INVESTIGATIONS ON EXCITATION AND PROPAGATION IN IONIZED MEDIA

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FOREWORD

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ABSTRACT

A study of the excitation and propagation of wave like disturbances in an ionized medium, such as the ionosphere, is made based on the linearized Euler's equation and Maxwell's equation. The local propagation constants of the basic modes of propagation are discussed. A computer program for the evaluation of these constants with given ionospheric properties is given. Methods of investigating the propagation of such waves in inhomogeneous and/or bounded media, such as ray tracing, invariant embedding, reflection and refraction, orthogonal expansion, and the use of a general matrix formulation are presented. A unified matrix-operator transform method for investigation of the excitation and propagation of disturbances in an ionized medium is proposed.

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EVALUATION

The subject report represents about 1 1/2 years effort on excitation and propagation of disturbances in the ionospheric plasma. Towards describing these phenomena, the contractor has applied matrix-operator techniques and developed generalized formulations applicable to a wide range of associated problems. However, particular problems though noted and discussed are not resolved. For instance, excitation is formulated for all possible souces but problems are solved only for extremely simplified media/source conditions.

The report significance is based on its scope rather than on detailed calculations. As indicated, the problem types are as broad as possible and the mathematical techniques powerful. However, the formulations are limited to linear macroscopic conditions. The significance of the contractor's results is that while little new physics is introduced in this report, the mathematical bases for several important problem classes have been developed.

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I

INTRODUCTION

This report presents the results of an investigation of the excitation and propagation of wave like disturbances in the ionosphere. Recognizing that the ionosphere is a complicated physical system such that the eventual solution of the problem has to rely heavily on approximate, numerical calculations, the primary efforts of our investigation were devoted to the choice of a model, the investigation of techniques that may yield the solution to the problem directly, and the solution of some idealized problems which may allow some insight into the validity of a particular approach or which may yield results useful to the eventual solution of the complete problem.

The model of the ionosphere is taken as a three-fluid plasma containing ions, electrons and neutral particles whose density, temperature and average mass are known. The problem is formulated mathematically by linearizing the coupled hydrodynamic and Maxwell equations. In order to clarify the approximations involved in the use of these equations, they are deduced from the Maxwell-Boltzmann equation in Chapter II.

The basic modes of propagation in the three-fluid plasma are discussed in Chapter III. It is found that if all the coupling effects are included, the propagation constants in a homogeneous medium satisfy a tenth order algebraic equation. The basic properties of these waves are introduced through the study of degenerate cases. Based on collected published data on the ionosphere (Appendix A), propagation constants were evaluated by use of a digital computer, for several discrete angular frequencies and directions of propagation, corresponding to an altitude of 100Km. The results of the computations are presented in Section 3.3. Due to a lack of time and funds, similar calculations were not made at other altitudes. The computer program used in the calculations is discussed in Appendix B.

A generalized operator transform approach was formulated as a unified approach to the problem of the excitation and propagation of waves in ionized media. This method, valid in principle for both homogeneous and inhomogeneous media, reduces the mathematical problem to that of solving a generalized integral equation in transform space. For homogeneous media, analytical solutions are possible. Some new results on the solution of the excitation problem in a homogeneous electron plasma have been presented by Wu (1965) in a separate report. A general discussion of the application of this approach to excitation problems in inhomogeneous media is given in Section 7.3.

The excitation problem in a bounded two-fluid plasma is considered in Section VI, using the conventional methods of orthogonal expansions. The formal solution of the fields excited in the plasma by a dipole source above a conducting plane and sphere is obtained in a form suitable for numerical integration.

Based on the assumption that the solution to the excitation problem in a homogeneous medium may be used near the source even in an inhomogeneous medium, the effect of the inhomogeneity on wave propagation is investigated in Section VII. Conventional ray theory, the technique of invariant embedding and the reflection and refraction of waves at a discontinuity, is formulated for an electron plasma. A novel approach, using Kron's method of large system analysis, is investigated in Appendix C. All of these methods, in principle, may be extended to more complicated problems. However, the feasibility of obtaining numerical solutions by these methods remains to be tested.

II

THE EQUATIONS OF MOTION

2.1 The Boltzmann Equation

Basically, the study of the excitation and propagation of disturbances in the ionosphere can be formulated in terms of the kinetics of a partially ionized gas mixture. From the microscopic point of view, therefore, one may start from the well known Boltzmann's equation with a source term, which is

$$\frac{\partial}{\partial t} f_j(\underline{r}, \underline{u}, t) + \underline{u} \cdot \frac{\partial}{\partial \underline{r}} f_j + \frac{\underline{X}_j}{m_j} \cdot \frac{\partial}{\partial \underline{u}} f_j = \sum_k I(f_j, \bar{f}_k) + S_j(\underline{r}, \underline{u}, t) \tag{2.1}$$

The terms in (2.1) are defined as follows:

$f_j(\underline{r}, \underline{u}, t)du^3$	is the average number of particles of the j^{th} component per unit volume in the velocity range $(\underline{u}, \underline{u}+d\underline{u})$;
m_j	is the mass per particle of the j^{th} component;
$\underline{X}_j$	is the force (external and long range) exerted on each particle of the j^{th} component;
$I(f_j, \bar{f}_k)$	is the rate of change of f_j due to short range interaction with the k^{th} component, and is known as the collision term;
$S_j(\underline{r}, \underline{u}, t)du^3$	is the source per unit volume per second of particles of j^{th} component externally injected into the mixture in the velocity range $(\underline{u}, \underline{u}+d\underline{u})$.

The long range forces of interaction between charged particles may be expressed in terms of the electric and magnetic field in the medium. Thus, the term $\underline{X}_j$ may be written as

$$\underline{X}_j = \underline{x}_j + q_j(\underline{E} + \underline{u} \times \underline{B}) \tag{2.2}$$

where

$\underline{x}_j$	is the external mechanical force;
q_j	is the charge per particle of the j^{th} component;
$\underline{E}$, and $\underline{B}$	are the macroscopic electric and magnetic field* respectively.

On the other hand, a detailed knowledge of the short range interaction, especially in an ionized media, is not known. Simple models for this interaction (or collision), are generally postulated in the investigation of Boltzmann's equation.

The solution of (2.1), even without the source, the external force and the collision term, is extremely difficult, and has been the subject of a great deal of research which, to date, has yielded only approximate solutions to some idealized problems. Since the pertinent macroscopic properties of the mixture, such as temperature, density, average velocity, etc., are all contained in the first three moments of Eq. (2.1), the present work is confined to the investigation of the macroscopic equations (magneto-hydrodynamic equations) governing the laws of mass, momentum, and energy conservation. Mathematically, these

*We assume here that the effective electric and magnetic fields acting on each particle are essentially the macroscopic field, see Ginzburg (1961), p. 28.

equations, which are interpreted as the first three moments of Eq. (2.1), do not represent a complete description of the system. In general, approximate relations between the higher moments and the first three moments are necessary. These relations are derived either by using perturbation theory (Chapman and Cowlings, 1939), an approximate distribution function (Everett, 1962), or variational methods (Marshall, 1957)*. The results obtained by these methods invariably introduce some macroscopic constants (or parameters) into the system of hydro-dynamic equations. For a complicated mixture, such as the ionosphere, some of these "constants" have never been determined experimentally or calculated theoretically with certainty. For an engineering approach, we shall, in this work, simplify these relations so that the "constants" involved in the resulting macroscopic equations are reasonably well known either from experimental measurement or approximate theoretical considerations.

2.2 The Macroscopic Equations

The macroscopic equations governing the average motion of each component in an ionized gas mixture can either be considered approximately as the first three moments of (2.1), or deduced directly from the application of the macroscopic conservation laws of mass, momentum, and energy. These laws, and some explanations concerning them, are given below.

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*Due to the vast amount of literature on this subject, a complete list of references cannot be given in this report and thus only representative references are listed.

(i) The conservation of mass and charge. If the mass density of each component in a mixture is ρ_j, and the macroscopic average of its velocity is $\underline{V}_j$, one then has

$$\frac{\partial}{\partial t} \rho_j + \nabla \cdot (\rho_j \underline{V}_j) = Q_j \tag{2.3}$$

where Q_j is the material source input. The charge density of each component is given by

$$\sigma_j = \frac{\rho_j q_j}{m_j} \tag{2.4}$$

and the current density is given by

$$\underline{I}_j = \sigma_j \underline{V}_j \tag{2.5}$$

Thus, the conservation of mass implies the conservation of charge.

(ii) The momentum equation. For each component of a mixture, we may define a pressure tensor to account for the transport of the fluctuation part of momentum. This pressure tensor may be written as

$$\underline{\underline{P}}_j = p_j \underline{\underline{I}} - \underline{\underline{\tau}}_j \tag{2.6}$$

where p_j is the partial pressure of the j^{th} component, $\underline{\underline{\tau}}_j$ is the traceless pressure tensor, and $\underline{\underline{I}}$ is the identity tensor. Similarly, the kinetic temperature T_j of each component can be defined in terms of the partial pressure by

$$p_j = \frac{\rho_j}{m_j} k T_j \tag{2.7}$$

where k is Boltzmann's constant. Due to short range interactions (collisions) between different components, the momentum of each component is not conserved. The usual momentum equation for each component is given by

$$\frac{\partial}{\partial t}(\rho_j \underline{V}_j) + \nabla \cdot \left[\rho_j \underline{V}_j \underline{V}_j + p_j \underline{\underline{I}} - \underline{\underline{\tau}}_j\right] = \sigma_j \underline{E} + \underline{I}_j \times \underline{B} + \underline{F}_j - \sum_i k_{ji}(\underline{V}_j - \underline{V}_i) \quad (2.8)$$

where $\underline{F}_j$ is the external mechanical force per unit volume acting on the j^{th} component, and $k_{ji}(\underline{V}_j - \underline{V}_i)$ is the effective force per unit volume acting on the j^{th} component due to short range interactions (collisions) with the i^{th} component. The precise calculation of the coefficients k_{ji} requires a detailed knowledge of the mechanism of short range interactions. Approximately, it can be shown that

$$k_{ji} = k_{ij} \cong \frac{2m_i m_j}{m_i + m_j} \times \text{(number of binary collisions in unit volume per sec. between particles of the two components)} \quad (2.9)$$

(iii) The energy equation. Due to collisions between particles of different components, the energy of each component in a mixture is not conserved. The transfer of energy due to a collision, again depends on the mechanism of the collision, which, of course, is unknown. Assuming a simple model of collisions, such as an inverse fifth power interaction, it can be shown that the energy equation for each component of the mixture is

$$\frac{\partial}{\partial t}\rho_j\left\{\frac{1}{2}V_j^2+U_j\right\}+\nabla\cdot\left[\rho_j(\frac{1}{2}V_j^2+U_j)\underline{V}_j+p_j\underline{V}_j-\underline{V}_j\cdot\underline{\underline{\tau}}_j+\underline{Q}_j\right]=E\cdot\underline{I}_j+\underline{F}_j\cdot\underline{V}_j+h_j$$

$$-\sum_i \nu_{ji}\left\{\left(\frac{m_j\underline{V}_j+m_i\underline{V}_i}{m_i+m_j}\right)\cdot(\underline{V}_j-\underline{V}_i)+\frac{3k}{m_i+m_j}(T_j-T_i)\right\} \qquad (2.10)$$

where $\underline{Q}_j$ is the conventional heat flux, h_j is the heat input to the j^{th} component, and U_j is the internal energy per unit mass of the j^{th} component.

Although the transfer of energy due to collisions given by the last term of (2.10) was derived from a very simple model, its physical implication is quite clear: collisions between particles of different components tend to equalize the average velocity of the components as well as the random energy (the term involving the temperature difference).

Besides these conservation laws, obtained from the mechanical point of view, the long range interaction between particles expressed in terms of the macroscopic electric field and magnetic field must also satisfy Maxwell's field equation. These are

$$\nabla\cdot(\epsilon_0\underline{E})=\sum_j\sigma_j+\sigma_s \qquad (2.11)$$

$$\nabla\cdot\underline{B}=0 \qquad (2.12)$$

$$\nabla\times\underline{E}-\frac{\partial\underline{B}}{\partial t}=0 \qquad (2.13)$$

$$\nabla \times \underline{B} - \mu_o \epsilon_o \frac{\partial E}{\partial t} = \sum_i \underline{I}_i + \underline{I}_s \qquad (2.14)$$

where μ_o and ϵ_o are, respectively, the free space permeability and permittivity $\left(\frac{1}{\sqrt{\mu_o \epsilon_o}} = c, \right.$ the velocity of light), σ_s, and $\underline{I}_s$ are the external sources of charge density and current density.

The solution of the equations obtained from the conservation laws together with Maxwell's equations, even with some known relations between $\underline{\underline{\tau}}$, $\underline{Q}$, ν_{jk} and the other macroscopic variables, is too complicated to yield any immediate result. Therefore, some approximations, neglecting secondary effects but retaining the basic features of primary physical importance, are necessary.

From the system point of view, the system of equations may be considered as several subsystems coupled together, i.e. the systems for the motion of each component, and the electromagnetic system. The motion of each component is coupled through (i) the collision terms and (ii) the electric and magnetic influence via Maxwell's equations. If this coupling is neglected, each component may be treated as a simple gas. For a disturbance in a simple gas, the conventional theory of acoustic waves applies. In acoustic theory we assume that, to a first order of approximation, the entropy remains constant and the acoustic velocity is given by

$$\left. \frac{\partial p}{\partial \rho} \right| \text{constant entropy} .$$

Based on the success of using linear theory and the constant entropy approximation in the investigation of acoustic disturbances, even in gas mixtures, our present investigation shall be concerned primarily with the effect of the coupling terms on the acoustic disturbances. Approximate equations that degenerate to the acoustic equations when coupling is neglected are given in the next section.

2.3 The Perturbation Equations

For the investigation of disturbances in the ionosphere, we shall simplify the macroscopic equations deduced above by using approximations consistent with our present day knowledge of the ionosphere. Therefore, the following idealizations are made in this investigation:

a) The ionosphere is assumed to contain three distinct components: electrons, ions (singly charged) and neutrals. The average mass per particle of each component is designated by m_e, m_i, and m_n, respectively.

b) When undisturbed, the average temperature and number density are known as a function of altitude. The temperatures of the components are denoted by T_e, T_i, and T_n, respectively, while the number densities are denoted by N_e, N_i, and N_n, respectively. In general, local electrical neutrality may be assumed, so that

$$N_e = N_i = N_o \quad . \tag{2.15}$$

The collision coefficients, k_{ei}, k_{en}, and k_{in}, given by Eq. (2.9) can therefore be estimated approximately.

c) The disturbances are assumed to be weak, so that linearized equations apply. The perturbation of the electron, ion, and neutral density is denoted by n_e, n_i, and n_n, respectively.

d) The thermal and viscous effects are considered to be of secondary importance, so that the stress tensor and heat flux can be neglected, and for each component j ,

$$\nabla \cdot p_j \cong m_j U_j^2 \nabla n_j \tag{2.16}$$

where U_j may be regarded as the local acoustic velocity of the j^{th} component.

e) The perturbed electric field is denoted by $\underline{E}$, and the perturbed magnetic field by $\mu_o \underline{h}$. Due to the presence of the earth's magnetic field, the total magnetic field is given by

$$\underline{B} = \underline{B}_o + \mu_o \underline{h} = B_o \hat{b} + \mu_o \underline{h} \tag{2.17}$$

where B_o is the magnitude, and $\hat{b}$ is the unit vector indicating the direction of the earth's magnetic field.

Based on the above assumptions, the equations in Section 2.2 may be linearized by neglecting the second order terms. Taking the Fourier transform (with respect to time) of the resulting linearized equations, the following equations governing the perturbed motion of electrons, ions, and neutral particles are obtained.

<u>Electron Motion</u>

$$- i\omega \underline{V}_e + \frac{U_e^2}{N_o} \nabla n_e = - \frac{e}{m_e} \left[\underline{E} + \underline{V}_e \times \underline{B}_o \right] - \frac{k_{ei}}{N_o m_e} (\underline{V}_e - \underline{V}_i) - \frac{k_{en}}{N_o m_e} (\underline{V}_e - \underline{V}_n) + \frac{\underline{F}_e}{N_o m_e} \tag{2.18}$$

$$-i\omega n_e + \nabla \cdot (N_o \underline{V}_e) = Q_e \tag{2.19}$$

where e is the electron charge (numerical value).

Ion Motion

$$-i\omega \underline{V}_i + \frac{U_i^2}{N_o} \nabla n_i = \frac{e}{m_i}\left[\underline{E} + \underline{V}_i \times \underline{B}_o\right]$$

$$- \frac{k_{ei}}{N_o m_i}(\underline{V}_e - \underline{V}_i) - \frac{k_{in}}{N_o m_i}(\underline{V}_i - \underline{V}_n) + \underline{F}_i \tag{2.20}$$

$$-i\omega n_i + \nabla \cdot (N_o \underline{V}_i) = Q_i \tag{2.21}$$

Neutral Particle Motion

$$-i\omega \underline{V}_n + \frac{U_n^2}{N_n} \nabla n_n = - \frac{k_{ni}}{N_n m_n}(\underline{V}_n - \underline{V}_i) - \frac{k_{ne}}{N_n m_n}(\underline{V}_n - \underline{V}_e) + \underline{F}_n \tag{2.22}$$

$$-i\omega n_n + \nabla \cdot (N_n \underline{V}_n) = Q_n \tag{2.23}$$

Maxwell's Equation

$$\nabla \times \underline{E} - i\omega\mu_o \underline{h} = -\underline{K} \tag{2.24}$$

$$\nabla \times \underline{h} + i\omega\epsilon_o \underline{E} = eN_o(\underline{V}_i - \underline{V}_e) + \underline{I}_s \tag{2.25}$$

where $\underline{I}_s$ is a current source, and where, for completeness, a magnetic current source $\underline{K}$ has been introduced.

The set of Eqs. (2.18) through (2.25) represents a coupled system of 18 first order partial differential equations relating the 18 field quantities (3 components each of $\underline{V}_i$, $\underline{V}_e$, $\underline{V}_n$, $\underline{E}$, $\underline{h}$, and the three scalar n_e, n_i, n_n) to the sources. The solution of such a system, especially in an inhomogeneous (stratified) medium such as the ionosphere is extremely complicated. Therefore, our investigation of this set of equations is limited to:

a) The investigation of the characteristics of the propagation of the disturbances in a locally homogeneous medium.

b) The development of techniques that may enable one to solve such large scale systems numerically or approximately.

c) The investigation of some idealized problems. For example, in some regions of the ionosphere, if we neglect collisions and ion motion, a relatively simple set of equations involving only the motion of electrons is obtained. These are given by

$$\nabla \times \underline{E} - i\omega\mu_o \underline{h} = -\underline{K} \tag{2.26}$$

$$\nabla \times \underline{h} + i\omega\epsilon_o \underline{E} + eN_o \underline{V} = \underline{I}_s \tag{2.27}$$

$$-i\omega m N_o \underline{V} + mU^2 \nabla n + eN_o \left[\underline{E} + \underline{V} \times \underline{B}_o\right] = \underline{F} \tag{2.28}$$

$$N_o \nabla \cdot \underline{V} + \nabla \cdot (N_o \underline{V}) - i\omega n = Q \tag{2.29}$$

where the subscript e has been deleted from all variables since no confusion would result. The excitation of disturbances in an electron plasma, based on certain limiting cases of the above equations, has been discussed by many authors but the most comprehensive investigation has been carried out by Y. K. Wu under this contract and the results presented in an earlier report (Wu, 1965).

d) Some idealized problems involving the excitation of disturbances in a bounded plasma.

The result of these investigations will be discussed in detail in the remainder of this report.

III

PROPAGATION CONSTANTS

3.1 The Dispersion Relation

In principle, in a homogeneous medium, the set of Eqs. (2.18) through (2.24) may be solved by the usual Fourier transform technique. Basically, for each function of the space variables $\underline{r}$, we may introduce the Fourier transform defined by

$$\tilde{f}(\underline{s}) = \frac{1}{(2\pi)^3} \int e^{-i\underline{s}\cdot\underline{r}} f(\underline{r})\, d\underline{r} \tag{3.1}$$

The transform of the set of Eqs. (2.18) through (2.24) yields a set of linear algebraic equations for the transformed field quantities, which can be solved by inverting the matrix formed by the coefficients of the algebraic equations (functions of $\underline{s}$). To carry out the inverse transform for the field components, it is then necessary to find the roots of the determinant of this matrix. This relation is known as the dispersion relation. The truth of this statement can be seen from the following discussion. Instead of applying a Fourier transform to Eqs. (2.18) through (2.24), assume that the components of the vectors and the three scalar quantities vary as $We^{i\underline{s}\cdot\underline{r}}$ where W is independent of r and represent the magnitude of V_{ex}, etc. This is the usual assumption of plane wave propagation where $\underline{s}$ is the propagation vector given by

$$\underline{s} = s\hat{s} \tag{3.2}$$

and $\underline{r}$ is a position vector in space. Making this substitution into Eqs. (2.18) through (2.24), assuming no sources, a system of linear homogeneous equations are obtained. In order that a non-trivial solution exist for this set it is necessary that the determinant of the coefficients be zero. This result yields the propagation vector $\underline{s}$ as a function of frequency and, therefore, is the dispersion relation. It is obvious that the determinant obtained in this way is identical to that obtained from the matrix resulting from the application of the Fourier transform.

Explicitly, if each field variable such as $\underline{V}_e(\underline{r})$ is replaced by $\underline{V}_e(\underline{r})e^{i\underline{s}\cdot\underline{r}}$ the following equations are obtained*.

$$\underline{s} \times \underline{E} = \omega\mu_o\underline{h} \tag{3.3}$$

$$\underline{s} \times \underline{h} = -\omega\epsilon_o\underline{E} - ieN_o(\underline{V}_i - \underline{V}_e) \tag{3.4}$$

$$n_e = \frac{N_o}{\omega}\,\underline{s}\cdot\underline{V}_e \tag{3.5}$$

$$n_i = \frac{N_o}{\omega}\,\underline{s}\cdot\underline{V}_i \tag{3.6}$$

$$n_n = \frac{N_n}{\omega}\,\underline{s}\cdot\underline{V}_n \tag{3.7}$$

$$\underline{V}_e(1+i\nu_{ei}+i\nu_{en}) = -i\frac{e}{m_e}\frac{\underline{E}}{\omega} - i\Omega_e\underline{V}_e \times \hat{b} + \underline{s}\frac{\underline{U}_e^2}{\underline{N}_o}\frac{n_e}{\omega} + i\nu_{ei}\underline{V}_i + i\nu_{en}\underline{V}_n \tag{3.8}$$

---

*No confusion should result if $\underline{V}_e$ is used to denote $\underline{V}_e(\underline{s})$ etc.

$$\underline{V}_i(1+i\nu_{ie}+\nu_{in}) = +i\frac{e}{m_i}\frac{\underline{\underline{E}}}{\omega}+i\Omega\underline{V}_i\times\hat{b}+\underline{s}\frac{U_i^2}{N_o}\frac{n_i}{\omega}+i\nu_{ie}\underline{V}_e+i\nu_{in}\underline{V}_n \tag{3.9}$$

$$\underline{V}_n(1+i\nu_{ni}+i\nu_{ne}) = \underline{s}\frac{U_n^2}{N_o}\frac{n_n}{\omega}+i\nu_{ne}\underline{V}_e+i\nu_{ni}\underline{V}_i \tag{3.10}$$

For simplicity, the normalized collision frequency ratio ν_{mn} which accounts for the loss of energy of particles of m^{th} kind due to collisions with those of the n^{th} kind given by

$$\nu_{mn} = \frac{k_{mn}}{\omega \times \text{density of } m^{th} \text{ component}}$$

has been used in Eqs. (3.3) through (3.10).

To obtain the dispersion relation from the above systems, we shall, without loss of generality, choose

$$\underline{s} = \hat{z}\, s$$

and

$$\underline{B}_o = (\hat{z}\cos\theta + \hat{y}\sin\theta)\, B_o$$

where θ is the angle between the direction of propagation and the d.c. magnetic field. By expanding the vector equations and quantities in (3.1) through (3.10) into components, the dispersion relation may be given in the following determinantal form:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = 0 \tag{3.11}$$

where the sub-determinants (minors) A_{ij} are given by the following equations.

$$
\left[A_{11}\right] = \begin{bmatrix}
(\beta_o^2 - s^2)(1 + i\nu_{ei} + i\nu_{en}) - \frac{1}{\beta_e^2}(\beta_o^2 - s^2)s^2 \sin^2\theta - \omega_e^2(\beta_o^2 - s^2 \sin^2\theta) & i\Omega_e(\beta_o^2 - s^2) & -\frac{1}{\beta_e^2}(\beta_o^2 - s^2)s^2 \sin\theta\cos\theta + \omega_e^2 s^2 \sin\theta\cos\theta \\
-i\Omega_e(\beta_o^2 - s^2) & (\beta_o^2 - s^2)(1 + i\nu_{ei} + i\nu_{en}) - \omega_e^2\beta_o^2 & 0 \\
-\frac{1}{\beta_e^2}(\beta_o^2 - s^2)s^2 \sin\theta\cos\theta + \omega_e^2 s^2 \sin\theta\cos\theta & 0 & (\beta_o^2 - s^2)(1 + i\nu_{ei} + i\nu_{en}) - \frac{1}{\beta_e^2}(\beta_o^2 - s^2)s^2 \cos^2\theta - \omega_e^2(\beta_o^2 - s^2\cos^2\theta)
\end{bmatrix}
\tag{3.12}
$$

$$
\left[A_{12}\right] = \left[\begin{array}{c|c|c}
\begin{matrix} -i(\beta_o^2 - s^2)\nu_{ei} \\ +\omega_e^2\left[\beta_o^2 - s^2 \sin^2\theta\right] \end{matrix} & 0 & -\omega_e^2 s^2 \sin\theta \cos\theta \\
\hline
0 & \begin{matrix} -i(\beta_o^2 - s^2)\nu_{ei} \\ +\omega_e^2 \beta_o^2 \end{matrix} & 0 \\
\hline
-\omega_e^2 s^2 \sin\theta \cos\theta & 0 & \begin{matrix} -i(\beta_o^2 - s^2)\nu_{ei} \\ +\omega_e^2(\beta_o^2 - s^2 \cos^2\theta) \end{matrix}
\end{array}\right] \tag{3.13}
$$

$$
\left[A_{13}\right] = \left[\begin{array}{c|c|c}
-i(\beta_o^2 - s^2)\nu_{en} & 0 & 0 \\
\hline
0 & -i(\beta_o^2 - s^2)\nu_{en} & 0 \\
\hline
0 & 0 & -i(\beta_o^2 - s^2)\nu_{en}
\end{array}\right] \tag{3.14}
$$

$$
\left[A_{21}\right] = \left[\begin{array}{c|c|c}
\begin{matrix} -i(\beta_o^2 - s^2)\nu_{ie} \\ +\omega_i^2(\beta_o^2 - s^2 \sin^2\theta) \end{matrix} & 0 & -\omega_i^2 s^2 \cos\theta \sin\theta \\
\hline
0 & \begin{matrix} -i(\beta_o^2 - s^2)\nu_{ie} \\ +\omega_i^2 \beta_o^2 \end{matrix} & 0 \\
\hline
-\omega_i^2 s^2 \sin^2\theta \cos^2\theta & 0 & \begin{matrix} -i(\beta_o^2 - s^2)\nu_{ie} \\ +\omega_i^2(\beta_o^2 - s^2 \cos^2\theta) \end{matrix}
\end{array}\right] \tag{3.15}
$$

$$
\left[A_{22}\right] = \left[\begin{array}{c|c|c}
\begin{gathered}(\beta_o^2-s^2)(1+i\nu_{ie}+i\nu_{in}) \\ -\frac{1}{\beta_i^2}(\beta_o^2-s^2)s^2\sin^2\theta \\ -\omega_i^2(\beta_o^2-s^2\sin^2\theta)\end{gathered} & -i\,\Omega_i(\beta_o^2-s^2) & \begin{gathered}-\frac{1}{\beta_i^2}(\beta_o^2-s^2)s^2\sin\theta\cos\theta \\ +\omega_i^2 s^2\sin\theta\cos\theta\end{gathered} \\
\hline
+i\,\Omega_i(\beta_o^2-s^2) & \begin{gathered}(\beta_o^2-s^2)(1+i\nu_{ie}+i\nu_{in}) \\ -\omega_i^2\,\beta_o^2\end{gathered} & 0 \\
\hline
\begin{gathered}-\frac{1}{\beta_i^2}(\beta_o^2-s^2)s^2\sin\theta\cos\theta \\ +\omega_i^2 s^2\sin\theta\cos\theta\end{gathered} & 0 & \begin{gathered}(\beta_o^2-s^2)(1+i\nu_{ie}+i\nu_{in}) \\ -\frac{1}{\beta_i^2}(\beta_o^2-s^2)s^2\cos^2\theta \\ -\omega_i^2(\beta_o^2-s^2\cos^2\theta)\end{gathered}
\end{array}\right]
\tag{3.16}
$$

$$
\left[A_{23}\right] = \left[\begin{array}{c|c|c}
-(\beta_o^2-s^2)\nu_{in} & 0 & 0 \\
\hline
0 & -(\beta_o^2-s^2)\nu_{in} & 0 \\
\hline
0 & 0 & -(\beta_o^2-s^2)\nu_{in}
\end{array}\right]
\tag{3.17}
$$

$$
[A_{33}] \quad \left[\begin{array}{c|c|c}
1+i\nu_{ni}+i\nu_{ne} - \frac{1}{\beta_n^2} s^2 \sin^2\theta & 0 & -\frac{1}{\beta_n^2} s^2 \sin\theta\cos\theta \\
\hline
0 & 1+i\nu_{ni}+i\nu_{ne} & 0 \\
\hline
-\frac{1}{\beta_n^2} s^2 \sin\theta\cos\theta & 0 & 1+i\nu_{ni}+i\nu_{ne} - \frac{1}{\beta_n^2} s^2 \cos^2\theta
\end{array}\right]
$$

(3.18)

$$
[A_{31}] = \left[\begin{array}{c|c|c}
-i\nu_{ne} & 0 & 0 \\
\hline
0 & -i\nu_{ne} & 0 \\
\hline
0 & 0 & -i\nu_{ne}
\end{array}\right]
$$

(3.19)

$$
[A_{32}] = \left[\begin{array}{c|c|c}
-i\nu_{ni} & 0 & 0 \\
\hline
0 & -i\nu_{ni} & 0 \\
\hline
0 & 0 & -i\nu_{ni}
\end{array}\right]
$$

(3.20)

For simplicity we have also introduced the following normalized quantities in Eqs. (4.12) through (4.20).

$$\beta_o \triangleq \omega \sqrt{\epsilon_o \mu_o} = \frac{\omega}{c}$$ (free space propagation constant)

$$\beta_{e,i,n} \triangleq \frac{\omega}{V_{e,i,n}}$$ (propagation constant of respective acoustic waves)

$$\omega_{e,i} = \frac{e^2 N_o}{\omega \epsilon_o m_{e,i}}$$ (normalized plasma frequency)

$$\Omega_{e,i} = \frac{eB_o}{\omega m_{e,i}}$$ (normalized gyro-frequency)

Equation (3.11), when developed, is a polynomial of 5th order in s^2, indicating the existence of five basic waves (modes). These modes may be interpreted as the result of collisional and other forms of coupling between the following basic types of waves: neutral acoustic, electron acoustic, ion acoustic, ordinary and extraordinary electromagnetic waves. In order to learn more about the effect of "coupling" on these five basic waves, the characteristics of wave propagation in the subsystem will be studied first. This, of course, implies the investigation of the roots of the subdeterminants of Eq. (3.11).

3.2 The Basic Modes of Propagation

Basically, there are five natural modes of propagation in the model of the ionosphere considered, corresponding to the five basic energy storage mechanisms. The dispersion relation, therefore, yields a polynomial of 5th degree in s^2, corresponding to the five basic types of waves (or modes of propagation). Due to the collisional coupling, and the collective action of charged components through the electrostatic and magnetic interaction, identification of each particular mode of propagation is somewhat ambiguous. In order to understand more clearly the roots of the dispersion relation, some general characteristics of the basic modes of propagation are discussed in this section.

Since the thermal energy exchange was neglected in the present first order analysis, the effect of collisions would be to cause attenuation of the waves due to the transfer of ordered energy into random energy. When the appropriate collision terms are small, $\nu \ll 1$, the effect of collisions on the phase velocity of propagation is small and the nature of the basic waves (or modes) of propagation can be determined by neglecting collisions. The effect of collisions can then be estimated by the following consideration. A particle of the i^{th} kind, with ordered motion, will lose some of its energy (ordered motion) when it collides with a different type of particle. This energy loss represents the maximum attenuation due to collisions since some of the ordered energy transferred to the second particle will eventually be returned (fedback) to a particle of the i^{th} kind by means of a series of collisions with other particles. Therefore, the energy lost in the initial collision can be considered as the attenuation due to collisions and the feedback of energy as the coupling effect of collisions.

As a result of the above discussion, the nature of the waves (modes) will be studied by first neglecting collisions and then the attenuation estimated by considering only the attenuation due to collisions.

a. Neutral Acoustic Waves

If collisions are neglected, the motion of neutrals can be considered separately. The equations describing the motion of neutrals are:

$$n_n = \frac{N_n}{\omega} \underline{s} \cdot \underline{V}_n \tag{3.7}$$

$$\underline{V}_n = \underline{s} \frac{U_n^2}{N_o} \frac{n_n}{\omega} = \frac{\underline{s}\, s}{\beta_n^2} \cdot \underline{V}_n \tag{3.10}$$

These equations yield the basic characteristics of the acoustic type waves, namely they are longitudinal waves in the sense that the velocity is in the direction of propagation. The propagation constant of the acoustic wave is given by

$$s = \beta_n = \frac{\omega}{U_n} \tag{3.21}$$

and is consistent with the definition of acoustic velocity. Collisional coupling would modify the above conclusion somewhat. To estimate the attenuation of the acoustic wave, we may modify (3.21) to

$$\begin{aligned} s &= \beta_n \sqrt{\left[1 + i\nu_{ni} + i\nu_{ne}\right]} \\ &= \frac{\omega}{U_n} \sqrt{1 + i\,\frac{k_{ni} + k_{ne}}{\omega N_n m_n}} \end{aligned} \tag{3.22}$$

Equation (3.22) gives the limiting value of the attenuation constant for acoustic waves at high frequency which is

$$\alpha \cong \frac{1}{2U_n} \frac{k_{ni} + k_{ne}}{N_n m_n} \tag{3.23}$$

b. Electron Plasma (Acoustic) Waves

The acoustic type of wave for electron motion is not as simple as that for neutrals.

Since the motion of charged particles necessarily causes current and bunching of charges, the mechanical equation of motion of the electrons should be considered together with Maxwell's equations. If we neglect collisions and the d. c. magnetic field, and assume that the motion of the ions is negligible due to their heavy mass, the coupled equations to be considered are

$$\underline{s} \times \underline{E} = \omega\mu_o \underline{h} \tag{3.3}$$

$$\underline{s} \times \underline{h} = -\omega\epsilon_o \underline{E} + ieN_o \underline{V}_e \tag{3.4'}$$

$$n_e = \frac{N_o}{\omega} \underline{s} \cdot \underline{V}_e \tag{3.5}$$

$$\underline{V}_e = -i \frac{e}{m_e} \frac{\underline{E}}{\omega} - \frac{\underline{s}}{N_o} \frac{n_e}{\omega} \tag{3.7'}$$

The above equations may be used to eliminate $\underline{h}$ and n_e, yielding

$$\left[1 - \frac{s^2}{\beta_o^2} + \frac{\underline{ss}}{\beta_o^2}\right] \underline{E} = \frac{ieN_o}{\omega\epsilon_o} \underline{V}_e \tag{3.24}$$

and

$$\left[1 - \frac{\underline{\underline{ss}}}{\beta_e^2}\right] \underline{V}_e = -\frac{ie}{m_e\omega} \underline{E} \tag{3.25}$$

The nature of the wave may be made clear by separating $\underline{E}$, and $\underline{V}_e$ into components in the direction of propagation, (the longitudinal components, E_L, V_{eL}) and the components transverse to the direction of propagation, E_T and V_{eT}. In terms of these components, we have

$$E_L = \frac{ieN_o}{\omega \epsilon_o} V_{eL} \tag{3.26}$$

$$(1 - \frac{s^2}{\beta_o^2}) E_T = \frac{ieN_o}{\omega \epsilon_o} V_{eT} \tag{3.27}$$

$$(1 - \frac{s^2}{\beta_e^2}) V_{eL} = - \frac{ie}{m_e \omega} E_L \tag{3.28}$$

$$V_{eT} = - \frac{ie}{m_e \omega} E_T \tag{3.29}$$

From Eqs. (3.26) and (3.28) we find that the longitudinal components of $\underline{V}_e$ and $\underline{E}$ are associated with the propagation constant,

$$s = \beta_e \sqrt{1 - \omega_e^2} \tag{3.30}$$

which has the characteristics of an acoustic wave, and is generally called the electron acoustic (plasma) wave.

On the other hand, from Eq. (3.27) and (3.29), we see that the transverse components are associated with the propagation constant

$$s = \beta_o \sqrt{1-\omega_e^2} \quad . \tag{3.31}$$

This transverse wave can be identified as the electromagnetic wave.

One may also estimate the attenuation of these waves by considering the attenuation effect of collisions and neglecting the coupling effect. For the electron acoustic wave, we have

$$s = \beta_e \sqrt{1-\omega_e^2 + i\nu_{ei} + i\nu_{en}} \quad . \tag{3.32}$$

For the electromagnetic wave, we have,

$$s = \beta_o^2 \sqrt{1-\frac{\omega_e^2}{1+i\nu_{ei}+i\nu_{en}}} \quad . \tag{3.33}$$

c. Ion Plasma (acoustic) Wave

When the ion motion is not negligible, such as in the case of a low frequency, collionless plasma, the coupled motion of electrons and ions is described by the following equations.

$$\left[1-\frac{s^2}{\beta_o^2}+\frac{\underline{s}\,\underline{s}_o}{\beta_o^2}\right]\underline{E} = -\frac{ieN_o}{\omega\epsilon_o}(\underline{V}_i-\underline{V}_e) \tag{3.34}$$

$$\left[1-\frac{\underline{s}\,\underline{s}_o}{\beta_e^2}\right]\underline{V}_e = -\frac{ie}{m_e\omega}\underline{E} \tag{3.35}$$

$$\left[1 - \frac{\underline{s} \cdot \underline{s}_o}{\beta_i^2}\right] \underline{V}_i = + \frac{ie}{m_i \omega} \underline{E} \tag{3.36}$$

Again, the longitudinal components are associated with acoustic types of waves. The propagation constants for these waves are determined by the equation

$$s^4 - \left[\beta_e^2(1-\omega_e^2) + \beta_i^2(1-\omega_i^2)\right] s^2 + \beta_e^2 \beta_i^2 (1-\omega_e^2-\omega_i^2) = 0 \tag{3.37}$$

The roots of this equation are given by

$$s^2 = \frac{1}{2}\left[\beta_e^2(1-\omega_e^2) + \beta_i^2(1-\omega_i^2) \pm \sqrt{\left[\beta_e^2(1-\omega_e^2) - \beta_i^2(1-\omega_i^2)\right]^2 + 4\beta_e^2\beta_i^2\omega_e^2\omega_i^2}\right. \tag{3.38}$$

Equation (3.38) shows that the two waves may be considered as the result of the coupling, by electric forces, of an ion acoustic wave and an electron acoustic wave and, in general, cannot be separated into an ion and electron acoustic wave. However, at high frequencies where the coupling is weak $(\omega_i^2 < \omega_e^2 \ll 1)$ the roots are approximately

$$s = \beta_i \sqrt{1-\omega_i^2} \tag{3.39}$$

and

$$s = \beta_e \sqrt{1-\omega_e^2} \tag{3.40}$$

where (3.39) is determined by the positive sign in (3.38) and may be identified as the ion acoustic wave and (3.40) is given by the negative sign and may be identified as the electron acoustic wave.

The effect of collisions at high frequencies can be estimated by neglecting the coupling effect of collision only and (3.39) and (3.40) become

$$s = \beta_i \sqrt{1-\omega_i^2+i\nu_{ie}+i\nu_{in}} \tag{3.41}$$

and

$$s = \beta_e \sqrt{1-\omega_e^2+i\nu_{ei}+i\nu_{en}} \tag{3.42}$$

At lower frequencies where the coupling effects are large the two waves, of course, cannot be identified as an ion or electron wave; however, since $\beta_i >> \beta_e$ we can say that an ion type of acoustic wave is given by the positive sign in Eq. (3.38) and an electron type of acoustic wave by the negative sign.

