6663-1-T

Copy____

THE UNIVERSITY OF MICHIGAN

COLLEGE OF ENGINEERING

DEPARTMENT OF ELECTRICAL ENGINEERING

Radiation Laboratory

UNIFIED APPROACH TO EXCITATION PROBLEMS IN COMPRESSIBLE PLASMA

Y-K Wu

July 1965

Prepared for

Contract No. AF 30(602)-3381
Project 5579 Task 557902
Contract Monitor, L. Strauss EMASA

6663-1-T = RL-2144

Contract With: Rome Air Development Center
Griffiss Air Force Base, New York 13442

Administered through:

OFFICE OF RESEARCH ADMINISTRATION • ANN ARBOR

ACKNOWLEDGMENT

It is a pleasure to acknowledge the assistance of Professor C-M Chu, who brought this problem to my attention and provided guidance and support for the investigation.

Thanks are also due to Mr. J. J. LaRue and Dr. D. B. vanHulsteyn for reading and correcting the manuscript, to many other colleagues of the Radiation Laboratory for stimulating discussions, and to the computation group of the Radiation Laboratory for assistance in numerical calculations.

ABSTRACT

This investigation is concerned primarily with a general, unified approach to the solution of excitation problems in a compressible plasma which may be anisotropic and inhomogeneous, and which may include different types of sources, e.g. electric current sources, magnetic current sources, fluid flux sources and mechanical body sources. A macroscopic, hydrodynamic approach is chosen and is based on the linearized Euler equations of motion and the Maxwell equations. The Maxwell-Euler equations are reformulated through linear operator and generalized transform techniques into an equivalent matrix integral equation. When the medium is homogeneous, this integral equation has an ideal kernel and the explicit solution can be easily obtained.

A thorough study is given for the excitation of disturbances due to different types of sources in a homogeneous electron plasma immersed in a constant magnetic field. Collisional damping effects are neglected and an adiabatic condition is assumed in the present study. As a preliminary requirement, the dispersion relation in the form of a cubic equation for the propagation constant square is analyzed as exactly as possible. Some illuminating graphs showing the propagation constants as functions of the normalized plasma frequency are employed for the above analysis and they are explained in conjunction with the so called Clemmow-Mullaly-Allis diagram. In due process, a proper terminology is introduced for the three types of waves involved in an electron plasma. The radiation field is then solved for both two- and three-dimensional excitation problems. Exact solutions are obtained for two-dimensional problems, and asymptotic solutions are obtained for three-dimensional problems by direct utilization of the dispersion curves. Some dispersion curves and the radiation field from a point current source oriented in the direction of a constant magnetic field are presented in graphical form, which are obtained numerically by a computer.

A proper ionospheric model is used for this calculation, which indicates comparatively strong excitation of modified plasma waves. Also, equivalence relations between different types of sources are obtained, which can be employed to express the fields excited by one type of source in terms of the fields excited by another type of source.

An illustration is given for the application of the operator transform formulation employed in this report to a three fluid plasma problem, and its application to the excitation problems in an inhomogeneous medium is also discussed.

PREFACE

In this work, a general, unified investigation of the excitation problems, considering different types of sources, in a compressible plasma with an externally impressed constant magnetic field is presented. A macroscopic, hydrodynamic approach is chosen, and the linearized Maxwell-Euler's equations are reformulated through linear operator and generalized transform techniques into an equivalent matric integral equation.

The candidate is indebted to his committee chairman, Professor C. M. Chu, who brought this problem to his attention and provided guidance and support for this investigation, and to other members of his committee for helpful suggestions and correcting the manuscript.

The author is thankful to Mr. J. J. LaRue and Dr. D. B. vanHulsteyn for reading and correcting the manuscript, and to many other colleagues of the Radiation Laboratory for stimulating discussions. Also, the author would like to express his appreciation to Dr. A. A. Nagy of Space Physics Research Laboratory, The University of Michigan, for his assistance in collecting latest data on ionospheric properties.

For computer calculations the author is thankful to the computation group of the Radiation Laboratory.

This investigation was supported by the Rome Air Development Center under Air Force Contract No. AF 30(602)-3381.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

Figure		Page

LIST OF APPENDICES

CHAPTER I

INTRODUCTION

This investigation is concerned primarily with the excitation of wave-like disturbances in a plasma, and is based on the linearized, coupled Euler's equations of motion and the Maxwell equations.

The propagation of plane waves in a plasma has been studied extensively by many investigators (e.g. Spitzer [1], Ratcliffe [2], Oster [3], Ginzburg [4], Budden [5], Pai [6], Tanenbaum and Mintzer [7], and Denisse and Delcroix [8]). In these studies the dispersion relation for the various waves was of primary concern and the excitation of these waves was not considered.

Recently, excitation problems in a plasma have attracted the attention of many investigators. Ginzburg [9] and Kolomenskii [10] considered the special case of the radiation of a point charge moving in a transparent anisotropic medium. Bunkin [11] studied the radiation field of a given distribution of external currents in an infinite homogeneous anisotropic medium and seems to be the first investigator to give a general solution to the excitation problem. Kogelnik [12], Arbel [13], Kuehl [14], Mittra [15], Mittra and Deschamps [16], Clemmow [17], Wu [18], Motz and Kogelnik [19], Arbel and Felsen [20], and Chow [21], analyzed similar radiation problems in an infinite homogeneous anisotropic medium. In all these works a single fluid plasma was considered and was assumed to be incompressible and thus could be characterized by a tensor dielectric constant.

With the assumption of incompressibility, the longitudinal plasma wave does not appear. Whale [22] discussed the importance of the radiation of energy as an electron plasma wave, and has shown that the calculated power radiated by this type of wave yielded results in good agreement with rocket observations. Hessel and Shmoys[23] have considered the excitation by a point current source in a compressible plasma in the absence of a static magnetic field, and found that most of the power goes into the plasma wave. Seshadri[24] treated the radiation from a line magnetic current source in a compressible plasma. However, a general, unified investigation of the excitation problem, considering different kinds of sources (e.g. electric current source, magnetic current source, fluid flux source, and mechanical body source) in a compressible plasma with a constant magnetic field is not available. The objective of the present investigation is to find this general solution by using a unified and systematic formulation.

In principle, the linearized Euler equations of motion and the Maxwell equations, including sources, may be considered as a linear operator relating the field quantities to the sources. Due to the large number of variables involved, the solution of the excitation problem, i.e., finding the inverse of the operator, in a medium which may be anisotropic and inhomogeneous, is very involved. In general, analytical solutions for such high-order systems can be obtained only in special cases. In this work, the formal operator method is used as a systematic approach to the excitation problem. Operator methods are a well-known and potent tool in quantum mechanics. The introduction of the operator method into electromagnetic fields has been explored by Bresler and Marcuvitz[25][26], Moses[27], and others. Recently, Diament[28] has introduced the formalism of an operator method combined with a generalized transform method in obtaining the formal solutions of Maxwell's equations for general linearized media. Because of the compact notation, systematic approach and convenience for numerical analysis his formal operator transform techniques are extended in the present work to the system of linearized equations describing the excitation of disturbances in a plasma.

In Chapter II, the general operator transform formalism for the linearized equations of plasma disturbances is developed and applied to a homogeneous, compressible electron fluid plasma immersed in a uniform magnetic field.

In Chapter III, the propagation of the three types of waves involved in an electron fluid plasma is studied carefully by analyzing the dispersion relation in the form of a cubic equation in propagation constant square obtained from Chapter II. Also, a consistent and general terminology for these three types of waves is developed, since there is no standard terminology available.

In Chapter IV, the wave characteristics obtained from Chapter III are utilized to solve the excitation problems in a compressible electron fluid plasma with a constant magnetic field. Equivalence relations between different types of sources are derived which can be applied to both two- and three-dimensional problems.

In Chapter V, a proper ionospheric model is used to calculate the dispersion curves, and then the radiation fields in the forms of asymptotic solutions. The numerical results are presented in graphical form.

The application of our formalism to a three-fluid plasma problem is illustrated in Appendix A.

Its application to the excitation problems in an inhomogeneous medium is discussed in Appendix F. As far as perturbation problems and general numerical solutions of the problems are concerned this method seems promising, but it does not look too promising to obtain exact solutions by using this method.

CHAPTER II

GENERAL FORMULATION

2.1 Basic Equations

In this section the basic equations governing weak disturbances produced by various kinds of sources in a neutral plasma composed of electrons, ions and neutral particles will be presented. The parameters and assumptions applicable to the undisturbed plasma are as follows:

(a) The number densities of electrons, ions and neutral particles are denoted by N_e, N_i and N_1, respectively. Assuming the ions are singly charged and a neutral plasma, the electron and ion number densities are equal and will be denoted by N_o, i. e. $N_e = N_i = N_o$. Negative ions are not considered in this investigation.

(b) The electron mass and the average mass for the ions and neutral particles are, respectively, m_e, m_i and m_n.

(c) The effective collision frequencies for momentum transfer between different types of particles is denoted by ν_{ab}, where the subscripts a and b refer to the types of particles. It is to be noted that these collision frequencies for momentum transfer satisfy the relations: $N_a m_a \nu_{ab} = N_b m_b \nu_{ba}$.

