the form

$$S_{mn}(c,\eta) \cong \alpha_{mn} J_m(\gamma_o^{mn}\rho) + \beta_{mn} N_m(\gamma_o^{mn}\rho) \tag{9}$$

where $\gamma_o^{mn} = (1-\eta_o)\sqrt{\lambda_{mn} - c^2\eta_o^2}$

and α_{mn}, β_{mn} are independent of η. If two values are given in the allowable range, e.g. values of S_{mn} at two points or values of S_{mn} and its derivative at one point, the constants α_{mn} and β_{mn} can be determined and the expression (9) used for any η within a certain neighborhood of the reference point or points, whose extent depends on these points and on the accuracy required. As an illustration of this procedure we compute an approximate value of $S_{oo}(c, \eta)$ for $c = 3.0$, $\eta = 0.5$, given the values of S_{oo} at $\eta_1 = \cos 50^o = .6428$, $\eta_2 = \cos 70^o = .3420$. On the basis of (9), we can write in general

$$S_{mn}(c, \eta_o) \cong \frac{1}{\Delta_{12}} \left[S_{mn}(c, \eta_1)\Delta_{02} - S_{mn}(c, \eta_2)\Delta_{01} \right]$$

where

$$\Delta_{ij} = J_m(\gamma_o^{mn} \rho_i) N_m(\gamma_o^{mn} \rho_j) - J_m(\gamma_o^{mn} \rho_j) N_m(\gamma_o^{mn} \rho_i).$$

Using the tabulated value of $\lambda_{oo}(3.0)$, $S_{oo}(3.0, .6428)$ and $S_{oo}(3.0, .3420)$ and the expression (10) for γ_o^{oo}, we obtain the value $S_{oo}(3.0, .5) \cong .7575$, which compares reasonably well with the tabulated value $S_{oo}(3.0, .5) = .7571$.

In the neighborhood of the origin, i.e. $|\eta| \ll 1$, the forms simplify somewhat. Since the parity of the angle functions is that of the quantity $m+n$, either the function or its derivative will vanish at $\eta = 0$, according as $m+n$ is odd or even, and we can thus write almost immediately for $|\eta| \ll 1$

$$S_{mn}(c, \eta) \cong \frac{\pi}{2} \sqrt{\lambda_{mn}}\, S_{mn}(c, 0) \left[J_m\left(\sqrt{\lambda_{mn}} \sqrt{\frac{1+\eta}{1-\eta}}\right) N_m'(\sqrt{\lambda_{mn}}) \right.$$

$$\left. - J_m'(\sqrt{\lambda_{mn}}) N_m\left(\sqrt{\lambda_{mn}} \sqrt{\frac{1+\eta}{1-\eta}}\right) \right] \qquad m+n \text{ even}$$

$$\cong -\frac{\pi}{2} S_{mn}'(c, 0) \left[J_m\left(\sqrt{\lambda_{mn}} \sqrt{\frac{1+\eta}{1-\eta}}\right) N_m(\sqrt{\lambda_{mn}}) \right.$$

$$\left. - J_m(\sqrt{\lambda_{mn}}) N_m\left(\sqrt{\lambda_{mn}} \sqrt{\frac{1+\eta}{1-\eta}}\right) \right] \qquad m+n \text{ odd.}$$

Various other particular applications of the above forms suggest themselves, some of which involve the zeros of functions such as the Δ_{ij}. A careful examination of the accuracy obtainable and the range of applicability of the methods should accompany investigation of these applications.

III. Conclusions

An adequate assessment of the forms and methods described in the preceding discussion will depend, of course, on some fairly voluminous computations, as well as some more detailed analytical work. Preferably these two lines of advance should be closely coordinated and the overall direction should be determined in accordance with the requirements of the unsolved or only partly solved physical problems. Although the present forms should give values sufficiently accurate for practical applications, it is clear that they are more suitable for filling in gaps in existing tables as the need arises than for large-scale production of new ones. One of the principal features of the Bessel function approximation is that it exhibits the dependence of the spheroidal functions on the wavelength-eccentricity parameter c and on the eigenvalue λ_{mn} more explicitly than the usual representations. Further exploitation of this feature should be of value in a number of circumstances.

Respectfully submitted

Frederick B. Sleator
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