As the coupling effects become larger, the motion of the electrons and ions become nearly equal and only a single wave propagates. This point can be determined from (3.37) and is

$$\omega_e^2+\omega_i^2 = 1$$

When $\omega_e^2+\omega_i^2 < 1$ two waves propagate while for $\omega_e^2+\omega_i^2 > 1$ only one wave can propagate. Similarly, the transverse component of the velocity and electric field is associated with the electromagnetic wave. This electromagnetic wave has a propagation constant given by

$$s = \beta_o \sqrt{1-\omega_e^2-\omega_i^2} \tag{3.43}$$

d. Electromagnetic Waves

The transverse characteristics of electromagnetic waves are modified due to the presence of the d. c. magnetic field. Inspection of Eq. (3. 3) reveals that the perturbed magnetic field is always transverse to the direction of propagation, while the electric field, in general, has a component in the direction of propagation. The presence of a d. c. magnetic field causes the electromagnetic field to split into two distinct modes. Based on the propagation characteristics of these waves, they are referred to as ordinary and extraordinary waves. However, in the analysis of a warm plasma problem, these waves are coupled together and, in general, lose their distinguishing characteristics.

An appropriate terminology for these waves has been given by Allis et al (1963) and discussed in detail by Wu (1965). This terminology is based on the characteristics of propagation of these waves when the direction of propagation is at a fixed angle with respect to the d. c. magnetic field. In this section, the well-known characteristics of these waves will be deduced in order to clarify the terminology used to describe them.*

The basic characteristics of electromagnetic waves in a plasma under the influence of d. c. magnetic field can be brought out by considering the waves in a cold, collisionless plasma. The equations to be solved are therefore,

$$\left(1-\frac{\underline{s}^2}{\beta_o^2}+\frac{\underline{s}\,\underline{s}_o}{\beta_o^2}\right)\underline{\underline{E}} = \frac{ieN_o}{\omega\epsilon_o}\underline{V}_e - \frac{ieN_o}{\omega\epsilon_o}\underline{V}_i \tag{3.44}$$

$$\underline{V}_e = -\frac{ie}{m_e}\frac{\underline{\underline{E}}}{\omega} - i\Omega_e\underline{V}_e \times \hat{b} \tag{3.45}$$

---

*This deduction follows the procedure of Stix (1962).

and

$$\underline{V}_i = +\frac{ie}{m_e}\frac{\underline{E}}{\omega} + i\Omega_i \underline{V}_i x \hat{b} \qquad (3.46)$$

Since the d. c. magnetic field tends to force the charged particles to have a circular motion in a plane perpendicular to the direction of the d. c. magnetic field, it is convenient to refer all directed quantities to the direction of the d. c. magnetic field. Therefore, without loss of generality, let

$$\hat{b} = \hat{z}$$

and

$$\underline{s} = s(\hat{z}\cos\theta - \hat{y}\sin\theta).$$

In terms of the components of $\underline{E}$, $\underline{V}_e$ and $\underline{V}_i$ Eqs. (3.44), (3.45) and (3.46) become

$$(1-\frac{s^2}{\beta_o^2})E_x = -\frac{ieN_o}{\omega\epsilon_o}V_{ix} + \frac{ieN_o}{\omega\epsilon_o}V_{ex} \qquad (3.47)$$

$$(1-\frac{s^2}{\beta_o^2}\sin^2\theta)E_z - \frac{s^2}{\beta_o^2}\sin\theta\cos\theta\, E_y = -\frac{ieN_o}{\omega\epsilon_o}V_{iz} + \frac{ieN_o}{\omega\epsilon_o}V_{ez} \qquad (3.48)$$

$$-\frac{s^2}{\beta_o^2}\sin\theta\cos\theta\, E_z + (1-\frac{s^2}{\beta_o^2}\cos^2\theta)E_y = -\frac{ieN_o}{\omega\epsilon_o}V_{iy} + \frac{ieN_o}{\omega\epsilon_o}V_{ey} \qquad (3.49)$$

$$V_{ez} = -\frac{ie}{m_e\omega}E_z \tag{3.50}$$

$$V_{ex} = -\frac{ie}{m_e}\frac{E_x}{\omega} - i\Omega_e V_{ey} \tag{3.51}$$

$$V_{ey} = -\frac{ie}{m_e}\frac{E_y}{\omega} + i\Omega_e V_{ex} \tag{3.52}$$

The equations for the ion velocities are similar to those given in Eqs. (3.50) to (3.52), and may be obtained from them by replacing e and Ω_e by -e and Ω_i, respectively.

Although the dispersion relations can be obtained directly by forming the determinants of the above equations, the basic characteristics of the waves can be made clear if we consider the component equations separately. Eliminating the z-component of velocity yields a relation between E_z and E_y:

$$(P - n^2 \sin^2\theta) E_z = n^2 \sin\theta \cos\theta \, E_y \tag{3.53}$$

where, for simplicity, we let

$$P \triangleq (1 - \omega_e^2 - \omega_i^2) \tag{3.54}$$

and

$$n^2 = \frac{s^2}{\beta_o^2} \tag{3.55}$$

Similarly, the transverse components of velocity may be expressed in terms of the transverse components of the electric fields by use of Eqs. (3.51) and (3.52). This is accomplished in simple manner by forming the right hand rotating and left hand rotating field components such as $E_x + iE_y$, $E_x - iE_y$, etc. Using the rotating field components in (3.51) and (3.52) we obtain

$$(1-\Omega_e)(V_{ex}+iV_{ey}) = \frac{ie}{m_e\omega}(E_x+iE_y) \tag{3.56}$$

$$(1-\Omega_e)(V_{ex}-iV_{ey}) = \frac{ie}{m_e\omega}(E_x-iE_y) \tag{3.57}$$

and similar equations for the ions. By substituting these results into (3.47) and (3.40) the following equations are obtained.

$$\left(\frac{R+L}{2} - n^2\right)E_x = iE_y\left(\frac{R-L}{2}\right) \tag{3.58}$$

and

$$-n^2\cos\theta\sin\theta\,E_z + \left(\frac{R+L}{2} - n^2\cos^2\theta\right)E_y = -i\,\frac{R-L}{2}\,E_x \tag{3.59}$$

where

$$R = 1 - \frac{\omega_e^2}{1-\Omega_e} - \frac{\omega_i^2}{1+\Omega_i}$$

$$L = 1 - \frac{\omega_e^2}{1+\Omega_e} - \frac{\omega_i^2}{1-\Omega_i}$$

From the determinant formed from the coefficients of Eqs. (3.53), (3.58) and (3.59) the equation for s, or n is given by

$$n^4(P\cos^2\theta+\frac{R+L}{2}\sin^2\theta)-n^2\left[P\frac{(R+L)}{2}(H\cos^2\theta)+RL\sin^2\theta\right]+RPL=0 \qquad (3.60)$$

Equation (3.60) indicates the anisotropic behavior of the propagation (direction dependence of propagation constant) of the electromagnetic waves, where as the acoustic type of waves discussed previously are isotropic. The directional characteristics, and the identification of these waves can be simply obtained by expressing Eq. (3.60) in the form

$$\tan^2\theta=\frac{P(n^2-R)(n^2-L)}{\frac{R+L}{2}(n^2-\frac{2RL}{R+L})(n^2-P)} \qquad (3.61)$$

Thus, in the direction $\theta=0$, the two waves have the following features.

a) $n^2=R$,

$$s=\beta_o\sqrt{1-\frac{\omega_e^2}{1-\Omega_e}-\frac{\omega_i^2}{1+\Omega_i}} \qquad (3.62)$$

and from Eq. (3.58)

$$E_x=-iE_y$$

which is the characteristic of a right hand circularly polarized wave, and

b) $n^2=L$,

$$s=\beta_o\sqrt{1-\frac{\omega_e^2}{1+\Omega_e}-\frac{\omega_i^2}{1+\Omega_i}} \qquad (3.63)$$

which corresponds to a left hand circularly polarized wave.

In the direction $\theta = 90^o$, (propagation transverse to the direction of the magnetic field) we have

a) $n^2 = P$

$$s = \beta_o \sqrt{1 - \omega_e^2 - \omega_i^2} \tag{3.64}$$

which is a wave uneffected by the magnetic field and, therefore, is generally referred to as the ordinary wave, and

b) $n^2 = \frac{2RL}{R+L}$

$$s = \beta_o \sqrt{\frac{\left(1 - \frac{\omega_e^2}{1-\Omega_e} - \frac{\omega_i^2}{1+\Omega_i}\right)\left(1 - \frac{\omega_e^2}{1+\Omega_e} - \frac{\omega_i^2}{1-\Omega_i}\right)}{\left(1 - \frac{\omega_e^2}{1-\Omega_e^2} - \frac{\omega_i^2}{1-\Omega_i^2}\right.}} \tag{3.65}$$

which is generally referred to as the extraordinary wave.

In general, for $0 < \theta < \frac{\pi}{2}$ the two waves are coupled and lose their distinguishing characteristics. However, they can be distinguished mathematically by the limiting condition as $\theta \longrightarrow 90^o$. Waves whose propagation constants are obtained from the same branch of the solution of Eq. (3.61) as the ordinary wave, (3.64) will be called the modified ordinary waves and the waves whose propagation constant is given by the same branch as the extraordinary wave, (3.65), will be called modified extraordinary waves.

The dispersion relation, given in the form of Eq. (3.61) can also be utilized to identify the regions where each of the waves can exist along a particular direction (or are heavily attenuated). Since $\tan^2\theta$ is positive along each mathematical branch of the solution, n^2 is a continuous function of θ, and waves are heavily attenuated if $n^2 < 0$. One may, according to the values of P, R, and L, investigate the possibility of the existence of unattenuated propagation by using the well-known CMA diagram. For the case of a cold plasma, such a diagram is given by Stix (1962). The application of such a diagram to the case of warm plasma was given by Wu (1965).

e. Waves in a Warm Plasma

The analysis of the dispersion relation pertaining to the propagation of waves in a warm plasma, where the effects of thermal velocities are considered, is generally quite complicated, even for the collisionless case. For an electron plasma, the determinantal equation was laboriously developed and discussed by Wu (1965). A simplified approach to this problem, based on the discussion given in the previous section, is to investigate the proportionality constant only in the directions $\theta = 0$ and $\theta = 90^o$. If the values of n^2 in these directions are known, say α_1, α_2 and α_3 for $\theta = 0$ and β_1, β_2 and β_3 for $\theta = 90^o$, the dispersion relation may be easily written in the form

$$\tan^2\theta = \text{constant}\ \frac{(n^2-\alpha_1)(n^2-\alpha_2)(n^2-\alpha_3)}{(n^2-\beta_1)(n^2-\beta_2)(n^2-\beta_3)} \qquad (3.66)$$

where the constant term can be obtained easily from the determinantal equation by a limiting process. For example, consider the case of a collisionless electron plasma. The set of Maxwell's are given by Eqs. (3.47), (3.48) and (3.49) while the equations of motion for the electrons are obtained by adding an acoustic velocity term to Eqs. (3.50), (3.51) and (3.52). The complete set of equations are

$$(1-n^2)E_x = \frac{ieN_o}{\omega\epsilon_o} V_{ex} \tag{3.67}$$

$$(1-n^2\sin^2\theta)E_z - n^2\sin\theta\cos\theta E_y = \frac{ieN_o}{\omega\epsilon_o} V_{ez} \tag{3.68}$$

$$-n^2\sin\theta\cos\theta E_z + (1-n^2\cos^2\theta)E_y = \frac{ieN_o}{\omega\epsilon_o} V_{ey} \tag{3.69}$$

$$(1-\frac{s^2}{\beta_e^2}\cos^2\theta)V_{ez} = -\frac{ie}{m_e}\frac{E_z}{\omega} - \frac{s^2}{\beta_o^2}\sin\theta\cos\theta V_{ey} \tag{3.70}$$

$$V_{ex} = -\frac{ie}{m_e}\frac{E_x}{\omega} - i\Omega_e V_{ey} \tag{3.71}$$

$$(1-\frac{s^2}{\beta_e^2}\sin^2\theta)V_{ey} = -\frac{ie}{m_e}\frac{E_y}{\omega} - \frac{s^2}{\beta_e^2}\sin\theta\cos\theta V_{ez} \tag{3.72}$$

For $\theta = 0$ we have from (3.68)

$$(1-n^2\frac{U_e^2}{\Delta^2})V_{ez} = -\frac{ie}{m_e}\frac{E_z}{\omega} \tag{3.73}$$

While the transverse components of the velocity are identical to those obtained for the case of a cold plasma and are given by Eqs. (3.64) and (3.65). The z-component of the system of equations is decoupled and yields one propagation constant as

$$s = \beta_e\sqrt{1-\omega_e^2} \tag{3.74}$$

which is easily identified as the electron plasma wave. The same procedure used to eliminate V_{ex} and V_{ey} for the cold plasma can be applied to the remaining equations yielding the propagation constants

$$s = \beta_o\sqrt{R} = \beta_o\sqrt{1-\frac{\omega_e^2}{1-\Omega_e}} \tag{3.75}$$

and

$$s = \beta_o\sqrt{L} = \beta_o\sqrt{1-\frac{\omega_e^2}{1+\Omega_e}} \tag{3.76}$$

which are easily identified as electromagnetic waves.

For $\theta = 90^o$ the components of the equation again decouple yielding the propagation constant

$$s = \beta_o \sqrt{1-\omega_e^2} \tag{3.31}$$

which is identified as the ordinary electromagnetic wave. For the transverse field, E_x and E_y can be eliminated yielding

$$V_{ex}\left(1-\frac{\omega_e^2}{1-\frac{s^2}{\beta_o^2}}\right) + i\Omega_e V_{ey} = 0 \tag{3.77}$$

$$-i\Omega_e V_{ex} + \left[1-\omega_e^2 - \frac{s^2}{\beta_e^2}\right] V_{ey} = 0 . \tag{3.78}$$

A second order equation in s^2 can be obtained from (3. 77) and (3. 78) and is

$$s^4 - s^2\left[(\beta_e^2+\beta_o^2)(1-\omega_e^2) - \Omega_e^2\beta_e^2\right] + \beta_o^2\beta_e^2\left[(1-\omega_e^2)^2 - \Omega_e^2\right] = 0 \tag{3.79}$$

The roots of (3. 79) can be investigated easily and correspond to waves obtained due to the coupling between the extraordinary electromagnetic wave and the electron acoustic wave.

The dispersion relation can then be obtained from the propagation constants and Eq. (3.66) and is

$$\tan^2\theta = - \frac{\left[n^2 - \frac{\beta_e^2}{\beta_o^2}(1-\omega_e^2)\right]\left[n^2 - R\right]\left[n^2 - L\right]}{\left[n^2 - P\right]\left[n^4 - n^2(1-\Omega_e^2) - \frac{\beta_e^2}{\beta_o^2}\left(\frac{R+L}{2} n^2 - RL\right)\right]} \quad . \tag{3.80}$$

The constant in (3.66) was evaluated by comparing the limit of (3.66) as $\beta_e \longrightarrow \infty$ to the cold plasma equation. A detailed discussion of Eq. (3.80) was given by Wu (1965) in a previous report.

The same approach, when applied to a warm plasma, considering both electron and ion motion, does not yield such a simple result. For the sake of completeness, the result is given below.

$$\tan^2\theta = - \frac{(s^2-\beta_o^2 L)(s^2-\beta_o^2 R)\left[s^4-\left[\beta_i^2(1-\omega_e^2)+\beta_e^2(1-\omega_e^2)\right] s^2+\beta_o^2\beta_i^2(1-\omega_e^2-\omega_i^2)\right]}{\left[(s^2-\beta_o^2 P) \;\; s^6+A_1 s^4+A_2 s^3+A_3\right]} \tag{3.81}$$

where

$$A_1 = - \left\{\beta_i^2(1-\omega_i^2-\Omega_i^2)+\beta_e^2(1-\omega_e^2-\Omega_e^2)+\beta_o^2(1-\omega_e^2-\omega_i^2)\right\}$$

$$A_2 = \beta_e^2\beta_i^2\left[1-\omega_e^2-\omega_i^2-\Omega_e^2(1-\omega_i^2)-\Omega_i^2(1-\omega_e^2)+\Omega_e^2\Omega_i^2\right]$$

$$+\beta_o^2\beta_i^2\left[(1-\omega_e^2-\omega_i^2)(1-\omega_i^2)-\Omega_i^2(1-\omega_e^2)\right]+\beta_o^2\beta_e^2\left[(1-\omega_e^2-\omega_i^2)(1-\omega_e^2)-\Omega_e^2(1-\omega_i^2)\right]$$

$$A_3 = -\beta_o^2\beta_e^2\beta_i^2\left[(1-\omega_e^2-\omega_i^2)^2+2\Omega_e\Omega_i\omega_e^2\omega_i^2-(1-\omega_e^2)\Omega_i^2-(1-\omega_i^2)\Omega_e^2+\Omega_e^2\Omega_i^2\right]$$

For the sake of comparison, Eq. (3.81) has been written in terms of s^2 rather than n^2.

It is easy to see, from Eq. (3.81), that for $\theta = 0$ two coupled electromagnetic waves are possible and two acoustic waves, due to the coupling between the electron and ion acoustic waves. For $\theta = 90^\circ$ the first term in the denominator is precisely the same as Eq. (3.64) and is the ordinary electromagnetic wave. The second term in the denominator yields three waves, an electromagnetic wave and two acoustic waves, which are the result of coupling between the extraordinary electromagnetic wave and the electron and ion acoustic waves.

3.3 Propagation Constants in the Ionosphere

The characteristic waves for the collisionless case discussed in the previous section yield the basic characteristics of propagation, in general, only for the direction of propagation parallel or perpendicular to the direction of the d. c. magnetic field. For intermediate directions of propagation, the basic modes are coupled and lose their identity. Indeed, even for the parallel and perpendicular case, the basic modes are coupled as indicated, for example, by Eq. (3.38). The effect of collisions, as mentioned in Section 3.1, can probably be considered as composed of two parts, attenuation due to collisions and coupling due to collisions. Examination of the curves in Appendix A shows that for the ionosphere, especially for charged particles, the coupling due to collisions is probably of secondary importance when compared to coupling due to electric forces and due to the d.c. magnetic field as exemplified by the plasma frequencies and gyro frequencies, respectively. Because of this, it is probable that the most important effect of collisions is the attenuation of the waves. The attenuation of the waves can be considered as due to two causes. The coupling due to electric and d.c. magnetic forces and that due to collisions. At frequencies and regions of the ionosphere where the coupling forces are small, the collisions terms (also small) have the greatest effect on the attenuation of the wave and have very little effect on the phase velocity of the waves. As the coupling forces become large, the attenuation due to collisions becomes small compared to electric and magnetic forces and, in this region, the collisions terms have their greatest effect on the phase velocity. This point can best be illustrated by examination of Eq. (3.32). The real and imaginary parts of this equation are

$$s_r = \beta_e \left[\frac{1-\omega_e^2 + \sqrt{(1-\omega_e^2)^2 + (\nu_{ei}+\nu_{en})^2}}{2} \right]^{\frac{1}{2}}$$

$$s_i = \beta_e \left[\frac{-(1-\omega_e)^2 + \sqrt{(1-\omega_e^2)^2 + (\nu_{ei}+\nu_{en})^2}}{2} \right]^{\frac{1}{2}} ,$$

Since for the regions of the ionosphere considered $\nu_{ei}+\nu_{en}\ll\omega_e^2$ it is evident that for $\omega_e^2<1$ s_r is a strong function of the plasma frequency and s_i is largely a function of the collision terms. For $\omega_e^2>1$, due to the factor $1-\omega_e^2$, the reverse situation is true.

It should be noted that the above analysis does not apply to the neutral wave since, of course, electric and magnetic forces do not affect the neutral particles directly. Because of this, the propagation constant for the neutral wave should be given to a good approximation, by Eq. (3.2) for the ionosphere.

One other fact of significance should be pointed out at this time in regard to the attenuation of the electron and ion types of waves. For loose coupling, $\omega_e^2\ll 1$, Eqs. (3.32) and (3.41) are approximately,

$$S\approx\beta_e+i\,\frac{(\nu_{ei}+\nu_{en})}{2U_e}$$

$$S\approx\beta_i+i\,\frac{(\nu_{ie}+\nu_{in})}{2U_i}$$

Thus, the attenuation of the electron and ion acoustic waves is given by the ratio of the collision frequency to the appropriate velocity. Again, referring to Appendix A, this number is relatively large and these waves are rapidly attenuated, in general, and the $\frac{1}{e}$ point for the magnitude of these waves is reached in a distance of one kilometer at the most and, in many cases, in a distance of a few meters.

A complete analysis of the effects of collision and direction of propagation on the modes propagating in the ionosphere would involve a detailed analysis of the dispersion relation, Eq. (3.11). Because of the complicated nature of this equation and the analytic expressions obtained for the coefficients of the resulting fifth order polynomial in s^2, the dispersion relation was programmed for analysis by a digital computer. A complete discussion of the program is given in Appendix B. Due to limitations of time and money, only a limited amount of data was obtained and the results tabulated in Table I. The propagation constants were computed for 15^o intervals of θ between 0 and 90^o at angular frequencies of 3×10^3, 3×10^4, 3×10^5, 3×10^6, 3×10^7, 3×10^8 and 3×10^9 radians per second, using parameters appropriate to an altitude of 100 Km in the ionosphere. The roots are presented in groups of five, corresponding to the five basic modes of propagation, the first two roots correspond to the two electromagnetic waves and the last three to the electron, ion and neutral acoustic wave, respectively. The columns in the table correspond to variations in direction of propagations and the rows, in groups of five, to frequency variation, the lowest frequency appearing at the top of the table and the highest frequency at the bottom. The left hand column of the table lists, in addition to frequency, the corresponding values of β_o, β_e, β_i and β_n while the two extreme right hand columns list the appropriate normalized values of the plasma frequencies, gyro-frequencies and the collision frequencies. Only three significant digits are listed for the real and imaginary parts of the propagation constants; however, many more digits were required to achieve the required accuracy. In this table, the imaginary parts of some of the roots are negative. One particular case of importance is that for the neutral wave at $\theta = 0$ and $\omega = 3 \times 10^3$. The negative sign was due to a lack of accuracy in the procedure for determining the roots of the polynomial in that not enough iterations were carried out in the process. The roots of this polynomial were evaluated a second time, using greater accuracy, and the following results obtained for the five roots.

TABLE I: VARIATION OF PROPAGATION CONSTANTS ALT 100 Km. WITH ANGLE θ AND FREQUENCY

	0°	15°	30°	45°	60°	75°	90°		
$\omega=.3\times10^{4}$	$.139\times10^{-4}+i.612\times10^{-3}$	$.143\times10^{-4}+i.622\times10^{-3}$	$.636\times10^{-3}+i.152\times10^{-4}$	$.703\times10^{-3}+i.188\times10^{-4}$	$.832\times10^{-3}+i.285\times10^{-4}$	$.785\times10^{-4}+i.122\times10^{-2}$	$.315.10^{-2}+i.321\times10^{-2}$	$\omega_e^2=.11\times10^{8}$	$k_{en}=.52\times10^{2}$
$\beta_o=10^{-5}$	$.592\times10^{-3}+i.132\times10^{-4}$	$.602\times10^{-3}+i.137\times10^{-4}$	$.159\times10^{-4}+i.657\times10^{-3}$	$.198\times10^{-4}+i.728\times10^{-3}$	$.307\times10^{-4}+i.867\times10^{-3}$	$.114\times10^{-2}+i.686\times10^{-4}$	$.227\times10^{-2}+i.115\times10^{-2}$	$\omega_i^2=.20\times10^{3}$	$k_{ie}=.19\times10^{-5}$
$\beta_e=.307\times10^{-1}$	$.702\times10^{-1}+i.141\times10^{3}$	$.702\times10^{-1}+i.141\times10^{3}$	$.703\times10^{-1}+i.141\times10^{3}$	$.704\times10^{-1}+i.141\times10^{3}$	$.706\times10^{-1}+i.141\times10^{3}$	$.534\times10^{1}+i.191\times10^{1}$	$.346\times10^{0}+i.141\times10^{3}$	$\Omega_e=.29\times10^{4}$	$k_{in}=.80\times10^{0}$
$\beta_i=.704\times10^{+1}$	$.533\times10^{1}+i.188\times10^{1}$	$.533\times10^{1}+i.188\times10^{1}$	$.533\times10^{1}+i.188\times10^{1}$	$.533\times10^{1}+i.188\times10^{1}$	$.533\times10^{1}+i.189\times10^{1}$	$.715\times10^{-1}+i.141\times10^{3}$	$.785\times10^{1}+i.619\times10^{1}$	$\Omega_i=.53\times10^{-1}$	$k_{ne}=.29\times10^{-8}$
$\beta_n=.103\times10^{2}$	$.103\times10^{2}-i.171\times10^{-6}$	$.103\times10^{2}-i.171\times10^{-6}$	$.103\times10^{2}-i.171\times10^{-6}$	$.103\times10^{2}-i.171\times10^{-6}$	$.103\times10^{2}-i.171\times10^{-6}$	$.103\times10^{2}-i.171\times10^{-6}$	$.103\times10^{2}-i.104\times10^{-5}$	$k_{ei}=10^{-1}$	$k_{ni}=.23\times10^{-8}$
$\omega=.3\times10^{5}$	$.173\times10^{-4}+i.190\times10^{-2}$	$.182\times10^{-4}+i.193\times10^{-2}$	$.214\times10^{-4}+i.204\times10^{-2}$	$.289\times10^{-4}+i.226\times10^{-2}$	$.484\times10^{-4}+i.268\times10^{-2}$	$.128\times10^{-3}+i.372\times10^{-2}$	$.899\times10^{-2}+i.109\times10^{-1}$	$\omega_e^2=.11\times10^{6}$	$k_{en}=.52\times10^{1}$
$\beta_o=10^{-4}$	$.190\times10^{-2}+i.175\times10^{-4}$	$.194\times10^{-2}+i.184\times10^{-4}$	$.204\times10^{-2}+i.217\times10^{-4}$	$.226\times10^{-2}+i.293\times10^{-4}$	$.269\times10^{-2}+i.492\times10^{-4}$	$.375\times10^{-2}+i.132\times10^{-3}$	$.976\times10^{-2}+i.101\times10^{-1}$	$\omega_i^2=.2\times10^{1}$	$k_{ie}=.19\times10^{-6}$
$\beta_e=.31\times10^{0}$	$.566\times10^{0}+i.134\times10^{3}$	$.567\times10^{0}+i.134\times10^{3}$	$.566\times10^{0}+i.134\times10^{3}$	$.567\times10^{0}+i.134\times10^{3}$	$.569\times10^{0}+i.134\times10^{3}$	$.582\times10^{0}+i.134\times10^{3}$	$.405\times10^{1}+i.134\times10^{3}$	$\Omega_e=.29\times10^{3}$	$k_{in}=.8\times10^{-1}$
$\beta_i=.7\times10^{2}$	$.528\times10^{2}+i.234\times10^{1}$	$.528\times10^{2}+i.234\times10^{1}$	$.528\times10^{2}+i.235\times10^{1}$	$.528\times10^{2}+i.235\times10^{1}$	$.528\times10^{2}+i.235\times10^{1}$	$.528\times10^{2}+i.237\times10^{1}$	$.522\times10^{2}+i.772\times10^{1}$	$\Omega_i=.53\times10^{-2}$	$k_{ne}=.28\times10^{-9}$
$\beta_n=.103\times10^{3}$	$.103\times10^{3}+i.00\times10^{0}$	$.103\times10^{3}+i.00\times10^{0}$	$.103\times10^{3}+i.00\times10^{0}$	$.103\times10^{3}+i.00\times10^{0}$	$.103\times10^{3}+i.00\times10^{0}$	$.103\times10^{3}+i.00\times10^{0}$	$.103\times10^{3}+i.00\times10^{0}$	$k_{ei}=10^{-2}$	$k_{ni}=.23\times10^{-9}$
$\omega=.3\times10^{6}$	$.519\times10^{-4}+i.583\times10^{-2}$	$.544\times10^{-4}+i.593\times10^{-2}$	$.634\times10^{-4}+i.625\times10^{-2}$	$.839\times10^{-4}+i.688\times10^{-2}$	$.135\times10^{-3}+i.809\times10^{-2}$	$.319\times10^{-3}+i.108\times10^{-1}$	$.733\times10^{-2}+i.298\times10^{-1}$	$\omega_e^2=.11\times10^{4}$	$k_{en}=.52\times10^{0}$
$\beta_o=10^{-3}$	$.620\times10^{-2}+i.559\times10^{-4}$	$.631\times10^{-2}+i.591\times10^{-4}$	$.668\times10^{-2}+i.705\times10^{-4}$	$.743\times10^{-2}+i.980\times10^{-4}$	$.896\times10^{-2}+i.173\times10^{-3}$	$.131\times10^{-1}+i.545\times10^{-3}$	$.338\times10^{-2}+i.236\times10^{-1}$	$\omega_i^2=.2\times10^{-1}$	$k_{ie}=.19\times10^{-7}$
$\beta_e=.31\times10^{1}$	$.325\times10^{-1}+i.101\times10^{3}$	$.343\times10^{-1}+i.101\times10^{3}$	$.407\times10^{-1}+i.101\times10^{3}$	$.572\times10^{-1}+i.101\times10^{3}$	$.107\times10^{0}+i.101\times10^{3}$	$.393\times10^{0}+i.1\times10^{3}$	$.129\times10^{2}+i.129\times10^{3}$	$\Omega_e=.29\times10^{2}$	$k_{in}=.8\times10^{-2}$
$\beta_i=.7\times10^{3}$	$.697\times10^{3}+i.286\times10^{1}$	$.697\times10^{3}+i.286\times10^{1}$	$.697\times10^{3}+i.286\times10^{1}$	$.697\times10^{3}+i.286\times10^{1}$	$.697\times10^{3}+i.286\times10^{1}$	$.697\times10^{3}+i.286\times10^{1}$	$.697\times10^{3}+i.286\times10^{1}$	$\Omega_i=.53\times10^{-3}$	$k_{ne}=.29\times10^{-10}$
$\beta_n=.1\times10^{4}$	$.103\times10^{4}+i.00\times10^{0}$	$.103\times10^{4}+i.00\times10^{0}$	$.103\times10^{4}+i.00\times10^{0}$	$.103\times10^{4}+i.00\times10^{0}$	$.103\times10^{4}+i.00\times10^{0}$	$.103\times10^{4}+i.00\times10^{0}$	$.103\times10^{4}+i.00\times10^{0}$	$k_{ei}=10^{-3}$	$k_{ni}=.23\times10^{-10}$
$\omega=.3\times10^{7}$	$.137\times10^{-3}+i.13\times10^{-1}$	$.141\times10^{-3}+i.132\times10^{-1}$	$.153\times10^{-3}+i.138\times10^{-1}$	$.175\times10^{-3}+i.149\times10^{-1}$	$.217\times10^{-3}+i.165\times10^{-1}$	$.218\times10^{-2}+i.426\times10^{-1}$	$.896\times10^{-3}+i.31\times10^{-1}$	$\omega_e^2=.11\times10^{2}$	$k_{en}=.52\times10^{1}$
$\beta_o=10^{-2}$	$.255\times10^{-1}+i.292\times10^{-3}$	$.262\times10^{-1}+i.327\times10^{-3}$	$.290\times10^{-1}+i.473\times10^{-3}$	$.364\times10^{-1}+i.104\times10^{-2}$	$.771\times10^{-1}+i.118\times10^{-1}$	$.297\times10^{-3}+i.191\times10^{-1}$	$.395\times10^{-3}+i.214\times10^{-1}$	$\omega_i^2=.2\times10^{-3}$	$k_{ie}=.19\times10^{-8}$
$\beta_e=.31\times10^{2}$	$.26\times10^{0}+i.953\times10^{2}$	$.289\times10^{0}+i.949\times10^{2}$	$.408\times10^{0}+i.933\times10^{2}$	$.863\times10^{0}+i.886\times10^{2}$	$.640\times10^{1}+i.625\times10^{2}$	$.104\times10^{2}+i.162\times10^{3}$	$.181\times10^{1}+i.131\times10^{3}$	$\Omega_e=.29\times10^{1}$	$k_{in}=.8\times10^{-3}$
$\beta_i=.7\times10^{4}$	$.704\times10^{4}+i.283\times10^{1}$	$.704\times10^{4}+i.283\times10^{1}$	$.704\times10^{4}+i.283\times10^{1}$	$.704\times10^{4}+i.283\times10^{1}$	$.704\times10^{4}+i.283\times10^{1}$	$.704\times10^{4}+i.283\times10^{1}$	$.704\times10^{4}+i.283\times10^{1}$	$\Omega_i=.53\times10^{-4}$	$k_{ne}=.29\times10^{-11}$
$\beta_n=.1\times10^{5}$	$.103\times10^{5}+i.00\times10^{0}$	$103\times10^{5}+i.00\times10^{0}$	$.103\times10^{5}+i.00\times10^{0}$	$.103\times10^{5}+i.00\times10^{0}$	$.103\times10^{5}+i.00\times10^{0}$	$.103\times10^{5}+i.00\times10^{0}$	$.103\times10^{5}+i.00\times10^{0}$	$k_{ei}=10^{-4}$	$k_{ni}=.23\times10^{-11}$
$\omega=.3\times10^{8}$	$.922\times10^{-1}+i.604\times10^{-4}$	$.923\times10^{-1}+i.595\times10^{-4}$	$.925\times10^{-1}+i.567\times10^{-4}$	$.928\times10^{-1}+i.525\times10^{-4}$	$.932\times10^{-1}+i.474\times10^{-4}$	$.937\times10^{-1}+i.425\times10^{-4}$	$.939\times10^{-1}+i.402\times10^{-4}$	$\omega_e^2=.11\times10^{0}$	$k_{en}=.52\times10^{-2}$
$\beta_o=10^{-1}$	$.958\times10^{-1}+i.174\times10^{-4}$	$.958\times10^{-1}+i.178\times10^{-4}$	$.956\times10^{-1}+i.191\times10^{-4}$	$.954\times10^{-1}+i.213\times10^{-4}$	$.951\times10^{-1}+i.242\times10^{-4}$	$.948\times10^{-1}+i.276\times10^{-4}$	$.945\times10^{-1}+i.294\times10^{-4}$	$\omega_i^2=.2\times10^{-5}$	$k_{ie}=.19\times10^{-9}$
$\beta_e=.31\times10^{3}$	$.290\times10^{3}+i.852\times10^{0}$	$.29\times10^{3}+i.862\times10^{0}$	$.287\times10^{3}+i.886\times10^{0}$	$.283\times10^{3}+i.917\times10^{0}$	$.28\times10^{3}+i.947\times10^{0}$	$277\times10^{3}+i.967\times10^{0}$	$.276\times10^{3}+i.974\times10^{0}$	$\Omega_e=.29\times10^{0}$	$k_{in}=.8\times10^{-4}$
$\beta_i=.7\times10^{5}$	$.704\times10^{5}+i.283\times10^{1}$	$.704\times10^{5}+i.283\times10^{1}$	$.704\times10^{5}+i.283\times10^{1}$	$.704\times10^{5}+i.283\times10^{1}$	$.704\times10^{5}+i.283\times10^{1}$	$.704\times10^{5}+i.283\times10^{1}$	$.704\times10^{5}+i.283\times10^{1}$	$\Omega_i=.53\times10^{-5}$	$k_{ne}=.29\times10^{-12}$
$\beta_n=.1\times10^{6}$	$.103\times10^{6}+i.00\times10^{0}$	$.103\times10^{6}+i.00\times10^{0}$	$.103\times10^{6}+i.00\times10^{0}$	$.103\times10^{6}+i.00\times10^{0}$	$.103\times10^{0}+i.00\times10^{0}$	$.103\times10^{0}+i.00\times10^{0}$	$.103\times10^{0}+i.00\times10^{0}$	$k_{ei}=10^{-5}$	$k_{ni}=.23\times10^{-12}$
$\omega=.3\times10^{9}$	$10^{0}+i.295\times10^{-6}$	$1\times10^{0}+i.295\times10^{-6}$	$1\times10^{0}+i.293\times10^{-6}$	$1\times10^{0}+i.290\times10^{-6}$	$1\times10^{0}+i.287\times10^{-6}$	$1\times10^{0}+i.283\times10^{-6}$	$1\times10^{0}+i.409\times10^{-6}$	$\omega_e^2=.11\times10^{-2}$	$k_{en}=.52\times10^{-3}$
$\beta_o=.1\times10^{1}$	$10^{0}+i.263\times10^{-6}$	$1\times10^{0}+i.263\times10^{-6}$	$1\times10^{0}+i.265\times10^{-6}$	$1\times10^{0}+i.267\times10^{-6}$	$1\times10^{0}+i.271\times10^{-6}$	$1\times10^{0}+i.274\times10^{-6}$	$1\times10^{0}+i.148\times10^{-6}$	$\omega_i^2=.2\times10^{-7}$	$k_{ie}=.19\times10^{-10}$
$\beta_e=.31\times10^{4}$	$.307\times10^{4}+i.806\times10^{0}$	$.307\times10^{4}+i.806\times10^{0}$	$.307\times10^{4}+i.807\times10^{0}$	$.307\times10^{4}+i.807\times10^{0}$	$.307\times10^{4}+i.807\times10^{0}$	$.307\times10^{4}+i.807\times10^{0}$	$.307\times10^{4}+i.807\times10^{0}$	$\Omega_e=.29\times10^{-1}$	$k_{in}=.8\times10^{-5}$
$\beta_i=.7\times10^{6}$	$.704\times10^{6}+i.283\times10^{1}$	$.704\times10^{6}+i.283\times10^{1}$	$.704\times10^{6}+i.283\times10^{1}$	$.704\times10^{6}+i.283\times10^{1}$	$.704\times10^{6}+i.283\times10^{1}$	$.704\times10^{6}+i.283\times10^{1}$	$.704\times10^{6}+i.283\times10^{1}$	$\Omega_i=.53\times10^{-6}$	$k_{ne}=.29\times10^{-13}$
$\beta_n=.1\times10^{7}$	$.103\times10^{7}+i.00\times10^{0}$	$.103\times10^{7}+i.00\times10^{0}$	$.103\times10^{7}+i.00\times10^{0}$	$.103\times10^{7}+i.00\times10^{0}$	$.103\times10^{7}+i.00\times10^{0}$	$.103\times10^{7}+i.00\times10^{0}$	$.103\times10^{7}+i.00\times10^{0}$	$k_{ei}=10^{-6}$	$k_{ni}=.23\times10^{-13}$
$\omega=.3\times10^{10}$	$1\times10^{1}+i.409\times10^{-8}$	$1\times10^{1}-i.124\times10^{-5}$	$1\times10^{1}+i.109\times10^{-6}$	$1\times10^{1}+i.146\times10^{-6}$	$1\times10^{1}+i.982\times10^{-7}$	$1\times10^{1}+i.109\times10^{-6}$	$1\times10^{1}-i.123\times10^{-5}$	$\omega_e^2=.11\times10^{-4}$	$k_{en}=.52\times10^{-4}$
$\beta_o=.1\times10^{2}$	$1\times10^{1}+i.149\times10^{-8}$	$1\times10^{1}+i.124\times10^{-5}$	$1\times10^{1}-i.104\times10^{-6}$	$1\times10^{1}-i.140\times10^{-6}$	$1\times10^{1}-i.926\times10^{-7}$	$1\times10^{1}-i.104\times10^{-6}$	$1\times10^{1}+i.124\times10^{-5}$	$\omega_i^2=.2\times10^{-9}$	$k_{ie}=.19\times10^{-11}$
$\beta_e=.31\times10^{5}$	$.307\times10^{5}+i.806\times10^{0}$	$.307\times10^{5}+i.806\times10^{0}$	$.307\times10^{5}+i.806\times10^{0}$	$.307\times10^{5}+i.806\times10^{0}$	$.307\times10^{5}+i.806\times10^{0}$	$.307\times10^{5}+i.806\times10^{0}$	$.307\times10^{5}+i.806\times10^{0}$	$\Omega_e=.29\times10^{-2}$	$k_{in}=.8\times10^{-6}$
$\beta_i=.7\times10^{7}$	$.704\times10^{7}+i.283\times10^{1}$	$.704\times10^{7}+i.283\times10^{1}$	$.704\times10^{7}+i.283\times10^{1}$	$.704\times10^{7}+i.283\times10^{1}$	$.704\times10^{7}+i.283\times10^{1}$	$.704\times10^{7}+i.283\times10^{1}$	$.704\times10^{7}+i.283\times10^{1}$	$\Omega_i=.53\times10^{-7}$	$k_{ne}=.29\times10^{-14}$
$\beta_n=.1\times10^{8}$	$103\times10^{8}+i.00\times10^{0}$	$.103\times10^{8}+i.00\times10^{0}$	$.103\times10^{8}+i.00\times10^{0}$	$.103\times10^{8}+i.00\times10^{0}$	$.103\times10^{8}+i.00\times10^{0}$	$.103\times10^{8}+i.00\times10^{0}$	$.103\times10^{8}+i.00\times10^{0}$	$k_{ei}=10^{-7}$	$k_{ni}=.23\times10^{-14}$

$$.139 \times 10^{-4} + i\,.612 \times 10^{-3}$$

$$.592 \times 10^{-3} + i\,.132 \times 10^{-4}$$

$$.702 \times 10^{-1} + i\,.141 \times 10^{3}$$

$$.533 \times 10^{1} + i\,.188 \times 10^{1}$$

$$.103 \times 10^{2} + i\,.251 \times 10^{-7}$$

It should be noted that all numbers agree except for the imaginary part of the root corresponding to the neutral wave which is now positive, as it should be.