(d) The acoustic velocities for ion, electron and neutral particle gas under adiabatic conditions are U_i, U_e and U_n, respectively.

(e) The plasma is constantly under the action of a d-c magnetic field $\overline{B}_o$.

(f) The plasma as a whole is stationary.

(g) Each gas obeys the ideal gas law.

In addition to the preceeding assumptions, it will be assumed that the sources of the disturbances are weak and thus the second order terms, such as the products of the perturbation terms, and the thermal and viscous effect can be neglected. In addition, it is assumed that the properties of the disturbed medium are nearly the same as the properties of the ambient medium. In this case, a set of linearized equations is usually considered to be adequate to relate the disturbances to their respective sources.

Considering just one Fourier component of the disturbances in the form of $e^{-i\omega t}$, and employing the rationalized mks system of units, this set of equations is the following linearized inhomogeneous Maxwell and Euler equations (Oster[3], Tanenbaum and Mintzer[7], Watanabe[29], Cohen[30], (31), (32), and Pai[33]):—

(a) The Maxwell equations:

$$\nabla \times \bar{E} - i\mu_o \omega \bar{h} = -\bar{K} \tag{2.1}$$

$$\nabla \times \bar{h} + i\epsilon_o \omega \bar{E} - eN_o(\bar{V}_i - \bar{V}_e) = \bar{J} \tag{2.2}$$

(b) The momentum transport equation and the mass transport equation for the electron gas:

$$-i\omega N_o m_e \bar{V}_e + m_e U_e^2 \nabla n_e + eN_o \left[\bar{E} + \bar{V}_e \times \bar{B}_o\right]$$
$$+ N_o m_e \nu_{ei}(\bar{V}_e - \bar{V}_i) + N_o m_e \nu_{en}(\bar{V}_e - \bar{V}_n) = \bar{F}_e \tag{2.3}$$

$$N_o \nabla \cdot \bar{V}_e + \bar{V}_e \cdot \nabla N_o - i\omega n_e = Q_e \tag{2.4}$$

(c) The momentum transport equation and the mass transport equation for the ion gas:

$$-i\omega N_o m_i \bar{V}_i + m_i U_i^2 \nabla n_i - eN_o \left[\bar{E} + \bar{V}_i x \bar{B}_o\right]$$

$$+ N_o m_i \nu_{ie}(\bar{V}_i - \bar{V}_e) + N_o m_i \nu_{in}(\bar{V}_i - \bar{V}_n) = \bar{F}_i \qquad (2.5)$$

$$N_o \nabla \cdot \bar{V}_i + \bar{V}_i \cdot \nabla N_o - i\omega n_i = Q_i \qquad (2.6)$$

(d) The momentum transport equation and the mass transport equation for the neutral particle gas:

$$-i\omega N_1 m_n \bar{V}_n + m_n U_n^2 \nabla n_n + N_1 m_n \nu_{ne}(\bar{V}_n - \bar{V}e)$$

$$+ N_1 m_n \nu_{ni}(\bar{V}_n - \bar{V}_i) = \bar{F}_n \qquad (2.7)$$

$$N_1 \nabla \cdot \bar{V}_n + \bar{V}_n \cdot \nabla N_1 - i\omega n_n = Q_n \qquad (2.8)$$

The following notation has been used in the above equations:

$\bar{h}$: varying component of the magnetic field

ϵ_o: dielectric constant of free space

$\bar{E}$: varying component of the electric field (constant component is not considered in this investigation)

$\bar{V}_{e,i,n}$: fluid velocity of the electron, ion, or neutral particle gas

$\bar{J}$: electric current source

μ_o: permeability of free space

$\bar{K}$: magnetic current source

$n_{e,i,n}$: varying component of the number density of the electron, ion or neutral particle gas

e: absolute value of the charge of an electron

$\bar{F}_{e,i,n}$: mechanical body source for the electron, ion or neutral particle gas

$Q_{e,i,n}$: fluid flux source for the electron, ion or neutral particle gas

The set of Eqs. (2.1) through (2.8) represents a system of partial differential equations relating 18 scalar functions. In the following section we shall present a formal operator transform method, which is convenient to solve this set of equations with source terms present.

2.2 Operator Transform Method

A formal solution to the set of Eqs. (2.1) through (2.8) can be obtained by an operator transform method. This method is an extension of that used by Diament[28] for the formal solution of Maxwell's equations in general linear media.

The procedure for obtaining the formal solution is as follows:

FIRST: For the purpose of exhibiting a general solution to a system of basic equations, it is convenient to reformulate them in the following single operator equation

$$\mathcal{W}\psi(r)=\phi(r) \tag{2.9}$$

where $\psi(r)$ is a field vector composed of the field variables such as the electric field $\bar{E}$, the velocity field $\bar{V}$, etc., $\phi(r)$ is the source vector containing various excitation sources such as the electric current source $\bar{J}$, the mechanical source $\bar{F}$, etc., and $\mathcal{W}$ is the system matrix differential operator relating the field to the sources. $\mathcal{W}$ contains all the properties of the medium and is a function of the space coordinate r. In general, without loss of generality, the system of basic equations can be rearranged so that some of the submatrices of $\overline{b}$ are identity matrices.

SECOND: Here, we introduce the generalized transform techniques, which amounts to choosing some convenient basis of representation for the solution and transforming the operator differential equation in real space to an operator integral equation in transform space. The generic summation symbol $\oint$, such as used in Quantum Mechanics[34], will be used, which requires that the expression following this symbol be integrated or summed over the entire range of the repeated variable. Formally, for any quantity a(r), we may introduce the following transform pair:

$$\text{Transform} \qquad A(s) = \oint d(s, r)a(r)$$
$$\text{Inverse} \qquad a(r) = \oint c(r, s)A(s) \tag{2.10}$$

with the property that

$$\oint c(r, s)d(s, p) = \overline{\mathbb{1}}(r, p)$$

and

$$\oint d(u, r)c(r, s) = \overline{\mathbb{1}}(u, s) \tag{2.11}$$

The idemfactor $\overline{\mathbb{1}}(u, s)$ comprises a Dirac delta function or a Kronecker delta and a unit dyadic, as required.

To illustrate the transform pair consider a rectangular coordinate system. The real space variables are coordinates (x, y, z) and the transform space variables may be considered as (s_1, s_2, s_3). The range of the real space and transform space variables is $-\infty$ to $+\infty$. In this case a Fourier transform is appropriate and d(s, r) and c(r, s) are

$$d(s, r) = \frac{1}{(2\pi)^3} e^{-ir \cdot s}$$
$$c(r, s) = e^{ir \cdot s} \tag{2.12}$$

If the nature of the problem requires the cylindrical coordinate system, we can apply a Fourier-Bessel transform given by

$$d(s, r) = e^{-i(n\phi + \beta z)} J_n(q\rho)$$
$$c(r, s) = \frac{1}{(2\pi)^2} e^{i(n\phi + \beta z)} J_n(q\rho) \tag{2.13}$$

The ranges of ρ and q are 0 to ∞, with weight functions ρ and q, respectively; the range of ϕ is 0 to 2π, and that of n is all integers; the ranges of z and β are $-\infty$ to ∞. The real space is expressed by the cylindrical coordinate system (ρ, ϕ, z), and the transform space is expressed by (q, n, β).

Now, we proceed to the transformation of the operator Eq. (2.9). Let $\Psi(s)$ and $\Phi(s)$ be the transforms of the vectors $\psi(r)$ and $\phi(r)$, respectively, i.e.,

$$\begin{cases} \Psi(s) = \oint d(s, r)\psi(r) \\ \psi(r) = \oint c(r, s)\Psi(s) \end{cases} \tag{2.14}$$

$$\begin{cases} \Phi(s) = \oint d(s, r)\phi(r) \\ \phi(r) = \oint c(r, s)\Phi(s) \end{cases} \tag{2.15}$$

Also, we take the transformation law for the matrix operator $\mathcal{W}$ as

$$\mathcal{W}(u, s) = \oint d(u, r)\mathcal{W}c(r, s) \tag{2.16}$$

Premultiplying both sides of Eq. (2.9) by $d(u, r)$, and then substituting the expansion for $\psi(r)$ as given by the transform pair in Eq. (2.14) and summing or integrating over the complete r-space the operator Eq. (2.9) in the real space becomes the operator integral equation in the transform space

$$\oint \mathcal{W}(u,s)\,\mathcal{V}(s) = \mathcal{J}(u) \tag{2.17}$$

This equation has the character of a generalized integral equation of the first kind, with $\mathcal{J}(s)$ as the forcing function, $\mathcal{V}(s)$ as the unknown function, and $\mathcal{W}(u,s)$ as the kernel. $\mathcal{W}(u,s)$ is a function of two composite variables of the transform space and retains all the pertinent information about the system.