Several other examples of roots with negative imaginary parts occur for $\omega = 3 \times 10^9$ for one of the electromagnetic waves. It is obvious that in this case the negative part is due to the numerical calculations and is not a representative number. Thus, it would seem reasonable that the imaginary parts of the roots are probably not reliable when they are much smaller than the corresponding real part.

In spite of the discussion, it is felt that the following conclusions are justified, based on Table I and the previous discussion.

1. For very high frequencies, the real part of the propagation constant for all modes is essentially given by β_o, β_e, β_i and β_n, respectively.

2. The electron and ion types of acoustic waves are highly attenuated, i.e. the magnitude of this type of wave decreases to $1/e$ of its initial value, in traveling only a few meters.

3. The attenuation of the electromagnetic waves is dependent on the collision frequencies for $\omega >> \omega_e^2$ but becomes large for $\omega \leq \omega_e^2$.

4. The attenuation of the neutral acoustic wave is relatively low and it will propagate over relatively long distances without appreciable loss(hundreds of kilometers).

IV

THE OPERATOR TRANSFORM METHOD

4.1 The Operator Approach

The set of linearized equations governing the excitation of waves in a three fluid plasma, (Eq. (2.3) through (2.14)), for a general inhomogeneous media with parameters depending on space coordinates, is close to impossible to solve. In the case of stratified media, one may, perhaps, with tedious algebraic manipulation, reduce these equations to a system of ordinary differential equations of higher order. General solutions for such higher order differential equations with variable coefficients are not known. Therefore, approximate numerical methods must be used to obtain the solution to such problems.

A formal procedure for reformulating the systems of equations which may introduce some simplification and offer the possibility of a numerical solution, is the general operator-transform method proposed by Wu (1965). This procedure is an extension of the operator method used by Diamet (1963) to obtain formal solutions for Maxwell's equations. The procedure for the formal reduction of the equations is as follows:

FIRST: For the purpose of exhibiting a general solution to a system of basic equations, it is convenient to reformulate them in the following single operator equation

$$\mathcal{W}\ \psi(r) = \phi\,(r) \tag{4.1}$$

where $\psi(r)$ is a field vector composed of the field variables such as the electric field $\underline{E}$, the velocity field $\underline{V}$, etc., $\phi(r)$ is the source vector containing various excitation sources such as the electric current source $\underline{J}$, the mechanical source $\underline{F}$, etc., and $\mathcal{W}$ is the system matrix differential operator relating the field to the sources. $\mathcal{W}$ contains all the properties of the medium and is a function of the space coordinate r. In general, without loss of generality, the system of basic equations can be rearranged so that some of the submatrices of $\mathcal{W}$ are identity matrices.

SECOND: Here, we introduce the generalized transform techniques, which amounts to choosing some convenient basis of representation for the solution and transforming the operator differential equation in real space to an operator integral equation in transform space. The generic summation symbol $\$$, such as used in Schiff (1955), will be used, which requires that the expression following this symbol be integrated or summed over the entire range of the repeated variable. Formally, for any quantity a(r), we may introduce the following transform pair:

$$\begin{aligned} &\text{Transform} \quad A(s) = \$\, d(s, r)a(r) \\ &\text{Inverse} \quad a(r) = \$\, c(r, s)A(s) \end{aligned} \tag{4.2}$$

with the property that

$$\$\, c(r, s)d(s, p) = \bar{\bar{1}}(r, p)$$

and

$$\$\, d(u, r)c(r, s) = \bar{\bar{1}}(u, s) \tag{4.3}$$

The idemfactor $\bar{\bar{1}}(u, s)$ comprises a Dirac delta function or a Kronecker delta and a unit dyadic, as required.

To illustrate the transform pair consider a rectangular coordinate system. The real space variables are coordinates (x, y, z) and the transform space variables may be considered as (s_1, s_2, s_3). The range of the real space and transform space variables is $-\infty$ to $+\infty$. In this case a Fourier transform is appropriate and d(s, r) and c(r, s) are

$$d(s, r) = \frac{1}{(2\pi)^3} e^{-ir \cdot s}$$

$$c(r, s) = e^{ir \cdot s} \qquad (4.4)$$

Now, we proceed to the transformation of the operator Eq. (4.1). Let $\bar{\psi}(s)$ and $\bar{\phi}(s)$ be the transforms of the vectors $\psi(r)$ and $\phi(r)$, respectively, i.e.,

$$\begin{cases} \bar{\psi}(s) = \sum\!\!\!\!\int d(s, r)\psi(r) \\ \psi(r) = \sum\!\!\!\!\int c(r, s)\bar{\psi}(s) \end{cases} \qquad (4.5)$$

$$\begin{cases} \bar{\phi}(s) = \sum\!\!\!\!\int d(s, r)\phi(r) \\ \phi(r) = \sum\!\!\!\!\int c(r, s)\bar{\phi}(s) \end{cases} \qquad (4.6)$$

Also, we take the transformation law for the matrix operator $\mathcal{W}$ as

$$\bar{\mathcal{W}}(u, s) = \sum\!\!\!\!\int d(u, r)\mathcal{W}c(r, s) \qquad (4.7)$$

Premultiplying both sides of Eq. (4.1) by d(u, r), and then substituting the expansion for $\psi(r)$ as given by the transform pair in Eq. (4.5) and summing or integrating over the complete r-space the operator Eq. (4.1) in the real space becomes the operator integral equation in the transform space

$$\sum\!\!\!\!\int \bar{\mathcal{W}}(u, s)\bar{\psi}(s) = \bar{\phi}(u) \qquad (4.8)$$

This equation has the character of a generalized integral equation of the first kind, with $\bar{\phi}(s)$ as the forcing function, $\bar{\psi}(s)$ as the unknown function, and $\bar{\mathcal{W}}(u, s)$ as the kernel. $\bar{\mathcal{W}}(u, s)$ is a function of two composite variables of the transform space and retains all the pertinent information about the system.

THIRD: Because of the earlier rearrangement and diagonalization, the dyadic kernel $\mathcal{W}(u, s)$ can be properly partitioned so that the order of the matrices to be manipulated may be reduced, by introducing coupled integral equations of the second kind, which in turn may be recombined into one integral equation of the second kind. For example, we can have for Maxwell's equations

$$\mathcal{W}(u, s) = \begin{bmatrix} \bar{\bar{1}}(u, s) & -Z(u, s) \\ -Y(u, s) & \bar{\bar{1}}(u, s) \end{bmatrix} \tag{4.9}$$

Then, the partitioning of $\bar{\Psi}(s)$ and $\bar{\Phi}(u)$ into two vectors

$$\bar{\Psi}(s) = \begin{bmatrix} V(s) \\ I(s) \end{bmatrix}, \qquad \bar{\Phi}(u) = \begin{bmatrix} W(u) \\ J(u) \end{bmatrix} \tag{4.10}$$

produces the following coupled integral equations

$$V(u) = W(u) + \mathop{\$}\!\!\!\!\!/\; Z(u, s)\, I(s) \tag{4.11}$$

$$I(u) = J(u) + \mathop{\$}\!\!\!\!\!/\; Y(u, s)\, V(s) \tag{4.12}$$

Let V(s) and I(s) correspond, respectively, to the transform of the electric field and the transform of the magnetic field, then these Eqs. (4.11) and (4.12) have the generalized forms of the telegraphist's equations of Schelkunoff (1955) if s is taken to indicate different modes in the waveguide.

The elimination of either the field vector V or I in the Eqs. (4.11) and (4.12) gives the general form of the Fredholm integral equation of the second kind, e.g.,

$$V(u) = F(u) + \mathop{\$}\!\!\!\!\!/\; K(u, s)\, V(s) \tag{4.13}$$

where

$$F(u) \equiv W(u) + \oint Z(u, s) J(s)$$
$$K(u, s) \equiv \oint Z(u, v) Y(v, s) \qquad (4.14)$$

are both known functions. For homogeneous media the kernel has the ideal form

$$K(u, s) = N(s) \bar{\bar{1}}(u, s) \qquad (4.15)$$

and the integral Eq. (4.13) can be explicitly solved as

$$V(s) = \left[1 - N(s)\right]^{-1} F(s). \qquad (4.16)$$

For inhomogeneous media, no such idealization is possible, but, in principle, solutions may be obtained by standard techniques of numerical analysis.

This generalized operator approach, therefore, may be called a "unified" approach in the sense that within the same mathematical framework, a technique is available which, in principle, is applicable to problems involving either homogeneous or inhomogeneous media.

4.2 The Integral Equation

The steps outlined in Section 4.1 for reducing the system of partial differential Eqs. (2.3) to (2.7) into integral equations in the transform domain is straightforward and has been carried out explicitly by Wu (1965). A sketch of this reduction is given in this section.

The matrix equation relating the fields $\underline{E}$, $\underline{V}_e$, $\underline{V}_i$, $\underline{V}_n$, $\underline{h}$, n_e, n_i, n_n to the sources $\underline{K}$, Q_e, Q_i, Q_n, I, $\underline{F}_e$, $\underline{F}_i$, $\underline{F}_n$, is rearranged in such a way that the resulting matrix operator contains two sub-matrices which are identity matrices. This rearrangement is essential in order to simplify the resulting integral equation by eliminating some of the field variables. The resulting matrix equation is given by Eq. (4.17), where $\underline{S}_{1,2,3,4}$ are related to the sources $\underline{I}$, $\underline{F}_e$, $\underline{F}_i$, and $\underline{F}_n$, while the A_{ij} are source independent operators. The explicit forms of $\underline{S}_i$, and A_{ij} are given by Wu (1965).

$$
\begin{bmatrix}
\bar{\bar{1}} & 0 & 0 & 0 & i\frac{\nabla\times\bar{\bar{1}}}{\omega\mu_o} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & \frac{iN_o}{\omega}(\nabla\cdot\bar{\bar{1}})'+\frac{i}{\omega}\nabla N_o\cdot\bar{\bar{1}} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & \frac{iN_o}{\omega}(\nabla\cdot\bar{\bar{1}})'+\frac{i}{\omega}\nabla N_o\cdot\bar{\bar{1}} & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{iN_1}{\omega}(\nabla\cdot\bar{\bar{1}})'+\frac{i}{\omega}\nabla N_1\cdot\bar{\bar{1}} \\
A_{11} & A_{12} & A_{13} & A_{14} & \bar{\bar{1}} & 0 & 0 & 0 \\
A_{21} & A_{22} & A_{23} & A_{24} & 0 & \bar{\bar{1}} & 0 & 0 \\
A_{31} & A_{32} & A_{33} & A_{34} & 0 & 0 & \bar{\bar{1}} & 0 \\
A_{41} & A_{42} & A_{43} & A_{44} & 0 & 0 & 0 & \bar{\bar{1}}
\end{bmatrix}
\begin{bmatrix}
\underline{h} \\ n_e \\ n_i \\ n_n \\ \underline{E} \\ \underline{V}_e \\ \underline{V}_i \\ \underline{V}_n
\end{bmatrix}
=
\begin{bmatrix}
-\frac{i\underline{K}}{\omega\mu_o} \\ \frac{iQ_e}{\omega} \\ \frac{iQ_i}{\omega} \\ \frac{iQ_n}{\omega} \\ \underline{S}_1 \\ \underline{S}_2 \\ \underline{S}_3 \\ \underline{S}_4
\end{bmatrix}
\tag{4.17}
$$

The matrix Eq. (4.17)can be put into an operator form as

$$\mathcal{W} \psi(r) = \phi(r) \tag{4.18}$$

where

$$\psi(r) \equiv \begin{bmatrix} \underline{h} \\ n_e \\ n_i \\ n_n \\ \underline{E} \\ \underline{V}_e \\ \underline{V}_i \\ \underline{V}_n \end{bmatrix} \qquad \phi(r) \equiv \begin{bmatrix} \frac{-i\underline{K}}{\omega\mu_o} \\ \frac{iQ_e}{\omega} \\ \frac{iQ_i}{\omega} \\ \frac{iQ_n}{\omega} \\ \underline{S}_1 \\ \underline{S}_2 \\ \underline{S}_3 \\ \underline{S}_4 \end{bmatrix} \tag{4.19}$$

Equation (4.18) can be considered as an abstract relation between the sources and the resultant fields. $\psi(r)$ is the eighteen-vector representing the field quantities, $\phi(r)$ is an eighteen-vector representing the source quantities, and $\mathcal{W}$ is the system matrix differential operator relating the field to the sources. The generalized Fourier transform, such as defined by Eqs. (4.2) and (4.3), may be used to obtain the transform of Eq. (4.18). The result may be formally expressed in the form of Eq. (4.8) which is repeated below.

$$\mathcal{S}\, \mathcal{W}(u, s)\, \psi(s) = \phi(u) \tag{4.20}$$

Equation (4.20) may be put into the generalized forms of the telegraphist's equation by partitioning the transform of the field vector, $\bar{\Psi}(s)$, the transform of the source vector, $\bar{\Phi}(s)$, and the transform of the matrix differential operator, $\mathcal{W}(u, s)$, as follows:

$$\bar{\Psi}(s) = \mathcal{S}\, d(s, r) \begin{bmatrix} \underline{h} \\ n_e \\ n_i \\ n_n \\ \underline{E} \\ \underline{V}_e \\ \underline{V}_i \\ \underline{V}_n \end{bmatrix} \equiv \begin{bmatrix} I_t(s) \\ V_e(s) \\ V_i(s) \\ V_n(s) \\ V_t(s) \\ I_e(s) \\ I_i(s) \\ I_n(s) \end{bmatrix} \tag{4.21}$$

$$\not{\mathcal{J}}(s) = \not{\mathcal{S}}\, d(s,r) \begin{bmatrix} \frac{-i\underline{K}}{\omega\mu_o} \\ \frac{iQ_e}{\omega} \\ \frac{iQ_i}{\omega} \\ \frac{iQ_n}{\omega} \\ \underline{S}_1 \\ \underline{S}_2 \\ \underline{S}_3 \\ \underline{S}_4 \end{bmatrix} = \begin{bmatrix} J_t(s) \\ W_e(s) \\ W_i(s) \\ W_n(s) \\ W_t(s) \\ J_e(s) \\ J_i(s) \\ J_n(s) \end{bmatrix} \tag{4.22}$$

where $I_t(s)$, $V_t(s)$, $I_e(s)$, $I_i(s)$, $I_n(s)$, $J_t(s)$, $W_t(s)$, $J_e(s)$, $J_i(s)$ and $J_n(s)$ are three by one column matrices, and $V_e(s)$, $V_i(s)$, $V_n(s)$, $W_e(s)$, $W_i(s)$ and $W_n(s)$ are scalars.

In view of the orthonomality of the transformation kernels, c(r, s) and d(s, r), the eighteen-dyadic kernel $\mathcal{W}(u, s)$ can be partitioned as

$$
\mathcal{W}(u,s) = \begin{bmatrix}
\bar{\bar{1}}(u,s) & 0 & 0 & 0 & -Y_t(u,s) & 0 & 0 & 0 \\
0 & 1(u,s) & 0 & 0 & 0 & -Z_e(u,s) & 0 & 0 \\
0 & 0 & 1(u,s) & 0 & 0 & 0 & -Z_i(u,s) & 0 \\
0 & 0 & 0 & 1(u,s) & 0 & 0 & 0 & -Z_n(u,s) \\
-Z_t(u,s) & -T_{te}(u,s) & -T_{ti}(u,s) & -T_{tn}(u,s) & \bar{\bar{1}}(u,s) & 0 & 0 & 0 \\
-T_{et}(u,s) & -Y_e(u,s) & -Y_{ei}(u,s) & -Y_{en}(u,s) & 0 & \bar{\bar{1}}(u,s) & 0 & 0 \\
-T_{it}(u,s) & -Y_{ie}(u,s) & -Y_i(u,s) & -Y_{in}(u,s) & 0 & 0 & \bar{\bar{1}}(u,s) & 0 \\
-T_{nt}(u,s) & -Y_{ne}(u,s) & -Y_{ni}(u,s) & -Y_n(u,s) & 0 & 0 & 0 & \bar{\bar{1}}(u,s)
\end{bmatrix} \quad (4.23)
$$

The elements in these equations are given explicitly by Wu (1965).

By substituting Eqs. (4.21), (4.22) and (4.23) into the integral Eq. (4.20) the following generalized telegraphist's equations can be obtained:

$$I_t(u) = J_t(u) + \fint Y_t(u,s)\,V_t(s) \tag{4.24}$$

$$V_t(u) = W_t(u) + \fint Z_t(u,s)\,I_t(s) + \fint T_{te}(u,s)\,V_e(s) + \fint T_{ti}(u,s)\,V_i(s) + \fint T_{tn}(u,s)\,V_n(s) \tag{4.25}$$

$$I_e(u) = J_e(u) + \fint Y_e(u,s)\,V_e(s) + \fint T_{et}(u,s)\,I_t(s) + \fint Y_{ei}(u,s)\,V_i(s) + \fint Y_{en}(u,s)\,V_n(s) \tag{4.26}$$

$$V_e(u) = W_e(u) + \fint Z_e(u,s)\,I_e(s) \tag{4.27}$$

$$I_i(u) = J_i(u) + \fint Y_i(u,s)\,V_i(s) + \fint T_{it}(u,s)\,I_t(s) + \fint Y_{ie}(u,s)\,V_e(s) + \fint Y_{in}(u,s)\,V_n(s) \tag{4.28}$$

$$V_i(u) = W_i(u) + \fint Z_i(u,s)\,I_i(s) \tag{4.29}$$

$$I_n(u) = J_n(u) + \fint Y_n(u,s)\,V_n(s) + \fint T_{nt}(u,s)\,I_t(s) + \fint Y_{ne}(u,s)\,V_e(s) + \fint Y_{ni}(u,s)\,V_i(s) \tag{4.30}$$

$$V_n(u) = W_n(u) + \fint Z_n(u,s)\,I_n(s) \tag{4.31}$$

By properly partitioning $\Psi(s)$, $\Phi(s)$ and $\mathcal{W}(u, s)$ as given by Eqs. (4.21), (4.22) and (4.23), the basic equations can be reformulated into the general form of the Fredholm Integral Equation of the second kind.

First of all, the transform of the field vector is partitioned into three column vectors each with six components as follows:

$$\Psi(s) \equiv \begin{bmatrix} \Psi_1(s) \\ \Psi_2(s) \\ \Psi_3(s) \end{bmatrix} \tag{4.32}$$

where

$$\Psi_1(s) \equiv \begin{bmatrix} I_t(s) \\ V_e(s) \\ V_i(s) \\ V_n(s) \end{bmatrix}, \quad \Psi_2(s) \equiv \begin{bmatrix} V_t(s) \\ I_e(s) \end{bmatrix}, \quad \Psi_3(s) \equiv \begin{bmatrix} I_i(s) \\ I_n(s) \end{bmatrix}.$$

Similarly, the transform of the source vector is partitioned as three six-column-vectors

$$\Phi(s) \equiv \begin{bmatrix} \Phi_1(s) \\ \Phi_2(s) \\ \Phi_3(s) \end{bmatrix} \tag{4.33}$$

where

$$\Phi_1(s) \equiv \begin{bmatrix} J_t(s) \\ W_e(s) \\ W_i(s) \\ W_n(s) \end{bmatrix}, \quad \Phi_2(s) \equiv \begin{bmatrix} W_t(s) \\ J_e(s) \end{bmatrix}, \quad \Phi_3(s) \equiv \begin{bmatrix} J_i(s) \\ J_n(s) \end{bmatrix}$$

Next, the transform of the matrix differential operator as given by Eq. (4.23) is partitioned in the following form

$$\mathcal{W}(u,s) \equiv \begin{bmatrix} \bar{\bar{I}}(u,s) & -\mathcal{W}_{12}(u,s) & -\mathcal{W}_{13}(u,s) \\ -\mathcal{W}_{21}(u,s) & \bar{\bar{I}}(u,s) & 0 \\ -\mathcal{W}_{31}(u,s) & 0 & \bar{\bar{I}}(u,s) \end{bmatrix} \qquad (4.34)$$

Substitution of Eqs. (4.32), (4.33) and (4.34) into the integral Eq. (4.20) gives three coupled integral equations which are

$$\bar{\Psi}_1(u) = \bar{\Phi}_1(u) + \oint \mathcal{W}_{12}(u,s)\bar{\Psi}_2(s) + \oint \mathcal{W}_{13}(u,s)\bar{\Psi}_3(s) \qquad (4.35)$$

$$\bar{\Psi}_2(u) = \bar{\Phi}_2(u) + \oint \mathcal{W}_{21}(u,s)\bar{\Psi}_1(s) \qquad (4.36)$$

$$\bar{\Psi}_3(u) = \bar{\Phi}_3(u) + \oint \mathcal{W}_{31}(u,s)\bar{\Psi}_1(s) \qquad (4.37)$$

where

$$\mathcal{W}_{12}(u,s) \equiv \begin{bmatrix} Y_t(u,s) & 0 \\ 0 & Z_e(u,s) \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad (4.38)$$

$$\mathcal{W}_{13}(u,s) \equiv \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ Z_i(u,s) & 0 \\ 0 & Z_n(u,s) \end{bmatrix} \qquad (4.39)$$

$$\mathcal{W}_{21}(u,s) = \begin{bmatrix} Z_t(u,s) & T_{te}(u,s) & T_{ti}(u,s) & T_{tn}(u,s) \\ T_{et}(u,s) & Y_e(u,s) & Y_{ei}(u,s) & Y_{en}(u,s) \end{bmatrix} \tag{4.40}$$

$$\mathcal{W}_{31}(u,s) = \begin{bmatrix} T_{it}(u,s) & Y_{ie}(u,s) & Y_i(u,s) & Y_{in}(u,s) \\ T_{nt}(u,s) & Y_{ne}(u,s) & Y_{ni}(u,s) & Y_n(u,s) \end{bmatrix} \tag{4.41}$$

Finally, the substitution of Eqs. (4.36) and (4.37) into Eq. (4.35) gives the desired Fredholm integral equation of the second kind for the field variable $\Psi_1(s)$ as

$$\Psi_1(u) = F(u) + \oint K(u,s)\Psi_1(s) \tag{4.42}$$

where we have defined

$$F(u) \equiv \phi_1(u) + \oint \mathcal{W}_{12}(u,s)\phi_2(s) + \oint \mathcal{W}_{13}(u,s)\phi_3(s) \tag{4.43}$$

and

$$K(u,s) \equiv \oint \mathcal{W}_{12}(u,v)\,\mathcal{W}_{21}(v,s) + \oint \mathcal{W}_{13}(u,v)\,\mathcal{W}_{31}(v,s) \tag{4.44}$$

which are both known functions.

Thus, the order of the matrices has been reduced from 18 x 18 to 6 x 6, representing a considerable simplification of the problem. No attempt has been made to obtain numerical solutions to the integral equation for a general inhomogeneous medium, however, application of the results to a homogeneous electron plasma has been illustrated by Wu (1965).

V

EXCITATION IN A HOMOGENEOUS PLASMA

5.1 General Formulation

The general operator formulation given in detail by Wu (1965) and outlined in Section IV, is, in principle, applicable to excitation problems in both homogeneous and inhomogeneous media. For the case of a homogeneous, unbounded media, of course, the set of equations given by (2.18) through (2.25) becomes a system of partial differential equations with constant coefficients. In this case, the direct use of a three-dimensional Fourier transform will reduce the system of differential equations in real space to a system of algebraic equations in transform space. Thus the Fourier transform of the field components can be expressed in terms of the Fourier transform of the sources.

Explicitly, if we define the Fourier transforms of the sources and fields by Eq. (4.1), and choose the coordinates such that

$$\underline{s} = \hat{z}\, s$$

and

$$\underline{B}_o = (\hat{z}\cos\theta + \hat{y}\sin\theta) B_o \quad ,$$

the following set of algebraic equations, relating the fields to the sources in transform space, is obtained.

Maxwell's Equations;

$$-sE_y - \omega\mu_o h_x = iK_x \tag{5.1}$$

$$sE_y - \omega\mu_o h_y = iK_y \tag{5.2}$$

$$-\omega\mu_o h_z = iK_z \tag{5.3}$$

$$-sh_y + \omega\epsilon_o E_x + ieN_o(V_{ix} - V_{ex}) = -iI_x \tag{5.4}$$

$$sh_y + \omega\epsilon_o E_y + ieN_o(V_{iy} - V_{ey}) = -iI_y \tag{5.5}$$

$$\omega\epsilon_o E_z + ieN_o(V_{iz} - V_{ez}) = -iI_z \tag{5.6}$$

Electron Motion:

$$n_e - \frac{N_o s}{\omega} V_{ez} = i\frac{Q_e}{\omega} \tag{5.7}$$

$$(1+i\nu_{ei}+i\nu_{en})V_{ex} + i\Omega_e V_{ey}\cos\theta - i\Omega_e V_{ez}\sin\theta + \frac{ie}{\omega m_e}E_x - i\nu_{ei}V_{ix} - i\nu_{en}V_{nx} = \frac{iF_{ex}}{\omega N_o m_e} \tag{5.8}$$

$$(1+i\nu_{ei}+i\nu_{en})V_{ey} - i\Omega_e V_{ex}\cos\theta + \frac{ie}{\omega m_e}E_y - i\nu_{ei}V_{iy} - i\nu_{en}V_{ny} = \frac{iF_{ey}}{\omega N_o m_e} \tag{5.9}$$

$$(1+i\nu_{ei}+i\nu_{en})V_{ez} + i\Omega_e V_{ex}\sin\theta - \frac{s}{\omega}\frac{U_e^2}{N_o}n_e + \frac{ie}{\omega m_e}E_z - i\nu_{ei}V_{iz} - i\nu_{en}V_{nz} = \frac{iF_{ez}}{\omega N_o m_e} \tag{5.10}$$

Ion Motion:

$$n_i - \frac{N_o s}{\omega}V_{iz} = i\frac{Q_i}{\omega} \tag{5.11}$$

$$(1+i\nu_{ie}+i\nu_{in})V_{ix} - i\Omega_i V_{iy}\cos\theta + i\Omega_i V_{iz}\sin\theta - \frac{ie}{\omega m_e}E_x - i\nu_{ie}V_{ex} - i\nu_{in}V_{nx} = \frac{iF_{ix}}{\omega N_o m_i} \tag{5.12}$$

$$(1+i\nu_{ie}+i\nu_{in})V_{iy} + i\Omega_i V_{ix}\cos\theta - \frac{ie}{\omega m_i}E_y - i\nu_{ie}V_{ey} - i\nu_{in}V_{ny} = \frac{iF_{iy}}{\omega N_o m_i} \tag{5.13}$$

$$(1+i\nu_{ie}+i\nu_{in})V_{iz} - i\Omega_i V_{ix}\sin\theta - \frac{s}{\omega}\frac{U_i^2}{N_o}n_i - \frac{ie}{\omega m_i}E_z - i\nu_{ie}V_{iz} - i\nu_{in}V_{nz} = \frac{iF_{iz}}{\omega N_o m_i} \tag{5.14}$$

Neutral Motion:

$$n_n - \frac{N_n s}{\omega}V_{nz} = i\frac{Q_n}{\omega} \tag{5.15}$$

$$(1+i\nu_{ni}+i\nu_{ne})V_{nx} - i\nu_{ni}V_{ix} - i\nu_{ne}V_{ex} = \frac{iF_{nx}}{\omega N_o m_n} \tag{5.16}$$

$$(1+i\nu_{ni}+i\nu_{ne})V_{ny} - i\nu_{ni}V_{iy} - i\nu_{ne}V_{ey} = \frac{iF_{ny}}{\omega N_n m_n} \tag{5.17}$$

$$(1+i\nu_{ni}+i\nu_{ne})V_{nz} - i\nu_{ni}V_{iz} - i\nu_{ne}V_{ez} - \frac{s}{\omega}\frac{U_n^2}{N_o}n_n = \frac{iF_{nz}}{\omega N_n m_n} \tag{5.18}$$

From the above equations, the transformed field quantities can be expressed in terms of the transformed sources. Formally, the system of Eqs. (5.1) through (5.18) can be represented by the matrix equation

$$[L(\underline{s})]\; f(\underline{s}) = S(\underline{s}) \;, \tag{5.19}$$

where $[L(s)]$ is the matrix of the coefficients of the equations and $f(\underline{s})$ and $S(\underline{s})$ are column vectors representing the field quantities and the sources respectively. The determinantal equation,

$$L(\underline{s}) = 0$$

the condition for the existence of fields in the source free region, has been discussed in detail for the collisional case in Section III.

The algebraic procedure involved in finding the inverse of the matrix $[L(\underline{s})]$, and then carrying out the inverse Fourier spatial transform to obtain the fields excited by various sources is straightforward, but, nevertheless, extremely tedious. Explicit expressions for the fields are generally obtained only for some ideal problems. The most general case carried out explicitly to this date is, perhaps, the work reported by Wu (1965) for the problem of excitation of waves in a collisionless electron plasma.

5.2 Collisionless Electron Plasma

For the case of a collisionless electron plasma, which is, perhaps, an adequate idealized model for the F-region of the ionosphere, the algebraic equations are considerably simplified. In this case, Eqs. (5.1) through (5.10) form a complete system, so that the inversion of the matrix is relatively simple. If one carries out the tedious algebraic steps, it can be shown that

$$E_i = \frac{1}{G}\left[\sum_j G_{ij}K_j + H_{ij}I_j + L_{ij}F_{ej} + M_iQ_e\right] \tag{5.20}$$

where i and j have the values x, y and z. Explicitly, the quantities in (5.20) are

$$G = -\frac{s^6}{\beta_e^2\beta_o^2}(1-\Omega_e^2\cos^2\theta) + s^4\left\{\frac{2}{\beta_o^2\beta_e^2}(1-\omega_p^2-\Omega_e^2\cos^2\theta)+\frac{1}{\beta_o^4}\left[(1-\omega_p^2)(1-\Omega_e^2\cos^2\theta)-\Omega_e^2\sin^2\theta\right]\right\}$$

$$-s^2\left\{\frac{1}{\beta_e^2}\left[(1-\omega_p^2)^2-\Omega_e^2\cos^2\theta\right]+\frac{1}{\beta_o^2}\left[2(1-\omega_p^2)(1-\omega_p^2-\Omega_e^2\cos^2\theta)-(2-\omega_p^2)\Omega_e^2\sin^2\theta\right]\right\}$$

$$+(1-\omega_p^2)\left[(1-\omega_p^2)^2-\Omega_e^2\right]$$

$$G_{11} = \frac{s}{\beta_o^2}\left[-\Omega_e\cos\theta a_{11} + i a_{12}\right]$$

$$G_{12} = -\frac{s}{\beta_o^2}\left[i a_{11} + \Omega_e\cos\theta\, a_{12} + \Omega_e\sin\theta\, a_{13}\right]$$

$$G_{13} = 0$$

$$G_{21} = \frac{s}{\beta_o^2}\left[-\Omega_e\cos\theta\, a_{21} + i a_{22}\right]$$

$$G_{22} = -\frac{s}{\beta_o^2}\left[i a_{21} + \Omega_e\cos\theta\, a_{22} + \Omega_e\sin\theta\, a_{23}\right]$$

$$G_{23} = 0$$

$$G_{31} = \frac{s}{\beta_o^2}\left[-\Omega_e \cos\theta\, a_{31} + i a_{32}\right]$$

$$G_{32} = -\frac{s}{\beta_o^2}\left[i a_{31} + \Omega_e \cos\theta\, a_{32} + \Omega_e \sin\theta\, a_{33}\right]$$

$$G_{33} = 0$$

$$H_{11} = -\frac{1}{\omega\epsilon_o}(i a_{11} - \Omega_e \cos\theta\, a_{12} - \Omega_e \sin\theta\, a_{13})$$

$$H_{12} = \frac{1}{\omega\epsilon_o}(\Omega_e \cos\theta\, a_{11} - i a_{12})$$

$$H_{13} = -\frac{1}{\omega\epsilon_o}\left[\Omega_e \sin\theta\, a_{11} + i\left(1 - \frac{s^2}{\beta_e^2}\right) a_{13}\right]$$

$$H_{21} = -\frac{1}{\omega\epsilon_o}\left[i a_{21} - \Omega_e \cos\theta\, a_{22} - \Omega_e \sin\theta\, a_{23}\right]$$

$$H_{22} = \frac{1}{\omega\epsilon_o}(\Omega_e \cos\theta\, a_{21} - i a_{22})$$

$$H_{23} = -\frac{1}{\omega\epsilon_o}\left[\Omega_e \sin\theta\, a_{21} + i\left(1 - \frac{s^2}{\beta_e^2}\right) a_{23}\right]$$

$$H_{31} = -\frac{1}{\omega\epsilon_o}\left[ia_{31} - \Omega_e \cos\theta\, a_{32} - \Omega_e \sin\theta\, a_{33}\right]$$

$$H_{32} = \frac{1}{\omega\epsilon_o}(\Omega_e \cos\theta\, a_{31} - ia_{32}$$

$$H_{33} = -\frac{1}{\omega\epsilon_o}\left[\Omega_e \sin\theta\, a_{31} + i(1 - \frac{s^2}{\beta_e^2})\, a_{33}\right]$$

$$L_{11} = -\frac{e}{\omega m_e \epsilon_o} a_{11}$$

$$L_{22} = -\frac{e}{\omega m_e \epsilon_o} a_{22}$$

$$L_{33} = -\frac{e}{\omega m_e \epsilon_o} a_{33}$$

$$L_{ij} = 0, \quad i \neq j,$$

$$M_1 = -\frac{eQ_e}{\beta_e^2 \omega\epsilon_o} s\, a_{13}$$

$$M_2 = -\frac{eQ_e}{\beta_e^2 \omega\epsilon_o} s\, a_{23}$$

$$M_3 = -\frac{eQ_e}{\beta_e^2 \omega \epsilon_o} s\, a_{33}$$

where

$$a_{11} = \frac{s^4}{\beta_o^2 \beta_e^2} - s^2 \left[\left(\frac{1}{\beta_o^2} + \frac{1}{\beta_e^2} \right)(1 - \omega_p^2) \right] + (1 - \omega_p^2)^2$$

$$a_{12} = -i\Omega_e \cos\theta \left\{ \frac{s^4}{\beta_o^2 \beta_e^2} - s^2 \left[\frac{1}{\beta_e^2} + \frac{1}{\beta_o^2} (1 - \omega_p^2) \right] + 1 - \omega_p^2 \right\}$$

$$a_{13} = -i\Omega_e \sin\theta \left[\frac{s^2}{\beta_o^2} - (1 - \omega_p^2) \right]$$

$$a_{21} = i\Omega_e \cos\theta \left\{ \frac{s^4}{\beta_e^2 \beta_o^2} - s^2 \left[\frac{1}{\beta_e^2} + \frac{1}{\beta_o^2} (1 - \omega_p^2) \right] + (1 - \omega_p^2) \right\}$$

$$a_{22} = \frac{s^4}{\beta_o^2 \beta_e^2} - s^2 \left[\left(\frac{1}{\beta_o^2} + \frac{1}{\beta_e^2} \right)(1 - \omega_p^2) - \frac{1}{\beta_o^2} \Omega_e^2 \sin^2\theta \right] + (1 - \omega_p^2)^2 - \Omega_e^2 \sin^2\theta$$

$$a_{23} = \Omega_e^2 \sin\theta \cos\theta \left(\frac{s^2}{\beta_o^2} - 1 \right)$$

$$a_{31} = -i\Omega_e \sin\theta \left\{ \frac{s^4}{\beta_e^2 \beta_o^2} - s^2 \left[\frac{1}{\beta_e^2} + \frac{1}{\beta_o^2} (1-\omega_p^2) \right] + (1-\omega_p^2) \right\}$$

$$a_{32} = \Omega_e^2 \sin\theta \cos\theta \left(\frac{s^2}{\beta_o^2} - 1 \right)$$

$$a_{33} = \frac{s^4}{\beta_o^4} (1-\Omega_e^2 \cos^2\theta) - \frac{2s^2}{\beta_o^2} (1-\omega_p^2-\Omega_e^2 \cos^2\theta) + (1-\omega_p^2)^2 - \Omega_e^2 \cos^2\theta$$

Using the expressions for the components of $\underline{E}$, the components of $\underline{h}$ may be obtained from Eqs. (5.1) through (5.3), while the components of $\underline{V}_e$ and n_e may be obtained from Eqs. (5.4) through (5.7).

The Fourier inversion of Eq. (5.20) to obtain expressions for the fields in real space, even for a unit impulse source, is quite complicated. Numerically, the asymptotic expression for the field can be evaluated by using the methods of stationary phase. Examples of such a procedure have been carried out by Wu (1965), where the electric field, due to current sources in the direction of the d. c. magnetic field, has been evaluated for numerical parameters appropriate to various regions of ionosphere. The numerical results obtained by Wu seem to indicate, and confirm the belief of several previous investigators, that a substantial amount of energy is excited in the form of plasma waves. However, from the numerical values of the propagation constants of the waves (Section 3.3) indicate that the plasma waves, when collisions are not neglected, attenuate quite rapidly.

Further detailed calculations are necessary to clear up this point, since this fact is very important in the investigation of radiation from current antennas in the ionosphere.

VI

SOURCES IN A BOUNDED PLASMA

6.1 Statement of the Problem

The following presentation is an attempt to describe the waves that might be generated by moving and stationary sources located in an ionized gas. Only the simplest model of an oscillating dipole is considered in any detail because of the complexity of the equations and the lack of time. It is felt, however, that the methods can eventually be extended to include more realistic and practical situations. For example, a moving vehicle can be considered, as a crude approximation, to resemble a charge moving along a prescribed path. Inclusion of this effect is simply a matter of appropriately selecting the source terms to describe the motion. In addition, the general formulation has been presented in a manner which will allow for spacial variations in the unperturbed charged particle density. An analytic description of the behavior of the fields when this feature is included has not been obtained in terms of known functions. It is felt, however, that the method is definitely amenable to numerical treatment, and this aspect of the problem is one which is definitely worth pursuing.

In this section we shall discuss the behavior of a macroscopically neutral plasma which is bounded by a perfect conductor in the interior region and which extends to infinity in the exterior. Since neutral particle effects will be ignored, we shall be concerned only with the ions and electrons, whose motions are governed by the first two moments of the Boltzmann equation together with the equation of state and Maxwell's equations. Certain simplifying assumptions will be made to assist in the analysis, namely that Landau damping and shielding effects may be ignored, that there is no external electric or magnetic field and that collision terms may be neglected. On this basis we can obtain two scalar equations whose solutions can specify completely the behavior of the plasma.

The perturbations which take place in the plasma will be considered as being produced by mechanical, electric or magnetic sources imbedded in the plasma. These sources, which are located a finite distance from the conductor cause the particles to move in some manner. A description of their motion is affected by the presence of the conductor which requires, then, a prescription of the boundary conditions to be satisfied. This is, in general, a very delicate matter and has never been well defined to everyone's satisfaction. In most cases, however, the conditions used by Cohen (1962), namely that the tangential electric field and the normal components of the particle velocities, and the magnetic field vanish at the surface of the conductor, are acceptable. Although these assumptions are not rigorously valid, they do have the advantage of simplicity. In addition, any waves stimulated by a source must satisfy some form of the radiation condition. A more exact statement of this behavior will be discussed when we are dealing with the appropriate Green's functions.

In the following three sections we shall discuss the manner in which the wave solutions may be obtained. The first section will be fairly general in that it will allow for variations in the particle densities, as well as for arbitrary sources and conductor geometry. In the second section, we shall consider one of the simplest problems, namely the oscillating electric dipole source located above a perfectly conducting plane. In the final section we shall investigate a somewhat more complicated problem involving the electric dipole in the presence of a conducting sphere. In both cases we will observe that the sources give rise to a transverse electromagnetic wave and two longitudinal plasma waves.

6.2 The Potential Functions

The situation we are considering involves the behavior of the particles of a fully ionized gas in an infinite half-space which is bounded by a conductor. Denoting the mechanical, electrical and magnetic sources by $\underline{F}(\underline{r})$, $S(\underline{r})$ and $\underline{J}(\underline{r})$, respectively, the Fourier transformed equations of motion and continuity together with Maxwell's equations

$$-i\omega \underline{V}_e + u_e^2 \, \underline{\nabla} n_e = -\frac{e}{m_e} N_o \underline{E} + \underline{F}_e \qquad \text{(electron motion)} \qquad (6.1)$$

$$-i\omega \underline{V}_i + u_i^2 \, \underline{\nabla} n_i = \frac{e}{m_i} N_o \underline{E} + \underline{F}_i \qquad \text{(ion motion)} \qquad (6.2)$$

$$\underline{\nabla} n_e = \frac{1}{i\omega} \underline{\nabla} (\underline{\nabla} \cdot \underline{V}_e) + \frac{1}{i\omega} \underline{\nabla} S_e \qquad \text{(electron continuity)} \qquad (6.3)$$

$$\underline{\nabla} n_i = \frac{1}{i\omega} \underline{\nabla} (\underline{\nabla} \cdot \underline{V}_i) + \frac{1}{i\omega} \underline{\nabla} S_i \qquad \text{(ion continuity)} \qquad (6.4)$$

$$\underline{\nabla} \times \underline{E} = \frac{i\omega}{c} \underline{H} \qquad (6.5)$$

$$\underline{\nabla} \times \underline{H} = -\frac{i\omega}{c} \underline{E} + \frac{e}{c} (\underline{V}_i - \underline{V}_e) + \frac{4\pi}{c} \underline{J} \qquad (6.6)$$

The subscripts (e) and (i) indicate terms relating to electrons and ions, respectively. The mechanical quantities $\underline{F}_e$ and $\underline{F}_i$ represent forces per unit mass, S_e and S_i are the rates at which electrons and ions are introduced into the plasma, and $\underline{V}_e = N_o \underline{v}_e$ and $\underline{V}_i = N_o \underline{v}_i$ are functions introduced for shorthand notation.

Substitution of (6.3) into (6.1) to eliminate n_e yields

$$-i\omega \underline{V}_e + \frac{u_e^2}{i\omega} \underline{\nabla} (\underline{\nabla} \cdot \underline{V}_e) = \frac{-e}{m_e} N_o \underline{E} + \underline{F}_e - \frac{u_e^2}{i\omega} \underline{\nabla} S_e \qquad (6.7)$$

while similarly, from (6.2) and (6.4) we have

$$- i\omega \underline{V}_i + \frac{u_i^2}{i\omega} \underline{\nabla}(\underline{\nabla} \cdot \underline{V}_i) = \frac{e}{m_i} N_o \underline{E} + \underline{F}_i - \frac{u_i^2}{i\omega} \underline{\nabla} S_i \quad . \tag{6.8}$$

From these two equations $\underline{E}$ may be eliminated to give us

$$m_e \left[- i\omega \underline{V}_e + \frac{u_e^2}{i\omega} \underline{\nabla}(\underline{\nabla} \cdot \underline{V}_e) \right] + m_i \left[- i\omega \underline{V}_i + \frac{u_i^2}{i\omega} \underline{\nabla}(\underline{\nabla} \cdot \underline{V}_i) \right]$$

$$= (m_e \underline{F}_e + m_i \underline{F}_i) - \left[\frac{m_e u_e^2}{i\omega} \underline{\nabla} S_e + \frac{m_i u_i^2}{i\omega} \underline{\nabla} S_i \right] \tag{6.9}$$

Taking the divergence of (6.6) and rearranging terms, we obtain

$$\underline{\nabla} \cdot \underline{V}_i = \underline{\nabla} \cdot \underline{V}_e - \frac{4\pi}{e} \underline{\nabla} \cdot \underline{J} + \frac{i\omega}{e} \underline{\nabla} \cdot \underline{E} \tag{6.10}$$

which, when substituted into (6.9) yields

$$m_e \left[- i\omega \underline{V}_e + \frac{u_e^2}{i\omega} \underline{\nabla}(\underline{\nabla} \cdot \underline{V}_e) \right] + m_i \left\{ - i\omega \underline{V}_i + \frac{u_i^2}{i\omega} \left[\underline{\nabla}(\underline{\nabla} \cdot \underline{V}_e) - \right. \right.$$

$$\left. \left. - \frac{4\pi}{e} \underline{\nabla}(\underline{\nabla} \cdot \underline{J}) + \frac{i\omega}{e} \underline{\nabla}(\underline{\nabla} \cdot \underline{E}) \right] \right\} = (m_e \underline{F}_e + m_i \underline{F}_i) -$$

$$\left[\frac{m_e u_e^2}{i\omega} \underline{\nabla} S_e + \frac{m_i u_i^2}{i\omega} \underline{\nabla} S_i \right] \quad . \tag{6.11}$$

Finally, from (6.5) and (6.6) we have

$$\underline{V}_i = \underline{V}_e + \frac{i\omega}{e} \underline{E} + \frac{c^2}{i\omega e} \underline{\nabla} \times (\underline{\nabla} \times \underline{E}) - \frac{4\pi}{e} \underline{J} \, , \tag{6.12}$$

so that $\underline{V}_i$ may be eliminated from (6.11) to give

$$-i\omega(m_e + m_i)\underline{V}_e + \frac{m_e u_e^2 + m_i u_i^2}{i\omega}\underline{\nabla}(\underline{\nabla}\cdot \underline{V}_e) =$$

$$-\frac{m_i c^2}{e}\left\{\frac{\omega^2}{c^2}\underline{E} - (\underline{\nabla}\times\underline{\nabla}\times\underline{E}) + \frac{u_i^2}{c^2}\underline{\nabla}\cdot\underline{E}\right\} - \frac{4\pi i\omega m_i}{e}\underline{J} + \frac{4\pi m_i u_i^2}{i\omega e}\underline{\nabla}(\underline{\nabla}\underline{J}) +$$

$$(m_e\underline{F}_e + m_i\underline{F}_i) - \left(\frac{m_e u_e^2}{i\omega}\underline{\nabla}S_e + \frac{m_i u_i^2}{i\omega}\underline{\nabla}S_i\right) \tag{6.13}$$

In (6.13) and (6.7), then, we have two equations which are similar in form and which may be written in tensor notation as

$$L_{j\alpha} V_\alpha^e = g_j$$

$$M_{j\alpha} V_\alpha^e = h_j \tag{6.14}$$

where

$$L_{j\alpha} = -i\omega(m_e + m_i)\delta_{j\alpha} + \frac{m_e u_e^2 + m_i u_i^2}{i\omega}\frac{\partial^2}{\partial x_j \partial x_\alpha} \tag{6.15}$$

$$g_j = \frac{m_i c^2}{e}\left[\left(1 - \frac{u_i^2}{c^2}\right)\frac{\partial^2}{\partial x_j \partial x_\alpha} - \left(\nabla^2 + \frac{\omega^2}{c^2}\right)\delta_{j\alpha}\right]E_\alpha - \frac{4\pi i\omega m_i}{e}J_j +$$

$$\frac{4\pi m_i u_i^2}{i\omega e}\frac{\partial^2 J_\alpha}{\partial x_j \partial x_\alpha} + (m_e F_j^e + m_i F_j^i) - \left(\frac{m_e u_e^2}{i\omega}\frac{\partial S_e}{\partial x_j} + \frac{m_i u_i^2}{i\omega}\frac{\partial S_i}{\partial x_j}\right)$$

(6.16)

$$M_{j\beta} = -i\omega\delta_{j\beta} + \frac{u_e^2}{i\omega}\frac{\partial^2}{\partial x_j \partial x_\beta} \qquad (6.17)$$

and

$$h_j = -\frac{e}{m_e} N_o \delta_{j\alpha} E_\alpha + F_j^e - \frac{u_e^2}{i\omega}\frac{\partial S_e}{\partial x_j} \qquad (6.18)$$

Since $L_{j\alpha}$ and $M_{j\alpha}$ commute,

$$M_{j\beta} g_\beta = L_{j\beta} h_\beta \qquad (6.19)$$

From (6.15) through (6.19), then, we obtain the vector equation

$$\frac{i\omega(m_e+m_i)}{m_e m_i}\frac{e^2 N_o}{c^2}\underline{E} - \frac{(m_e u_e^2 + m_i u_i^2)}{i\omega m_e m_i}\frac{e^2}{c^2}\underline{\nabla}(\underline{\nabla}\cdot N_o \underline{E}) = -i\omega\left(1-\frac{u_i^2}{c^2}\right)\underline{\nabla}(\underline{\nabla}\cdot\underline{E}) +$$

$$i\omega\left(\nabla^2+\frac{\omega^2}{c^2}\right)\underline{E} - \frac{u_e^2 u_i^2}{i\omega c^2}\underline{\nabla}(\nabla^2\underline{\nabla}\cdot\underline{E}) + i\omega\frac{u_e^2}{c^2}\underline{\nabla}(\underline{\nabla}\cdot\underline{E}) + \underline{X} + \underline{\nabla}R \qquad (6.20)$$

In this relationship we have introduced the source terms

$$\underline{X} = -4\pi\frac{\omega^2}{c^2}\underline{J} - \frac{i\omega e}{m_i c^2}(m_e \underline{F}_e + m_i \underline{F}_i) + \frac{i\omega e}{m_i c^2}(m_e+m_i)\underline{F}_e \qquad (6.21)$$

and

$$R(\underline{r}) = -\frac{4\pi}{c^2}(u_e^2+u_i^2)\underline{\nabla}\cdot\underline{J} - \frac{4\pi u_e^2 u_i^2}{\omega^2 c^2}\nabla^2\underline{\nabla J} + \frac{e u_e^2}{i\omega m_i c^2}\left[m_e\underline{\nabla}\cdot\underline{F}_e + m_i\underline{\nabla}\cdot\underline{F}_i\right] +$$

$$\frac{e}{m_i c^2}(m_e u_e^2 S_e + m_i u_i^2 S_i) + \frac{e u_e^2}{m_i \omega^2 c^2}\left[m_e u_e^2 \nabla^2 S_e + m_i u_i^2 \nabla^2 S_i\right] \tag{6.22}$$

We now assume that the electric field is composed of a longitudinal and transverse part, to wit

$$\underline{E} = -\underline{\nabla}\phi + \frac{i\omega}{c}\underline{\nabla}\mathrm{x}\,\underline{Z} = -\underline{\nabla}\phi + \frac{i\omega}{c}\underline{A} \tag{6.23}$$

and

$$\underline{H} = \underline{\nabla}\mathrm{x}(\underline{\nabla}\mathrm{x}\,\underline{Z}) = \underline{\nabla}\mathrm{x}\,\underline{A} \tag{6.24}$$

With this notation, (6.20) becomes

$$\frac{i\omega(m_e+m_i)}{m_e m_i}\frac{e^2 N_o}{c^2}\left[-\underline{\nabla}(N_o\phi) + (\underline{\nabla}N_o)\phi + \frac{i\omega}{c}\underline{A}\right] -$$

$$-\frac{m_e u_e^2 + m_i u_i^2}{i\omega m_e m_i}\frac{e^2}{c^2}\underline{\nabla}\,\underline{\nabla}\cdot\left[-N_o\underline{\nabla}\phi + \frac{i\omega}{c}N_o\underline{A}\right] =$$

$$i\omega\left(1-\frac{u_e^2+u_i^2}{c^2}\right)\underline{\nabla}\,\nabla^2\phi + i\omega\left(\nabla^2+\frac{\omega^2}{c^2}\right)\left[-\underline{\nabla}\phi + \frac{i\omega}{c}\underline{A}\right] +$$

$$\frac{u_e^2 u_i^2}{i\omega c^2}\underline{\nabla}\nabla^4\phi + \underline{X} + \underline{\nabla}R \tag{6.25}$$

The gauge condition is thus

$$-\frac{i\omega(m_e+m_i)}{m_e m_i}\frac{e^2 N_o}{c^2}\phi - \frac{m_e u_e^2+m_i u_i^2}{i\omega m_e m_i}\frac{e^2}{c^2}\left[-N_o\nabla^2\phi - N_o'\phi' + \frac{i\omega}{c}N_o'A_z\right] =$$

$$\frac{u_e^2 u_i^2}{i\omega c^2}\nabla^4\phi - \frac{i\omega}{c^2}(u_e^2+u_i^2)\nabla^2\phi - \frac{i\omega^3}{c^2}\phi + R(\underline{r}) \qquad (6.26)$$

where we have assumed that N_o varies only in the direction normal to the conducting surface. The remaining part of (6.25) yields

$$\frac{i\omega(m_e+m_i)}{m_e m_i}\frac{e^2}{c^2}N_o'\underline{n}\phi - \frac{m_e+m_i}{m_e m_i}\frac{i\omega e^2}{c^3}N_o\underline{A} = -\frac{\omega^2}{c}(\nabla^2+\frac{\omega^2}{c^2})\underline{A}+\underline{X} \qquad (6.27)$$

In these two equations we observe that coupling of the longitudinal and transverse waves will occur if $N_o' \neq 0$, so that the two types of wave propagation are independent only in a homogeneous plasma. In addition, it can be seen that $R(\underline{r})$ and $\underline{X}(\underline{r})$ as defined in (6.22) and (6.21) are, essentially, sources of longitudinal and transverse propagations, respectively.

It is evident that (6.26) and (6.27) can be uncoupled with little difficulty but that solving the resulting equations analytically with variable N_o can be done only for the simplest of cases. In the following discussion, then, we choose $N_o' = 0$ and write (6.26) and (6.27) as

$$\nabla^4\phi + \frac{b}{a_1}\nabla^2\phi + \frac{c}{a_1}\phi = (\nabla^2+\alpha^2)(\nabla^2+\beta^2)\phi = -\frac{i\omega c^2}{u_e^2 u_i^2}R(\underline{r}) \qquad (6.28)$$

and

$$(\nabla^2+\sigma^2)\underline{A} = \frac{c}{\omega^2}\underline{X}(\underline{r}) \qquad (6.29)$$

where

$$\alpha^2 = \frac{b}{2a_1} - \frac{1}{2}\left[\left(\frac{b}{a_1}\right)^2 - \frac{4c}{a_1}\right]^{1/2} \tag{6.30}$$

$$\beta^2 = \frac{b}{2a_1} + \frac{1}{2}\left[\left(\frac{b}{a_1}\right)^2 - \frac{4c}{a_1}\right]^{1/2} \tag{6.31}$$

and

$$\sigma^2 = \frac{\omega^2}{c^2} - \frac{m_e + m_i}{m_e m_i} \frac{e^2 N_o}{c^2} \tag{6.32}$$

If we assume that the electron kinetic energy is at least as great as the ion kinetic energy, say

$$m_e u_e^2 = \lambda m_i u_i^2 , \qquad \lambda > 1 \tag{6.33}$$

then from (6.26)

$$\frac{b}{a_1} = \frac{\omega^2}{u_i^2}\left(1 + \frac{m_e}{\lambda m_i}\right) - \frac{(1+\lambda)}{u_e^2}\frac{e^2 N_o}{m_e} \simeq \frac{\omega^2}{u_i^2}\left[1 - \frac{(1+\lambda)}{\lambda}\frac{m_e}{m_i}\frac{\omega_{pe}^2}{\omega^2}\right] \tag{6.34}$$

and

$$\frac{c}{a_1} = \frac{m_e}{\lambda m_i}\frac{\omega^4}{u_i^4}\left[1 - \frac{\omega_{pe}^2 + \omega_{pi}^2}{\omega^2}\right] \simeq \frac{m_e}{\lambda m_i}\frac{\omega^4}{u_i^4}\left[1 - \frac{\omega_{pe}^2}{\omega^2}\right] \tag{6.35}$$

Since $\left(\frac{b}{a_1}\right)^2$ is greater than $\left(\frac{c}{a_1}\right)$ by the order of $\frac{m_e}{\lambda m_i}$ (except when $\omega^2 = (1+\lambda)\frac{m_e}{m_i}\omega_{pe}^2$)

$$\alpha^2 \simeq \frac{c/a_1}{b/a_1} = \frac{\omega^2}{u_e^2} \frac{\left(1 - \frac{\omega_{pe}^2}{\omega^2}\right)}{\left[1 - \frac{(1+\lambda)m_e}{\lambda m_i} \frac{\omega_{pe}^2}{\omega^2}\right]} \tag{6.36}$$

and

$$\beta^2 \simeq \frac{b}{a_1} = \frac{\omega^2}{u_i^2} \left[1 - \frac{(1+\lambda)m_e}{\lambda m_i} \frac{\omega_{pe}^2}{\omega^2}\right] \tag{6.37}$$

Also,

$$\sigma^2 = \frac{1}{c^2} \left[\omega^2 - (\omega_{pe}^2 + \omega_{pi}^2)\right] \simeq \frac{\omega^2 - \omega_{pe}^2}{c^2} \tag{6.38}$$

(In subsequent sections, we shall delete the subscript (e) on ω_{pe}^2 since $\omega_p^2 \simeq \omega_{pe}^2 + \omega_{pi}^2 \simeq \omega_{pe}^2$.) We thus have, in (6.36-6.38) three propagation constants which vary inversely as u_e, u_i and c respectively.

Turning now to the boundary conditions, we seek to determine what effect the vanishing of $\underline{n} \cdot \underline{V}_i = \underline{n} \cdot \underline{V}_e = 0$ will have upon $\underline{A}$ and ϕ. From (6.7) and (6.8) we obtain

$$\underline{\nabla}(\underline{\nabla} \cdot \underline{V}_e) + \frac{\omega^2}{u_e^2} \underline{V}_e = -\frac{i\omega e}{m_e u_e^2} N_o \underline{E} + \frac{i\omega}{u_e^2} \underline{F}_e - \underline{\nabla} S_e \tag{6.39}$$

and

$$\underline{\nabla}(\underline{\nabla} \cdot \underline{V}_i) + \frac{\omega^2}{u_i^2} \underline{V}_i = \frac{i\omega e}{m_i u_i^2} N_o \underline{E} + \frac{i\omega}{u_i^2} \underline{F}_i - \underline{\nabla} S_i \tag{6.40}$$

Subtraction of (6.39) from (6.40) yields

$$\underline{\nabla}\left[\underline{\nabla}\cdot(\underline{V}_i-\underline{V}_e)\right] + \omega^2\left[\frac{\underline{V}_i}{u_i^2}-\frac{\underline{V}_e}{u_e^2}\right] = i\omega e\left[\frac{1}{m_e u_e^2}+\frac{1}{m_i u_i^2}\right] N_o \underline{E} + i\omega\left[\frac{\underline{F}_i}{u_i^2}-\frac{\underline{F}_e}{u_e^2}\right] - (\underline{\nabla S}_i-\underline{\nabla S}_e) \tag{6.41}$$

From (6.10), however,

$$\underline{\nabla}\left[\underline{\nabla}\cdot(\underline{V}_i-\underline{V}_e)\right] = \frac{i\omega}{e}\underline{\nabla}(\underline{\nabla}\cdot\underline{E}) - \frac{4\pi}{e}\underline{\nabla}(\underline{\nabla}\cdot\underline{J}) \tag{6.42}$$

which, when substituted into (6.41) yields

$$\omega^2\left[\frac{\underline{V}_i}{u_i^2}-\frac{\underline{V}_e}{u_e^2}\right] = i\omega e\left[\frac{1}{m_e u_e^2}+\frac{1}{m_i u_i^2}\right] N_o \underline{E} - \frac{i\omega}{e}\underline{\nabla}(\underline{\nabla}\cdot\underline{E}) + \frac{4\pi}{e}\underline{\nabla}(\underline{\nabla}\cdot\underline{J}) + i\omega\left[\frac{\underline{F}_i}{u_i^2}-\frac{\underline{F}_e}{u_e^2}\right] - (\underline{\nabla S}_i-\underline{\nabla S}_e) \tag{6.43}$$

Finally, $\underline{V}_i$ from (6.12) may be used in (6.43) to obtain

$$\omega^2\left(\frac{1}{u_i^2}-\frac{1}{u_e^2}\right)\underline{V}_e = i\omega e\left[\frac{1}{m_i u_i^2}+\frac{1}{m_e u_e^2}\right] N_o \underline{E} - \frac{i\omega}{e}\underline{\nabla}(\underline{\nabla}\cdot\underline{E}) + \frac{i\omega}{e}\frac{c^2}{u_i^2}\left[\underline{\nabla}\times(\underline{\nabla}\times\underline{E}) - \frac{\omega^2}{c^2}\underline{E}\right] + \frac{4\pi}{e}\underline{\nabla}(\underline{\nabla}\cdot\underline{J}) + i\omega\left[\frac{\underline{F}_i}{u_i^2}-\frac{\underline{F}_e}{u_e^2}\right] - (\underline{\nabla S}_i-\underline{\nabla S}_e) + \frac{4\pi}{e}\frac{\omega^2}{u_i^2}\underline{J} \tag{6.44}$$

Similarly using (6.12) to eliminate $\underline{V}_e$ in (6.43) results in

$$\omega^2\left(\frac{1}{u_i^2} - \frac{1}{u_e^2}\right)\underline{V}_i = i\omega e\left[\frac{1}{m_i u_i^2} + \frac{1}{m_e u_e^2}\right] N_o \underline{E} - \frac{i\omega}{e}\underline{\nabla}(\underline{\nabla}\cdot\underline{E}) + \frac{i\omega}{e}\frac{c^2}{u_e^2}\left[\underline{\nabla}\mathrm{x}(\underline{\nabla}\mathrm{x}\underline{E}) - \frac{\omega^2}{c^2}\underline{E}\right] +$$

$$\frac{4\pi}{e}\underline{\nabla}(\underline{\nabla}\cdot\underline{J}) + i\omega\left[\frac{\underline{F}_i}{u_i^2} - \frac{\underline{F}_e}{u_e^2}\right] - (\underline{\nabla S}_i - \underline{\nabla S}_e) + \frac{4\pi}{e}\frac{\omega^2}{u_e^2}\underline{J} \qquad (6.45)$$

If the normal components of $\underline{V}_i$ and $\underline{V}_e$ are to vanish at the surface, then the two following conditions must hold on the boundary

$$i\omega e\left[\frac{1}{m_i u_i^2} + \frac{1}{m_e u_e^2}\right] N_o E_n - \frac{i\omega}{e}\frac{\partial}{\partial n}(\underline{\nabla}\cdot\underline{E}) + \frac{4\pi}{e}\frac{\partial}{\partial n}\underline{\nabla}\cdot\underline{J} + i\omega\underline{n}\cdot\left[\frac{\underline{F}_i}{u_i^2} - \frac{\underline{F}_e}{u_e^2}\right] - \frac{\partial}{\partial n}(S_i - S_e) = 0 \qquad (6.46)$$

and

$$i\omega\underline{n}\cdot\left[\underline{\nabla}\mathrm{x}(\underline{\nabla}\mathrm{x}\underline{E}) - \frac{\omega^2}{c^2}\underline{E}\right] + 4\pi\frac{\omega^2}{c^2}J_n = 0 \qquad (6.47)$$

It is a simple matter to use (6.23) to obtain two mixed boundary conditions for A_n and ϕ at the surface. The third condition, namely that $(\underline{n}\mathrm{x}\underline{E}) = 0$ on the conductor, merely requires that

$$-\frac{\partial\phi}{\partial t} + \frac{i\omega}{c}A_t = 0 \ , \qquad (6.48)$$

while for the final one, $\underline{n}\cdot\underline{H} = 0$, we have

$$\underline{n}\cdot(\underline{\nabla}\mathrm{x}\underset{\sim}{A}) = 0 \qquad (6.49)$$

6.3 Vertical Oscillating Dipole Above A Perfectly Conducting Plane

For an oscillating vertical dipole we have

$$\underline{J} = \underline{i}_z J = e\eta\omega_o \sin\omega_o t\, \delta(x)\, \delta(y)\, \delta(z-z_o) \qquad (6.50)$$

Thus, $\underline{X}$ in (6.21) has only a z- component which we denote by $Z(\underline{r})$, so that (6.29) may be written

$$(\nabla^2+\sigma^2) A_x = 0$$

$$(\nabla^2+\sigma^2) A_y = 0 \qquad (6.51)$$

$$(\nabla^2+\sigma^2) A_z = \frac{c}{\omega^2} Z(\underline{r})$$

The Fourier transform of (6.51), assuming that a variable $u(\underline{r})$ transforms according to

$$u(\underline{k}, z) = \iint_{-\infty}^{\infty} u(\underline{r}) e^{-i\underline{k}\cdot\underline{r}} dx\, dy \qquad (6.52)$$

gives us

$$A_x'' - (k^2-\sigma^2) A_x = 0 \qquad (6.53a)$$

$$A_y'' - (k^2-\sigma^2) A_y = 0 \qquad (6.53b)$$

$$A_z'' - (k^2-\sigma^2) A_z = \frac{c}{\omega^2} Z(\underline{k}, z) \qquad (6.53c)$$

Likewise, from (6.28) we obtain

$$L\phi = \left[\frac{d^2}{dz^2} - (k^2-\alpha^2)\right]\left[\frac{d^2}{dz^2} - (k^2-\beta^2)\right]\phi = -\frac{i\omega c^2}{u_e^2 u_i^2} R(\underline{k}, z) \qquad (6.54)$$

The variables A_x and A_y must satisfy the radiation condition. Thus

$$A_x(k, z) = c_1 e^{-\sqrt{k^2-\sigma^2}\, z}$$

$$A_y(k, z) = c_2 e^{-\sqrt{k^2-\sigma^2}\, z} \tag{6.55}$$

The condition $\underline{n} \cdot \underline{H} = 0$ requires that $k_y c_1 = k_x c_2$. This, however, means that $\underline{n} \cdot \underline{H} \equiv 0$, and since this adds nothing to our solution, we may choose $c_1 = c_2 = 0$.

It remains, then, for us to solve (6.53c) and (6.54) for A_z and ϕ. Taking the first of these, we consider the inhomogeneous equation

$$\frac{d^2 g}{dz^2} - (k^2 - \sigma^2) g = \delta(z-\xi) \tag{6.56}$$

which satisfies the condition $g(0, \xi) = 0$ and the radiation condition. Thus

$$g(z, \xi) = \begin{cases} \dfrac{\sinh \sqrt{k^2-\sigma^2}\, z}{\sqrt{k^2-\sigma^2}}\, e^{-\sqrt{k^2-\sigma^2}\, \xi} & z < \xi \\[2ex] \dfrac{\sinh \sqrt{k^2-\sigma^2}\, \xi}{\sqrt{k^2-\sigma^2}}\, e^{-\sqrt{k^2-\sigma^2}\, z} & \xi < z \end{cases} \tag{6.57}$$

Then

$$A_z(z,k^2) = \frac{c}{\omega^2}\int_0^\infty g(z,\xi)\,Z(\xi)\,d\xi + \psi(0)e^{-\sqrt{k^2-\sigma^2}\,z} \tag{6.58}$$

where the constant $\psi(0)$ must be determined from the boundary conditions.

For the solution of (6. 54) which satisfies homogeneous boundary conditions, we consider the auxiliary equation

$$LG = \left[\frac{d^2}{dz^2} - (k^2-\alpha^2)\right]\left[\frac{d^2}{dz^2} - (k^2-\beta^2)\right] G(\xi, z) = \delta(z-\xi) \tag{6.59}$$

which satisfies the radiation condition and the requirements

$$\begin{aligned} &G(0,\xi) = 0 \\ &\frac{\partial G}{\partial z}(0,\xi) = 0 \end{aligned} \tag{6.60}$$

$G(z, \xi)$ is constructed from the four solutions of

$$L\phi = \left[\frac{d^2}{dz^2} - (k^2-\alpha^2)\right]\left[\frac{d^2}{dz^2} - (k^2-\beta^2)\right]\phi = 0, \tag{6.61}$$

namely

$$\phi_1 = e^{-\sqrt{k^2-\alpha^2}\,z} = e^{-m_1 z} \tag{6.62a}$$

$$\phi_2 = e^{\sqrt{k^2-\alpha^2}\,z} = e^{m_1 z} \tag{6.62b}$$

$$\phi_3 = e^{-\sqrt{k^2-\beta^2}\,z} = e^{-m_2 z} \tag{6.62c}$$

$$\phi_4 = e^{\sqrt{k^2-\beta^2}\, z} = e^{m_2 z} \tag{6.62d}$$

In order to interpret the radiation condition, we note from (6.54) and (6.59) that we obtain

$$\left[\phi LG - GL\phi\right] = \phi\, \delta(z-\xi) + \frac{i\omega c^2}{u_e^2 u_i^2} RG \tag{6.63}$$

Integrating from $\xi=0$ to ∞, we have

$$\phi(z) = \frac{-i\omega c^2}{u_e^2 u_i^2} \int_0^\infty G(\xi, z) R(\xi)\, d\xi + \int_0^\infty \left[\phi(\xi) L_\xi G - G L_\xi \phi(\xi)\right] d\xi \tag{6.64}$$

The final term in this equation will vanish if $G(\xi, z)$ varies as a linear combination of $\phi_1(z)$ and $\phi_3(z)$ as $z \longrightarrow \infty$. Thus

$$G(\xi, z) = \begin{cases} a_1\phi_1(z) + a_2\phi_2(z) + a_3\phi_3(z) + a_4\phi_4(z) & z < \xi \\ b_1\phi_1(z) + b_3\phi_3(z) & z > \xi \end{cases} \tag{6.65}$$

This, then, is the form which satisfies the radiation condition.

Using the continuity requirements at $z=\xi$ and the conditions stated in (6.60), it can be shown that

$$G(z,\xi)=\frac{1}{2(m_2-m_1)^2(m_1+m_2)m_1m_2}\Big\{\Big[m_2(m_2-m_1)e^{m_1z}-m_2(m_1+m_2)e^{-m_1z}$$

$$+2m_1m_2e^{-m_2z}\Big]e^{-m_1\xi}+\Big[m_1(m_1-m_2)e^{m_2z}-m_1(m_1+m_2)e^{-m_2z}$$

$$+2m_1m_2e^{-m_1z}\Big]e^{-m_2\xi}\Big\} \tag{6.66}$$

for $z<\xi$. When $z>\xi$, the roles of z and ξ in (6.66) are interchanged. Hence

$$\phi(z)=\frac{-i\omega c^2}{u_e^2u_i^2}\int_0^\infty G(\xi,z)R(\xi)\,d\xi \tag{6.67}$$

is the solution of (6.54) which satisfies the homogeneous boundary conditions. In order to satisfy the inhomogeneous boundary conditions we merely add two functions such that

$$\phi(z)=-\frac{i\omega c^2}{u_e^2u_i^2}\int_0^\infty G(\xi,z)R(\xi)\,d\xi+\phi(0)G_1(z)+\phi'(0)G_2(z)\,, \tag{6.68}$$

where $G_1(z)$ and $G_2(z)$ satisfy (6.60) and

$$G_1(0)=1 \qquad G_1'(0)=0 \tag{6.69}$$

$$G_2(0)=0 \qquad G_2'(0)=1 \tag{6.70}$$

The three boundary conditions in (6.46-6.48) require that

$$\phi'''(0)-\left[(1+\lambda)\frac{\omega_p^2}{u_e^2}+K^2\right]\phi'(0)+\frac{i\omega}{c}(1+\lambda)\frac{\omega_p^2}{u_e^2}A_z(0)=0 \qquad (6.71)$$

$$\frac{i\omega}{c}\phi'(0)+\frac{\omega_p^2}{c^2}A_z(0)=-\frac{c}{\omega^2}Z(0,k^2) \qquad (6.72)$$

$$\phi(0)=0 \qquad (6.73)$$

The last of these allows us to set $\phi(0)$ on the right hand side of (6.68) equal to zero. The problem remains, then, to determine the other two constants. To do this, we consider $\phi(z)$ when $z<z_o$. From (6.22), (6.50) and (6.68) we have

$$\phi(z)\simeq\frac{2\pi^2 e\eta\omega\omega_o\left[\delta(\omega+\omega_o)-\delta(\omega-\omega_o)\right]}{u_i^2}\frac{\omega_p^2}{\omega^2}\left\{\frac{m_1t_1(z)e^{-m_1z_o}}{\left[1-\left(\frac{1+\lambda}{\lambda}\right)\frac{m_e}{m_i}\frac{\omega_p^2}{\omega^2}\right]}+\right.$$

$$\left.\left(\frac{1+\lambda}{\lambda}\right)\frac{m_e}{m_i}m_2t_2(z)e^{-m_2z_o}\right\}+\frac{\phi'(0)}{m_2-m_1}\left[e^{-m_1z}-e^{-m_2z}\right] \qquad (6.74)$$

For simplicity, in (6.74) we have let $G(\xi,z)$, as specified in (6.66), be denoted by

$$G(\xi,z)=t_1(z)e^{-m_1\xi}+t_2(z)e^{-m_2\xi} \qquad (6.75)$$

To evaluate $\phi'''(0)$ we note that

$$t_1'''(0) = \frac{-m_2}{m_1 + m_2} \tag{6.76a}$$

and

$$t_2'''(0) = \frac{m_1}{m_1 + m_2} \tag{6.76b}$$

Thus,

$$\phi'''(0) = (m_1^2 + m_1 m_2 + m_2^2)\phi'(0) + e\eta\omega\omega_o \frac{\left[\delta(\omega+\omega_o) - \delta(\omega-\omega_o)\right]}{u_i^2} \frac{\omega_p^2}{\omega^2} \frac{m_1 m_2}{m_1 + m_2}$$

$$\left\{ \frac{1+\lambda}{\lambda} \frac{m_e}{m_i} e^{-m_2 z_o} - \frac{e^{-m_1 z_o}}{\left[1 - \frac{1+\lambda}{\lambda} \frac{m_e}{m_i} \frac{\omega_p^2}{\omega^2}\right]} \right\} = (m_1^2 + m_1 m_2 + m_2^2)\phi'(0) + F(\omega, k^2, z_o) \tag{6.77}$$

With the aid of (6.77), the conditions in (6.71) and (6.72) reduce to

$$\begin{bmatrix} m_1^2 + m_1 m_2 + m_2^2 - \left[\frac{(1+\lambda)\omega_p^2}{u_e^2} + k^2\right] , & \frac{i\omega}{c} \frac{(1+\lambda)\omega_p^2}{u_e^2} \\ \frac{i\omega}{c} & \frac{\omega_p^2}{c^2} \end{bmatrix} \begin{bmatrix} \phi'(0) \\ A_z(0) \end{bmatrix} = \begin{bmatrix} -F(\omega, k^2, z_o) \\ -\frac{c}{\omega^2} Z(0, k^2) \end{bmatrix} \tag{6.78}$$

For the dipole at $z=z_o$, it is clear from (6.21) that $Z(0,k^2)=0$, which will simplify matters somewhat.

If we denote the determinant of the matrix in (6.78) by Δ, we have

$$\Delta = \frac{\omega_p^2}{c^2}\left\{k^2-\alpha^2-\frac{\omega^2}{u_i^2}+\sqrt{(k^2-\alpha^2)(k^2-\beta^2)}+(1+\lambda)\frac{\omega^2}{u_e^2}\right. \tag{6.79}$$

Thus, for example,

$$A_z(0)=\frac{i\omega}{c}\,\frac{F(\omega,k^2,z_o)}{\Delta} \tag{6.80}$$

A complete analytic solution with this rather complicated expression is very difficult and has not been successfully obtained so far. We proceed, however, to obtain a formal solution which may be evaluated numerically if necessary.

For $z<z_o$ as we are assuming,

$$A_z(z) = -4\pi e\,\eta\omega\omega_o\frac{\pi}{i}\left[\delta(\omega+\omega_o)-\delta(\omega-\omega_o)\right]\frac{\sinh\left[\sqrt{(k^2-\sigma^2)}\;z\right]}{\sqrt{k^2-\sigma^2}}\;\times$$

$$e^{-\sqrt{k^2-\sigma^2}\;z_o}+A_z(0)\,e^{-\sqrt{k^2-\sigma^2}\;z} \tag{6.81}$$

Performing the spacial inverse transform first, we have

$$A_z(\omega,\underline{r},z)=\frac{1}{4\pi^2}\iint_{-\infty}^{\infty}A_z(\omega,\underline{k},z)\,e^{i\underline{k}\cdot\underline{r}}dk_x dk_y \tag{6.82}$$

Since $\underline{k}$ appears in $A_z(\omega, \underline{k}, z)$ only as k^2, it is more convenient to let

$$k_x = k \cos t \tag{6.83a}$$

$$k_y = k \sin t \tag{6.83b}$$

$$x = r \cos \phi \tag{6.83c}$$

$$y = r \sin \phi \tag{6.83d}$$

so that

$$\underline{k} \cdot \underline{r} = kr \cos (\phi - t) \tag{6.84}$$

and

$$dk_x dk_y = kdkdt \tag{6.85}$$

Thus, (6.82) becomes

$$A_z(\omega, \underline{r}, z) = \frac{1}{4\pi^2} \int_0^\infty kdkA_z(\omega, k^2, z) \int_0^{2\pi} e^{ikr \cos(t-\phi)} dt =$$

$$\frac{1}{2\pi} \int_0^\infty A_z(\omega, k^2, z) J_0(kr) kdk \tag{6.86}$$

Substitution of the expression (6.81) into (6.86) yields

$$A_z(\omega, \underline{r}, z) = -e\eta\omega\omega_o \frac{\pi}{i}\left[\delta(\omega+\omega_o) - \delta(\omega-\omega_o)\right] \int_0^\infty \left\{ \frac{e^{-(z_o - z)\sqrt{k^2-\sigma^2}}}{\sqrt{k^2-\sigma^2}} - \frac{e^{-(z_o + z)\sqrt{k^2-\sigma^2}}}{\sqrt{k^2-\sigma^2}} \right\} \times$$

$$J_o(kr)kdk + \frac{1}{2\pi}\int_0^\infty A_z(0)\,e^{-\sqrt{k^2-\sigma^2}\,z} J_o(kr)kdk = -e\eta\omega\omega_o \frac{\pi}{i} \quad \times$$

$$\left[\delta(\omega+\omega_o) - \delta(\omega-\omega_o)\right]\left\{ \frac{e^{i\sigma\sqrt{r^2+(z_o-z)^2}}}{\sqrt{r^2+(z_o-z)^2}} - \frac{e^{i\sigma\sqrt{r^2+(z_o+z)^2}}}{\sqrt{r^2+(z_o+z)^2}} \right. \quad +$$

$$\frac{1}{2\pi}\int_0^\infty A_z(0)\,e^{-\sqrt{k^2-\sigma^2}\,z} J_o(kr)\,kdk \qquad (6.87)$$

To obtain the time dependent expression for $A_r(\underline{r}, z, t)$, we perform the inverse transformation of (6.87) which yields

$$A_z(\underline{r}, z, t) = e\eta\omega_o^2 \sin\omega_o t \left\{ \frac{e^{i\sigma_o \sqrt{r^2+(z_o-z)^2}}}{\sqrt{r^2+(z_o-z)^2}} - \frac{e^{i\sigma_o \sqrt{r^2+(z_o+z)^2}}}{\sqrt{r^2+(z_o+z)^2}} \right\} +$$

$$\left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \int_0^{\infty} A_z(0) e^{-\sqrt{k^2+\sigma^2}\, z} J_o(kr) k dk \qquad (6.88)$$

In this equation ,

$$\sigma_o = \frac{1}{c}\sqrt{\omega^2 - \omega_p^2} \qquad (6.89)$$

is the plasma propagation constant for this two-fluid plasma. The first two terms in (6.88), then, correspond to the spherical waves which are generated by the source dipole at $z=z_o$ and its image dipole located at $z=-z_o$. The third term is more difficult to analyze exactly, owing to the complicated nature of the integrand.