THIRD: Because of the earlier rearrangement and diagonalization, the dyadic kernel $\mathcal{W}(u,s)$ can be properly partitioned so that the order of the matrices to be manipulated may be reduced, by introducing coupled integral equations of the second kind, which in turn may be recombined into one integral equation of the second kind. For example, we can have for Maxwell's equations

$$\mathcal{W}(u,s) = \begin{bmatrix} \bar{I}(u,s) & -Z(u,s) \\ -Y(u,s) & \bar{I}(u,s) \end{bmatrix} \tag{2.18}$$

Then, the partitioning of $\mathcal{V}(s)$ and $\mathcal{J}(u)$ into two vectors

$$\mathcal{V}(s) = \begin{bmatrix} V(s) \\ I(s) \end{bmatrix}, \qquad \mathcal{J}(u) = \begin{bmatrix} W(u) \\ J(u) \end{bmatrix} \tag{2.19}$$

produces the following coupled integral equations

$$V(u) = W(u) + \oint Z(u,s)\,I(s) \tag{2.20}$$

$$I(u) = J(u) + \oint Y(u,s)\,V(s) \tag{2.21}$$

Let V(s) and I(s) correspond, respectively, to the transform of the electric field and the transform of the magnetic field, then these Eqs. (2.20) and (2.21) have the generalized forms of the telegraphist's equations of Schelkunoff[35] if s is taken to indicate different modes in the waveguide.

The elimination of either the field vector V or I in the Eqs. (2.20) and (2.21) gives the general form of the Fredholm integral equation of the second kind, e.g.,

$$V(u)=F(u)+\oint K(u,s)V(s) \tag{2.22}$$

where

$$\begin{aligned} F(u) &\equiv W(u)+\oint Z(u,s)J(s) \\ K(u,s) &\equiv \oint Z(u,v)Y(v,s) \end{aligned} \tag{2.23}$$

are both known functions. For homogeneous media the kernel has the ideal form

$$K(u,s)=N(s)\bar{\bar{1}}(u,s) \tag{2.24}$$

and the integral Eq. (2.22) can be explicitly solved as

$$V(s)=\left[1-N(s)\right]^{-1}F(s). \tag{2.25}$$

Another degenerate case exists when the dyadic kernel can be expressed in factored form as

$$K(u,s)=A(u)B(s) \tag{2.26}$$

and again we have explicit solutions. For inhomogeneous media, in general, an explicit formal solution to the integral equation can be obtained recursively by the application of the general theory studied by Diament[28]. If approximate solutions are sufficient, a general kernel can be approximated by a degenerate kernel.

2.3 <u>One-Fluid Plasma</u>

The operator/transform method presented in the previous section will be applied to one-fluid plasma problems. Its application to three-fluid plasma problems is very complicated, and is illustrated in Appendix A. By one-fluid plasma we will consider only the electron gas, and the motions of ions and neutral particles will be neglected. Our intention is to study the applicability of our method to simpler problems thus paving the way to more difficult three-fluid problems. Besides, there are many practical situations in which we can neglect the effect of heavy particles, e.g., the ratio propagation in the most part of the ionosphere except, maybe, D-region. Although electron fluid plasma problems have been studied by many investigators (Ginzburg[4], Budden[5], Bunkin[11], Arbel[13]), there are still many important problems to be solved. One of such problems is the excitation problem in the compressible electron fluid plasma immersed in a constant magnetic field. This problem will be given a full treatment in the subsequent chapters.

[A] Operator Form

The basic equations for the electron fluid plasma can be obtained from Eqs. (2.1), (2.2), (2.3) and (2.4) as

$$\nabla \times \bar{E} - i\mu_0 \omega \bar{h} = -\bar{K} \tag{2.27}$$

$$\nabla \times \bar{h} + i\epsilon_0 \omega \bar{E} + eN_0 \bar{V} = \bar{J} \tag{2.28}$$

$$-i\omega N_0 m\bar{V} + mU^2 \nabla n + eN_0 \left[\bar{E} + \bar{V} \times \bar{B}_0\right] = \bar{F} \tag{2.29}$$

$$N_0 \nabla \cdot \bar{V} + \bar{V} \cdot \nabla N_0 - i\omega n = Q \tag{2.30}$$

where all subscripts e have been dropped, and also we have neglected the collisional dissipation effects.

In order to be able to obtain a proper matrix form of the Eqs. (2.27) through (2.30), we will express $\bar{E}$ and $\bar{V}$ in terms of $\bar{h}$, n, $\bar{J}$ and $\bar{F}$ by employing Eqs. (2.28) and (2.29). Firstly, $\bar{E}$ is eliminated between these two equations to get

$$\left(\frac{\omega^2-\omega_p^2}{i\omega}\right)\bar{V}+\omega_c\bar{V}x\hat{b}=\frac{e}{i\omega\epsilon_o m}\nabla x\bar{h}-\frac{U^2}{N_o}\nabla n-\frac{e}{i\omega\epsilon_o m}\bar{J}+\frac{1}{mN_o}\bar{F} \qquad (2.31)$$

where $\hat{b}$ is the unit vector in the direction of the externally applied constant magnetic field, and also use is made of the conventional electron cyclotron frequency, $\omega_c=eB_o/m$, and the electron plasma frequency, $\omega_p^2=e^2N_o/\epsilon_o m$. Secondly, Eq. (2.31) is explicitly solved for $\bar{V}$ by taking scalar product and vector product of Eq. (2.31) with $\hat{b}$, and then $\bar{E}$ can be solved from Eq. (2.28). Their results are

$$A_{11}\bar{h}+A_{12}n+\bar{E}=\bar{S}_1 \qquad (2.32)$$

$$A_{21}\bar{h}+A_{22}n+\bar{V}=\bar{S}_2 \qquad (2.33)$$

where

$$A_{11}=\frac{\omega_p^2\omega/i\epsilon_o}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\left[\frac{i\omega_c}{\omega}\hat{b}x(\nabla x\bar{\bar{1}})+\frac{(\omega^2-\omega_p^2)}{\omega^2}\nabla x\bar{\bar{1}}-\frac{\omega_c^2}{\omega^2-\omega_p^2}\hat{b}\hat{b}\cdot\nabla x\bar{\bar{1}}\right]-\frac{i\nabla x\bar{\bar{1}}}{\epsilon_o\omega} \qquad (2.34)$$

$$A_{12}=\frac{eU^2\omega/i\epsilon_o}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\left[\omega_c\hat{b}x(\nabla\cdot\bar{\bar{1}})+\frac{(\omega^2-\omega_p^2)}{i\omega}\nabla\cdot\bar{\bar{1}}+\frac{i\omega\omega_c^2}{(\omega^2-\omega_p^2)}\hat{b}\hat{b}\cdot(\nabla\cdot\bar{\bar{1}})\right] \qquad (2.35)$$

$$A_{21}=\left[A_{11}+\frac{i\nabla x\bar{\bar{1}}}{\epsilon_o\omega}\right]\frac{\epsilon_o\omega}{ieN_o} \qquad (2.36)$$

$$A_{22}=A_{12}\frac{\epsilon_o\omega}{ieN_o} \qquad (2.37)$$

$$\bar{S}_1=\frac{\omega/i\epsilon_o}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\left\{\frac{e}{m}\left[\omega_c\hat{b}x\bar{F}+\frac{(\omega^2-\omega_p^2)}{i\omega}\bar{F}+\frac{i\omega\omega_c^2}{(\omega^2-\omega_p^2)}\hat{b}\hat{b}\cdot\bar{F}\right]\right.$$
$$\left.+\omega_p^2\left[i\frac{\omega_c}{\omega}\hat{b}x\bar{J}+\frac{(\omega^2-\omega_p^2)}{\omega^2}\bar{J}-\frac{\omega_c^2}{(\omega^2-\omega_p^2)}\hat{b}\hat{b}\cdot\bar{J}\right]\right\}-i\frac{\bar{J}}{\epsilon_o\omega} \qquad (2.38)$$

$$\bar{S}_2 = \left[\bar{S}_1 + i\frac{\bar{J}}{\epsilon_o \omega}\right]\frac{\epsilon_o \omega}{ieN_o} \tag{2.39}$$

In the expressions given by Eqs. (2.34) through (2.37) we have employed some dyadic operations with their associated matrices as follows:

$$\bar{\bar{1}} \equiv \hat{x}\hat{x}+\hat{y}\hat{y}+\hat{z}\hat{z} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.40}$$

$$\nabla \times \bar{\bar{1}} \equiv \left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right) \times (\hat{x}\hat{x}+\hat{y}\hat{y}+\hat{z}\hat{z}) \longrightarrow \begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \tag{2.41}$$

$$\nabla \cdot \bar{\bar{1}} \equiv \left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right) \cdot (\hat{x}\hat{x}+\hat{y}\hat{y}+\hat{z}\hat{z}) \longrightarrow \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \tag{2.42}$$

$$\hat{b}\hat{b} \equiv (\hat{x}b_x+\hat{y}b_y+\hat{z}b_z)(\hat{x}b_x+\hat{y}b_y+\hat{z}b_z) \longrightarrow \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} \begin{bmatrix} b_x & b_y & b_z \end{bmatrix} \tag{2.43}$$

$$\hat{b} \times \bar{\bar{1}} \equiv (\hat{x}b_x+\hat{y}b_y+\hat{z}b_z) \times (\hat{x}\hat{x}+\hat{y}\hat{y}+\hat{z}\hat{z}) \longrightarrow \begin{bmatrix} 0 & -b_z & b_y \\ b_z & 0 & -b_x \\ -b_y & b_x & 0 \end{bmatrix} \tag{2.44}$$

Now the original Eqs. (2.27) and (2.30), together with the rearranged Eqs. (2.32) and (2.33), can be put into the following desirable matrix form:

$$\begin{bmatrix} \bar{\bar{1}} & 0 & i\frac{\nabla x\bar{\bar{1}}}{\omega\mu_o} & 0 \\ 0 & 1 & 0 & \frac{iN_o}{\omega}(\nabla\cdot\bar{\bar{1}})'+\frac{i}{\omega}\nabla N_o\cdot\bar{\bar{1}} \\ A_{11} & A_{12} & \bar{\bar{1}} & 0 \\ A_{21} & A_{22} & 0 & \bar{\bar{1}} \end{bmatrix} \begin{bmatrix} \bar{h} \\ n \\ \bar{E} \\ \bar{V} \end{bmatrix} = \begin{bmatrix} -i\bar{K}/\omega\mu_o \\ iQ/\omega \\ \bar{S}_1 \\ \bar{S}_2 \end{bmatrix} \tag{2.45}$$

where $(\nabla\cdot\bar{\bar{1}})'$ is the transpose of the matrix given by Eq. (2.42).