6.4 Vertical Oscillating Dipole Above a Perfectly Conducting Sphere

As in (6.50), we have

$$\underline{J} = \underline{i}_r e\eta \sin\omega_o t \frac{\delta(r-r_o)\delta(\theta)}{2\pi r^2 \sin\theta} = \underline{i}_z e\eta \sin\omega_o t \frac{\delta(r-r_o)\delta(\theta)}{2\pi r^2 \sin\theta} \qquad (6.90)$$

where we have assumed the dipole to be oriented along the z-axis. This provides us with cylindrical symmetry and it develops, as in the previous section, that $A_\rho = A_\phi \equiv 0$. Thus, we again have to deal only with A_z. There will, then, be r and θ components of $\underline{A}$, which are given by

$$
\begin{aligned}
A_r &= A_z \cos\theta \\
A_\phi &= A_z \sin\theta
\end{aligned}
\tag{6.91}
$$

We introduce now the Legendre transform

$$
T = 2\pi \int_0^\pi d\theta \sin\theta \, P_n(\cos\theta) \tag{6.92}
$$

such that $T: f(r,\theta) = f_n(r)$. The inverse of T, applied to $f_n(r)$, is found to be

$$
T^{-1} f_n(r) = f(r,\theta) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{2n+1}{2} P_n(\cos\theta) f_n(r) = -\frac{1}{2} \int_c \frac{\nu P_{\nu-1/2}(-\cos\theta) f_{\nu-1/2}(r) d\nu}{\cos \pi\nu} \tag{6.93}
$$

For the purpose of solving (6.28) and (6.29) we note that

$$
T: \nabla^2 f = \left[\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) - \frac{n(n+1)}{r^2} \right] f_n(r) \tag{6.94}
$$

and

$$
T: \nabla^4 f = \left[\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) - \frac{n(n+1)}{r^2} \right] \left[\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) - \frac{n(n+1)}{r^2} \right] f_n(r) \tag{6.95}
$$

We thus have to solve the equations

$$\left\{\frac{1}{r^2}\frac{d}{dr}(r^2\frac{d}{dr})+\left[\alpha^2-\frac{n(n+1)}{r^2}\right]\right\}\left\{\frac{1}{r^2}\frac{d}{dr}(r^2\frac{d}{dr})+\left[\beta^2-\frac{n(n+1)}{r^2}\right]\right\}\phi_n=-\frac{i\omega c^2}{u_e^2 u_i^2}R_n(r) \tag{6.96}$$

and

$$\left\{\frac{1}{r^2}\frac{d}{dr}(r^2\frac{d}{dr})+\left[\sigma^2-\frac{n(n+1)}{r^2}\right]\right\}\psi_n(r)=\frac{c^2}{\omega}X_n(r) \tag{6.97}$$

where we will let $\psi(r,\theta) = A_z(r,\theta)$ in order to avoid subscript confusion. The four independent solutions to the homogeneous equation corresponding to (6.96) are

$$\begin{aligned} p_1(r) &= h_n^{(1)}(\alpha r) \\ p_2(r) &= h_n^{(2)}(\alpha r) \\ p_3(r) &= h_n^{(1)}(\beta r) \\ p_4(r) &= h_n^{(2)}(\beta r) \end{aligned} \tag{6.98}$$

while for (6.97) we have

$$\begin{aligned} q_1(r) &= h_n^{(1)}(\sigma r) \\ q_2(r) &= h_n^{(2)}(\sigma r) \end{aligned} \tag{6.99}$$

The constriction of the Green's function for (6.96) which satisfies the two conditions

$$G_n(a) = 0$$
$$\frac{\partial G_n}{\partial r}(a) = 0 \qquad (6.100)$$

is a straightforward, though complicated, process. It develops, after several pages of algebra that will be omitted, that

$$G_n(\xi, r) = \frac{\pi i \xi^2}{4(\beta^2 - \alpha^2)\left[p_1(a)p_3'(a) - p_3(a)p_1'(a)\right]} \left\{\lambda_1(r)p_1(\xi) + \lambda_3(r)p_3(\xi)\right\} \qquad (6.101)$$

for $\xi < r$, where

$$\lambda_1(r) = \left[p_2(a)p_3'(a) - p_3(a)p_2'(a)\right] p_1(r) - \left[p_1(a)p_3'(a) - p_3(a)p_1'(a)\right] p_2(r)$$
$$+ \left[p_1(a)p_2'(a) - p_2(a)p_1'(a)\right] p_3(r) \qquad (6.102)$$

and

$$\lambda_3(r) = \left[p_3(a)p_4'(a) - p_4(a)p_3'(a)\right] p_1(r) - \left[p_1(a)p_4'(a) - p_4(a)p_1'(a)\right] p_3(r)$$
$$+ \left[p_1(a)p_3'(a) - p_3(a)p_1'(a)\right] p_4(r) \qquad (6.103)$$

When $\xi > r$, the roles of ξ and r in the bracketed expression of (6.101) are interchanged.

The corresponding Green's function for (6.97) satisfying $g_n(a) = 0$ is given by

$$g_n(\xi, r) = \frac{\pi\xi^2}{4i} \frac{q_1(\xi)}{q_1(a)} \left[q_2(a)q_1(r) - q_1(a)q_2(r)\right] \tag{6.104}$$

when $r < \xi$, while

$$g_n(\xi, r) = \frac{\pi\xi^2}{4i} \frac{q_1(r)}{q_1(a)} \left[q_2(a)q_1(\xi) - q_1(a)q_2(\xi)\right] \tag{6.105}$$

for $r > \xi$. The complete expressions for $\phi_n(r)$ and $\psi_n(r)$ are obtained in manner analogous to the one used in treating the planar case, and it may be shown that

$$\phi_n(r) = -\frac{i\omega c^2}{u_e^2 u_i^2} \int_a^\infty G_n(\xi, r)R_n(\xi)d\xi + \phi_n(a)G_n^{(1)}(r) + \phi_n'(a)G_n^{(2)}(r) \tag{6.106}$$

where

$$G_n^{(1)}(r) = \frac{p_3'(a)p_1(r) - p_1'(a)p_3(r)}{p_1(a)p_3'(a) - p_3(a)p_1'(a)} \tag{6.107}$$

and

$$G_n^{(2)}(r) = -\frac{p_3(a)p_1(r) - p_1(a)p_3(r)}{p_1(a)p_3'(a) - p_3(a)p_1'(a)} \tag{6.108}$$

Likewise ,

$$\psi_n(r) = \frac{c^2}{\omega} \int_a^\infty g_n(\xi, r)Z_n(\xi) + \frac{h_n^{(1)}(\sigma r)}{h_n^{(1)}(\sigma a)} \psi_n(a) \; . \tag{6.109}$$

From (6.22), (6.90) and (6.92) we note that

$$-\frac{i\omega c^2}{u_e^2 u_i^2} R_n(\xi) = e\eta\omega_o \frac{\pi}{i}\left[\delta(\omega+\omega_o)-\delta(\omega-\omega_o)\right]\left\{4\pi i\omega\left(\frac{1}{u_e^2}+\frac{1}{u_i^2}\right)\frac{\delta'(\xi-r_o)}{\xi^2} - \right.$$

$$\left. \frac{4\pi}{i\omega}\left[\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d}{d\xi}\right)-\frac{n(n+1)}{\xi^2}\right]\frac{\delta'(\xi-r_o)}{\xi^2}\right\} \qquad (6.110)$$

so that

$$-\frac{i\omega c^2}{u_e^2 u_i^2}\int_a^\infty G_n(\xi, r)R_n(\xi)d\xi = -\frac{\pi^3 ie\eta\omega\omega_o\left[\delta(\omega+\omega_o)-\delta(\omega-\omega_o)\right]}{(\beta^2-\alpha^2)\left[p_1(a)p_3'(a)-p_3(a)p_1'(a)\right]} \times$$

$$\left\{\left[\frac{1}{u_e^2}+\frac{1}{u_i^2}-\frac{\alpha^2}{\omega^2}\right] p_1'(r_o)\lambda_1^n(r)+\left[\frac{1}{u_e^2}+\frac{1}{u_i^2}-\frac{\beta^2}{\omega^2}\right]p_3'(r_o)\lambda_3^n(r)\right\}$$

$$= c_1^n(a, r_o)\lambda_1^n(r)+c_3^n(a, r_o)\lambda_3^n(r) \qquad (6.111)$$

Similarly ,

$$\frac{c^2}{\omega} Z_n(\xi) = -\frac{4\pi^2}{i} e\,\eta\omega\omega_o\left[\delta(\omega+\omega_o)-\delta(\omega-\omega_o)\right]\frac{\delta(\xi-r_o)}{\xi^2} \qquad (6.112)$$

whereby

$$\frac{c^2}{\omega}\int_a^\infty g_n(\xi, r)Z_n(\xi) = \pi^3 e\eta\omega\omega_o\left[\delta(\omega+\omega_o)-\delta(\omega-\omega_o)\right] \quad \frac{q_1(r_o)}{q_1(a)}\alpha$$

$$\left[q_2(a)q_1(r)-q(a)q_2(r)\right] = K_1^n(a, r_o)q_1^n(r)-K_2^n(a, r_o)q_2^n(r) \qquad (6.113)$$

Using the definition

$$\phi(r,\theta)=\sum_{n=0}^{\infty}\frac{2n+1}{2}\,P_n(\cos\theta)\,\phi_n(r)\ , \tag{6.114}$$

we have, from (6.106) and (6.111) the result

$$\phi(r,\theta)=\sum_{n=0}^{\infty}\frac{2n+1}{2}\,P_n(\cos\theta)\left\{c_1^n\lambda_1^n(r)+c_3^n\lambda_3^n(r)+\phi_n(a)G_n^{(1)}(r)+\phi_n'(a)G_n^{(2)}(r)\right\} \tag{6.115}$$

Similarly , we obtain from (6.109) and (6.113)

$$\psi(r,\theta)=\sum_{n=0}^{\infty}\frac{2n+1}{2}\,P_n(\cos\theta)\left\{K_1^n q_1^n(r)+K_2^n q_2^n(r)+\frac{h_n^{(1)}(\sigma r)}{h_n^{(1)}(\sigma a)}\,\psi_n(a)\right\}\ . \tag{6.116}$$

Equations (6.115) and (6.116) represent a formal solution. The problem remains, however, to determine the arbitrary constants $\phi_n(a)$, $\phi_n'(a)$ and $\psi_n(a)$. This may be done by considering the boundary conditions stated in (6.46) through (6.48). Substituting in the functional expression for $\underline{E}$ from (6.23), these requirements reduce to

$$\frac{i\omega}{c}\frac{\partial\phi}{\partial r}+\frac{\omega_p^2}{c^2}\cos\theta\,A_z=0 \qquad \text{at } r=a \tag{6.117}$$

$$\nabla^2\frac{\partial\phi}{\partial r}+\frac{i\omega}{c}(1+\lambda)\frac{\omega_p^2}{u_e^2}\left[1-\frac{\omega_p^2}{\omega^2}\right]\cos\theta\,A_z=0 \qquad \text{at } r=a \tag{6.118}$$

$$\frac{1}{\sin\theta}\frac{\partial\phi}{\partial\theta}-\frac{i\omega}{c}\,a\,A_z=0 \qquad \text{at } r=a \tag{6.119}$$

The term which causes the most difficulty is $\nabla^2 \frac{\partial \phi}{\partial r}$. After some algebraic manipulation, we find from (6.107), (6.108), (6.111) and (6.115) that

$$\nabla^2 \frac{\partial \phi}{\partial r}\bigg|_a = \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{2n+1}{2} P_n(\cos\theta) \Bigg\{ -\frac{4i}{\pi a^2}(\beta^2-\alpha^2)\left[c_1^n p_3'(a)+c_3 p_1'(a)\right]$$

$$+ \frac{(\beta^2-\alpha^2)p_1'(a)p_3'(a)}{(p_1 p_3' - p_3 p_1')_{r=a}} \phi_n(a) - \frac{(\beta^2 p_1 p_3' - \alpha^2 p_3 p_1')_{r=a}}{(p_1 p_3' - p_3 p_1')_{r=a}} \phi_n'(a) =$$

$$\frac{1}{2\pi} \sum \frac{2n+1}{2} P_n(\cos\theta)\left[F_n(a,r_o)+b_n\phi_n(a) - d_n\phi_n'(a)\right] \qquad (6.120)$$

where $F_n(a, r_o)$, $b_n(a, r_o)$ and $c_n(a, r_o)$ are known quantities.

Turning now to the boundary conditions and taking (6.119) first, we note that if $x = \cos\theta$,

$$\frac{1}{\sin\theta}\frac{\partial P_n(\cos\theta)}{\partial\theta} = -\frac{\partial P_n(x)}{\partial x} = -\sum_{m=0}^{\left(\frac{n-1}{2}\right),\ n\ \text{odd};\ \left(\frac{n}{2}-1\right),\ n\ \text{even}} \left[2n-(4m+1)\right] P_{n-(2m+1)}(x) \qquad (6.121)$$

Thus

$$-\frac{1}{\sin\theta}\frac{\partial\phi(a,\theta)}{\partial\theta}=\frac{1}{2\pi}\left\{\sum_{n=0}^{\infty}\frac{4n+5}{2}\phi_{2n+2}(a)\sum_{m=0}^{n}\left[4n-(4m-3)\right]P_{2n-(2m-1)}(x)\right.+$$

$$\left.\sum_{n=0}^{\infty}\frac{4n+3}{2}\phi_{2n+1}(a)\sum_{m=0}^{n}\left[4n-(4m-1)\right]P_{2n-2m}(x)\right\}=$$

$$\frac{1}{2\pi}\left\{\sum_{n=0}^{\infty}\frac{4n+5}{2}\phi_{2n+2}(a)\sum_{S=0}^{n}(4S+3)P_{2S+1}(x)+\sum_{n=0}^{\infty}\frac{4n+3}{2}\phi_{2n+2}(a)\sum_{S=0}^{n}(4S+1)P_{2S}(x)\right\}=$$

$$\frac{1}{2\pi}\left\{\sum_{S=0}^{\infty}(4S+3)P_{2S+1}(x)\sum_{n=S}^{\infty}\frac{4n+5}{2}\phi_{2n+2}(a)+\sum_{S=0}^{\infty}(4S+1)P_{2S}(x)\sum_{n=S}^{\infty}\frac{4n+3}{2}\phi_{2n+1}(a)\right\}$$

(6.122)

Thus, from (6.119) together with (6.116) and (6.122),

$$\sum_{S=0}^{\infty}\left\{(4S+3)P_{2S+1}(x)\sum_{n=S}^{\infty}\frac{4n+5}{2}\phi_{2n+2}(a)+(4S+1)P_{2S}(x)\sum_{n=S}^{\infty}\frac{4n+3}{2}\phi_{2n+1}(a)\right\}+$$

$$\frac{i\omega a}{c}\sum_{S=0}^{\infty}\frac{2S+1}{2}P_S(x)\psi_S(a) \qquad (6.123)$$

From the orthogonality of the Legendre polynomials, we obtain the first recursion expression, namely

$$\begin{cases} \dfrac{i\omega a}{c}\psi_m(a) + \displaystyle\sum_{n=\frac{m-1}{2}}^{\infty} (4n+5)\phi_{2n+2}(a) = 0 & m \text{ odd} \\ \\ \dfrac{i\omega a}{c}\psi_m(a) + \displaystyle\sum_{n=m/2}^{\infty} (4n+3)\phi_{2n+1}(a) = 0 & m \text{ even} \end{cases} \qquad (6.124)$$

To show that the two sines expansions do indeed converge, we note that

$$\phi(a, 0) + \phi(a, \pi) = \frac{1}{2\pi} \sum_{n=0}^{\infty} (4n+1)\, \phi_{2n}(a) \qquad (6.125a)$$

and

$$\phi(a, 0) - \phi(a, \pi) = \frac{1}{2\pi} \sum_{n=0}^{\infty} (4n+3)\, \phi_{2n+1}(a) \qquad (6.125b)$$

Thus, in place of (6.124) we could have written

$$\psi_o(a) = -\frac{c}{i\omega a}\, 2\pi\left[\phi(a, 0) - \phi(a, \pi)\right] \qquad (6.126a)$$

$$\psi_m(a) = -\frac{c}{i\omega a}\left\{2\pi\left[\phi(a,0)-\phi(a,\pi)\right] - \sum_{n=0}^{(\frac{m}{2}-1)} (4n+3)\phi_{2n+1}(a)\right. \tag{6.126b}$$

for m even, $m \neq 0$

$$\psi_m(a) = -\frac{c}{i\omega a}\left\{2\pi\left[\phi(a,0)+\phi(a,\pi)\right] - \sum_{n=0}^{(\frac{m-1}{2})} (4n+1)\phi_{2n}(a)\right. \tag{6.126c}$$

for m odd

The condition in (6.117) is handled in much the same manner, although the situation is simpler. Omitting the details, we find that

$$\frac{i\omega}{c}\phi_0'(a) + \frac{\omega_p^2}{c^2}\psi_1(a) = 0 \tag{6.127a}$$

$$\frac{i\omega}{c}\phi_m'(a) + \frac{\omega_p^2}{c^2}\frac{1}{2m+1}\left[m\psi_{m-1}(a) + (m+1)\psi_{m+1}(a)\right] \quad , \quad m > 0 \tag{6.127b}$$

Using the expressions from (6.126) we thus obtain

$$\phi_0'(a) = -\frac{\omega_p^2}{2\omega^2 a}\left\{2\pi\left[\phi(a,0)+\phi(a,\pi)\right] - \phi_0(a)\right\} \tag{6.128a}$$

$$\phi_1'(a) = -\frac{\omega_p^2}{2\omega^2 a}\left\{2\pi\left[\phi(a,0)-\phi(a,\pi)\right] - 2\phi_1(a)\right. \tag{6.128b}$$

$$\phi'_m(a) = -\frac{\omega_p^2}{2\omega^2 a}\left\{2\pi\left[\phi(a,0)+\phi(a,\pi)\right] - (m+1)\phi_m(a) - \sum_{n=0}^{(\frac{m}{2}-1)} (4n+1)\phi_{2n}(a)\right\}$$

m even, m≠0 (6.128c)

$$\phi'_m(a) = -\frac{\omega_p^2}{2\omega^2 a}\left\{2\pi\left[\phi(a,0)-\phi(a,\pi)\right] - (m+1)\phi_m(a) - \sum_{n=0}^{(\frac{m-3}{2})} (4n+3)\phi_{2n+1}(a)\right\}$$

m odd, m≠1 (6.128d)

Finally, from (6.118), together with (6.116), (6.120) and (6.127) we have

$$F_o(a,r_o)+b_o\phi_o(a)-\left[d_o+\frac{c^2}{u_e^2}(1+\lambda)(1-\frac{\omega_p^2}{\omega^2})\right]\phi'_o(a) = 0 \qquad (6.129a)$$

$$F_m(a,r_o)+b_m\phi_m(a)-\left[d_m+\frac{c^2}{u_e^2}(1+\lambda)(1-\frac{\omega_p^2}{\omega^2})\right]\phi'_m(a) = 0 \qquad (6.129b)$$

Substitution of the expressions for $\phi'_m(a)$ from (6.128) into (6.129) thus yields a recursion scheme for $\phi_m(a)$ in terms of known functions. Once these values are determined, we can obtain $\psi_m(a)$ and $\phi'_m(a)$ from (6.126) and (6.128), respectively.

We have thus been able to establish a formal solution for $\phi(\underline{r})$ and $\psi(\underline{r})$ which may be evaluated for particular cases of interest. It is not too difficult to understand that the process for any given situation is not particularly simple. Nevertheless, it should be possible to obtain solutions in the shadow region for example, by using a Watson transformation, or a near zone solution by retaining just the first few terms of the Legendre polynomial expansion.

6.5 Discussion of Result

Two points are worth noting in regard to the preceding discussion. In the first place, it is quite evident that the difficulties encountered in the analysis were due to the fact that ion motion was taken into account. This effect served to give us a fourth order equation in ϕ rather than the second order equation we would have obtained had we considered the ions to be stationary. More devastating from the standpoint of a rigorous treatment, however, was the fact that the boundary conditions were vastly more complicated. It was illustrated in the third part of this section, for example, that even in the case of an oscillating dipole above a perfectly conducting plane, we obtain a solution which cannot be integrated in any simple manner.

The second point that should be mentioned regards the matter as to just how realistic the boundary conditions really are. The assumption that the particles undergo elastic reflection when striking a perfectly conducting object has been used by practically everyone treating this type of problem. The fact remains, however, that an electron striking a metallic surface might either be absorbed or undergo some diffuse reflection. An accurate description of the actual phenomenon has never, to our knowledge, been described in a satisfactory manner. This is one facet of the problem, then, that definitely bears further investigation.

VII

INHOMOGENEOUS MEDIA

7.1 Introduction

Although the propagation of waves in an infinite homogeneous ionized medium has been extensively investigated in the literature, the excitation of waves and their subsequent propagation in an inhomogeneous media, such as the ionsphere, has been scarcely touched, due to the complicated nature of the mathematics involved. In this section, we shall discuss possible methods, approximate or exact, for treating such problems.

If the local dispersion relation (and propagation constants) are known as a function of the spacial variables, the excitation problem may be solved approximately by considering the fields excited in the vicinity of a source and extending these fields to other points in space by ray tracing techniques. A general discussion of ray tracing in ionized media is given in Section 7.2 .

For drastic changes in the properties of the medium, the ray tracing technique may not give accurate results due to the neglect of intermode coupling. For these regions the reflection and refraction of the waves should be used. A general discussion of the reflection and refraction of waves, due to discontinuities in the medium, is discussed in Section 7.3 .

For a general variation of the properties of the medium in one direction (i.e., stratified), the recently developed technique of invariant embedding, probably, would also apply. The application of this technique to wave propagation in ionized media is discussed in Section 7.4 .

Our primary goal in this work is to develop a unified approach to deal with excitation and propagation problems in both homogeneous and inhomogeneous media. The generalized operator transform technique developed during the course of this investigation is designed to accomplish this goal. The application of this technique to excitation problems in homogeneous media has been demonstrated but due to lack of time and the complexity of the mathematics involved, the application of this method to problems involving inhomogeneous media remains to be investigated. In Section 7.5 a general discussion of the application of the operator transform method to problems involving inhomogeneous media is given.

7.2 Ray Tracing

If the propagation constants for different "modes" of propagation are known as a function of space, the propagation of disturbances initiated at any point may be approximately calculated by the well known technique of ray tracing. The application of ray tracing techniques to anisotropic (propagation constant depends on direction) and dissipative media has been discussed in detail by Brandstatter (1963). If the propagation constant does not change appreciably in a wave length, and the inter-mode coupling is negligible, then knowing the intital amplitude and direction of a ray at one point, the ray path and amplitude can be obtained by numerical integration.

In terms of the propagation constant written in the form (assume real, for the time being)

$$s = \frac{\omega}{c} n(\underline{r}, \hat{\sigma})$$

where $\hat{\sigma}$ is a unit vector indicating the direction of propagation, the standard form of the ray equation may be easily shown to be

$$\frac{d\underline{r}}{d\tau} = \frac{1}{n^2} \frac{\partial G(\underline{r}, \underline{\sigma})}{\partial \underline{\sigma}} \tag{7.1}$$

$$\frac{d\underline{\sigma}}{d\tau} = -\frac{1}{n^2} \frac{\partial G(\underline{r}, \underline{\sigma})}{\partial \underline{r}} \tag{7.2}$$

where

$$\underline{\sigma} \triangleq \hat{\sigma}\, n(\underline{r}, \hat{\sigma}) \tag{7.3}$$

$$G(\underline{r}, \hat{\sigma}) = \frac{1}{2}\left[\underline{\sigma} \cdot \underline{\sigma} - n^2(\underline{r}, \hat{\sigma})\right] \tag{7.4}$$

and

τ is the time of propagation.

Equations (7.1) and (7.2) are a set of six (6) coupled equations. Knowing the initial position $\underline{r}$, and direction $\hat{\sigma}$, these equations may be integrated numerically in steps to yield the ray path in the parametric form $\underline{r}(\tau)$. The direction of phase propagation at any point along the ray path is also obtained in the parametric form $\underline{\sigma} = \underline{\sigma}(\tau)$.

For a stratified medium such as the ionosphere, the ray equations can be simplified further. Following our discussion on the propagation constants for each mode, we have a value of n in the form

$$n(\underline{r}, \mu)$$

where $\mu = \cos\theta$ is the cosine of the angle between the d.c. magnetic field and the direction of propagation. In order to adapt these results to the ray equation, we shall choose a new coordinate system such that the medium is stratified in the $\hat{z}$ direction, while the d.c. magnetic field is in the direction

$$\hat{b} = \hat{z} \cos \theta_o + \hat{y} \sin \theta_o \quad . \tag{7.5}$$

If the direction of phase propagation is designated by

$$\hat{\sigma} = \hat{z}\cos\alpha + \hat{x}\sin\alpha\cos\beta + \hat{y}\sin\alpha\sin\beta$$

then

$$\mu = \cos\alpha\cos\theta_o + \sin\alpha\sin\beta\sin\theta_o \tag{7.6}$$

Introducing these equations into (7.1) and (7.2), and simplifying, one finds that

$$\beta = \text{constant} \tag{7.7}$$

$$\frac{d\alpha}{d\tau} = -\frac{1}{n^2}\frac{\partial n(z,\mu)}{\partial z} \tag{7.8}$$

$$\frac{dx}{d\tau} = \frac{1}{n^2}\left\{n\sin\alpha\cos\beta + \mu\sin\alpha\cos\beta\frac{\partial n}{\partial\mu}\right\} \tag{7.9}$$

$$\frac{dy}{d\tau} = \frac{1}{n^2}\left\{n\sin\alpha\sin\beta + (\mu\sin\alpha\sin\beta - \sin\theta)\frac{\partial n}{\partial\mu}\right\} \tag{7.10}$$

and

$$\frac{dz}{d\tau} = \frac{1}{n^2}\left\{n\cos\alpha + (\mu\cos\alpha - \cos\theta_o)\frac{\partial n}{\partial\mu}\right\} . \tag{7.11}$$

With some initial values of α and β these equations can be integrated in a straightforward manner. If several rays are traced so that the variation of the cross section of a tube of rays can be calculated, then, by conservation of energy, in the lossless medium, the amplitude of each field component can be computed from the fact that they are inversely proportional to the cross sectional area of the ray tube.

When collisions are not neglected, the propagation constant is, in general, complex. We may write the real and imaginary components of s in the form

$$s = \frac{\omega}{c}\left[n(z,\mu) + i\chi(z,\mu)\right] \tag{7.12}$$

Then, approximately, the ray path is obtained from the real part of s , $\left[\text{i.e., } n(z,\mu)\right]$. The fields, however, are now attenuated. The attenuation along a ray may be approximately given by

$$\exp\left[-\int^{S'} \chi(z,\mu)\hat{\sigma}\cdot\frac{dr}{ds}\,ds\right]$$

where the integration is along a ray path.

In general, the ray tracing technique is valid when n does not change drastically, such as near a discontinuity. Near the discontinuity, intermode coupling necessarily exists, and the investigation of such reflection phenomena shall be treated in Section 7.3.

The ray tracing technique is also not valid near the regions where n is near zero or infinity and changes sign. Near the infinity of n , the wave is totally absorbed, while near a zero of n , the wave is reflected (Budden, 1961).

7.3 Reflection and Refraction

The technique of ray tracing, applicable when the properties of the medium do not vary appreciably in a distance of one wavelength, neglects the coupling of the various modes due to the inhomogeneity of the medium. For strong changes in the properties of the medium, especially for large gradients of temperature or density, such that the medium may be idealized by means of a discontinuity, the intermode coupling may be investigated by the conventional methods of reflection of waves. Assuming that the surface of the discontinuity can be represented by a plane, the calculation of the reflection and refraction of plane waves can be carried out, in principle, by assuming the forms of reflected and refracted waves at the discontinuity, and determining the amplitudes of these waves in terms of the incident wave by use of the boundary conditions. Although such methods are standard for acoustic waves and electromagnetic waves (in dielectric medium), their application to an ionized medium becomes very involved due to the existence of several modes simultaneously. Existing investigations have been carried out only for the case of a plasma without a d. c. magnetic field, such as reported by Kritz and Mintzer (1960), Haynes and Kahn (1965), etc. In this section, the general formulation of the reflection and refraction problem in the presence of d. c. magnetic field is given. However, due to the complicated nature of the expressions involved, no attempt is made to derive explicit formulas for the reflection coefficients.

Consider the discontinuity of a medium to be in the plane $z = 0$. For clarity, we shall use the superscript $(+)$ to denote all quantities in the region $z > 0$, and superscript $(-)$ to denote all quantities in the region $z < 0$. When a wave is incident on the boundary, say from the region $z < 0$, waves would be partially reflected in the region $z < 0$, and partially transmitted into the region $z > 0$. The amplitude of the reflected and transmitted waves are determined by the boundary conditions at $z=0$. Assuming that the velocity components associated with the waves are small (in conformity with the perturbation approximation), so that the effect of the distortion of the boundary is negligible, then the following set of boundary conditions is sufficient to determine the reflected and refracted amplitudes:

a-The electromagnetic conditions,

$$E_x^+ = E_x^- \qquad (7.13)$$

$$E_y^+ = E_y^- \qquad (7.14)$$

$$h_x^+ = h_x^- \qquad (7.15)$$

$$h_y^+ = h_y^- \qquad (7.16)$$

b-The kinematic conditions,

$$V_{ez}^+ = V_{ez}^- \qquad (7.17)$$

$$V_{iz}^+ = V_{iz}^- \qquad (7.18)$$

$$V_{nz}^+ = V_{nz}^- \qquad (7.19)$$

c-The dynamic conditions,

$$(U_e^+)^2 n_e^+ = (U_e^-)^2 n_e^- \qquad (7.20)$$

$$(U_i^+)^2 n_i^+ = (U_i^-)^2 n_i^- \tag{7.21}$$

$$(U_n^+)^2 n_n^+ = (U_n^-)^2 n_n^- \quad . \tag{7.22}$$

Explicitly, let us consider the case where electron motion is dominant and collisions are negligible. Then, assuming that the magnetic field is in the y - z plane, such that

$$\hat{b} = \hat{z} \cos\theta_o + \hat{y} \sin\theta_o \quad , \tag{7.23}$$

the propagation constant in any direction

$$\hat{s} = \hat{z} \cos\alpha + \hat{x} \sin\alpha \cos\gamma + \hat{z} \sin\alpha \sin\gamma$$

may be determined through the dispersion relation

$$\beta_o^2 \left[s^2 - \beta_e^2 (1 - \omega_p^2)\right] \left[s^2 - \beta_o^2 (1 - \omega_p^2)\right] - \Omega^2 \cos^2\theta \left[s^2 - \beta_o^2\right] \left[s^2 - \beta_e^2 (1 - \omega_p^2)\right] \beta_o^2$$

$$+ \Omega^2 \sin^2\theta \left[s^2 - \beta_o^2\right] \left[s^2 - \beta_o^2 (1 - \omega_p^2)\right] \beta_o^2 \beta_e^2 = 0 \tag{7.24}$$

where

$$\cos\theta = \cos\theta_o \cos\alpha + \sin\theta_o \sin\alpha \sin\gamma \tag{7.25}$$

By algebraic manipulation, the field quantities associated with each mode, i.e. each s satisfying Eq. (7.25), are determined within one constant.

This means that for each s, we have

$$\frac{E_x}{f_1} = \frac{E_y}{f_2} = \frac{h_x}{f_3} = \frac{h_y}{f_4} = \frac{V_{ez}}{f_5} = \frac{n_e}{f_6} \tag{7.26}$$

where the functions f_i are given by

$$f_1(\alpha, \gamma, s, \omega_p, \beta_e) = \frac{\sin\theta_o \cos\alpha - \cos\theta_o \sin\alpha \sin\gamma}{\cos\theta_o} \left[s^2 - \beta_e^2(1-\omega_p^2)\right]\left[s^2 - \beta_o^2(1-\omega_p^2)\right] \beta_o^2$$

$$- i\Omega \sin\alpha \cos\gamma (s^2 - \beta_o^2) \left\{ \cos\theta \left[s^2 - \beta_e^2(1-\omega_p^2)\right]\beta_o^2 + \sin\theta \left[s^2 - \beta_o^2(1-\omega_p^2)\right]\beta_e^2 \right\} \tag{7.27}$$

$$f_2(\alpha, \gamma, s, \omega_p, \beta_e) = \frac{\cos\theta_o \sin\alpha \sin\gamma}{\cos\theta} \left[s^2 - \beta_e^2(1-\omega_p^2)\right]\left[s^2 - \beta_o^2(1-\omega_p^2)\right] \beta_o^2$$

$$+ i\Omega(s^2 - \beta_o^2) \left\{ (\sin\theta_o - \sin\alpha \sin\gamma \cos\theta) \left[s^2 - \beta_e^2(1-\omega_p^2)\right]\beta_o^2 - \sin\alpha \sin\gamma \sin\theta \left[s^2 - \beta_o^2(1-\omega_p^2)\right]\beta_e^2 \right\} \tag{7.28}$$

$$f_3(\alpha, \gamma, s, \omega_p, \beta_e) = \frac{\beta_o^2 s}{\omega\mu_o} \left\{ -i\Omega(\sin\theta_o \cos\alpha - \cos\theta_o \sin\alpha \sin\gamma)\left[s^2 - \beta_o^2\right]\left[s^2 - \beta_e^2(1-\omega_p^2)\right] \right.$$

$$\left. - \sin\alpha \cos\gamma \left[s^2 - \beta_e^2(1-\omega_p^2)\right]\left[s^2 - \beta_o^2(1-\omega_p^2)\right] \right\} \tag{7.29}$$

$$f_4(\alpha, \gamma, s, \omega_p, \beta_e) = \frac{\beta_o^2 s}{\omega \mu_o} \left\{ -i\Omega \cos\theta_o \sin\alpha \sin\gamma (s^2 - \beta_o^2) \left[s^2 - \beta_e^2(1-\omega_p^2)\right] \right.$$

$$\left. + \frac{\sin\theta_o \sin\alpha \sin\gamma}{\cos\theta} \left[s^2 - \beta_o^2(1-\omega_p^2)\right] \left[s^2 - \beta_e^2(1-\omega_p^2)\right] \right\} \tag{7.30}$$

$$f_5(\alpha, \gamma, s, \omega_p, \beta_e) = -\frac{ie}{\omega m} \frac{1}{\omega_p^2} (s^2 - \beta_o^2) \left\{ \frac{\sin\theta_o \sin\alpha \cos\gamma}{\cos\theta} \left[s^2 - \beta_o^2(1-\omega_p^2)\right]\left[s^2 - \beta_e^2(1-\omega_p^2)\right] \right.$$

$$\left. -i\Omega(\cos\theta_o - \cos\alpha\cos\theta)(s^2-\beta_o^2)\left[s^2 - \beta_e^2(1-\omega_p^2)\right] - i\Omega\cos\alpha\sin\theta\left[s^2 - \beta_o^2(1-\omega_p^2)\right]\beta_e^2 \right\} \tag{7.31}$$

$$f_6(\alpha, \gamma, s, \omega_p, \beta_e) = -\frac{\omega \epsilon_o}{e} \Omega s \sin\theta \left[s^2 - \beta_o^2\right]\left[s^2 - \beta_o^2(1-\omega_p^2)\right] \beta_e^2 \tag{7.32}$$

where we have explicitly written down the arguments of the f's to indicate their dependence on the properties of the medium. For any mode corresponding to a propagation constant s_i, the associated field E_x, E_y, h_x, h_y, V_{ez} and n_e can then be represented by a column matrix

$$A_i \left[f(\alpha_i, \gamma_i, s_i, \omega_p, \beta_e)\right] e^{is_i(\cos\alpha_i z + \sin\alpha_i \cos\gamma_i x + \sin\alpha_i \sin\gamma_i y)} \tag{7.33}$$

where A_i is the "amplitude" of the wave, and

$$[f] = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix} \tag{7.34}$$

are the appropriate components. Across any discontinuity, the total field must be continuous.

If a wave is incident from the negative region on the boundary, and the incident field is given by mode i, corresponding to s_{io}^-, α_{io}^-, γ_{io}^- then we have the following reflected and refracted fields.

Incident field:

$$\left[f(\alpha_{io}^-, \gamma_{io}^-, s_{io}^-, \omega_p, \beta_e^-)\right] e^{i s_i^- (\cos\alpha_i^- z + \sin\alpha_i^- \cos\gamma_i^- x + \sin\alpha_i^- \sin\gamma_i^- y)}$$

Reflected fields: corresponding to $j = 1, 2, 3$:

$$R_{ij}\left[f(\alpha_j^-, \gamma_j^-, s_j^-, \omega_p, \beta_e^-)\right] e^{i s_j^- (\cos\alpha_j^- z + \sin\alpha_j^- \cos\gamma_j^- x + \sin\alpha_j^- \sin\gamma_j^- y)}$$

and the transmitted (refracted) fields, corresponding to $j = 1, 2, 3$,

$$T_{ij}\left[f(\alpha_j^+, \gamma_j^+, s_j^+, \omega_p, \beta_e^+)\right] e^{i s_j^+ (\cos\alpha_j^+ z + \sin\alpha_j^+ \cos\gamma_j^+ x + \sin\alpha_j^+ \cos\gamma_j^+ y)},$$

The continuity of fields at $z=0$, therefore, yields the following relation on the direction of the reflected and refracted waves

$$s_{io}^{-}\sin\alpha_{i}^{-}\cos\gamma_{i}^{-} = s_{j}^{-}\sin\alpha_{j}^{-}\cos\gamma_{j}^{-} = s_{j}^{+}\sin\alpha_{j}^{+}\cos\gamma_{j}^{+} \tag{7.35}$$

$$s_{io}^{-}\sin\alpha_{i}^{-}\sin\gamma_{i}^{-} = s_{j}^{-}\sin\alpha_{j}^{-}\sin\gamma_{j}^{-} = s_{j}^{+}\sin\alpha_{j}^{+}\cos\gamma_{j}^{+} \tag{7.36}$$

where s_{io}^{-}, α_{io}^{-}, γ_{io}^{-} are fixed by the incident wave and $j=1, 2, 3$, corresponding to three reflected and refracted waves. From Eqs. (7.35) and (7.36), it is evident that γ is the same for all waves, while the α's are related by

$$s_{io}^{-}\sin\alpha_{io}^{-} = s_{j}^{-}\sin\alpha_{j}^{-} = s_{j}^{+}\sin\alpha_{j}^{+} \tag{7.37}$$

where on physical grounds, we restrict

$$\alpha_{j}^{+} < \frac{\pi}{2}$$

and (7.38)

$$\alpha_{j}^{-} > \frac{\pi}{2}$$

For the particular $j = i$, we have immediately

$$\alpha_{i}^{-} = \pi - \alpha_{io}^{-},$$

(7.39)

$$s_{i}^{-} = s_{io}^{-},$$

for other values of j, s_j, α_j have to be determined by the dispersion relation. After the values of s, and α are determined, the following vector equation enables us to determine the coefficients of reflection and refraction

$$\left[f(\alpha_{io}^{-}, \gamma_{io}, s_{io}^{-}, \omega_p^{-}, \beta_e^{-})\right] + \sum_{j=1}^{3} R_{ij} \left[f(\alpha_j^{-}, \gamma_{io}, s_j^{-}, \omega_p^{-}, \beta_e^{-})\right]$$

$$= \sum_{j=1}^{3} T_{ij} \left[f(\alpha_j^{+}, \gamma_{io}, s_j^{+}, \omega_p^{+}, \beta_e^{+})\right] \qquad (7.40)$$

Since there are six component equations in Eq. (7.40), the six coefficients R_{ij}, T_{ij} may be solved by inverting a 6x6 matrix. This can be done by a computer with relative ease.

Extension of this formulation to a collisionless two-fluid plasma is straightforward. In this case, the dispersion relation has four roots, and the column matrix has eight components. The general form of the result should be the same as that discussed above.

7.4 Invariant Embedding

The effect of the intermode coupling on waves propagating in a region where the properties of the medium are changing rapidly, can be investigated by the familiar principle of invariant embedding. This technique has found wide application in the fields of light and neutron scattering and can be applied to the present problem. In this method the region where the properties of the medium are changing rapidly is considered as a layer rather than as an idealized discontinuity as discussed in Section 7.2. The mathematical formulation of "invariant embedding" has been summarized in an article by Baily and Wing (1963). In the section, the problem of evaluating the reflection and transmission coefficients of a layer of stratified, inhomogeneous electron plasma will be formulated in a manner suitable for numerical computation. The formulation may be extended, in a straightforward manner, to include the case of ion and neutral particle motion, but, of course, the numerical computations are much more involved.