This matrix equation can, then, be put into an operator equation

$$\mathcal{W}\,\psi(r) = \phi(r) \tag{2.46}$$

where

$$\psi(r) = \begin{bmatrix} \bar{h} \\ n \\ \bar{E} \\ \bar{V} \end{bmatrix}, \qquad \phi(r) = \begin{bmatrix} -i\bar{K}/\omega\mu_o \\ iQ/\omega \\ \bar{S}_1 \\ \bar{S}_2 \end{bmatrix} \tag{2.47}$$

Thus, the basic Maxwell-Euler's Eqs. (2.27) through (2.30) have been reformulated into a single abstract relation between the sources and the resultant fields. $\psi(r)$ is a ten-vector containing the field quantities, $\phi(r)$ is a ten-vector representing the source quantities, and $\mathcal{W}$ is the system matrix differential operator relating the fields to the sources. Two identity submatrices of $\mathcal{W}$, as can be seen from Eq. (2.45), are highly significant in deriving an integral equation of the second kind in an inhomogeneous medium.

[B] Generalized Telegraphist's Equations

Generalized Fourier transform as given by Eqs. (2.10) and (2.11) will be used to transform the operator Eq. (2.46). Then, Eqs. (2.14) and (2.15) give the transform pairs for the field vector and the source vector given in Eq. (2.47), and Eq. (2.16) gives the transform for the matrix differential operator, $\mathcal{W}$.

The resultant integral equation of the first kind as given by Eq. (2.17) is

$$\mathop{\int\!\!\!\!\!\!\sum} \mathcal{W}(u, s)\, \bar{\Psi}(s) = \bar{\Phi}(u) \tag{2.48}$$

This equation may be put into the generalized forms of the telegraphist's equations by partitioning the transform of the field vector, $\bar{\Psi}(s)$, the transform of the source vector, $\bar{\Phi}(s)$, and the transform of the matrix differential operator, $\mathcal{W}(u, s)$, as follows:

$$\bar{\Psi}(s) = \mathop{\int\!\!\!\!\!\!\sum} d(s, r) \begin{bmatrix} \bar{h} \\ n \\ \bar{E} \\ \bar{V} \end{bmatrix} = \begin{bmatrix} I_t(s) \\ V_e(s) \\ V_t(s) \\ I_e(s) \end{bmatrix} \tag{2.49}$$

$$\bar{\Phi}(s) = \mathop{\int\!\!\!\!\!\!\sum} d(s, r) \begin{bmatrix} -i\bar{K}/\omega\mu_o \\ iQ/\omega \\ \bar{S}_1 \\ \bar{S}_2 \end{bmatrix} = \begin{bmatrix} J_t(s) \\ W_e(s) \\ W_t(s) \\ J_e(s) \end{bmatrix} \tag{2.50}$$

where $I_t(s)$, $V_e(s)$, $V_t(s)$ and $I_e(s)$ are, respectively, the transform of the magnetic field, $\bar{h}$, the transform of the density variation, n, the transform of the electric field, $\bar{E}$, and the transform of the fluid velocity, $\bar{V}$, and also $J_t(s)$, $W_e(s)$, $W_t(s)$ and $J_e(s)$ correspond, respectively, to the transform of the magnetic current source, $\bar{K}$, the transform of the fluid flux source, Q, the transform of the three-vector source function, $\bar{S}_1$, and the transform of the three-vector source function, $\bar{S}_2$. Taking advantage of the orthonormality property of the transformation kernels as given by Eq. (2.11), the ten-dyadic kernel, $\mathcal{W}(u, s)$, can be partitioned as

$$\mathcal{W}(u,s) = \begin{bmatrix} \bar{\bar{I}}(u,s) & 0 & -Y_t(u,s) & 0 \\ 0 & 1(u,s) & 0 & -Z_e(u,s) \\ -Z_t(u,s) & -T_{te}(u,s) & \bar{\bar{I}}(u,s) & 0 \\ -T_{et}(u,s) & -Y_e(u,s) & 0 & \bar{\bar{I}}(u,s) \end{bmatrix} \tag{2.51}$$

where 1(u, s) is a Dirac or Kronecker delta function which is the same as the scalar form of the idemfactor $\bar{\bar{I}}(u,s)$. The three-dyadic immittance functions $Y_t(u, s)$ and $Z_t(u, s)$; the three-dyadic transfer function $T_{et}(u, s)$; the three-row-vector impedance function $Z_e(u, s)$; the three-column-vector admittance function $Y_e(u, s)$; the three-column-vector transfer function $T_{te}(u, s)$ are defined as:

$$-Y_t(u,s) = \oint d(u,r)\, i\, \frac{\nabla \times \bar{\bar{I}}}{\omega\mu_o}\, c(r,s) \tag{2.52}$$

$$-Z_t(u,s) = \oint d(u,r)\, A_{11}\, c(r,s) \tag{2.53}$$

$$-T_{et}(u,s) = \oint d(u,r)\, A_{21}\, c(r,s) \tag{2.54}$$

$$-Z_e(u,s) = \oint d(u,r) \left[\frac{iN_o}{\omega}(\nabla \cdot \bar{\bar{I}})' + \frac{i}{\omega} \nabla N_o \cdot \bar{\bar{I}} \right] c(r,s) \tag{2.55}$$

$$-Y_e(u,s) = \oint d(u,r)\, A_{22}\, c(r,s) \tag{2.56}$$

$$-T_{te}(u,s) = \oint d(u,r)\, A_{12}\, c(r,s). \tag{2.57}$$

In Eqs. (2.52), (2.53) and (2.54) both transformation kernels, d(u, r), and inverse transformation kernels, $c(r, s)$, are three-diagonal-dyadics; in Eq. (2.55) $d(u, r)$ can be taken as a scalar and $c(r, s)$ taken as a three-diagonal-dyadic; in Eqs. (2.56) and (2.57), three-diagonal-dyadics can be used for $d(u, r)$ and scalars can be used for $c(r, s)$.

Substitution of the partitioned matrices as given by Eqs. (2.49), (2.50) and (2.51) into the transform integral Eq. (2.48) decomposes this equation into the following set which, due to their form, will be called the generalized telegraphist's equations.

$$I_t(u) = J_t(u) + \oint Y_t(u, s)\, V_t(s) \tag{2.58}$$

$$V_t(u) = W_t(u) + \oint Z_t(u, s)\, I_t(s) + \oint T_{te}(u, s)\, V_e(s) \tag{2.59}$$

$$I_e(u) = J_e(u) + \oint T_{et}(u, s)\, I_t(s) + \oint Y_e(u, s)\, V_e(s) \tag{2.60}$$

$$V_e(u) = W_e(u) + \oint Z_e(u, s)\, I_e(s) \tag{2.61}$$

Equations (2.58) to (2.61) contain in a compact form, in the transform space, the laws governing the excitation and propagation of "fields" in the linearized medium. The set of immittances and transfer functions, which can be evaluated from Eqs. (2.52) through (2.57), contain all the intrinsic properties of the medium, while four components of the source vector, $W_t(u)$, $J_t(u)$, $J_e(u)$, $W_e(u)$ represent all the sources. For Maxwell's equations only, if ϵ_0 and μ_0 in Eqs. (2.58) and (2.59) are replaced by appropriate tensor permittivity and permeability, the pertinent equations are

$$\begin{cases} I_t(u) = J_t(u) + \oint Y_t(u, s)\, V_t(s) & (2.62) \\ V_t(u) = W_t(u) + \oint Z_t(u, s)\, I_t(s). & (2.63) \end{cases}$$

This set of equations is the original form given by Diament[28]. They may be compared with the telegraphist's equations of Schelkunoff[35] or the network equations of Marcuvitz[36]. Z_t may be interpreted as an impedance function while Y_t may be interpreted as an admittance function. Hessel, Marcuvitz and Shmoys[37] have explored some aspects of the application of transmission line equations to a problem involving a compressible plasma and air and the associated boundary between the plasma and air. However, they did not consider the effect of a constant magnetic field. The results of their investigation yield some versions of Eqs. (2.58) to (2.61). Equations (2.58) and (2.59) give the transmission line system for the transverse electromagnetic wave, and Eqs. (2.60) and (2.61) give the transmission line system for the electron acoustic type of wave.