For some background on the principle of invariant embedding, consider the set of coupled differential equations given by

$$\frac{dU(z)}{dz} = A(z)\,U(z) + B(z)\,W(z)$$
$$-\frac{dW(z)}{dz} = C(z)\,U(z) + D(z)\,W(z) \qquad (7.41)$$

where $U(z)$ and $W(z)$ are nxl column matrices, while A, B, C and D are nxn matrices. If we interpret $U(z)$ as the field quantities associated with the forward propagating wave, and $W(z)$ as the field quantities associated with the backward propagating wave, then the reflection and transmission characteristics of a layer in a region $0 \leq z \leq z_o$ may be investigated by solving Eq. (7.41), subject to the following boundary conditions

$$U(0) = U_o$$
$$W(z_o) = 0 \qquad (7.42)$$

where U_o is the vector, whose components are the amplitude of the incident wave. The overall effect on wave propagation, due to the presence of this layer, can then be expressed in terms of a reflection matrix R and a transmission matrix T. In terms of these matrices, we may evaluate the reflected wave at $z = 0$ and transmitted wave at $z = z_o$ by the relation

$$\begin{aligned} U(z_o) &= T\,U_o \\ W(0) &= R\,U_o \end{aligned} \qquad (7.43)$$

By using the principle of invariant embedding, it can easily be shown that, in order to obtain T and R, it is not necessary to solve the linear boundary value problem (Eq. (7.41) and Eq. (7.42)). In fact, T(z) and R(z) satisfy the first order nonlinear differential equations given by the following

$$\frac{d}{dz} R(z) = R(z)B(z)R(z) + R(z)A(z) + D(z)R(z) + C(z) \qquad (7.44)$$

$$\frac{d}{dz} T(z) = T(z)B(z)R(z) + T(z)A(z) \qquad (7.45)$$

This set of nonlinear Eqs. (7.44) and (7.45), can be integrated from $z=z_o$ to $z=0$ to obtain the reflection and transmission matrices. The obvious boundary conditions, for the integration, are $R=0$ and $T=1$ at $z=z_o$.

To apply the above formulation in the calculation of transmission and reflection matrices for an inhomogeneous slab, let us consider an electron plasma with non-uniform properties in the region defined by $z_o \geq z \geq 0$.

The regions for $z < 0$, and $z > z_o$, are assumed to be filled with electron plasmas of uniform properties (or free space which can be considered as an electron plasma with zero electron density). Let waves be incident from the region $z < 0$. From the discussion of the reflection and refraction of waves in Section 7.3, we note that the propagation constant in the x direction ($s \sin\alpha \cos\gamma$) and in the y direction ($s \cos\alpha \sin\gamma$) are constants. Therefore, each field component, such as E_x, may be represented in the form

$$E_x \sim E_x(z) e^{i s_x \chi} e^{i s_y y}$$

where the amplitudes of the field components are functions of z only. The source free equations for the collisionless plasma, where the direction of the d.c. magnetic field is the same as given in the Section 7.3, can then be reduced to a set of coupled differential equations. The explicit forms of these equations are:

$$
\begin{bmatrix} \frac{dh_x}{dZ} \\ \frac{dh_y}{dZ} \\ \frac{dE_x}{dZ} \\ \frac{dE_y}{dZ} \\ \frac{dV_z}{dZ} \\ \frac{dn}{dZ} \end{bmatrix}
=
\begin{bmatrix}
0 & 0 & i\omega\mu_o \frac{s_x s_y}{\beta_o^2} & i\omega\mu_o\left[1-\frac{s_x^2}{\beta_o^2}\right] & -\frac{\omega m}{e}\omega_p^2 s_x & 0 \\
0 & 0 & -i\omega\mu_o\left[1-\frac{s_y^2}{\beta_o^2}\right] & -i\omega\mu_o \frac{s_x s_y}{\beta_o^2} & -\frac{\omega m}{e}\omega_p^2 s_y & 0 \\
-i\omega\epsilon_o\left[\frac{s_x s_y}{\beta_o^2} - \frac{i\omega_p^2\Omega\cos\theta}{1-\Omega^2\cos^2\theta}\right] & -i\omega\epsilon_o\left[1-\frac{s_x^2}{\beta_o^2} - \frac{\omega_p^2}{1-\Omega^2\cos^2\theta}\right] & 0 & 0 & \frac{\omega m\epsilon_o}{e}\frac{\omega_p^2\Omega^2\sin\theta\cos\theta}{(1-\Omega^2\cos^2\theta)} & -\frac{e\omega}{\beta_e^2}\frac{(\Omega\cos\theta s_x + s_y)}{(1-\Omega^2\cos^2\theta)} \\
i\omega\epsilon_o\left[1-\frac{\omega_p^2}{1-\Omega^2\cos^2\theta} - \frac{s_y^2}{\beta_o^2}\right] & i\omega\epsilon_o\left[\frac{s_x s_y}{\beta_o^2} + \frac{i\omega_p^2\Omega\cos\theta}{1-\Omega^2\cos^2\theta}\right] & 0 & 0 & \frac{i\omega^2 m\epsilon_o\omega_p^2\Omega\sin\theta}{e(1-\Omega^2\sin^2\theta)} & \frac{e\omega}{\beta_e^2}\frac{(s_x - \Omega\cos\theta s_y)}{1-\Omega^2\cos^2\theta} \\
-\frac{e(s_x + is_y\Omega\cos\theta)}{\omega m(1-\Omega^2\cos^2\theta)} & -\frac{e}{\omega m}\frac{(s_y - is_x\Omega\cos\theta)}{1-\Omega^2\cos^2\theta} & 0 & 0 & -\frac{1}{\omega_p^2}\frac{\partial\omega_p^2}{\partial z} + \frac{\Omega\sin\theta}{1-\Omega^2\cos^2\theta}(is_y\Omega\cos\theta + s_x) & -\frac{ie^2(s_x^2+s_y^2)}{\omega m\epsilon_o\omega_p^2\beta_e^2(1-\Omega^2\cos^2\theta)} \\
i\omega_p^2\beta_e^2\frac{\epsilon_o}{e}\frac{\Omega\sin\theta}{(1-\Omega^2\cos^2\theta)} & \frac{\epsilon_o}{e}\omega_p^2\beta_e^2\frac{\Omega^2\sin\theta\cos\theta}{1-\Omega^2\cos^2\theta} & -\frac{\omega_p^2\beta_e^2}{\omega e}s_y & \frac{\omega_p^2\beta_e^2}{\omega e}s_x & \frac{i\omega m\epsilon_o\omega_p^2\beta_e^2}{e^2}\left(\frac{1-\Omega^2}{1-\Omega^2\cos^2\theta} - \omega_p^2\right) & -\frac{\Omega\sin\theta(s_x - \Omega\cos\theta s_y)}{(1-\Omega^2\cos^2\theta)}
\end{bmatrix}
\begin{bmatrix} h_x \\ hy \\ E_x \\ E_y \\ V_z \\ n \end{bmatrix}
\qquad (7.46)
$$

All the field components in Eq. (7.46) are continuous across any boundary.

To reduce the above equations to a form that is adaptable for use in the principle of invariant embedding, let us assume that for the region $z<0$, the propagation constants of the three modes have been calculated and denote their values by s_a, s_b and s_c, respectively. For any fixed direction of the incident wave, s_x and s_y are fixed. If we denote

$$U_a = +\sqrt{s_a^2 - s_x^2 - s_y^2}\ , \tag{7.47}$$

and similar expressions for U_b and U_c, then the amplitudes of the incident waves may be written as

$$P_a\left[f(U_a)\right]e^{iU_a z} + P_b\left[f(U_b)\right]e^{iU_b z} + P_c\left[f(U_c)\right]e^{iU_c z} \tag{7.48}$$

where the f's are the column vectors for the field quantities. The explicit forms of the f's are given in Eqs. (7.26) through (7.32), and a simple algebraic substitution will yield the expression for f in terms of the U's, instead of s, α and γ. Similarly, the reflected waves are represented by

$$Q_a\left[f(-U_a)\right]e^{-iU_a z} + Q_b\left[f(-U_b)\right]e^{-iU_b z} + Q_c\left[f(-U_c)\right]e^{-iU_c z} \tag{7.49}$$

The "amplitudes" of the reflected wave can, then, be expressed in terms of the reflection matrix, such that

$$\begin{bmatrix} Q_a \\ Q_b \\ Q_c \end{bmatrix} = \left[R\right] \begin{bmatrix} P_a \\ P_b \\ P_c \end{bmatrix} \tag{7.50}$$

Note that $[R]$ in Eq. (7.50) is not the same $[R]$ given in Eq. (7.43). Similarly, for the region $z \geq z_o$, we may denote the values of the propagation constants as $\tilde{s}_a$, $\tilde{s}_b$ and $\tilde{s}_c$, respectively. From the values of s_x and s_y, fixed by the incident field, we can calculate

$$\tilde{U}_a = + \sqrt{\tilde{s}_a^2 - s_x^2 - s_y^2} \quad . \tag{7.51}$$

The transmitted fields in the region $z > z_o$ are, therefore, given by the general form

$$\tilde{Q}_a\left[\tilde{f}(\tilde{U}_a)\right] e^{-i\tilde{U}_a(z-z_o)} + \tilde{Q}_b\left[\tilde{f}(+\tilde{U}_b)\right] e^{-i\tilde{U}_b(z-z_o)} + \tilde{Q}_c\left[\tilde{f}(+\tilde{U}_c)\right] e^{-i\tilde{U}_c(z-z_o)} \tag{7.52}$$

where the $\tilde{f}$'s are the appropriate functions for the field components corresponding to the properties of the medium in the region $z > z_o$. In terms of a transmission matrix, the "amplitudes" of the transmitted waves can be calculated from

$$\begin{bmatrix} \tilde{Q}_a \\ \tilde{Q}_b \\ \tilde{Q}_c \end{bmatrix} = \left[\tau(z_o)\right] \begin{bmatrix} P_a \\ P_b \\ P_c \end{bmatrix} \quad . \tag{7.53}$$

The transmission and reflection matrices may be calculated by the principle of invariant embedding, if we can reformulate the set of Eqs. (7.46) in terms of forward and backward propagating waves. Utilizing the property that the field components must be continuous, we see that the boundary conditions for the differential Eqs. (7.46) are

(i) at Z = 0 ,

$$\begin{bmatrix} E_x \\ E_y \\ h_x \\ h_y \\ V_z \\ n \end{bmatrix} = P_a f(U_a)\Big] + P_b f(U_b)\Big] + P_c f(U_c)\Big] + Q_a f(-U_a)\Big] + Q_b f(-U_b)\Big] + Q_c f(-U_c)\Big] \tag{7.54}$$

and (ii) at $Z = z_o$,

$$\begin{bmatrix} E_x \\ E_y \\ h_x \\ h_y \\ V_z \\ n \end{bmatrix} = \tilde{Q}_a \tilde{f}(\tilde{U}_a)\Big] + \tilde{Q}_b \tilde{f}(\tilde{U}_b)\Big] + \tilde{Q}_c \tilde{f}(\tilde{U}_c)\Big] \tag{7.55}$$

We may eliminate τ and R from Eqs. (7.50), (7.53), (7.54) and (7.55), by forming linear combinations of E_x, E_y, h_x, h_y, V_z and n that may be identified as the forward and backward propagating waves. Such algebraic derivations are somewhat complicated, but are, nevertheless straightforward. The procedure is stated here:

a) Form the determinant

$$\Delta = \begin{vmatrix} f_1(U_a) & f_2(U_a) & f_3(U_a) & f_4(U_a) & f_5(U_a) & f_6(U_a) \\ f_1(U_b) & f_2(U_b) & f_3(U_b) & f_4(U_b) & f_5(U_b) & f_6(U_b) \\ f_1(U_c) & f_2(U_c) & f_3(U_c) & f_4(U_c) & f_5(U_c) & f_6(U_c) \\ f_1(-U_a) & f_2(-U_a) & f_3(-U_a) & f_4(-U_a) & f_5(-U_a) & f_6(-U_a) \\ f_1(-U_b) & f_2(-U_b) & f_3(-U_b) & f_4(-U_b) & f_5(-U_b) & f_6(-U_b) \\ f_1(-U_c) & f_2(-U_c) & f_3(-U_c) & f_4(-U_c) & f_5(-U_c) & f_6(-U_c) \end{vmatrix} \qquad (7.56)$$

and obtain the cofactors corresponding to each of the elements of the first three rows and denote these cofactors by a_i, b_i, c_i $(i = 1, 2, 3, 4, 5, 6)$.

b) Form the determinant

$$\widetilde{\Delta} = \begin{vmatrix} f_1(-U_a) & f_2(-U_a) & f_3(-U_a) & f_4(-U_a) & f_5(-U_a) & f_6(-U_a) \\ f_1(-U_b) & f_2(-U_b) & f_3(-U_b) & f_4(-U_b) & f_5(-U_b) & f_6(-U_b) \\ f_1(-U_c) & f_2(-U_c) & f_3(-U_c) & f_4(-U_c) & f_5(-U_c) & f_6(-U_c) \\ \widetilde{f}_1(\widetilde{U}_a) & \widetilde{f}_2(\widetilde{U}_a) & \widetilde{f}_3(\widetilde{U}_a) & \widetilde{f}_4(\widetilde{U}_a) & \widetilde{f}_5(\widetilde{U}_a) & \widetilde{f}_6(\widetilde{U}_a) \\ \widetilde{f}_1(\widetilde{U}_b) & \widetilde{f}_2(\widetilde{U}_b) & \widetilde{f}_3(\widetilde{U}_b) & \widetilde{f}_4(\widetilde{U}_b) & \widetilde{f}_5(\widetilde{U}_b) & \widetilde{f}_6(\widetilde{U}_b) \\ \widetilde{f}_1(\widetilde{U}_c) & \widetilde{f}_2(\widetilde{U}_c) & \widetilde{f}_3(\widetilde{U}_c) & \widetilde{f}_4(\widetilde{U}_c) & \widetilde{f}_5(\widetilde{U}_c) & \widetilde{f}_6(\widetilde{U}_c) \end{vmatrix} \tag{7.57}$$

and obtain the cofactors of the elements in the first three rows. Denote these cofactors by $\widetilde{a}_i$, $\widetilde{b}_i$, $\widetilde{c}_i$ $(i = 1, 2, 3, 4, 5, 6)$.

c) Define $[U]$, $[W]$ by the transformation,

$$\begin{bmatrix} U \\ \\ W \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ \widetilde{a}_1 & \widetilde{a}_2 & \widetilde{a}_3 & \widetilde{a}_4 & \widetilde{a}_5 & \widetilde{a}_6 \\ \widetilde{b}_1 & \widetilde{b}_2 & \widetilde{b}_3 & \widetilde{b}_4 & \widetilde{b}_5 & \widetilde{b}_6 \\ \widetilde{c}_1 & \widetilde{c}_2 & \widetilde{c}_3 & \widetilde{c}_4 & \widetilde{c}_5 & \widetilde{c}_6 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ h_x \\ hy \\ V_z \\ n \end{bmatrix} \tag{7.58}$$

d) It is easily seen that, at $z = 0$,

$$[U] = \Delta \begin{bmatrix} P_a \\ P_b \\ P_c \end{bmatrix} \quad (7.59)$$

and

$$[W] = \begin{bmatrix} \tilde{\Delta}_{1a} & \tilde{\Delta}_{1b} & \tilde{\Delta}_{1c} \\ \tilde{\Delta}_{2a} & \tilde{\Delta}_{2b} & \tilde{\Delta}_{2c} \\ \tilde{\Delta}_{3a} & \tilde{\Delta}_{3b} & \tilde{\Delta}_{3c} \end{bmatrix} \begin{bmatrix} P_a \\ P_b \\ P_c \end{bmatrix} + \tilde{\Delta} \begin{bmatrix} Q_a \\ Q_b \\ Q_c \end{bmatrix} \quad (7.60)$$

where, $\tilde{\Delta}_{1a}$ is the determinate formed by replacing the first row of Eq. (7.57) by $f_1(U_a)$, $f_2(U_a)$, $f_3(U_a)$, $f_4(U_a)$, $f_5(U_a)$, $f_6(U_a)$. The other determinants are similarly defined.

The boundary conditions at $z = z_o$ are

$$[W] = 0 \quad (7.61)$$

and

$$[U] = \begin{bmatrix} \Delta_{1a} & \Delta_{1b} & \Delta_{1c} \\ \Delta_{2a} & \Delta_{2b} & \Delta_{2c} \\ \Delta_{3a} & \Delta_{3b} & \Delta_{3c} \end{bmatrix} \begin{bmatrix} \tilde{Q}_a \\ \tilde{Q}_b \\ \tilde{Q}_c \end{bmatrix} \quad (7.62)$$

where Δ_{1a} is the determinant formed by replacing the first row of Δ by $\tilde{f}_1(\tilde{U}_a)$, $\tilde{f}_2(\tilde{U}_a)$ . . . $\tilde{f}_6(\tilde{U}_a)$. The other determinants are similarly defined.

e) From the transformation given by Eq. (7.58), the differential equation for U and W can be obtained. This set of equations is in the standard form of Eq. (7.41), with the same type of boundary conditions as Eq. (7.42). Thus, Eqs. (7.44) and (7.45) may be integrated to obtain $[R]$ and $[\tau]$ appropriate for $[U]$ and $[W]$.

f) For the reflection and transmission of the actual fields, we see that from Eq. (7.62)

$$\begin{bmatrix} \Delta_{1a} & \Delta_{1b} & \Delta_{1c} \\ \Delta_{2a} & \Delta_{2b} & \Delta_{2c} \\ \Delta_{3a} & \Delta_{3b} & \Delta_{3c} \end{bmatrix} [\tau] \begin{bmatrix} P_a \\ P_b \\ P_c \end{bmatrix} = \Delta [T] \begin{bmatrix} P_a \\ P_b \\ P_c \end{bmatrix} \tag{7.63}$$

Thus,

$$[\tau] = \Delta \begin{bmatrix} \Delta_{1a} & \Delta_{1b} & \Delta_{1c} \\ \Delta_{2a} & \Delta_{2b} & \Delta_{2c} \\ \Delta_{3a} & \Delta_{3b} & \Delta_{3c} \end{bmatrix}^{-1} [T] \tag{7.64}$$

Similarly, from Eq. (7.60),

$$\begin{bmatrix} \tilde{\Delta}_{1a} & \tilde{\Delta}_{1b} & \tilde{\Delta}_{1c} \\ \tilde{\Delta}_{2a} & \tilde{\Delta}_{2b} & \tilde{\Delta}_{2c} \\ \tilde{\Delta}_{3a} & \tilde{\Delta}_{3b} & \tilde{\Delta}_{3c} \end{bmatrix} \begin{bmatrix} P_a \\ P_b \\ P_c \end{bmatrix} + \tilde{\Delta} [R] \begin{bmatrix} P_a \\ P_b \\ P_c \end{bmatrix} = \Delta [T] \begin{bmatrix} P_a \\ P_b \\ P_c \end{bmatrix} \tag{7.65}$$

Thus,

$$[R] = \frac{\Delta}{\widetilde{\Delta}} [T] - \frac{1}{\widetilde{\Delta}} \begin{bmatrix} \widetilde{\Delta}_{1a} & \widetilde{\Delta}_{1b} & \widetilde{\Delta}_{1c} \\ \widetilde{\Delta}_{2a} & \widetilde{\Delta}_{2b} & \widetilde{\Delta}_{2c} \\ \widetilde{\Delta}_{3a} & \widetilde{\Delta}_{3b} & \widetilde{\Delta}_{3c} \end{bmatrix} \tag{7.66}$$

The procedure outlined above seems to be complicated, but numerical results can be obtained in a fairly straightforward manner.

7.5 Operator Transform Method

A general and, perhaps, more basic approach to the investigation of the excitation and propagation problem in an inhomogeneous, ionized medium is probably the generalized operator method outlined in Section IV. It is shown there, that for the most general excitation problem, this formulation reduces to the solution of an integral equation of Fredholm type, i.e.,

$$\Psi_1(u) = F(u) + \oint k(u, s) \Psi_1(s) \tag{7.67}$$

where the kernel k(u, s) is a 6x6 matrix. For stratified media, this equation involves only a one-dimensional integral, so that a numerical solution of such an equation may be within the reach of presént day computer capacity. Additional simplifications obtained by taking advantage of the symmetry of the sources may further simplify the problem. In this section, some possible approximate solutions obtained by utilizing this formulation are discussed.

a) The Perturbation Method. If the excitation problem in a medium is solved, for example, for the case of plane waves in a stratified media, then the effect of a slight disturbance of the medium on the resulting wave can be investigated approximately by perturbation methods. Mathematically, this means that if the solution $\Psi_o(u)$ of

$$\Psi_o(u) = F(u) + \oint k_o(u, s)\Psi_o(s) \tag{7.68}$$

is known, then for a slight perturbation of the medium, the kernel $k(u, s)$ may be written as

$$k(u, s) = k_o(u, s) + \Delta k(u, s) \ .$$

Assuming a solution of the form

$$\Psi(u) = \Psi_o(u) + \Delta\Psi(u)$$

and using the familiar Born approximation $\Delta\Psi(u)$ is given approximately by

$$\Delta\Psi(u) = \oint \Delta k(u, s)\, \Psi_o(s) \ . \tag{7.69}$$

b) Series Solutions. If we rewrite Eq. (7.76) in the form

$$\Psi(u) = F(u) + \lambda \int k(u, s)\Psi(s) \tag{7.70}$$

by either considering Eq. (7.67) as a special case of (7.70) for $\lambda = 1$, or by scaling k so that λ is a small parameter, then a series approximation for obtaining the resolvent of the integral equation may apply. Formally, we may write the solution of Eq. (7. 70) as ,

$$\Psi(s) = F(u) + \lambda \oint H(u, s, \lambda) F(u) \tag{7.71}$$

The formal procedure of obtaining the resolvent $H(u, s, \lambda)$ has been discussed by Diament (1963). Following the derivations for a one-dimensional integral equation, it is suggested that if we write

$$H(u, s, \lambda) = \frac{c(u, s, \lambda)}{p(\lambda)} \tag{7.72}$$

where $p(\lambda)$ and $c(u, s, \lambda)$ are entire functions of λ , and expand $p(\lambda)$ as

$$p(\lambda) = \sum_{n=0}^{\infty} p_n \lambda^n \tag{7.73}$$

and $c(u, s, \lambda)$ as

$$c(u, s, \lambda) = \sum_{n=0}^{\infty} c_n(u, s) \lambda^n \tag{7.74}$$

then a numerical scheme may be employed to obtain the various coefficients of Eqs. (7. 73) and (7. 74) by the following recurrence relations,

$$p_o = 1 \ ,$$

$$c_o(u, s) = k(u, s)$$

$$-np_n = \oint \mathrm{Tr}\; c_{n-1}(s, s)$$

$$c_n(u, s) = k(u, s)\, p_n + \oint k(u, v)\; c_{n-1}(v, s)$$

where $\mathrm{Tr}\, c$ denotes the trace of the matrix c. The rate of convergence of this scheme, however, has not been investigated.

Other possible approaches for obtaining approximate, numerical solutions of integral equations, such as the variational technique, Gelerkin's method, or the approximation of the equation by a linear algebraic equation, which have been used in solving one-dimensional equations, in principle, can be applied to Eq. (7.67). However, unless the equation is further reduced, the labor of computation is, perhaps, prohibitive.

In order to take full advantage of this formulation, perhaps, the most realistic approach is to obtain complete solutions for some ideal kernels, which closely approximate actual physical problems, where the solution may be carried out, in part, analytically. Solutions of a variety of realistic problems can then be treated as perturbations of the solutions of these standard problems.

Due to limitations of time, no computations were made for inhomogeneous media.

VIII

CONCLUSIONS

The primary goal in this work was to investigate the excitation and propagation of wave like disturbance in the ionosphere. As a necessary step in carrying out such an investigation, a realistic three-fluid model of the ionosphere was developed. The fundamental modes, which can propagate in an infinite homogeneous media, have been investigated and a computer program developed to compute the propagation constant corresponding to parameters appropriate to the ionosphere.

Various approaches to the problem of the excitation and propagation of disturbances in an inhomogeneous medium have been investigated. As a result of this investigation, it is suggested that there are two realistic approaches to this problem.

1. Solution of the excitation problem, if the medium is locally homogeneous near the source. From the known field excited near the source, the fields in other regions may be obtained by using the appropriate formulation for a stratified medium.

2. Solution of a set of standard problems, using the operator transform method, by idealizing the kernel of the integral equation so that the complete solution of a set of standard problems can be carried out, at least in part, by analytical methods. The solution for a real problem can then be obtained by perturbation methods from the known solution of an appropriate standard problem.

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APPENDIX A

MACROSCOPIC PROPERTIES OF THE IONOSPHERE

A.1 Discussion of the Model

For the purpose of this report, a three-fluid model of the ionosphere containing an electron gas, positive ion gas and a neutral gas was developed which would contain parameters appropriate to an average ionosphere in the altitude range of 100 Km to 700 Km. Since this model was not intended to represent the real ionosphere at any particular geographic location or time, some assumptions were made which would simplify analytic procedures based on this model but which would maintain the salient features of the real ionosphere. For example, the ratio of the specific heats, γ, for the neutral gas was assumed to be constant and equal to the sea level value of 1.4. Similarly, the magnitude of the earth's magnetic field was assumed to be constant and equal to 1/2 Gauss. In addition, thermal equilibrium among the gases was assumed. Both theoretical and experimental evidence is available which indicates that the electron and ion temperature, in some cases, may be greater than the neutral particle temperature and thus, for example, the electron and ion "acoustic velocities" predicted by this model may be lower than corresponding values for the real ionosphere (Nagy et al, 1963).

A brief description of source of the parameters used in the model ionosphere follows. Since the acoustic velocities and the collision frequencies were computed from the parameters of the model ionosphere, these quantities will be discussed separately.

A.2 Macroscopic Properties

a. Neutral Gas

The properties of the neutral gas were taken from the U. S. Standard Atmosphere (1962). Since the ionosphere is weakly ionized, it was assumed that these parameters would provide a reasonable representation of the values for the neutral gas. Curves of the temperature, molecular weight, neutral particle density and collision frequency are shown in Fig. 1 through 4, respectively.

b. Electron and Ion Gas

Single ionization was assumed and thus, the number densities of the electrons and ions are equal. In addition, the molecular weight of the positive ions was assumed to be equal to that of the neutral particles. Also, thermal equilibrium was assumed and thus, the electron and ion temperatures are taken to be the same as the neutral gas temperature. The curve of electron density versus altitude is a composite curve taken from recent data appearing in the literature and is the same curve as used by Wu (1965). This curve is shown in Fig. 5.

c. Acoustic Velocities

The acoustic velocities for the r^{th} gas were computed from the usual equation

$$U_r = \sqrt{\frac{\gamma_r R T_r}{M_r}}$$

where γ_r is the ratio of the specific heats, R is the gas constant with the value 8314 joules $(^oK^{-1})(hg\text{-}mol)^{-1}$, T_r is the temperature in degrees Kelvin and M_r is the molecular weight.

The value of γ for the neutral gas was taken as the value given by the U.S. Standard Atmosphere (1962) for the altitude between 0 and 90 Km., i.e., 1.4. The motion in the electron and ion gases is supported by electric forces involving 1 degree of freedom only, and thus, γ for the charged particle gases was taken as 3 (Spitzer, 1962). The acoustic velocities in the electron, ion and neutral gas are shown in Fig. 6.

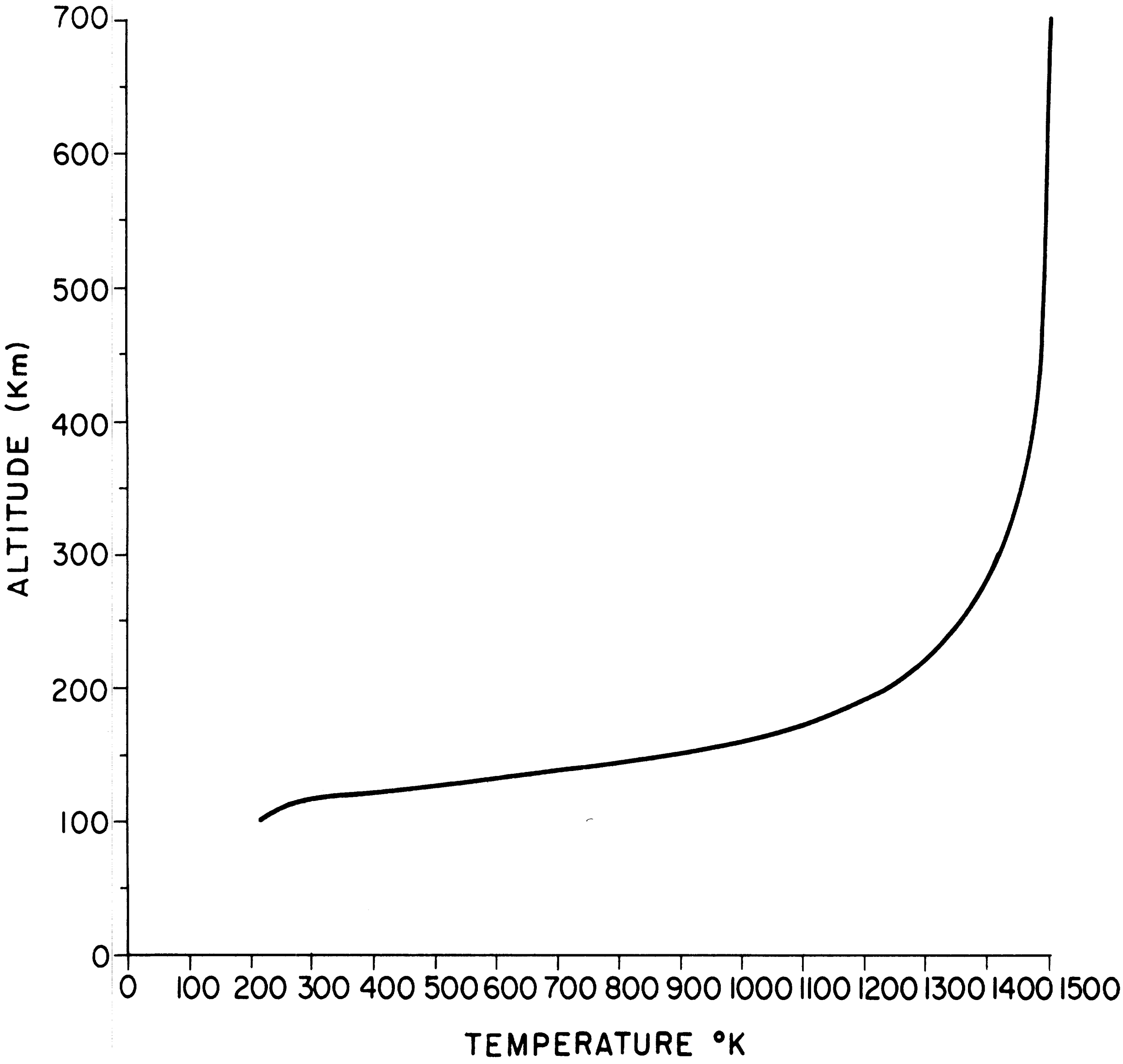

FIG. 1: NEUTRAL GAS TEMPERATURE AS A FUNCTION OF GEOMETRIC ALTITUDE.

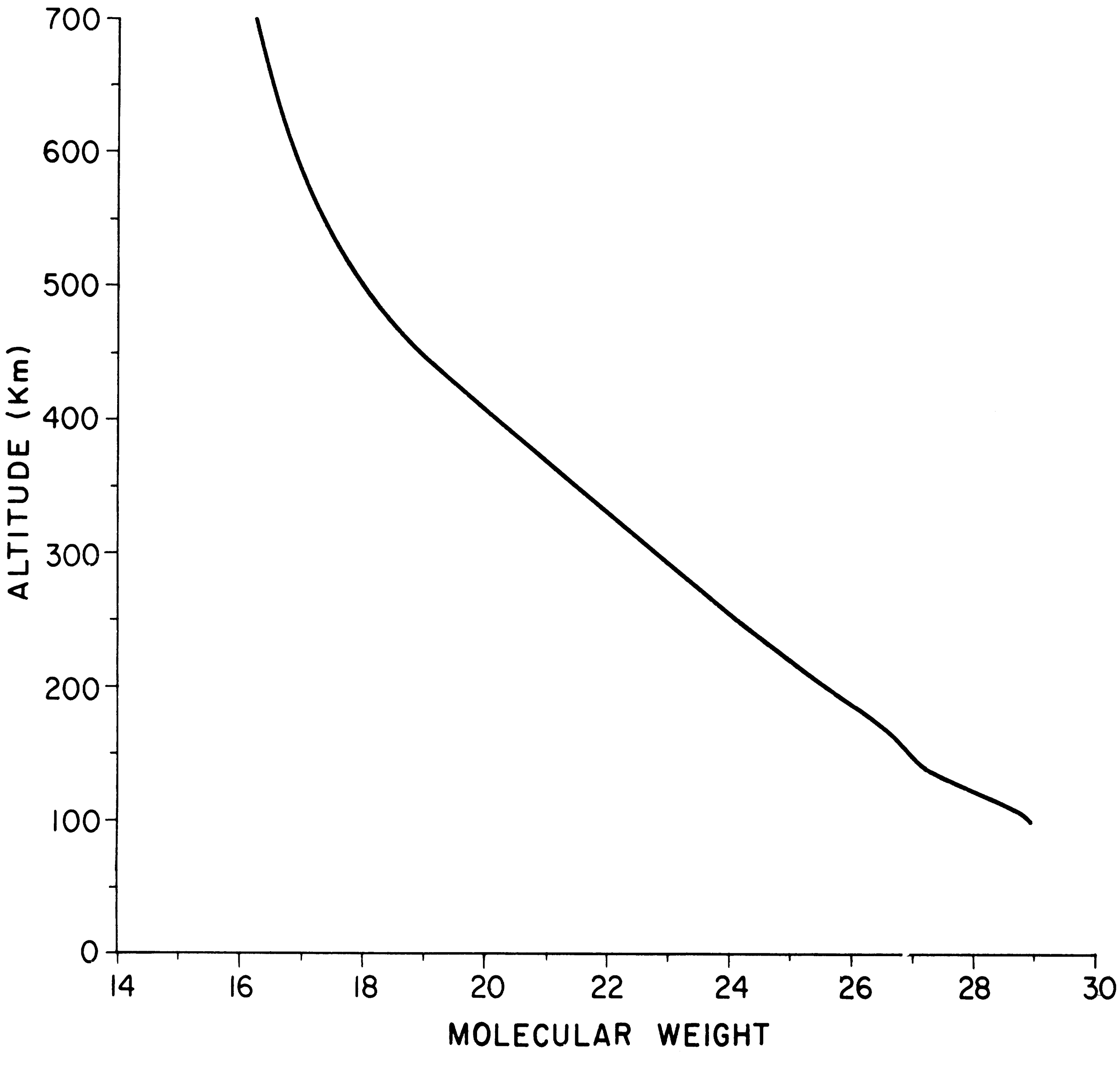

FIG. 2: MOLECULAR WEIGHT AS A FUNCTION OF GEOMETRIC ALTITUDE.

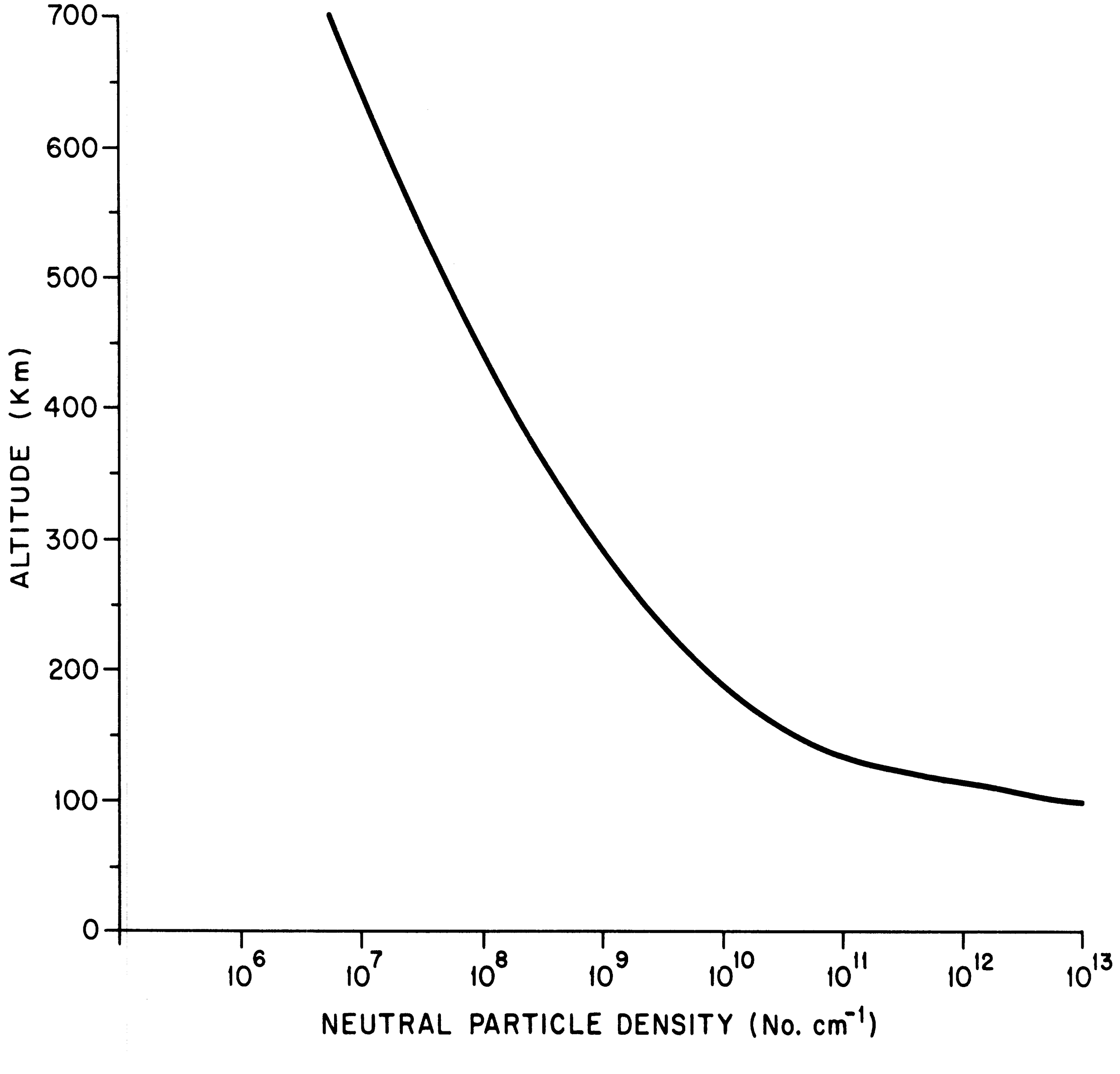

FIG. 3: NEUTRAL PARTICLE DENSITY AS A FUNCTION OF GEOMETRIC ALTITUDE.