[C] Fredholm Integral Equation

The general Fredholm integral equation of the first kind, Eq. (2.48), which is equivalent to the original Maxwell-Euler's equations, will now be reformulated into a general Fredholm integral equation of the second kind which is more amenable to analysis. At the same time we have reduced the order of the matrices to be manipulated from 10 x 10 to 4 x 4. This step can be easily performed for Maxwell's equations in the form of Eqs. (2.62) and (2.63), but for our Maxwell-Euler's equations we can not directly reduce the generalized telegraphist's Equations (2.58) through (2.61) into the integral equation of the second kind. Thus, in order to effect this reduction $\underline{\psi}(s)$, $\underline{\phi}(s)$ and $\mathcal{W}(u, s)$ will be partitioned in the following way.

$$\Psi(s)=\begin{bmatrix}\Psi_1(s)\\ \Psi_2(s)\end{bmatrix}, \qquad \Phi(s)=\begin{bmatrix}\Phi_1(s)\\ \Phi_2(s)\end{bmatrix} \tag{2.64}$$

where

$$\Psi_1(s)=\begin{bmatrix}I_t(s)\\ V_e(s)\end{bmatrix}, \quad \Psi_2(s)=\begin{bmatrix}V_t(s)\\ I_e(s)\end{bmatrix}, \quad \Phi_1(s)=\begin{bmatrix}J_t(s)\\ W_e(s)\end{bmatrix}, \quad \Phi_2(s)=\begin{bmatrix}W_t(s)\\ J_e(s)\end{bmatrix} \tag{2.65}$$

and

$$\mathcal{W}(u,s)=\begin{bmatrix}\bar{\bar{I}}(u,s) & -\mathcal{W}_{12}(u,s)\\ -\mathcal{W}_{21}(u,s) & \bar{\bar{I}}(u,s)\end{bmatrix} \tag{2.66}$$

where

$$\mathcal{W}_{12}(u,s)=\begin{bmatrix}Y_t(u,s) & 0\\ 0 & Z_e(u,s)\end{bmatrix}, \quad \mathcal{W}_{21}(u,s)=\begin{bmatrix}Z_t(u,s) & T_{te}(u,s)\\ T_{et}(u,s) & Y_e(u,s)\end{bmatrix} \tag{2.67}$$

The introduction of these partitioned matrices into the integral Eq. (2.48) gives the following coupled integral equations

$$\Psi_1(u)=\Phi_1(u)+\oint \mathcal{W}_{12}(u,s)\Psi_2(s) \tag{2.68}$$

$$\Psi_2(u)=\Phi_2(u)+\oint \mathcal{W}_{21}(u,s)\Psi_1(s) \tag{2.69}$$

and the substitution of Eq. (2.69) into Eq. (2.68) gives rise to the desired integral equation of the second kind

$$\Psi_1(u)=F(u)+\oint K(u,s)\Psi_1(s) \tag{2.70}$$

where the compound source is

$$F(u)=\Phi_1(u)+\oint \mathcal{W}_{12}(u,s)\Phi_2(s) \tag{2.71}$$

and the four-dyadic kernel is

$$K(u, s) = \oint \mathcal{W}_{12}(u, v)\, \mathcal{W}_{21}(v, s) \tag{2.72}$$

[D] Formal Solution in A Homogeneous Plasma

The integral Eq. (2.70) can be easily solved for a homogeneous plasma because the kernel has the ideal form $K(u, s) = N(s)\, \bar{\bar{1}}(u, s)$.

Choosing a Fourier transform and thus using the transformation kernels as given by Eq. (2.12), we can obtain from the defining Eqs. (2.52) through (2.57)

$$-Y_t(u, s) = -\frac{\bar{\bar{s}}}{\omega\mu_o}\, 1(u, s) \tag{2.73}$$

$$-Z_e(u, s) = -\frac{N_o}{\omega}\, s'\, 1(u, s) \tag{2.74}$$

$$-Z_t(u, s) = \frac{\omega\omega_p^2/i\epsilon_o}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\left[\frac{-\omega_c}{\omega}\bar{\bar{b}}\,\bar{\bar{s}} + i\,\frac{(\omega^2-\omega_p^2)}{\omega^2}\,\bar{\bar{s}} - i\,\frac{\omega_c^2}{(\omega^2-\omega_p^2)}\, bb'\,\bar{\bar{s}}\right] 1(u, s) + \frac{\bar{\bar{s}}}{\epsilon_o\omega}\, 1(u, s) \tag{2.75}$$

$$-T_{te}(u, s) = \frac{e\omega U^2/i\epsilon_o}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\left[i\omega_c\,\bar{\bar{b}}\, s + \frac{(\omega^2-\omega_p^2)}{\omega}\, s - \frac{\omega\omega_c^2}{(\omega^2-\omega_p^2)}\, bb'\, s\right] 1(u, s) \tag{2.76}$$

$$-T_{et}(u, s) = \frac{\epsilon_o\omega}{ieN_o}\left[-Z_t(u, s) - \frac{\bar{\bar{s}}}{\epsilon_o\omega}\, 1(u, s)\right] \tag{2.77}$$

$$-Y_e(u, s) = -\frac{\epsilon_o\omega}{ieN_o}\, T_{te}(u, s) \tag{2.78}$$

where

$$\bar{\bar{b}} \equiv \begin{bmatrix} 0 & -b_z & b_y \\ b_z & 0 & -b_x \\ -b_y & b_x & 0 \end{bmatrix}, \quad b \equiv \begin{bmatrix} b_x \\ by \\ b_z \end{bmatrix}, \quad b' \equiv \begin{bmatrix} b_x & b_y & b_z \end{bmatrix}$$

$$s \equiv \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}, \quad s' \equiv \begin{bmatrix} s_1 & s_2 & s_3 \end{bmatrix}, \quad \bar{\bar{s}} \equiv \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix} \tag{2.79}$$

The kernel of the integral Eq. (2.70) is

$$K(u,s) = \oint \mathcal{W}_{12}(u,v)\, \mathcal{W}_{21}(v,s)$$

$$= \begin{bmatrix} \oint Y_t(u,v) Z_t(v,s) & \oint Y_t(u,v) T_{te}(v,s) \\ \oint Z_e(u,v) T_{et}(v,s) & \oint Z_e(u,v) Y_e(v,s) \end{bmatrix} \tag{2.80}$$

with

$$\oint Y_t(u,v) Z_t(v,s)$$

$$= \frac{-c^2\omega_p^2/\omega}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\, \bar{\bar{s}} \left[i\omega_c \bar{\bar{b}} + \frac{\omega(\omega^2-\omega_p^2-\omega_c^2)}{\omega_p^2} \bar{\bar{I}} - \frac{\omega\omega_c^2}{(\omega^2-\omega_p^2)}\, bb' \right] \bar{\bar{s}}\, 1(u,s) \tag{2.81}$$

$$\oint Y_t(u,v) T_{te}(v,s)$$

$$= \frac{iec^2U^2}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\, \bar{\bar{s}} \left[i\omega_c \bar{\bar{b}} - \frac{\omega\omega_c^2}{(\omega^2-\omega_p^2)}\, bb' \right] s\, 1(u,s) \tag{2.82}$$

$$\oint Z_e(u,v)T_{et}(v,s)$$

$$= \frac{i\omega_p^2/e}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\, s' \left[i\omega_c \bar{\bar{b}} - \frac{\omega\omega_c^2}{(\omega^2-\omega_p^2)} bb' \right] \bar{\bar{s}}\, 1(u,s) \tag{2.83}$$

$$\oint Z_e(u,v)Y_e(v,s)$$

$$= \frac{\omega U^2}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\, s' \left[i\omega_c \bar{\bar{b}} + \frac{(\omega^2-\omega_p^2)}{\omega}\bar{\bar{1}} - \frac{\omega\omega_c^2}{(\omega^2-\omega_p^2)} bb' \right] s\, 1(u,s) \tag{2.84}$$

where $c = 1/\sqrt{\mu_o \epsilon_o}$ is the velocity of light in free space. Thus, the kernel has the ideal form

$$\oint \mathcal{W}_{12}(u,v)\, \mathcal{W}_{21}(v,s) = N(s)\, 1(u,s) \tag{2.85}$$

where N(s) is a 4x4 matrix.

Substitution of this ideal form of the kernel given by Eq. (2.85) into the integral equation will produce the solution of the integral equation directly as

$$\begin{bmatrix} I_t(s) \\ V_e(s) \end{bmatrix} = \left[1-N(s)\right]^{-1} F(s) \tag{2.86}$$

In real space the magnetic field and the density fluctuation field are given by

$$\begin{bmatrix} \bar{h}(r) \\ n(r) \end{bmatrix} = \oint \left[1-N(s)\right]^{-1} F(s)\, e^{ir\cdot s} \tag{2.87}$$

which is usually evaluated by the method of residues at the zeros of the determinant

$$\det. \left[1-N(s)\right] = 0 \tag{2.88}$$

Equation (2.88) is the conventional dispersion relation when s is interpreted as the propagation constant. Thus, the importance of the dispersion relation in finding the excited fields is obvious.

$\underline{\Psi}_2(s)$, which is a six-column-vector composed of the transform of the electric field, $V_t(s)$, and the transform of the velocity field, $I_e(s)$, can now be obtained from Eq. (2.69).