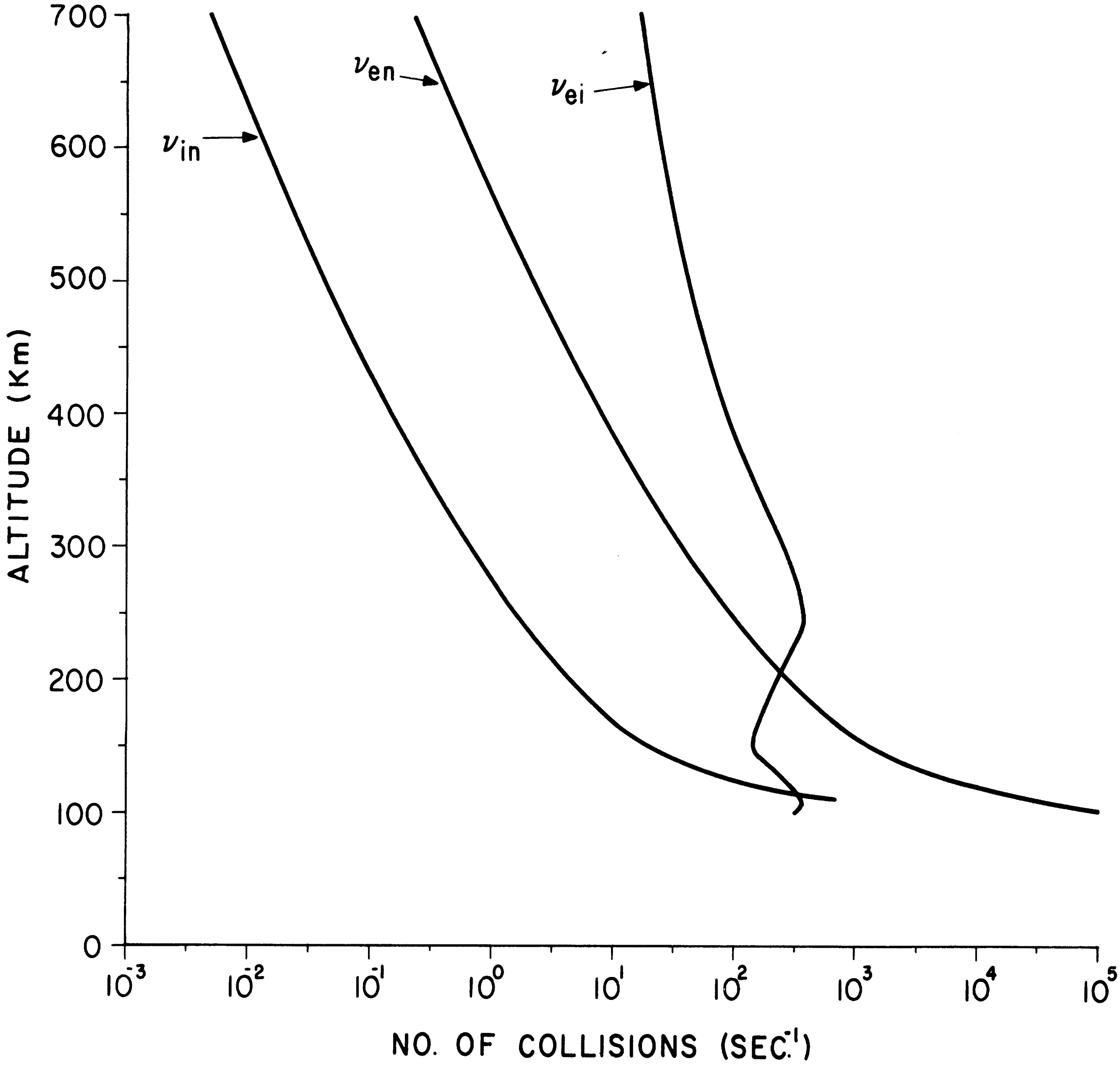

FIG. 4: ELECTRON-ION, ELECTRON NEUTRAL AND ION NEUTRAL COLLISION FREQUENCY AS A FUNCTION OF GEOMETRIC ALTITUDE.

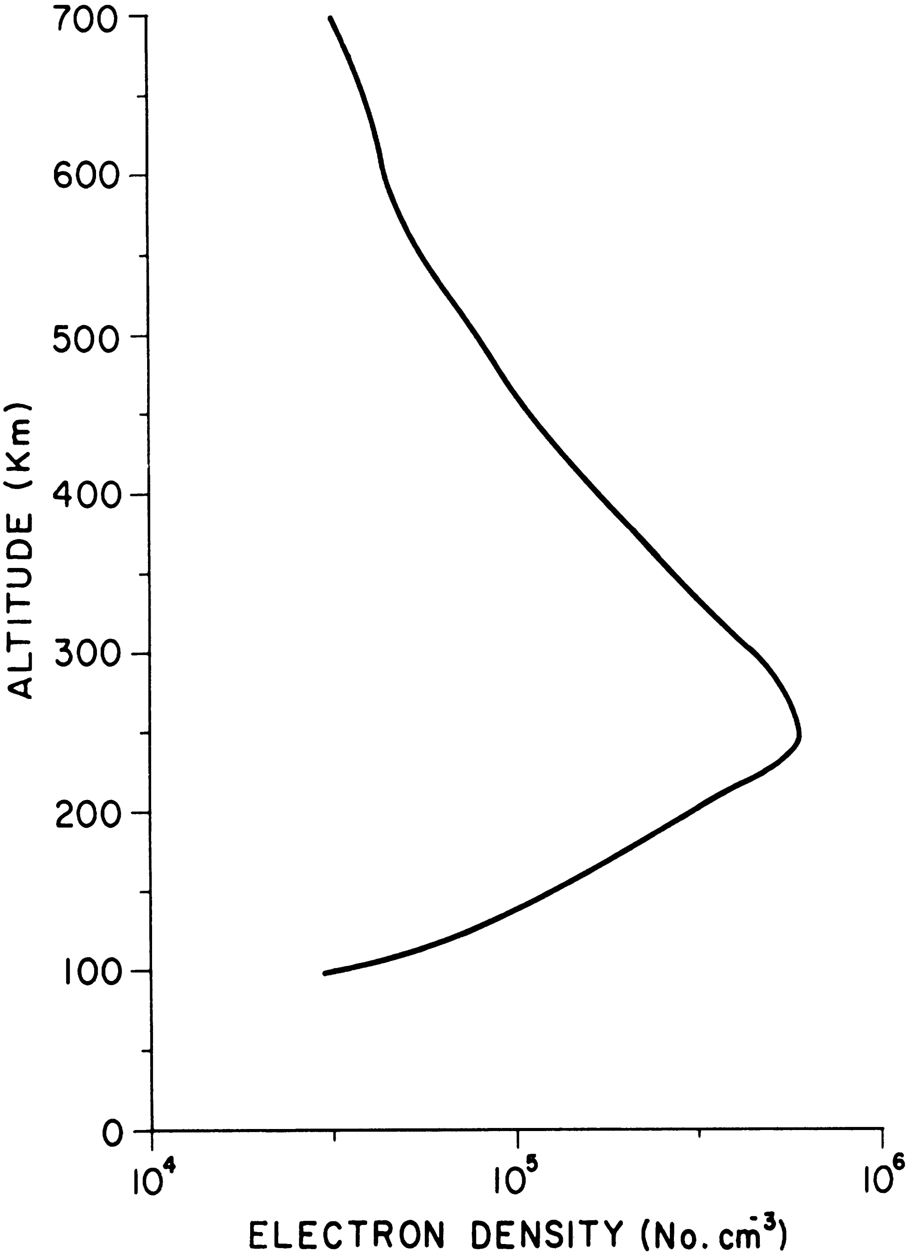

FIG. 5: ELECTRON DENSITY AS A FUNCTION OF GEOMETRIC ALTITUDE.

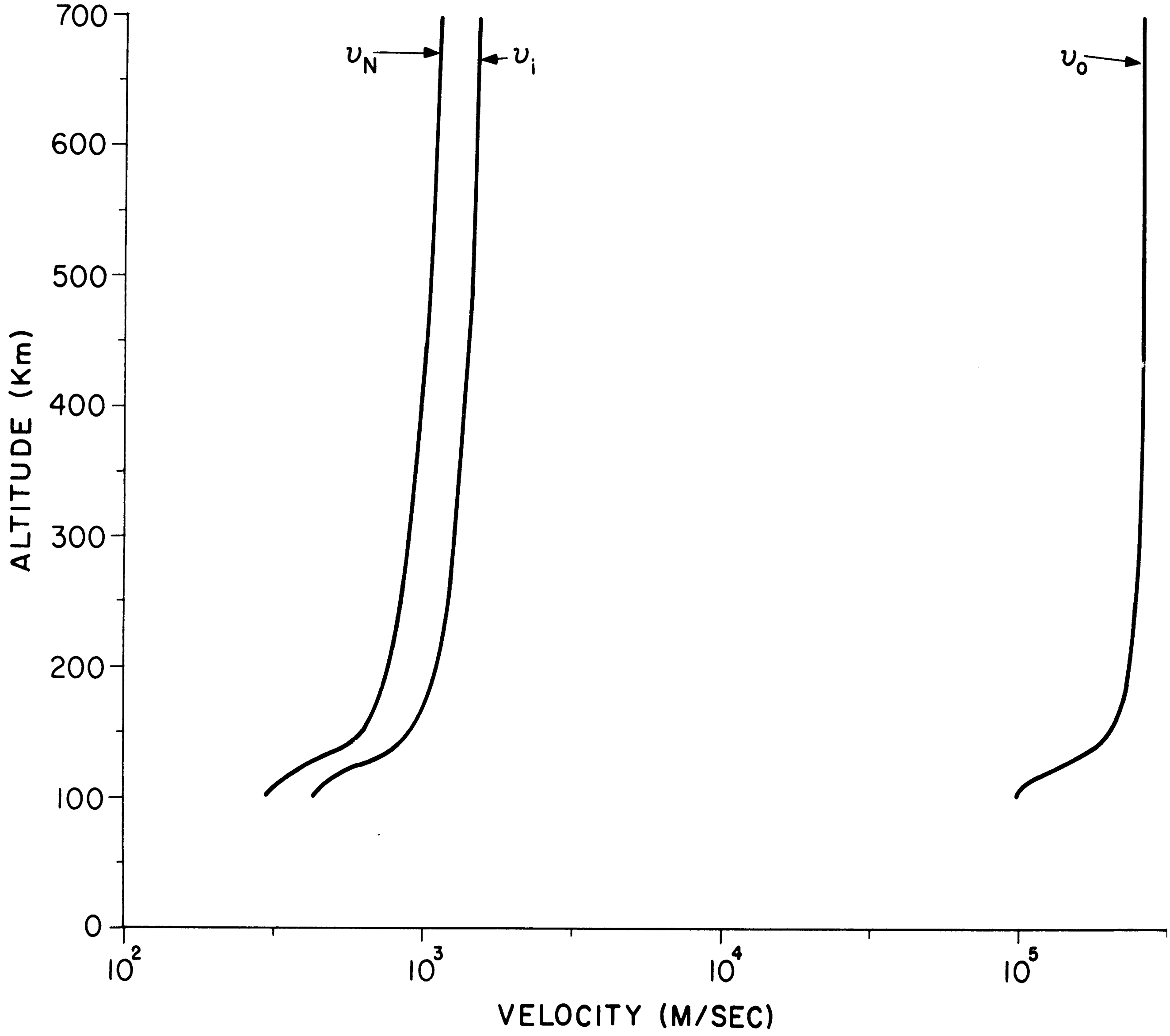

FIG. 6: ACOUSTIC VELOCITY IN ELECTRON, ION AND NEUTRAL GAS AS A FUNCTION OF GEOMETRIC ALTITUDE.

d. Collision Frequency

Since the ionosphere is weakly ionized, the value of the ion neutral collision frequency, ν_{in} was taken to be the same as the collision frequency given in the U.S. Standard Atmosphere (1962). The electron-neutral and electron-ion collision frequency were computed from the following equations due to Cowling (1945).

$$\nu_{en} = 1.8 \times 10^{-8} \left(\frac{T}{300}\right)^{\frac{1}{2}} N_n$$

$$\nu_{ei} = 6.1 \times 10^{-3} \left(\frac{T}{300}\right)^{-\frac{3}{2}} N_i$$

where ν_{en} and ν_{ei} are the number of collisions per second of an electron with neutral particles and positive ions, respectively. T is the temperature in degrees Kelvin and N_n and N_i are the number density per cubic centimeter of neutrals and positive ions, respectively.

Since collisions are a momentum transfer process, conservation of momentum requires that the following equation be satisfied

$$N_r m_r \nu_r s = N_s m_s \nu_s r \quad .$$

Thus, the ion-electron, neutral-electron and neutral-ion collision frequencies are given by the following equations, respectively.

$$\nu_{ie} = \frac{N_e}{N_i} \frac{m_e}{m_i} \nu_{ei}$$

$$\nu_{ne} = \frac{N_e}{N_n} \frac{m_e}{m_n} \nu_{en}$$

$$\nu_{ni} = \frac{N_i}{N_n} \frac{m_i}{m_n} \nu_{in} \quad .$$

The values of the ion-neutral, electron-neutral and electron-ion collision frequencies are shown in Fig. 4 and the neutral-electron, neutral-ion and ion-electron collision frequencies are shown in Fig. 7.

e. The Electron and Ion Plasma Frequencies

The electron and ion plasma frequencies were computed from the usual relation

$$\omega_{pr} = \sqrt{\frac{N_r^2 q^2}{m_r \epsilon_o}}$$

where N_r is the particle density per cubic meter, q is the electronic charge in coulombs, m_r is the mass of the particle in K_g and ϵ_o the dielectric constant of free space in farads per meter. Values of the electron and ion plasma frequency are shown in Fig. 8.

f. The Electron and Ion Gyro Frequency

The electron and ion gyro frequencies were computed from the equation

$$\omega_{gr} = \frac{q\beta_o}{m_r}$$

where q is the numerical value of the electronic charge, β_o is the magnitude of the earth's magnetic field and m_r is the mass of the particle. Since the magnitude of the earth's magnetic field was taken to be constant, the electron gyro frequency is a constant with a value of 8.76×10^7 cycles/second. The ion gyro frequency which varies with the mass of the positive ions is shown in Fig. 9

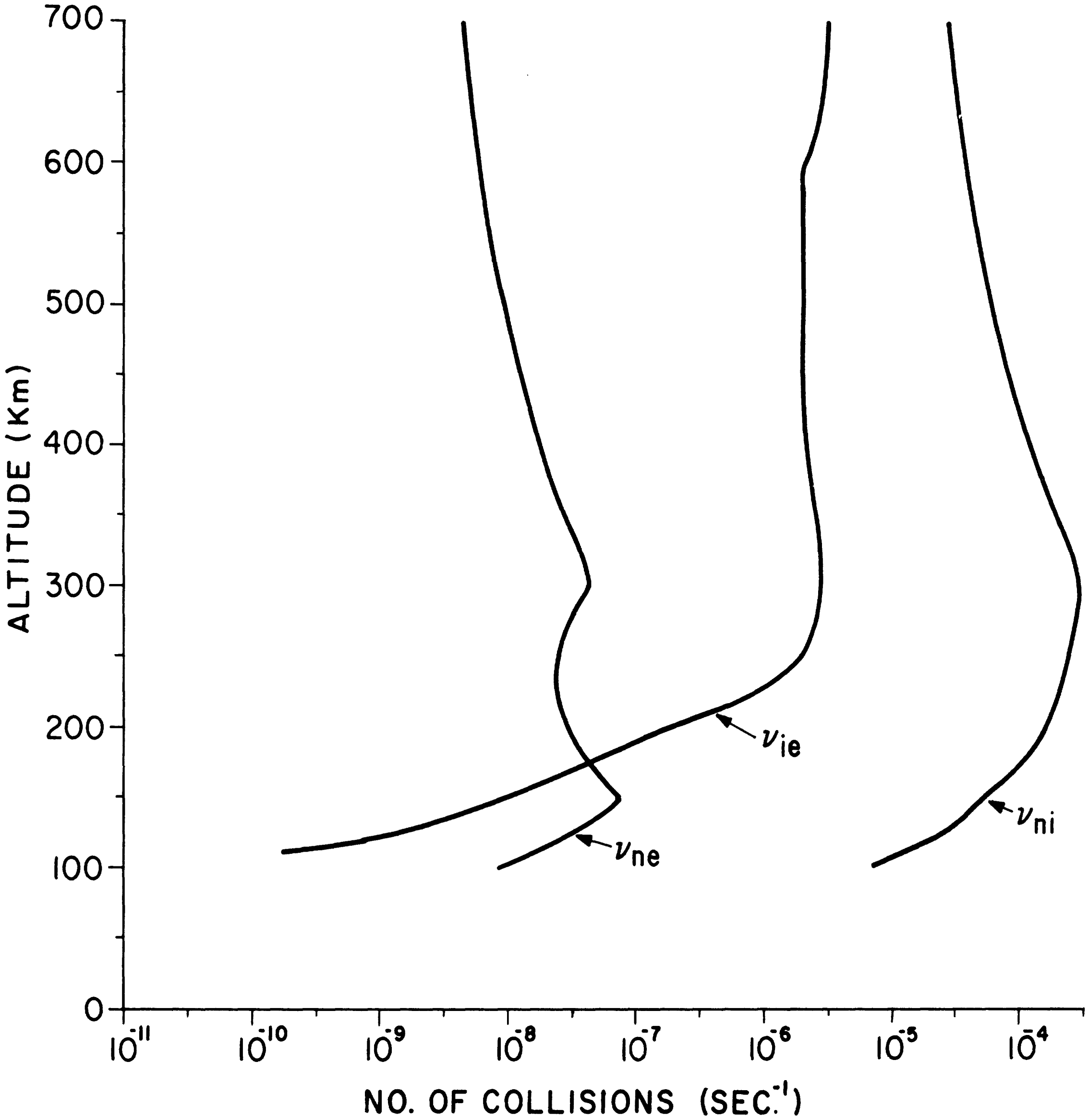

FIG. 7: ION-ELECTRON, NEUTRAL-ELECTRON AND NEUTRAL-ION COLLISION FREQUENCY AS A FUNCTION OF GEOMETRIC ALTITUDE.

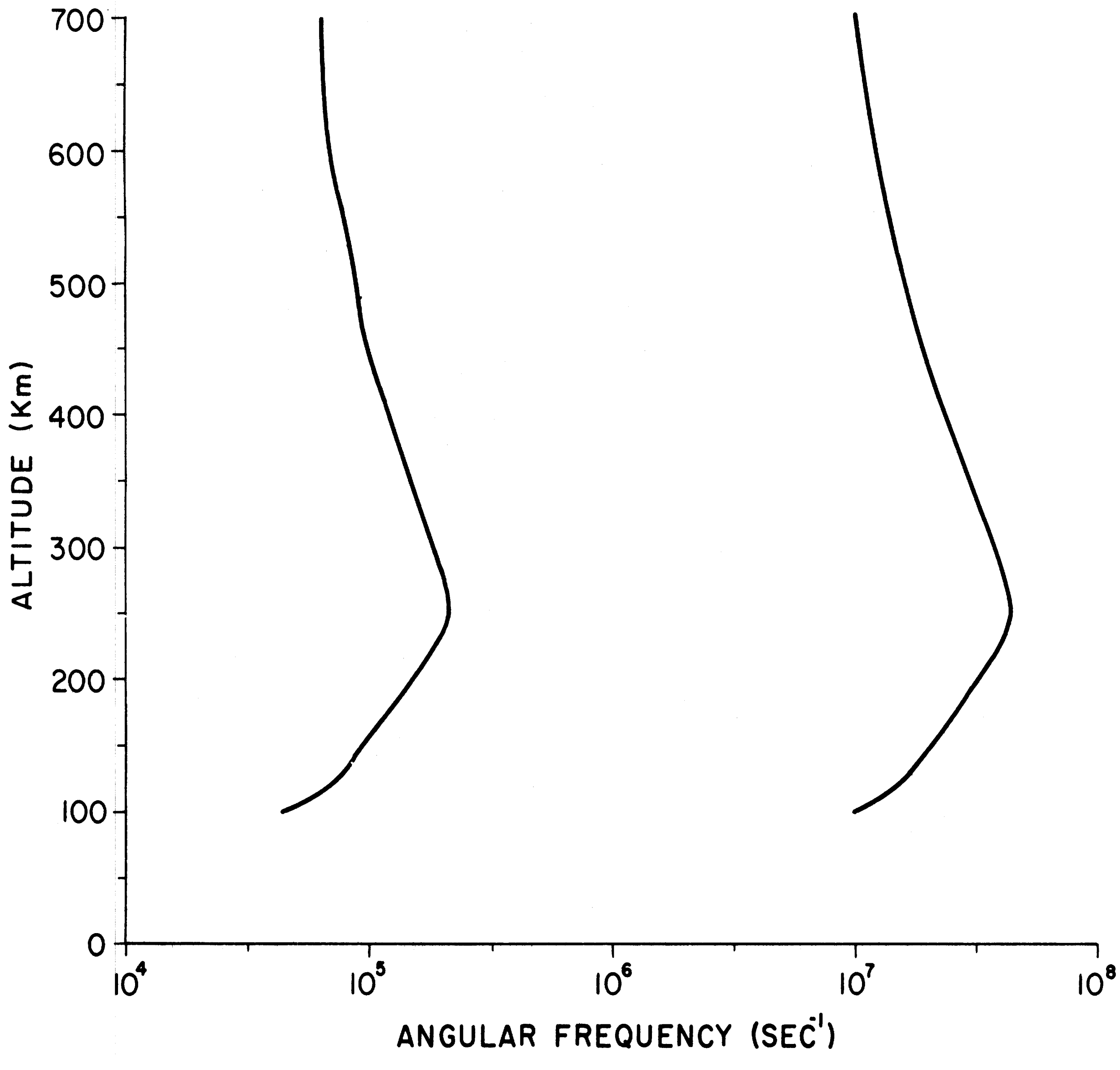

FIG. 8: ELECTRON AND ION PLASMA FREQUENCY AS A FUNCTION OF GEOMETRIC ALTITUDE.

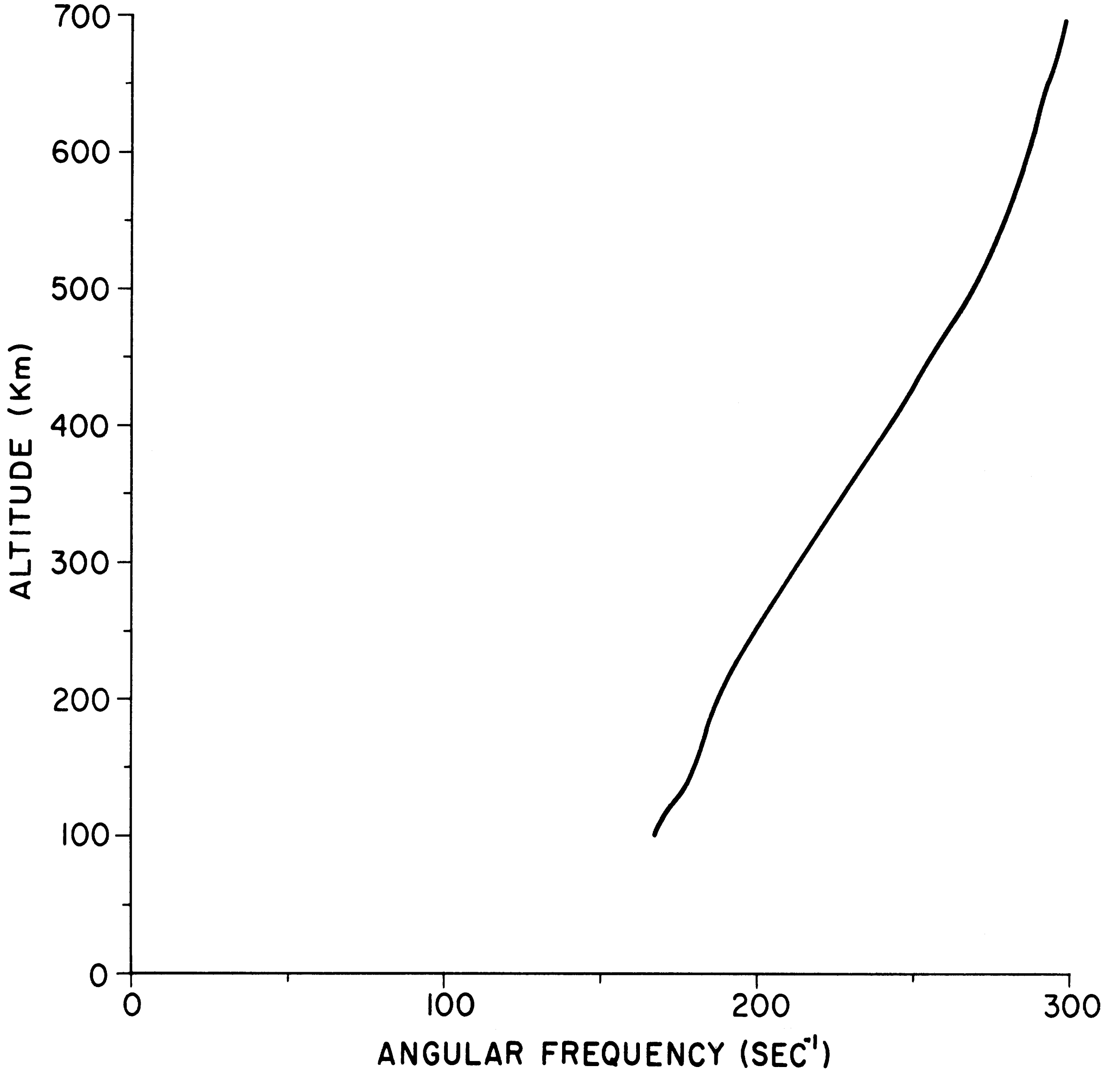

FIG. 9: ION GYRO FREQUENCY AS A FUNCTION OF GEOMETRIC ALTITUDE.

APPENDIX B

COMPUTER PROGRAM FOR NUMERICAL EVALUATION OF THE PROPAGATION CONSTANT

Due to the complicated nature of the dispersion relation, Eq. (3.11), the problem of evaluating the propagation constant was programmed for solution by a digital computer.

The data read in was taken from the ionospheric model in Appendix A. Calculation is then done in single precision to find the elements for two distinct three-dimensional matrices called A and B. The third dimension is to allow for each element being complex and in double precision. Matrix B has two complete columns of zeros. Each element of a determinant is of the form $a+bs^2$, where a is an element of the matrix A, and b is an element of the matrix B. When this determinant is evaluated it will give a fifth-degree equation in s^2.

To obtain the constant term of the polynomial the matrix A is evaluated as a determinant. All determinant evaluations are done using a Gauss-Jordan pivotal matrix inversion with the diagonal elements used as the pivotal elements and the value of the determinant found by taking the product of all the pivotal elements. The method is further described by Arden (1963). The computation is done in double-precision complex arithmetic and the value returned is in double-precision complex form. To introduce s^2 into the determinant, one column of the A matrix is replaced by the corresponding column of the B matrix and then used as the determinant. The coefficient of the s^{2n} term of the polynomial is the sum of the values of all possible determinants with n columns of A replaced by the corresponding columns of B.

The roots of the polynomial are found by a specialized version of Newton's method. All computation is done using double-precision complex arithmetic. Synthetic division was used to find the values of the polynomial and its first derivative. As each root is found, the equation is reduced. When it is reduced to a quadratic, the quadratic formula is used to find the last two roots. The initial approximations used for roots, the order in which the roots were extracted and the decision, in some cases, to extract the inverse of the root from the inverted polynomial, were all obtained as the result of previous analysis and hand computation done by Harold Hunter.

The external routines used were: DCXINV, previously described which evaluates the determinant; PROP, which does double-precision complex multiplication; DIV, which does double-precision complex division; ADD, which form the double-precision complex sum of two products of a real single-precision number and a double-precision complex number; DPMOIV, which raises a double-precision complex number to a real single-precision power; NEWT, an almost separate program linked to the main program by program common which, as previously described, finds the roots of the polynomial; QUAD, which finds the roots of a double-precision complex quadratic equation.

System routines used, but not displayed are: DPFA, DPFM, DPFDV, which perform, respectively, double-precision addition, multiplication and division; SIN and COS, which find the single-precision sine and cosine; ZERO, which zeros out a specified block of storage; MOVER, which copies a block of storage into a second block of storage; FTRAP, which sets to zero floating point underflows; DPSIN, DPCOS, DPEXP, DPELOG, DPSQRT, ATAN1, which find, respectively, the double-precision sine, cosine, power of e, logarithm to the base e, square-root and single valued arctangent.

The coding is done in MAD, a compiler similar to FORTRAN. A manual for the MAD can be obtained from The University of Michigan Computing Center.

```
              REFERENCES ON                                                                 *001
              BOOLEAN BS                                                                    *001
              PROGRAM COMMON CF,R1,R2,R3                                                    *002
              VECTOR VALUES HEAD=$1H1S20,12HPAGE NUMBER I3/14H FOR ALTITUDE=I3,S5,6HO      *003
            1 MEGA=E9.3,S5,10HAND ANGLE=F6.2/4H WE=E14.8,S2,3HWI=E14.8,S2,3                 *003
            1 HBO=E14.8,S2,3HBI=E14.8,S2,3HBE=E14.8,S2,3HBN=E14.8/5H KIN=E1                 *003
            1 4.8,S2,4HKEI=E14.8,S2,4HKEN=E14.8,S2,4HKNI=E14.8,S2,4HKIE=E14                 *003
            1 .8,S2,4HKNE=E14.8/4H OE=E14.8,S2,3HOI=E14.8*$                                 *003
              INTEGER L,M                                                                   *004
              VECTOR VALUES INPUT =$7E10.2*$                                                *005
              INTEGER ALT                                                                   *006
              NUM=0                                                                         *007
              VECTOR VALUES BS=0B                                                           *008
              FTRAP.                                                                        *009
START         READ FORMAT INPUT,VIN,VEI,VEN,VNI,VIE,VNE,UN,UI,UE,WEB,WIB,OIB               *010
              READ DATA                                                                     *011
              UE=UE*10.                                                                     *012
              VECTOR VALUES OEB=.88E7                                                       *013
              INTEGER NUM                                                                   *014
              DIMENSION (R1,R2,R3)(4)                                                       *015
              THROUGH END,FOR VALUES OF W=3E3                                               *016
              WE=WEB/W/W                                                                    *017
              WI=WIB/W/W                                                                    *018
              BO=W/3E8                                                                      *019
              BO2=BO*BO                                                                     *020
              BI=W/UI                                                                       *021
              BE=W/UE                                                                       *022
              BN=W/UN                                                                       *023
              R1(1)=1./BN/BN                                                                *024
              R2(1)=1./BI/BI                                                                *025
              R3(1)=1./BE/BE                                                                *026
              ZERO.(R1(2)...R1(4),R2(2)...R2(4),R3(2)...R3(4))                              *027
              KIN=VIN/W                                                                     *028
              KEI=VEI/W                                                                     *029
              KEN=VEN/W                                                                     *030
              KNI=VNI/W                                                                     *031
              KIE=VIE/W                                                                     *032
              KNE=VNE/W                                                                     *033
              OE=OEB/W                                                                      *034
              OI=OIB/W                                                                      *035
              WHENEVER BS                                                                   *036
              KNI=0.                                                                        *037
              KIE=0.                                                                        *038
              KNE=0.                                                                        *039
              OI=0.                                                                         *040
              OE=0.                                                                         *041
              END OF CONDITIONAL                                                            *042
              VECTOR VALUES PI=3.1415927                                                    *043
              THROUGH END,FOR VALUES OF ANG=0.                                              *044
              NUM=NUM+1                                                                     *045
              PRINT FORMAT HEAD,NUM,ALT,W,ANG,WE,WI,BO,BI,BE,BN,KIN,KEI,KEN,KNI,KIE,        *046
            1 KNE,OE,OI                                                                     *046
              DIMENSION (A,B,D)(7*7*4),CF(6*4)                                              *047
```

```
ZERO.(A...A(7,7,4),B...B(7,7,4))
ZERO.(CF...CF(6,4))
THETA=ANG/180.*PI
SINT=SIN.(THETA)
COST=COS.(THETA)
A(1,1,1)=B02*WE
A(1,1,3)=-B02*(KEI-KNI)
A(1,2,1)=B02*(1.-WI)
A(1,2,3)=B02*(KIE+KIN+KNI)
OEC=OE*COST
A(1,3,1)=-B02*OEC*KNI
OIC=OI*COST
OES=OE*SINT
OIS=OI*SINT
A(1,4,1)=B02*OIC*KNI
A(1,4,3)=-B02*OIC
A(2,1,1)=-1.
A(2,1,3)=-KNI-KNE-KEN
A(2,2,1)=-1.
A(2,2,3)=-KNI-KNE-KIN
A(2,3,1)=OEC*(KNI+KNE)
A(2,3,3)=-OEC
A(2,4,1)=-OIC*(KNI+KNE)
A(2,4,3)=OIC
A(3,1,1)=B02*OEC*KNI
A(3,2,1)=-B02*OIC*KNI
A(3,2,3)=B02*OIC
A(3,3,1)=B02*WE
A(3,3,3)=-B02*(KEI-KNI)
A(3,4,1)=B02*(1.-WI)
A(3,4,3)=B02*(KIE+KIN+KNI)
A(3,5,1)=-B02*OES*KNI
A(3,6,1)=B02*OIS*KNI
A(3,6,3)=-B02*OIS
A(4,1,1)=-OEC*(KNI+KNE)
A(4,1,3)=OEC
A(4,2,1)=OIC*(KNI+KNE)
A(4,2,3)=-OIC
A(4,3,1)=-1.
A(4,3,3)=-KNI-KNE-KEN
A(4,4,1)=-1.
A(4,4,3)=-KNI-KNE-KIN
A(4,5,1)=OES*(KNI+KNE)
A(4,5,3)=-OES
A(4,6,1)=-OIS*(KNI+KNE)
A(4,6,3)=OIS
A(5,3,3)=-OES
A(5,5,1)=1.-WE
A(5,5,3)=KEI+KEN
A(5,6,1)=WI
A(5,6,3)=-KIE
A(5,7,3)=-KNE
A(6,4,3)=OIS
A(6,5,1)=WE
A(6,5,3)=-KEI
A(6,6,1)=1.-WI
A(6,6,3)=KIE+KIN
A(6,7,3)=-KNI
A(7,5,3)=-KEN
A(7,6,3)=-KIN
```

```
            A(7,7,1)=1.
            A(7,7,3)=KNI+KNE
            B(1,1,3)=KEI-KNI
            B(1,2,1)=-1.
            B(1,2,3)=-KIE-KIN-KNI
            B(1,3,1)=DEC*KNI
            B(1,4,1)=-OIC*KNI
            B(1,4,3)=OIC
            B(3,1,1)=-DEC*KNI
            B(3,2,1)=OIC*KNI
            B(3,2,3)=-OIC
            B(3,3,3)=KEI-KNI
            B(3,4,1)=-1.
            B(3,4,3)=-KIE-KIN-KNI
            B(3,5,1)=DES*KNI
            B(3,6,1)=-OIS*KNI
            B(3,6,3)=OIS
            B(5,5,1)=-1./BE/BE
            B(6,5,1)=-1./BI/BI
            B(7,7,1)=-1./BN/BN
            SETUP.
            DCXINV.(7,D,DET)
            STORE.(0)
            THROUGH A1,FOR TL=1,1,TL.G.5
            L=GS(TL)
            SETUP.
            MOVER.(A(L,1,1)...A(L,7,4),D(L,1,1)...D(L,7,4))
            DCXINV.(7,D,DET)
            STORE.(1)
            MOVER.(A(1,1,1)...A(7,7,4),D(1,1,1)...D(7,7,4))
            MOVER.(B(L,1,1)...B(L,7,4),D(L,1,1)...D(L,7,4))
            DCXINV.(7,D,DET)
            STORE.(4)
A1          CONTINUE
            THROUGH A2,FOR TL=1,1,TL.G.5
            L=GS(TL)
            THROUGH A2,FOR TM=TL+1,1,TM.G.5
            M=GS(TM)
            SETUP.
            MOVER.(A(L,1,1)...A(L,7,4),D(L,1,1)...D(L,7,4))
            MOVER.(A(M,1,1)...A(M,7,4),D(M,1,1)...D(M,7,4))
            DCXINV.(7,D,DET)
            STORE.(2)
            MOVER.(A(1,1,1)...A(7,7,4),D(1,1,1)...D(7,7,4))
            MOVER.(B(L,1,1)...B(L,7,4),D(L,1,1)...D(L,7,4))
            MOVER.(B(M,1,1)...B(M,7,4),D(M,1,1)...D(M,7,4))
            DCXINV.(7,D,DET)
            STORE.(3)
A2          CONTINUE
            MOVER.(A(1,1,1)...A(7,7,4),D(1,1,1)...D(7,7,4))
            DCXINV.(7,D,DET)
            STORE.(5)
            THROUGH A3,FOR L=0,1,L.G.5
            PRINT FORMAT $1HOS20,1HAI1,1H=DE22.16,5H    +IDE22.16*$,L,CF(L+1,1)...C          F(L+1,4)
          1 F(L+1,4)
A3           CONTINUE
            NEWT.
END         CONTINUE
            TRANSFER TO START
            DIMENSION DET(4)
```

```
INTEGER TM,TL
INTERNAL FUNCTION SETUP.
MOVER.(A(2,1,1)...A(2,7,4),D(2,1,1)...D(2,7,4))
MOVER.(A(4,1,1)...A(4,7,4),D(4,1,1)...D(4,7,4))
MOVER.(B(1,1,1)...B(1,7,4),D(1,1,1)...D(1,7,4))
MOVER.(B(3,1,1)...B(3,7,4),D(3,1,1)...D(3,7,4))
MOVER.(B(5,1,1)...B(7,7,4),D(5,1,1)...D(7,7,4))
FUNCTION RETURN
END OF FUNCTION
INTERNAL FUNCTION STORE.(X)
INTEGER I,K
I=4*X
DPFA.(CF(I+1),CF(I+2),DET(1),DET(2),CF(I+1),CF(I+2))
DPFA.(CF(I+3),CF(I+4),DET(3),DET(4),CF(I+3),CF(I+4))
FUNCTION RETURN
INTEGER GS,X
END OF FUNCTION
VECTOR VALUES GS(1)=1,3,5,6,7
END OF PROGRAM
```

```
          EXTERNAL FUNCTION (N,B,DET)
          ENTRY TO CCXINV.
          ZERO.(DET(1)...DET(4))
          DET(1)=1.
          INTEGER I,J,K,N,L
          THROUGH S1, FOR K=1,1,K.G.N
          PROD.(DET,B(K,K,0),DET)
          F2.(B1,K,K,K,K)
          THROUGH S2, FORJ=1,1,J.G.N
          WHENEVER J.NE.K
          F2.(B2, K,J,K,K)
          DPFDV.(B2(1),B2(2),B1(1),B1(2),B(K,J,1),B(K,J,2))
          DPFDV.(B2(3),B2(4),B1(1),B1(2),B(K,J,3),B(K,J,4))
S2        END OF CONDITIONAL
          DPFDV.(B(K,K,1),B(K,K,2),B1(1),B1(2),B(K,K,1),B(K,K,2))
          DPFDV.(B(K,K,3),B(K,K,4),B1(1),B1(2),B(K,K,3),B(K,K,4))
          B(K,K,3)=-B(K,K,3)
          B(K,K,4)=-B(K,K,4)
          THROUGH S1, FOR I=1,1,I.G.N
          WHENEVER I.E.K,TRANSFER TOS1
          THROUGH S3, FOR J=1,1,J.G.N
          WHENEVER J.NE.K
          F1.(B2,I,K,K,J)
          THROUGHS5, FOR L=1,1,L.G.4
S5        B3(L)=-B2(L)
          DPFA.(B(I,J,1),B(I,J,2),B3(1),B3(2),B(I,J,1),B(I,J,2))
          DPFA.(B(I,J,3),B(I,J,4),B3(3),B3(4),B(I,J,3),B(I,J,4))
S3        END OF CONDITIONAL
          F1.(B2,I,K,K,K)
          THROUGH S4,FORL=1,1,L.G.4
S4        B(I,K,L)=-B2(L)
S1        CONTINUE
          DIMENSION B1(4),B2(4),B3(4)
          FUNCTION RETURN
          INTERNAL FUNCTION (Z,P,Q,R,S)
          DIMENSION TA(4), TB(4), TC(4)
          INTEGER P,Q,R,S,T
          ENTRY TO F1.
          TB(3)=B(R,S,3)
          TB(4)=B(R,S,4)
          TRANSFER TO CONT
          ENTRY TO F2.
          TB(3)=-B(R,S,3)
          TB(4)=-B(R,S,4)
CONT      TB(1)=B(R,S,1)
          TB(2)=B(R,S,2)
          THROUGH SETA, FOR T=1,1,T.G.4
SETA      TA(T)=B(P,Q,T)
          EXECUTE PROD. (TA,TB,TC)
          THROUGH STOC, FOR T=1,1,T.G.4
STOC      Z(T)=TC(T)
          FUNCTION RETURN
          END OF FUNCTION

          END OF FUNCTION
```

```
       EXTERNAL FUNCTION NEWT.
       REFERENCES ON
       PROGRAM COMMON A,R1,R2,R3
       VECTOR VALUES ONE(1)=1.,0.,0.,0.
       INTEGER DEG,K  ,J,L
       DIMENSION (C,A)((0...5)*4),(T,R1,R2,R3,R4,R5,F,DEL)(4)
       VECTOR VALUES CHEK=$1H 6(S5,DE15.9)*$
       DEG=5
       ROOT.(R1)
       DIV.(ONE,R1,RT)
       PRINT COMMENT $0    THE SQUARE OF THE ROOT$
       PRINT FORMAT RES,RT(1)...RT(4)
       DPMOIV.(.5,RT,RT)
       PRINT COMMENT $        THE ROOT$
       PRINT FORMAT RES,RT(1)...RT(4)
       VECTOR VALUES RES=$1H S20,DE22.16,S5,5H+I    DE22.16*$
       ROOT.(R2)
       DIV.(ONE,R2,RT)
       PRINT COMMENT $0    THE SQUARE OF THE ROOT$
       PRINT FORMAT RES,RT(1)...RT(4)
       DPMOIV.(.5,RT,RT)
       PRINT COMMENT $        THE ROOT$
       PRINT FORMAT RES,RT(1)...RT(4)
       ROOT.(R3)
       DIV.(ONE,R3,RT)
       PRINT COMMENT $0    THE SQUARE OF THE ROOT$
       PRINT FORMAT RES,RT(1)...RT(4)
       DPMOIV.(.5,RT,RT)
       PRINT COMMENT $        THE ROOT$
       PRINT FORMAT RES,RT(1)...RT(4)
       (K=1,1,K.G.2,DIV.(A(K,0),A,A(K,0)))
       QUAD.(A(4),A(8),DUM)
       DIMENSION DUM(8)
       (K=1,1,K.G.4,R4(K)=DUM(K))
       PRINT COMMENT $0    THE SQUARE OF THE ROOT$
       PRINT FORMAT RES,R4(1)...R4(4)
       DPMOIV.(.5,R4,RT)
       PRINT COMMENT $        THE ROOT$
       PRINT FORMAT RES,RT(1)...RT(4)
       (K=1,1,K.G.4,RT(K)=DUM(4+K))
       PRINT COMMENT $0    THE SQUARE OF THE ROOT$
       PRINT FORMAT RES,RT(1)...RT(4)
       DPMOIV.(.5,RT,RT)
       PRINT COMMENT $        THE ROOT$
       DIMENSION RT(4)
       PRINT FORMAT RES,RT(1)...RT(4)
       FUNCTION RETURN
       INTERNAL FUNCTION ROOT.(R)
AGAIN  (K=1,1,K.G.4,T(K)=A(DEG,K))
       THROUGH A1,FOR K=DEG-1,-1,K.L.0
       PROD.(T,R,T)
       ADD.(1.,A(K,0),1.,T,T)
       (J=1,1,J.G.4,C(K,J)=T(J))
```

```
A1          CONTINUE                                                                   *054
            (K=1,1,K.G.4,F(K)=T(K),T(K)=A(DEG,K))                                      *055
            THROUGH A2,FOR K=DEG-1,-1,K.L.1                                            *056
            PROD.(T,R,T)                                                               *057
            ADD.(1.,A(K,0),1.,T,T)                                                     *058
A2          CONTINUE                                                                   *059
            DIV.(F,T,DEL)                                                              *060
            WHENEVER .ABS.DEL(1).LE..ABS.(1E-12*R(1)).AND..ABS.DEL(3).LE..ABS.         *061
          1 (1E-12*R(3)),TRANSFER TO OUT                                               *061
            ADD.(1.,R,-1.,DEL,R)                                                       *062
            TRANSFER TO AGAIN                                                          *063
 OUT        (K=1,1,K.G.4,A(DEG-1,K)=A(DEG,K))                                          *064
            THROUGH A3,FOR K=DEG-1,-1,K.L.1                                            *065
            (L=1,1,L.G.4,A(K-1,L)=C(K,L))                                              *066
A3          CONTINUE                                                                   *067
            DEG=DEG-1                                                                  *068
            FUNCTION RETURN                                                            *069
            END OF FUNCTION                                                            *070
            END OF FUNCTION                                                            *071
```

```
EXTERNAL FUNCTION QUAD.(B,C,R)
REFERENCES ON
PROD.(B,B,B2)
DIMENSION (B2,SQ)(4)
ADD.(1.,B2,-4.,C,SQ)
DPMDIV.(.5,SQ,SQ)
ADD.(-.5,B,-.5,SQ,R)
ADD.(-.5,B,.5,SQ,R(4))
FUNCTION RETURN
END OF FUNCTION
```

```
EXTERNAL FUNCTION PROD.(A,B,C)
INTEGER M
DIMENSION D(8)
DPFM.(A(1),A(2),B(1),B(2),D(1),D(2))
DPFM.(A(3),A(4),B(3),B(4),D(3),D(4))
DPFM.(A(3),A(4),B(1),B(2),D(5),D(6))
DPFM.(A(1),A(2),B(3),B(4),D(7),D(8))
D(3)=-D(3)
D(4)=-D(4)
DPFA.(D(1),D(2),D(3),D(4),C(1),C(2))
DPFA.(D(5),D(6),D(7),D(8),C(3),C(4))
FUNCTION RETURN
END OF FUNCTION
```

```
EXTERNAL FUNCTION ADD.(X,A,Y,B,C)
VECTOR VALUES Z=0.
DPFM.(X,Z,A(1),A(2),RT1,RT2)
DPFM.(X,Z,A(3),A(4),IT1,IT2)
DPFM.(Y,Z,B(1),B(2),RS1,RS2)
DPFM.(Y,Z,B(3),B(4),IS1,IS2)
DPFA.(RT1,RT2,RS1,RS2,C(1),C(2))
DPFA.(IT1,IT2,IS1,IS2,C(3),C(4))
FUNCTION RETURN
END OF FUNCTION
```

```
EXTERNAL FUNCTION DIV.(A,B,C)
T(1)=B(1)
T(2)=B(2)
T(3)=-B(3)
T(4)=-B(4)
PROD.(A,T,N)
PROD.(B,T,T)
DPFDV.(N(1),N(2),T(1),T(2),C(1),C(2))
DPFDV.(N(3),N(4),T(1),T(2),C(3),C(4))
DIMENSION (T,N)(4)
FUNCTION RETURN
END OF FUNCTION
```

```
EXTERNAL FUNCTION DPMDIV.(N,SQ,RS)
DIMENSION TA(4),(R,ANG,COS,SIN)(1)
TA(1)=SQ(1)
TA(2)=SQ(2)
TA(3)=-SQ(3)
TA(4)=-SQ(4)
PROD.(SQ,TA,TA)
DPSQRT.(TA(1),R)
ATAN1.(SQ(3),SQ(1),ANG)
DPFM.(N,0.,ANG,ANG(1),ANG,ANG(1))
DPELOG.(R,R)
DPFM.(N,0.,R,R(1),R,R(1))
DPEXP.(R,R)
DPCOS.(ANG,COS)
DPSIN.(ANG,SIN)
DPFM.(R,R(1),COS,COS(1),RS(1),RS(2))
DPFM.(R,R(1),SIN,SIN(1),RS(3),RS(4))
FUNCTION RETURN
END OF FUNCTION
```