CHAPTER III

WAVE PROPAGATION IN ONE-FLUID PLASMA

3.1 Introduction

The close relationship existing between the dispersion relation, which describes the propagation characteristics of waves, and the excited fields is apparent from the fact that various poles of the inverse transformation integrals give the dispersion relations for the different types of waves. Lighthill[38] and Felsen[39], all stressed the importance of the direct application of wave surfaces obtained from the dispersion relation in finding radiation fields. Thus, in this chapter we will analyze the dispersion relation and discuss the propagation characteristics of those waves existing in an electron fluid plasma, which is the preliminary requirement for solving the excitation problems. Collisional dissipation effects are neglected in order to show the salient features. This should be practically permissible for high frequencies and in higher ionospheric regions.

To facilitate the analysis it is convenient to give a proper terminology to the waves whose propagation constant squares are given by the roots of the dispersion relation. For an electron plasma a standard terminology has not yet been established for the three waves corresponding to the three roots. Judging waves by their frequency characteristics, Denisse and Delcroix[8] have used the terms, ordinary waves, extraordinary waves and electron waves. Allis, Buchsbaum and Bers[40] have adopted the optical criteria of judging waves by their local characteristics, i.e., by the shapes of the phase velocity surface and the polarization, thus they have used the terms, ordinary waves, extraordinary waves, right-handed circularly polarized waves, left-handed circularly polarized waves, and plasma waves. The terminology used in this work will be developed in a manner similar to that of Denisse and Delcroix[8]. However, since the motion of the ions has been neglected, a more exact analysis of the roots is possible and the result can be related directly to the work of Allis, Buchsbaum and Bers[40], and Stix[41]. Thus, the names, "modified ordinary wave", "modified extraordinary wave" and "modified plasma wave" have been associated with each branch of the root of the dispersion relation for the intermediate inclination of the constant magnetic field to the propagation direction. The point of view adopted here is that the ordinary and extraordinary electromagnetic waves in magnetoionic theory (Ratcliffe[2]) and the plasma wave are coupled together and modified by each other due to the constant magnetic field.

3.2 Dispersion Relation

Without loss of generality, the coordinate axes can be chosen such that the externally applied constant magnetic field is in the $\hat{y}$ direction and given by

$$\bar{B}_o = \hat{y} B_o \, . \tag{3.1}$$

Thus $$b_x = b_z = 0, \quad b_y = 1 .$$

Applying Eq. (3.1) to Eq. (2.81) through (2.84) the following matrix for the kernel function N(s) is obtained:

$$N(s) = \begin{bmatrix} N_{11} & N_{12} & N_{13} & N_{14} \\ N_{21} & N_{22} & N_{23} & N_{24} \\ N_{31} & N_{32} & N_{33} & N_{34} \\ N_{41} & N_{42} & N_{43} & N_{44} \end{bmatrix} \times \frac{1}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2} \quad (3.2)$$

with

$$N_{11} = c^2\left\{ s_3^2 \frac{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}{\omega^2-\omega_p^2} + s_2^2(\omega^2-\omega_p^2-\omega_c^2)\right\} \quad (3.3)$$

$$N_{12} = -c^2 s_2\left\{\frac{i\omega_c\omega_p^2}{\omega} s_3 + s_1(\omega^2-\omega_p^2-\omega_c^2)\right\} \quad (3.4)$$

$$N_{13} = c^2\left\{\frac{i\omega_c\omega_p^2}{\omega} s_2^2 - s_1 s_3 \frac{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}{\omega^2-\omega_p^2}\right\} \quad (3.5)$$

$$N_{14} = iec^2U^2\omega_c s_2\left\{s_3\frac{\omega\omega_c}{\omega^2-\omega_p^2} - is_1\right\} \quad (3.6)$$

$$N_{21} = c^2 s_2\left\{\frac{i\omega_c\omega_p^2}{\omega} s_3 - s_1(\omega^2-\omega_p^2-\omega_c^2)\right\} \quad (3.7)$$

$$N_{22} = c^2(s_1^2+s_3^2)(\omega^2-\omega_p^2-\omega_c^2) \quad (3.8)$$

$$N_{23} = -c^2 s_2\left\{\frac{i\omega_c\omega_p^2}{\omega} s_1 + s_3(\omega^2-\omega_p^2-\omega_c^2)\right\} \quad (3.9)$$

$$N_{24} = -ec^2U^2\omega_c(s_1^2+s_3^2) \quad (3.10)$$

$$N_{31} = -c^2\left\{\frac{i\omega_c\omega_p^2}{\omega} s_2^2 + s_1 s_3 \frac{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}{\omega^2-\omega_p^2}\right\} \quad (3.11)$$

$$N_{32} = c^2 s_2 \left\{ \frac{i\omega_c \omega_p^2}{\omega} s_1 - s_3 (\omega^2 - \omega_p^2 - \omega_c^2) \right\} \tag{3.12}$$

$$N_{33} = c^2 \left\{ s_1^2 \frac{(\omega^2-\omega_p^2)^2 - \omega_c^2 \omega^2}{\omega^2 - \omega_p^2} + s_2^2 (\omega^2 - \omega_p^2 - \omega_c^2) \right\} \tag{3.13}$$

$$N_{34} = -iec^2 U^2 \omega_c s_2 \left\{ s_1 \frac{\omega\omega_c}{\omega^2 - \omega_p^2} + i s_3 \right\} \tag{3.14}$$

$$N_{41} = -\frac{i\omega_c \omega_p^2}{e} s_2 \left\{ s_3 \frac{\omega\omega_c}{\omega^2 - \omega_p^2} + i s_1 \right\} \tag{3.15}$$

$$N_{42} = -\frac{\omega_c \omega_p^2}{e} (s_1^2 + s_3^2) \tag{3.16}$$

$$N_{43} = \frac{i\omega_c \omega_p^2}{e} s_2 \left\{ s_1 \frac{\omega\omega_c}{\omega^2 - \omega_p^2} - i s_3 \right\} \tag{3.17}$$

$$N_{44} = \omega U^2 \left\{ \frac{\omega^2 - \omega_p^2}{\omega} s^2 - \frac{\omega\omega_c^2}{\omega^2 - \omega_p^2} s_2^2 \right\} \tag{3.18}$$

where $s^2 \equiv s_1^2 + s_2^2 + s_3^2$.

The required dispersion relation is obtained from the determinant of the four by four matrix $[1 - N(s)]$ as given by Eq. (2.88). After some manipulation the dispersion relation can be expressed as

$$\det.\left[1-N(s)\right] = \frac{\omega^4(\omega^2-\omega_c^2)}{(\omega^2-\omega_p^2)\left[(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2\right]}\left[1-\frac{c^2}{\omega^2}(s_1^2+s_3^2)-\frac{\omega_p^2+U^2s_2^2}{\omega^2}\right] \times$$

$$\left[1-\frac{c^2}{\omega^2}(s_1^2+2s_2^2+s_3^2)+\frac{c^4}{\omega^4}s_2^2s^2-\frac{2\omega^2\omega_p^2+U^2(s_1^2+s_3^2)(\omega^2-\omega_p^2-c^2s^2)-\omega_p^4-c^2\omega_p^2(s_1^2+2s_2^2+s_3^2)}{\omega^2(\omega^2-\omega_c^2)}\right]$$

$$+\frac{s_2^2(s_1^2+s_3^2)}{(\omega^2-\omega_p^2)\left[(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2\right]}\left[c^4(\omega_p^2-\omega^2-\omega_c^2)+2c^2U^2(\omega^2-\omega_p^2-c^2s^2)+U^4(\omega_p^2-\omega^2)-\omega_c^2c^2U^2\right]$$

$$+\frac{c^2s_2^2(s^4-s^2s_2^2)}{\omega^2(\omega^2-\omega_p^2)\left[(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2\right]}\left(c^4\omega^2-c^4\omega_c^2+c^2\omega_c^2U^2+\omega^2U^4\right)=0\,. \qquad (3.19)$$

To facilitate the analysis, the following notation will be used:

$$\sqrt{s_1^2 + s_3^2} = s \sin\theta , \qquad s_2 = s \cos\theta$$

$$\frac{\omega}{c} = \beta_o , \quad \frac{\omega}{U} = \beta_e , \quad \frac{\omega_c}{\omega} = \Omega , \quad \frac{\omega_p}{\omega} = \omega_o$$

where θ is the angle between the direction of the magnetic field and the propagation direction. Using this notation, the general three-dimensional dispersion relation as given by Eq. (3.19) becomes

$$s^6(\Omega^2\cos^2\theta - 1) + s^4\left[(1-\omega_o^2)(\beta_e^2+2\beta_o^2) - \Omega^2(\beta_e^2+2\beta_o^2\cos^2\theta - \beta_e^2\omega_o^2\cos^2\theta)\right]$$

$$+ s^2\beta_o^2\left[-(1-\omega_o^2)^2(2\beta_e^2+\beta_o^2) + \Omega^2(2\beta_e^2+\beta_o^2\cos^2\theta - \beta_e^2\omega_o^2\cos^2\theta - \beta_e^2\omega_o^2)\right]$$

$$+ \beta_e^2\beta_o^4(1-\omega_o^2)\left[(1-\omega_o^2)^2 - \Omega^2\right] = 0 . \qquad (3.20)$$

3.3 Basic Types of Waves

Simple systems are considered first, thus introducing plasma waves, ordinary waves and extraordinary waves. Next, Cardan's solution for a cubic equation will be used to obtain an exact solution to the dispersion relation, Eq. (3.20), and these three roots are identified as modified plasma waves, modified ordinary waves and modified extraordinary waves.

[A] Waves in Absence of Magnetic Field

Since the externally applied constant magnetic field is the main reason for complicating the nature of the waves, the case without magnetic field is considered first. The dispersion relation, Eq. (3.20) reduces to

$$s^6 - s^4(1-\omega_o^2)(\beta_e^2 + 2\beta_o^2) + s^2\beta_o^2(1-\omega_o^2)^2(2\beta_e^2+\beta_o^2) - \beta_e^2\beta_o^4(1-\omega_o^2)^3 = 0 \qquad (3.21)$$

This relation can be factored into

$$\left[s^2 - \beta_o^2(1-\omega_o^2)\right]^2 \left[s^2 - \beta_e^2(1-\omega_o^2)\right] = 0 . \qquad (3.22)$$

This first factor yields

$$s = \beta_o \sqrt{1-\omega_o^2} \ , \qquad (3.23)$$

which is the propagation constant for the electromagnetic wave modified by the effect of space charge.

The second factor yields

$$s = \beta_e \sqrt{1-\omega_o^2} \ , \qquad (3.24)$$

which is the propagation constant for the plane plasma wave.

Some authors would prefer using the terminologies, "electro-acoustic wave", "electron acoustic wave" or "electron wave" (Denisse and Delcroix[(8)]) instead of "plasma wave", but we use the term "plasma wave" which has more historical importance. In the event that the ion motions can not be neglected, we can still use the modified terms, "electron plasma wave" and "ion plasma wave".

[B] Cold Plasma

The complicated effect of the static magnetic field is considered for the simple system, where

$$\frac{1}{\beta_e} = \frac{U}{\omega} \longrightarrow 0 . \qquad (3.25)$$

Physically this situation corresponds to the case that the electron gas temperature is so low as to be negligible, and the plasma wave does not propagate. In general, the major effect in this system is to split the electromagnetic wave into two components which are well known in the field of Magnetoionic Theory (Ratcliffe[2], Budden[5]).

The dispersion relation, Eq. (3.20), reduces to a quadratic equation in s^2

$$s^4\left[(1-\omega_o^2)-\Omega^2(1-\omega_o^2\cos^2\theta)\right]$$

$$+\ s^2\beta_o^2\left[-2(1-\omega_o^2)^2+\Omega^2(2-\omega_o^2\cos^2\theta-\omega_o^2)\right]$$

$$+\beta_o^4(1-\omega_o^2)\left[(1-\omega_o^2)^2-\Omega^2\right] = 0. \tag{3.26}$$

This equation shows that s^2 has one root equal to zero at

$$\omega_o^2 = 1\,, \qquad \omega_o^2 = 1\pm\Omega \tag{3.27}$$

and also one root goes to infinity at

$$\omega_o^2 = \frac{1-\Omega^2}{1-\Omega^2\cos^2\theta}\,. \tag{3.28}$$

The two roots which characterize the two components of the electromagnetic wave are

$$s^2 = \beta_o^2\left[1-\frac{\omega_o^2}{1-\dfrac{\Omega^2\sin^2\theta}{2(1-\omega_o^2)}+\left[\dfrac{\Omega^4\sin^4\theta}{4(1-\omega_o^2)^2}+\Omega^2\cos^2\theta\right]^{1/2}}\right] \tag{3.29}$$

$$s^2 = \beta_o^2\left[1-\frac{\omega_o^2}{1-\dfrac{\Omega^2\sin^2\theta}{2(1-\omega_o^2)}-\left[\dfrac{\Omega^4\sin^4\theta}{4(1-\omega_o^2)^2}+\Omega^2\cos^2\theta\right]^{1/2}}\right] \tag{3.30}$$

These are the Appleton-Hartree formula for the collisionless case.

If the propagation direction is perpendicular to the direction of the constant magnetic field, $\theta=\frac{\pi}{2}$, then the formula (3.29) becomes

$$s^2 = \beta_o^2\left[1-\omega_o^2\right]\,, \tag{3.31}$$

and the formula (3. 30) becomes

$$s^2 = \beta_o^2 \left[1 - \frac{\omega_o^2 (1-\omega_o^2)}{1-\omega_o^2-\Omega^2} \right] \tag{3.32}$$

The wave characterized by Eq. (3. 31) yields the same propagation constant given by Eq. (3. 23) which is unaffected by the magnetic field, and is for this reason, usually called the ordinary wave. The other wave characterized by Eq. (3. 32) is usually called the extraordinary wave. Equations (3. 31) and (3. 32) go to zero and infinity at those points given by Eqs. (3. 27) and (3. 28).

Conventionally (Ratcliffe[(2)]), for $\theta \neq \frac{\pi}{2}$, the wave characterized by Eq. (3. 29) is also called the ordinary wave, and the wave characterized by Eq. (3. 30) is called the extraordinary wave, but this definition breaks down at $\theta = 0$, since for the propagation parallel to the magnetic field, i. e. $\theta = 0$, the formula (3. 29) yields

$$s^2 = \beta_o^2 \left[1 - \frac{\omega_o^2}{1+\Omega} \right] , \tag{3.33}$$

and the formula (3. 30) yields

$$s^2 = \beta_o^2 \left[1 - \frac{\omega_o^2}{1-\Omega} \right] , \tag{3.34}$$

which do not display the zero at $\omega_o^2 = 1$, nor the infinity which is given by (3. 28) also at $\omega_o^2 = 1$. This is because the process of obtaining Eqs. (3. 33) and (3. 34) from Eqs. (3. 29) and (3. 30) by setting $\theta = 0$ is not valid at $\omega_o^2 = 1$, since the dispersion relation (3. 26) has a common factor $(1-\omega_o^2)$ for $\theta = 0$. Still, it is true that Eq. (3. 33) corresponds to the ordinary wave and Eq. (3. 34) corresponds to the extraordinary wave for $\omega_o^2 < 1$. What happens near $\theta = 0$ is that the ordinary wave is taken over by the extraordinary wave completely at $\omega_o^2 = 1$, and only the extraordinary wave is present for $\omega_o^2 > 1$. The true state of affairs is illustrated more clearly by Fig. 1 and Fig. 2.

[C] Warm Plasma

The three roots of the dispersion relation, Eq. (3.20), will be expressed by Cardan's formula. These three roots characterize three types of waves. Or, more specifically, these three roots are the squares of the propagation constants for three types of waves.

One of these roots will reduce to the propagation constant square of the ordinary wave at $\theta = \frac{\pi}{2}$, thus, we will call the wave characterized by this special root the "modified ordinary wave". Another root will reduce to the propagation constant square of the plasma wave at $\theta = 0$, and we will call the wave characterized by this root the "modified plasma wave". Then, the wave characterized by the remaining third root should be called the "modified extraordinary wave". Since the electron plasma is assumed to be dissipationless, the three roots of Eq. (3.20), that is the squares of the propagation constants, are either positive or negative real numbers, corresponding to propagating or evanescent waves respectively. This fact can also be proved from the original ten-by-ten system matrix obtained from Eqs. (2.27), (2.28), (2.29) and (2.30) (Denisse and Delcroix[(8)]).

The mathematical details follow next. The dispersion relation (3.20) can be rewritten in the following standard form

$$s^6 + ps^4 + qs^2 + r = 0\,, \qquad (3.35)$$

where

$$p = -2\beta_o^2 + \beta_e^2\omega_o^2 + \frac{\beta_e^2(1-\Omega^2) - 2\omega_o^2\beta_o^2}{\Omega^2\cos^2\theta - 1} \tag{3.36}$$

$$q = \beta_o^2\left[\beta_o^2 - \beta_e^2\omega_o^2 + \frac{\beta_e^2(1-\Omega^2)(\omega_o^2-2) + 2\beta_e^2\omega_o^2(1-\omega_o^2) + \beta_o^2\omega_o^2(2-\omega_o^2)}{\Omega^2\cos^2\theta - 1}\right] \tag{3.37}$$

$$r = \frac{\beta_e^2\beta_o^4(1-\omega_o^2)\left[(1-\omega_o^2)^2 - \Omega^2\right]}{\Omega^2\cos^2\theta - 1} \quad . \tag{3.38}$$

The three wave propagation constants are given by the three roots of Eq. (3.35) as

$$k_1^2 = A + B - \frac{p}{3} \tag{3.39}$$

$$k_2^2 = -\frac{A+B}{2} + \frac{A-B}{2}\sqrt{-3} - \frac{p}{3} \tag{3.40}$$

$$k_3^2 = -\frac{A+B}{2} - \frac{A-B}{2}\sqrt{-3} - \frac{p}{3} \tag{3.41}$$

where

$$A = \left[-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right]^{1/3} \tag{3.42}$$

$$B = \left[-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right]^{1/3} \tag{3.43}$$

$$a = \frac{1}{3}(3q - p^2)\,, \qquad b = \frac{1}{27}(2p^3 - 9pq + 27r) \quad . \tag{3.44}$$

Also, there is the relation $3AB = -a$, which determines the choice of one of the three roots for A and B to be used in order to make the function "a" real. The expressions for the three roots as given by Eqs. (3.39), (3.40) and (3.41) in terms of the original coefficients of Eq. (3.20) are given in Appendix C.