APPENDIX C

MATRIX ANALYSIS

C.1 Discussion of the Method

In this appendix a method for the piecewise inversion of matrices of large order will be developed. The result obtained will be similar to the method of "tearing" large scale systems developed by Kron (1963) but the work presented here will begin with a system of linear algebraic equations which, it will be assumed, have been obtained as the result of the analysis of a physical system where as Kron's technique begins with a consideration of the system itself and the applicable basic physical laws (Kron, 1963, Branin, 1959). The reason for the different approach is that in some cases of interest, such as the analysis of the plasma source problem discussed in Section II, a direct approach, such as Kron's, may not be fruitful.

A simple example of Kron's approach to the problem of piecewise analysis of large systems will be given in terms of electrical network theory and results of the two methods compared.

The technique discussed in this section will be applied to the excitation of disturbances in a homogeneous plasma and the extension to a stratified media discussed.

C.2 General Formulation

Consider the system of linear equations given by

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{in}x_n \\ &\vdots \\ y_n &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{aligned} \qquad (C.1)$$

where $a_{ii} \neq 0$.

The systems of Eqs. (C.1) can be written in matrix form as

$$Y = A_s X \ . \tag{C.2}$$

The matrix A of order n can be expressed as a product of matrices as follows

$$A_s = PBQ \tag{C.3}$$

where B is a diagonal matrix and p_{ij} and q_{ij} have values of +1, -1 or 0, and the indices i and j take all values from 1 to n .

From Eq. (C.3) an element of A_s is given by

$$a_{ij} = \sum_r \sum_s p_{ir} b_{rs} q_{sj} \ . \tag{C.4}$$

Due to the fact that B is diagonal matrix, Eq. (C.4) reduces to

$$a_{ij} = p_{ir} b_{rr} q_{rj} \tag{C.5}$$

A particularly useful form of the matrices P and Q can be obtained by arranging the first n elements of B such that

$$b_{ii} = a_{ii} \tag{C.6}$$

where

$$a_{ii} = f_{ii} + \ell_{ii} \quad . \qquad f_{ii} \neq 0 \tag{C.7}$$

P, B and Q can now be partitioned so that Eq. (C. 3) has the form

$$A_s = \begin{bmatrix} U_n, & G \end{bmatrix} \begin{bmatrix} F_o + L & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} U_n \\ K \end{bmatrix} \tag{C.8}$$

where the elements of F_o and L are f_{ii} and ℓ_{ii} defined by (C. 7) and the elements of B_2 are the a_{ij} , U_n is a unit matrix of order n and 0 is the zero matrix.

Equation (C. 8) can be written in terms of P and Q as

$$A_s = F_o + PM\,Q \tag{C.9}$$

$$M = \begin{bmatrix} L & 0 \\ 0 & B_2 \end{bmatrix}$$

At this point it must be recognized that some of the entries in M may be zero. Since, in later analysis, it will be necessary that the inverse of M, or some submatrices of M, exist it will be convenient to eliminate the zero entries at this time. This can be done easily, since M is a diagonal matrix, by partitioning M in such a fashion that all zero entries on the main diagonal of M appear in a submatrix of M. The resulting expression is as follows

$$\begin{bmatrix} P_1, P_2 \dots, P_m \end{bmatrix} \begin{bmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_m \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_m \end{bmatrix} \qquad (C.10)$$

Equation (C. 10) can be written as a sum as

$$\sum_{i=1}^{m} P_i M_i Q_i \quad . \qquad (C.11)$$

Performing the multiplication indicated in (C. 10) or the summation in (C. 11) the zero entries contribute nothing to the product and the following result is obtained.

$$GHK = \sum_{i=1}^{m} P_i M_i Q_i \qquad (C.12)$$

Where H is obtained from M by deleting the appropriate rows and columns of M corresponding to zero elements on the main diagonal. Thus, H is a diagonal matrix with no zeros appearing on the main diagonal. G and K , of course, are obtained from P and Q by deleting the columns of P and the rows of Q corresponding to the columns or rows deleted from M.

Equation (C. 12) can now be substituted into Eq. (C. 9) and the following equation obtained

$$A_s = F_o + GHK \tag{C.13}$$

H in Eq. (C.13) can now be partitioned into s submatrices, all of which are diagonal, and the matrix product written as a sum as follows

$$A_s = F_o + \sum_{i=1}^{s} G_i H_i K_i \quad . \tag{C.14}$$

Summing over j of s terms ($j \leq s$) of Eq. (C.14) an expression for A_j is obtained as follows

$$A_j = F_o + \sum_{i=1}^{j} G_i H_i K_i \quad . \tag{C.15}$$

From Eq. (C.15) it follows immediately that

$$A_j = A_{j-1} + G_j H_j K_j \quad . \tag{C.16}$$

C.3 Matrix Inversion

One method of inverting the matrix A_s is to make use of the following matrix identity (Branin, 1959)

$$(F+GHK)^{-1} = F^{-1} - F^{-1}G(KF^{-1}G+H^{-1})^{-1}KE^{1} \tag{C.17}$$

Applying the identity (C. 17) to Eq. (C. 16) the following result is obtained.

$$A_j^{-1} = (A_{j-1} + G_j H_j K_j)^{-1} = A_{j-1}^{-1} - A_{j-1}^{-1} G_j (K_j A_{j-1}^{-1} G_j + H_j^{-1})^{-1} K_j A_{j-1}^{-1} \quad (C.18)$$

Equation (C. 18) expresses the inverse of the matrix A_j in terms of the inverse of A_{j-1} and H_j, G_j, K_j. Therefore, repeated application of Eq. (C. 18) will yield the desired inverse of A_s.

By means of Eq. (C. 18) the problem of inverting a matrix of order n has been changed to that of inverting a matrix of order r, the rank of H_i and multiplication and addition of matrices. Two extreme cases are of interest. If $r = 1$, the inversion indicated on the right side of Eq. (C. 1) amounts to taking the reciprocal of a number,while if none of the a_{ij} in Eq. (C. 1) are zero, Eq. (C. 12) with no partitioning becomes

$$A_s^{-1} = (A_o + GHK)^{-1} = A_o^{-1} - A_o^{-1} G (K A_o^{-1} G + H^{-1})^{-1} K A_o^{-1} \quad (C.19)$$

and thus involves finding the inverse of a matrix of order n^2. Thus, as is usual when the inverse of a matrix is desired, the technique to be used for the inversion process is dependent on the form of the particular matrix to be inverted.

C. 4 Physical Interpretation

Assuming that the system of Eq. (C. 1) is related to a physical system, the a_{ii} may be considered as the sum of the self impedance or admittance of the i^{th} portion of the system plus the coupling impedance or admittance from other parts of the system. Thus, f_{ii} refers to the self impedances and ℓ_{ii} to sum of the coupling impedances. The a_{ij}, of course, refer to the coupling impedances or admittances between the various parts of the system. This interpretation can be easily extended to the matrix Eq. (C. 14). If the systems of Eq. (C. 1) are related to a coupled mechanical-electrical system, for example, and the coupling effects between the systems are to be studied, F_o of Eq. (C. 14) can be partitioned into two parts

$$F_o = F_{o1} + F_{o2}$$

Where F_{o1} refers to the mechanical system and F_{o2} refers to the electrical system. Similarly, the summation on the right side of Eq. (C. 14) can be considered to be composed of three parts, that is

$$\sum_{i=1}^{3} G_i H_i K_i = G_1 H_1 K_1 + G_2 H_2 K_2 + G_3 H_3 K_3$$

where H_1 contains all the mechanical coupling terms, H_2 contains all the electrical coupling terms and H_3 contains the electro-mechanical coupling terms. Thus, if $G_3 H_3 K_3$ is not included in the summation in Eq. (C. 14) the mechanical and electrical systems are not coupled and their behavior in the absence of coupling can be determined by repeated application of Eq. (C. 16) or any standard matrix inversion technique. The effect of coupling the two systems can then be determined by adding $G_3 H_3 K_3$ by means of Eq. (C. 16). It should be noted that H_3 can also be partitioned and the effect of any coupling term or any group of these terms determined by repeated applications of Eq. (C. 16).

C.5 Kron's Method

The result of the analysis of the preceding sections is similar to that obtained by Kron (1963). However, Kron's method of dealing with large systems begins with fundamental physical laws and yields the equations of the system in the form of (C.13) as a natural result. Because of the basic importance of Kron's technique from the standpoint of the physical laws involved, a brief discussion of this method will be given in this section. The development is couched in the terminology of electric networks, primarily because of the ease with which the final result may be obtained. This in no way implies a limitation on Kron's methods and for a more detailed discussion of the application of this method the reader is referred to references given above. The development of electric network theory is, of necessity, brief and includes only material necessary to achieve the final result.

C.6 Graph Theory

Several excellent treatments of linear graph theory as applied to electrical networks exist in the literature and thus only a brief summary will be given in this report. The development given here will follow roughly that given by Reed (1961) or Reed and Seshu (1961).

It is assumed that the reader is familiar with such terms as oriented graph, tree, tree complement, node, etc.

The number of elements in the graph will be denoted by e and the number of nodes by n. Elements in a tree of the graph will be called branches and elements in the tree complement will be called chords. Thus, for any connected graph there will be n-1 brances and e-(n-1) chords.

It is well known that the connectivity relations of a linear oriented graph can be specified by various matrices formed according to different rules (Branin, 1959). For purposes of this report, three of these matrices will be of particular interest and will be defined in the following paragraphs.

In all cases the columns of the matrices will correspond to elements of the graph, and the columns will be arranged such that reading from left to right the first n-1 columns will correspond to branches and the last e-(n-1) columns will correspond to chords. Thus, all of the matrices will contain e columns.

The incidence matrix a with elements a_{ij}. The rows of this matrix correspond to n-1 of the nodes of the graph, one of the nodes, called the reference node, being omitted.

The elements a_{ij} are defined as follows:

$a_{ij} = (+1, -1, 0)$ if the j^{th} element is positively, negatively, not incident upon the i^{th} node.

The matrix A can be partitioned according to trees and chords as

$$A = \left[A_T A_c \right]$$

where A_T has an inverse (Reed and Seshu, 1961).

The segregate matrix S with elements s_{ij}. This matrix contains n-1 rows corresponding to the branches of a tree. Each row of this matrix corresponds to a set of segregate elements defined as follows. If the i^{th} branch is removed the tree is divided into two subgraphs denoted by G_1 and G_2. Let the i^{th} branch be positively incident on G_1 (directed toward G_1) contain the nodes $a_1, b_1, c_1 \ldots k_1$ and G_2 contain the nodes $a_2, b_2, c_2 \ldots k_2$. Then the j^{th} chord belongs to the i^{th} segregate set if it is connected between nodes j_1 and j_2.

Thus, $s_{ij} = (+1, -1)$ if the j^{th} element belongs to the i^{th} segregate set and is positively, negatively incident on G_1 and zero if does not belong to the i^{th} segregate set.

Because of the manner in which S is formed, it may be partitioned

$$S = \begin{bmatrix} U_T & S_c \end{bmatrix} \tag{C.20}$$

where U_T is a unit matrix containing n-1 rows or columns.

The mesh matrix B with elements b_{ij} has e-(n-1) rows corresponding to the chords of the graph. When all chords but the i^{th} are removed from the graph a single closed path is formed, the i^{th} path, and the orientation of this path is determined by the i^{th} chord. Thus, b_{ij} is $b_{ij} = (+1_{j-1})$ if the j^{th} branch belongs to the i^{th} path and is positively (negatively) oriented with respect to this path and is zero otherwise.

Because of the way it is formed B can be partitioned

$$B = \begin{bmatrix} B_T & U_c \end{bmatrix} \tag{C.21}$$

where U_c is a unit matrix containing e-(n-1) rows.

Although other connectivity relations may be defined (Branin, 1961), the above are sufficient for the purposes of this report. Indeed, the matrices A and S are closely related, and it can be shown that one can be obtained from the other by simple transformations which can be obtained from the graph.

A very important relation which exists between S(A) and B is as follows:

$$S\,B' \equiv 0 \tag{C.22}$$

$$A\,B' \equiv 0\ . \tag{C.23}$$

The proof of this relation is well known and given in many references in the literature (Branin, 1959) (Reed and Seshu, 1961).

C.7 Electrical Network Theory

The mesh and node equations of electrical network theory can be developed in terms of the graph theory discussed in Section C.6. In order to do so, however, it is necessary first to define the elements of an electrical network.

In general, three elements are necessary to describe the behavior of an electrical network. These are:

1. Arbitrary voltage source denoted by the symbol e. For this element there is no relation between the terminal voltage of the element and the current through the element. The terminal voltage is an arbitrary function. A one column matrix containing B of these elements will be denoted by the symbol

$$\mathcal{E} = \begin{bmatrix} e_1 \\ e_2 \\ e_k \end{bmatrix}$$

2. Arbitrary current source h. In this element the current is an arbitrary function and there is no relationship between the terminal voltage and current. A one column matrix containing k of these elements is represented by the symbol H, i.e.

$$H = \begin{bmatrix} h_1 \\ \cdot \\ \cdot \\ h_k \end{bmatrix}$$

3. Impedance or admittance elements denoted by z_i or y_i. For these elements the voltage v_i and current $_i$ for the i^{th} element are related by

$$v_i = z_i i_i$$

or

$$i_i = y_i v_i$$

where

$$z_i y_i = 1 \ .$$

A matrix representation of these elements is given by

$$V = ZI \qquad (C.24)$$

or

$$I = YV \qquad (C.25)$$

where $ZY = U$ and where U is a unit matrix. It should be noted that the use of the terms impedance or admittance implies "steady state" or the use of a Fourier or Laplace transform.

It is evident from the definition of the incidence matrix A that Kirchhoff's current law may be expressed as:

$$A I_e = 0 \qquad (C.26)$$

where I_e is a column matrix representing the currents in the elements of the network. From the fact that $A = \left[A_T A_c\right]$ and A_T has an inverse, we can multiply (C.26) by A_T^{-1} to obtain

$$A_T^{-1} A I_e = 0 = U_T A_T^{-1} A_c \ I_e = U_T S_c = S I_e = 0 \ . \qquad (C.27)$$

Thus the segregate equations may be considered as a generalized form of current law which may be stated as follows: the algebraic sum of the currents in a segregate set must be zero.

If the matrix of the element voltage is represented by I_e, then Kirchhoff's voltage law is given by

$$B\,V_e = 0\ . \tag{C.28}$$

It is convenient to petition the element voltage and current matrices as follows:

$$V_e = \begin{bmatrix} \mathcal{E} \\ V_T \\ V_c \\ V_H \end{bmatrix} \qquad I_e = \begin{bmatrix} I_{\mathcal{E}} \\ I_T \\ I_c \\ H \end{bmatrix} \tag{C.29}$$

where the symbols $\mathcal{E}$ and H have been defined previously. When these symbols are used as subscripts, they refer to the matrix of the currents of an arbitrary voltage source and the voltage of an arbitrary current source, respectively.

In addition, the subscripts T and c refer to impedance or admittance elements located in the tree or tree complement, respectively. Thus, ohms law, Eqs. (C.24) and (C.25) may conveniently be written as

$$\begin{bmatrix} V_T \\ V_c \end{bmatrix} = \begin{bmatrix} Z_T & 0 \\ 0 & Z_c \end{bmatrix} \begin{bmatrix} I_c \\ I_T \end{bmatrix} \tag{C.30}$$

and

$$\begin{bmatrix} I_T \\ I_c \end{bmatrix} = \begin{bmatrix} Y_T & 0 \\ 0 & Y_c \end{bmatrix} \begin{bmatrix} V_T \\ V_c \end{bmatrix} \tag{C.31}$$

The segregate matrix S can be partitioned in the following manner:

$$\begin{bmatrix} U & 0 & S_{13} & S_{14} \\ 0 & U_T & S_{23} & S_{24} \end{bmatrix} \tag{C.32}$$

where the first n_e rows correspond to arbitrary voltage sources.

Also, the mesh matrix B can be partitioned

$$B = \begin{bmatrix} B_{11} & B_{12} & U_c & 0 \\ B_{21} & B_{22} & 0 & U_{11} \end{bmatrix} \tag{C.33}$$

where the last n_h rows correspond to arbitrary current sources.

The incidence matrix A can be partitioned in a similar fashion except that unit matrices do not necessarily appear. Since the S matrix can be considered as associated with a generalized form of Kirchhoff's current low the A matrix can be obtained as a special case of a particular S matrix and thus will not be discussed further in this report.

The mesh equations can be obtained from (C.28) as follows. The partitioned form of this equation is

$$\begin{bmatrix} B_{11} & B_{12} & U_c & 0 \\ B_{21} & B_{22} & 0 & U_{11} \end{bmatrix} \begin{bmatrix} \mathcal{E} \\ V_T \\ V_c \\ V_H \end{bmatrix} = 0 \qquad \text{(C.34)}$$

From Eq. (C.34), we obtain

$$B_{11}\mathcal{E} + \begin{bmatrix} B_{12} & u_c \end{bmatrix} \begin{bmatrix} V_T \\ V_c \end{bmatrix} = 0 \qquad \text{(C.35)}$$

Using ohm laws in Eq. (C.35) we obtain

$$B_{11}\mathcal{E} + \begin{bmatrix} B_{12} & U_c \end{bmatrix} \begin{bmatrix} Z_T & 0 \\ 0 & Z_c \end{bmatrix} \begin{bmatrix} I_T \\ I_c \end{bmatrix} = 0 \qquad \text{(C.36)}$$

At this point we assume

$$I_e = B' I_a \qquad \text{(C.37)}$$

where the currents I_a are as yet unknown. Writing Eq. (C.37) in detail

$$\begin{bmatrix} I_{\mathcal{E}} \\ I_T \\ I_c \\ H \end{bmatrix} = \begin{bmatrix} B'_{11} & B'_{21} \\ B'_{12} & B'_{22} \\ U_c & 0 \\ 0 & U_4 \end{bmatrix} \begin{bmatrix} I_a \end{bmatrix} \qquad \text{(C.38)}$$

From Eq. (C.38) it is obvious that

$$\begin{bmatrix} I_a \end{bmatrix} = \begin{bmatrix} I_c \\ \\ H \end{bmatrix} \quad \text{thus} \qquad \text{(C.39)}$$

thus

$$\begin{bmatrix} I_T \\ \\ I_c \end{bmatrix} = \begin{bmatrix} B'_{11} & B'_{22} \\ \\ U_c & 0 \end{bmatrix} \begin{bmatrix} I_c \\ \\ H \end{bmatrix} \qquad \text{(C.40)}$$

Using this result in Eq. (C.36) the following equation is obtained

$$B_{11}\mathcal{E} + \begin{bmatrix} B_{12} & U_c \end{bmatrix} \begin{bmatrix} Z_T & 0 \\ \\ 0 & Z_c \end{bmatrix} \begin{bmatrix} B'_{12} \\ \\ U_c \end{bmatrix} \begin{bmatrix} I_c \end{bmatrix} + B_{12} Z_T B'_{22} H = 0 \qquad \text{(C.41)}$$

The only unknowns in Eq. (C.41) are the chord currents I_c of which there are $e-(n-1)-nh$ in number and thus the I_c may be determined if

$$\left[\begin{bmatrix} B_{12} & U_c \end{bmatrix} \begin{bmatrix} Z_T & 0 \\ \\ 0 & Z_c \end{bmatrix} \begin{bmatrix} B'_{12} \\ \\ U_c \end{bmatrix} \right]^{-1} = \begin{bmatrix} Z_c + B_{12} Z + B'_{12} \end{bmatrix}^{-1} \qquad \text{(C.42)}$$

exists. The above sets are the so called mesh equations of ordinary network theory. In order that the equations be valid, it must be shown that Kirchhoff's current low is satisfied. This can be shown easily by premultiplying (C.23) by S thus

$$SI_e = SB' I_a$$

Since $SB' \equiv 0$, Kirchhoff's current law is satisfied.

Segregate equations. The segregate (and thus node) equations may be developed as follows. The partitioned form of Eq. (C.27) is

$$\begin{bmatrix} U_c & 0 & S_{13} & S_{14} \\ 0 & U_c & S_{23} & S_{24} \end{bmatrix} \begin{bmatrix} I \\ I_T \\ I_c \\ H \end{bmatrix} = 0 \qquad (C.43)$$

From Eq. (C.44) we obtain

$$\begin{bmatrix} U_c & S_{23} \end{bmatrix} \begin{bmatrix} I_T \\ I_c \end{bmatrix} + S_{24} H = 0 . \qquad (C.44)$$

Using Eq. (C.25) in Eq. (C.44)

$$\begin{bmatrix} U_c & S_{23} \end{bmatrix} \begin{bmatrix} Y_T & 0 \\ 0 & Y_c \end{bmatrix} \begin{bmatrix} V_T \\ V_c \end{bmatrix} + S_{24} H = 0 \qquad (C.45)$$

Assume $V_e = S' V_a$ in detail

$$\begin{bmatrix} \mathcal{E} \\ V_T \\ V_c \\ V_H \end{bmatrix} = \begin{bmatrix} U & 0 \\ 0 & U_T \\ S'_{13} & S'_{23} \\ S'_{14} & S'_{24} \end{bmatrix} V_a \qquad (C.46)$$

From the above it is obvious that

$$V_a = \begin{bmatrix} \mathcal{E} \\ V_T \end{bmatrix} \tag{C.47}$$

Thus

$$\begin{bmatrix} V_T \\ V_c \end{bmatrix} = \begin{bmatrix} 0 & U_T \\ S'_{13} & S'_{23} \end{bmatrix} V_T \tag{C.48}$$

Using Eq. (C.48) in Eq. (C.45)

$$\begin{bmatrix} U_c & S_{23} \end{bmatrix} \begin{bmatrix} Y_T & 0 \\ 0 & Y_c \end{bmatrix} \begin{bmatrix} U_T \\ S'_{23} \end{bmatrix} V_T + S_{23} Y_c S'_{13} + S_{24} H = 0 \tag{C.49}$$

Equation (C.49) can be solved for the V_T providing

$$\left[\begin{bmatrix} U_c & S_{23} \end{bmatrix} \begin{bmatrix} Y_T & 0 \\ 0 & Y_c \end{bmatrix} \begin{bmatrix} U_T \\ S'_{23} \end{bmatrix} \right]^{-1} = \begin{bmatrix} Y_T + S_{23} Y_c S'_{23} \end{bmatrix}^{-1} \tag{C.50}$$

This result is the usual form of the segregate equations.

The form of the inverse required for the mesh and segregate (node) equations, (C.42) and (C.50) respectively, are of the same form as Eq. (C.13) and, thus, the same inversion technique will apply. There is one significant difference between the network equations developed in this section and the development in Section C.2. This difference lies in the coupling terms a_{ij} and $F_{ij}(Y_{ij})$. In the case of the electrical network theory it has been assumed that $F_{ij} = F_{ji}$. Thus, if the matrix product $B_{12}F_TB'_{12}$ is partitioned such that F_{T1} contains only F_{ij} and the matrix product taken F_{ij} will appear in four positions. It will appear as the coupling term between the i^{th} and j^{th} mesh, as the coupling term between the j^{th} and i^{th} mesh and also in such a position that it is added to the self impedance of the i^{th} and j^{th} meshes. The condition the $F_{ij} = F_{ji}$ is not always true in network theory. Consider, for example, the mesh equations for linear analysis of a vacuum tube or transistor. (Lo et al, 1955). This situation is usually accounted for by the addition of a voltage source at an appropriate place in the equivalent circuit. In contrast to this situation, the general development presented in Section II adds each element in separately, regardless of whether it appears as a self impedance or admittance or as a coupling term. This allows somewhat more flexibility in the way the terms are added to obtain the final result.

C.8 Application to the Excitation Problem

The method for the piecewise inversion of matrices can be applied to the excitation problem discussed in Section V. The system of Eqs. (5.1) through (5.18) written in matrix form is given by Eq. (5.19) which is

$$[L(\underline{s})]\, f(\underline{s}) = S(\underline{s}) \qquad (5.19)$$

where

$$f(\underline{s}) = \begin{bmatrix} h_x \\ h_y \\ h_z \\ E_x \\ E_y \\ E_z \\ n_e \\ n_i \\ n_n \\ V_{ex} \\ V_{ey} \\ V_{ez} \\ V_{ix} \\ V_{iy} \\ V_{iz} \\ V_{nx} \\ V_{ny} \\ V_{nz} \end{bmatrix} \qquad S(\underline{s}) = \begin{bmatrix} iK_x \\ iK_y \\ iK_z \\ -iJ_x \\ -iJ_y \\ -iJ_z \\ i\frac{Q_e}{\omega} \\ i\frac{Q_i}{\omega} \\ i\frac{Q_n}{\omega} \\ i\frac{F_{ex}}{\omega N_o m_e} \\ i\frac{F_{ey}}{\omega N_o m_e} \\ i\frac{F_{ez}}{\omega N_o m_e} \\ i\frac{F_{ix}}{\omega N_o m_i} \\ i\frac{F_{iy}}{\omega N_o m_i} \\ i\frac{F_{iz}}{\omega N_o m_i} \\ i\frac{F_{nx}}{\omega N_n m_n} \\ i\frac{F_{ny}}{\omega N_n m_n} \\ i\frac{F_{nz}}{\omega N_n m_n} \end{bmatrix}$$

and

$$
\left[L(s)\right] = \begin{bmatrix}
-\omega\mu_o & 0 & 0 & 0 & -s & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\omega\mu_o & 0 & 0 & s & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\omega\mu_o & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -s & 0 & \omega\epsilon_o & 0 & 0 & 0 & 0 & 0 & -ieN_o & 0 & 0 & ieN_o & 0 & 0 & 0 & 0 & 0 \\
0 & s & 0 & 0 & \omega\epsilon_o & 0 & 0 & 0 & 0 & 0 & -ieN_o & 0 & 0 & ieN_o & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \omega\epsilon_o & 0 & 0 & 0 & 0 & 0 & -ieN_o & 0 & 0 & ieN_o & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\frac{N_o}{\omega}s & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{N_o s}{\omega} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{N_n s}{\omega} \\
0 & 0 & 0 & \frac{ie}{\omega m_e} & 0 & 0 & 0 & 0 & 0 & a_e & i\Omega_e\cos\theta & -i\Omega_e\sin\theta & -i\nu_{ei} & 0 & 0 & -i\nu_{en} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{ie}{\omega m_e} & 0 & 0 & 0 & 0 & -i\Omega_e\cos\theta & a_e & 0 & 0 & -i\nu_{ei} & -i\nu_{en} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{ie}{\omega m_e} & -\frac{s}{\omega}\frac{U_e^2}{N_o} & 0 & 0 & i\Omega_e\sin\theta & 0 & a_e & 0 & 0 & -i\nu_{ei} & -i\nu_{en} & 0 & 0 \\
0 & 0 & 0 & -\frac{ie}{\omega m_i} & 0 & 0 & 0 & 0 & 0 & -i\nu_{ie} & 0 & 0 & a_i & -i\Omega_i\cos\theta & i\Omega_i\sin\theta & -i\nu_{in} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{ie}{\omega m_i} & 0 & 0 & 0 & 0 & 0 & -i\nu_{ie} & 0 & i\Omega_i\cos\theta & a_i & 0 & 0 & -i\nu_{in} & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{ie}{\omega m_i} & 0 & -\frac{s}{\omega}\frac{U_i^2}{N_o} & 0 & 0 & 0 & -i\nu_{ie} & -i\Omega_i\sin\theta & 0 & a_i & 0 & 0 & -i\nu_{in} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i\nu_{ne} & 0 & 0 & -i\nu_{ni} & 0 & 0 & a_n & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i\nu_{ne} & 0 & 0 & -i\nu_{ni} & 0 & 0 & a_n & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{sU_n^2}{\omega\epsilon_o} & 0 & 0 & -i\nu_{ne} & 0 & 0 & -i\nu_{ni} & 0 & 0 & a_n
\end{bmatrix}
$$

where

$$a_e = 1 + i\nu_{ei} + i\nu_{en}$$

$$a_i = 1 + i\nu_{ie} + i\nu_{in}$$

$$a_n = 1 + i\nu_{ni} + i\nu_{ne}$$

Denoting the matrix $[L(\underline{s})]$, for simplicity, bv L , the matrix L can be written as

$$L = L_o + L_T + L_E + L_\Omega + L_\nu \qquad \text{(C.51)}$$

where L_o is a diagonal matrix composed of the elements on the main diagonal of L with the exception of the collision terms contained in a_e, a_i and a_n . L_T is a matrix containing the thermal velocities U_e, U_i and U_n , L_E is the matrix containing the electric force terms, L_Ω is the matrix containing the magnetic force terms and L_ν is a matrix containing the collision terms. These matrices are obtained in straightforward manner from L .

Each of the matrices L_T, L_E, L_Ω and L_ν can now be written as a product in the form

$$L_T = G_T L_{Td} K_T$$

$$L_E = G_E L_{Td} K_E$$

$$L_\Omega = G_\Omega L_{\Omega d} K_\Omega$$

$$L_\nu = G_\nu L_{\nu d} K_\nu$$

where the subscript d has been used to denote a diagonal matrix. These matrix products are easily formed, although somewhat tedious to write in detail.

In order to illustrate the procedure, consider the matrix L_Ω. Examination of the matrix L shows that there are eight entries which contain the electron and ion gyro frequency, thus, the matrix $L_{\Omega d}$ will be a diagonal matrix of order 8. Therefore, G_Ω will be an 18 x 8 matrix and K_Ω will be 8 x 18. The arrangement of the elements in $L_{\Omega d}$ is usually determined by the results desired but is not critical since $L_{\Omega d}$ can be partitioned and thus, L_Ω expressed as a sum. In this case, $L_{\Omega d}$ will be arranged as follows

$$L_{\Omega d} = \begin{bmatrix} i\Omega_e\cos\theta &&&&&&& \\ & -i\Omega_e\sin\theta &&&&&& \\ && -i\Omega_e\cos\theta &&&&& \\ &&& i\Omega_e\sin\theta &&&& \\ &&&& -i\Omega_i\cos\theta &&& \\ &&&&& i\Omega_i\sin\theta && \\ &&&&&& i\Omega_i\cos\theta & \\ &&&&&&& -i\Omega_i\sin\theta \end{bmatrix}$$

The algebraic signs of the entries in L have been preserved in $L_{\Omega d}$ and thus the entries in G_Ω and K_Ω will be +1 or zero. The matrices G_Ω and K_Ω can now be determined by inspection as follows. Consider the element in the third row and columnar of $L_{\Omega d}$, i.e., $-i\Omega_e\cos\theta$. This term was obtained from the eleventh row and tenth column of L, hence, g_{11}, 3= 1 and k_3, 10= 1. Similarly, the non-zero entries in G_Ω and K_Ω are

g_{10}, 1=1	K_1, 11=1
g_{10}, 2=1	K_{10}, 12=1
g_{11}, 3=1	K_3, 10=1
g_{12}, 4=1	K_4, 10=1
g_{13}, 5=1	K_5, 14=1
g_{13}, 6=1	K_6, 15=1
g_{14}, 7=1	K_7, 13=1
g_{15}, 13=1	K_{15}, 13=1

The matrix L in the form of (C. 51) can now be inverted by application of Eq. (C. 18) or the terms in (C. 51) can be partitioned further. For example, again considering the L_Ω, since $\Omega_e >> \Omega_i$ the matrix can be partitioned such that the Ω_e and Ω_i are separated. In this case, a solution could be obtained neglecting Ω_i and then this solution modified by the addition of Ω_i and, thus, the effect of the ion gyro-frequency determined.

Another useful result can be obtained from (C. 51) by neglecting the collision matrix L_ν. The matrices L_T, L_E and L_Ω can then be partitioned in such a manner that all terms belonging to the electron plasma are in the first part of the partitioned matrix, all other terms are in the second part of the matrix. The matrix for the collisionless electron plasma has been inverted by Wu and, thus, is known. The inverse of this matrix can then be modified by use of Eq. (C. 18) to yield the general case of a three-fluid plasma including collisions.

One additional remark should be made at this time about the collision matrix L_ν. Since a_e, a_i and a_n contain collision terms it is evident that the matrix L_{ν} also contains entries on the main diagonal. These main diagonal terms are, of course, coupling terms which appear in the "self impedance" and correspond to the ℓ_{ii} of Section C.1. More specifically, these terms are the attenuation collision terms discussed in Section 3.3 while the off diagonal terms are the coupling portion of the collisions. Thus, the method of piecewise inversion of matrices allows these two effects to be easily separated.

The technique for the piecewise inversion of matrices is, perhaps, most useful for numerical work. Assuming that appropriate numerical values, including ω and θ, have been inserted in L, this matrix is still a function of the Fourier space transform variable s. Because of this, the work involved in finding L^{-1} has been magnified considerably and the process of inversion can be carried out by one of two different methods.

a. The inverse of L can be evaluated for a sufficient number of discrete values of s to achieve the desired result.

b. The coefficients of the powers of s can be computed and thus L^{-1} obtained as a function of s.

The method selected would depend, of course, on the form in which the result is desired and the number of mathematical operations involved.

A relatively simple example of the latter procedure is discussed in Appendix B in connection with the computer program for evaluating the propagation constants. In this case, the coefficients of the polynomial were evaluated directly from the determinant defining the dispersion relation.

The procedure for the piecewise inversion of matrices can be formally extended to the case of a stratified media as follows. Assume the medium to be stratified into n layers in the Z direction and the boundaries of the layers located at the points $Z_o, < Z_1, < Z_2 \ldots . < Z_n$. The solution to the set of

Eqs. (C.51) can be obtained for the i^{th} layer ($Z_{i-1} \leq Z \leq Z_i$) in the form of a matrix equation. This equation may then be evaluated at the edges of the layer, i.e., at $Z = Z_{i-1}$ and Z_i. If this procedure is carried out for each layer and the boundary conditions matched at $Z = Z_o$, $Z_1 \ldots Z_n$ a matrix equation in the form of (1) would result. The matrix inversion technique discussed could then, in principle, be applied and the solution obtained. However, the complexity of the system of equations obtained in this manner is such that, in all probability, this type of analysis may be prohibitively expensive.

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A study of the excitation and propagation of wave like disturbances in an ionized medium, such as the ionosphere, is made based on the linearized Euler's equation and Maxwell's equation. The local propagation constants of the basic modes of propagation are discussed. A computer program for the evaluation of these constants with given ionospheric properties is given. Methods of investigating the propagation of such waves in inhomogeneous and/or bounded media, such as ray tracing, invariant embedding, reflection and refraction, orthogonal expansion, and the use of a general matrix formulation are presented. A unified matrix-operator transform method for investigation of the excitation and propagation of disturbances in an ionized medium is proposed.

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