For the case of transverse propagation, $\theta = \frac{\pi}{2}$, and so we have

$$p = -\left[\beta_e^2(1-\Omega^2-\omega_o^2)+2\beta_o^2(1-\omega_o^2)\right]$$

$$q = \beta_o^2\beta_e^2\left[(1-\omega_o^2)(1-\omega_o^2-\Omega^2)+(1-\omega_o^2)^2-\Omega^2\right]+\beta_o^4(1-\omega_o^2)^2$$

$$r = -\beta_o^4\beta_e^2(1-\omega_o^2)\left[(1-\Omega^2)-\omega_o^2(2-\omega_o^2)\right]$$

and

$$a = \frac{1}{3}\left\{2\beta_o^2\beta_e^2\left[(1-\omega_o^2)^2-\Omega^2\right]-\beta_o^2\beta_e^2\omega_o^2\Omega^2-\beta_e^4(1-\Omega^2-\omega_o^2)^2-\beta_o^4(1-\omega_o^2)^2\right\}$$

$$b = \frac{1}{27}\Big\{-3\beta_o^2\beta_e^4(1-\omega_o^2)(1-\Omega^2-\omega_o^2)^2-6\beta_o^4\beta_e^2(1-\omega_o^2)^2(1-\Omega^2-\omega_o^2)$$

$$+2\beta_o^6(1-\omega_o^2)^3-2\beta_e^6(1-\Omega^2-\omega_o^2)^3+9\beta_o^4\beta_e^2\Omega^2\omega_o^2(1-\omega_o^2)$$

$$+9\beta_o^2\beta_e^4(1-\Omega^2-\omega_o^2)\left[(1-\omega_o^2)^2-\Omega^2\right]\Big\}$$

$$\sqrt{\frac{b^2}{4}+\frac{a^3}{27}} = \frac{3\sqrt{-3}\,\beta_o^2\beta_e^2\omega_o^2\Omega^2}{27}\sqrt{4\beta_o^4\beta_e^4\left[\frac{(1-\omega_o^2-\Omega^2)}{\beta_o^2}+\frac{(1-\omega_o^2)}{\beta_e^2}\right]^2-\beta_o^2\beta_e^2\left[(1-\omega_o^2)^2-\Omega^2\right]}\;.$$

Thus we have

$$A^3 = -\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}$$

$$= \frac{1}{27}\Bigg\{-\beta_o^6(1-\omega_o^2)^3 + \beta_e^6(1-\omega_o^2-\Omega^2)^3 - \frac{3}{2}\beta_o^4\beta_e^2(1-\omega_o^2-\Omega^2)(1-\omega_o^2)^2$$

$$+\frac{3}{2}\beta_o^2\beta_e^4(1-\omega_o^2-\Omega^2)^2(1-\omega_o^2) - \frac{9}{2}\beta_o^2\beta_e^4(1-\omega_o^2-\Omega^2)\left[(1-\omega_o^2)^2-\Omega^2\right] + \frac{9}{2}\beta_o^4\beta_e^2(1-\omega_o^2)\left[(1-\omega_o^2)^2-\Omega^2\right]$$

$$+3\sqrt{-3}\,\beta_o^2\beta_e^2\omega_o^2\Omega^2\sqrt{4\beta_o^4\beta_e^4\left[\frac{(1-\omega_o^2-\Omega^2)}{\beta_o^2} + \frac{(1-\omega_o^2)}{\beta_e^2}\right]^2 - \beta_o^2\beta_e^2\left[(1-\omega_o^2)^2-\Omega^2\right]}\;\Bigg\}$$

$$A = \left(-\frac{1}{\sqrt{-3}} - \frac{1}{3}\right)\Bigg\{\frac{1}{2}\beta_o^2(1-\omega_o^2) - \frac{1}{4}\beta_o^2\beta_e^2\left[(1-\omega_o^2)\left(\frac{1}{\beta_o^2}+\frac{1}{\beta_e^2}\right) - \frac{\Omega^2}{\beta_o^2}\right]$$

$$-\frac{\sqrt{-3}}{2}\sqrt{\frac{1}{4}\beta_o^4\beta_e^4\left[(1-\omega_o^2)\left(\frac{1}{\beta_o^2}+\frac{1}{\beta_e^2}\right) - \frac{\Omega^2}{\beta_o^2}\right]^2 - \beta_o^2\beta_e^2\left[(1-\omega_o^2)^2-\Omega^2\right]}\Bigg\}$$

Similarly we can obtain

$$B = \left(-\frac{1}{3}+\frac{1}{\sqrt{-3}}\right)\left\{\frac{1}{2}\beta_o^2(1-\omega_o^2) - \frac{1}{4}\beta_o^2\beta_e^2\left[(1-\omega_o^2)\left(\frac{1}{\beta_o^2}+\frac{1}{\beta_e^2}\right)-\frac{\Omega^2}{\beta_o^2}\right]\right.$$

$$\left.+\frac{\sqrt{-3}}{2}\sqrt{\frac{1}{4}\beta_o^4\beta_e^4\left[(1-\omega_o^2)\left(\frac{1}{\beta_o^2}+\frac{1}{\beta_e^2}\right)-\frac{\Omega^2}{\beta_o^2}\right]^2-\beta_o^2\beta_e^2\left[(1-\omega_o^2)^2-\Omega^2\right]}\right\}.$$

Finally three propagation constants will be given by

$$s_I^2 = A + B - \frac{p}{3}$$

$$= \frac{1}{2}\beta_o^2\beta_e^2\left[(1-\omega_o^2)\left(\frac{1}{\beta_o^2}+\frac{1}{\beta_e^2}\right)-\frac{\Omega^2}{\beta_o^2}\right]$$

$$+\sqrt{\frac{1}{4}\beta_o^4\beta_e^4\left[(1-\omega_o^2)\left(\frac{1}{\beta_o^2}+\frac{1}{\beta_e^2}\right)-\frac{\Omega^2}{\beta_o^2}\right]^2-\beta_o^2\beta_e^2\left[(1-\omega_o^2)^2-\Omega^2\right]} \qquad (3.45)$$

$$s_{II}^2 = -\frac{A+B}{2}+\frac{A-B}{2}\sqrt{-3}-\frac{p}{3}$$

$$= \frac{1}{2}\beta_o^2\beta_e^2\left[(1-\omega_o^2)\left(\frac{1}{\beta_o^2}+\frac{1}{\beta_e^2}\right)-\frac{\Omega^2}{\beta_o^2}\right]$$

$$-\sqrt{\frac{1}{4}\beta_o^4\beta_e^4\left[(1-\omega_o^2)\left(\frac{1}{\beta_o^2}+\frac{1}{\beta_e^2}\right)-\frac{\Omega^2}{\beta_o^2}\right]^2-\beta_o^2\beta_e^2\left[(1-\omega_o^2)^2-\Omega^2\right]} \qquad (3.46)$$

$$s_{III}^2 = -\frac{A+B}{2}-\frac{A-B}{2}\sqrt{-3}-\frac{p}{3}$$

$$= \beta_o^2(1-\omega_o^2) \qquad (3.47)$$

Equation (3.47) gives the dispersion relation for the ordinary wave, and since Eq. (3.47) is obtained by applying the same formula $\left(-\frac{A+B}{2} - \frac{A-B}{2}\sqrt{-3} - \frac{p}{3}\right)$, which is used to define k_3^2 in Eq. (3.41), the wave with propagation constant k_3 will be called the "modified ordinary wave". Also, the Eqs. (3.45), (3.46) and (3.47) display all the zeros of the dispersion relation, (3.20), at $\omega_o^2 = 1$ and $\omega_o^2 = 1 \pm \Omega$. Thus, this process of defining the modified ordinary wave is similar to that of defining ordinary wave in magnetoionic theory, and there is no contradiction of this process in obtaining Eq. (3.47) from Eq. (3.41).

In the case of propagation along the magnetic field, $\theta = 0$, and we have

$$p = -\beta_e^2(1-\omega_o^2) - 2\beta_o^2 + \frac{2\beta_o^2\omega_o^2}{1-\Omega^2}$$

$$q = \beta_o^4 - \beta_o^4\omega_o^2\,\frac{2-\omega_o^2}{1-\Omega^2} - 2\beta_o^4\beta_e^2(1-\omega_o^2)\left[-\frac{1}{\beta_o^2} + \frac{\omega_o^2}{\beta_o^2(1-\Omega^2)}\right]$$

$$r = -\beta_o^4\beta_e^2(1-\omega_o^2)\left[1 - \frac{\omega_o^2(2-\omega_o^2)}{1-\Omega^2}\right]$$

$$a = -\frac{1}{3}\left\{\left[\beta_e^2(1-\omega_o^2) - \beta_o^2\,\frac{(1-\Omega^2-\omega_o^2)}{1-\Omega^2}\right]^2 + \frac{3\beta_o^4\omega_o^4\Omega^2}{(1-\Omega^2)^2}\right\}$$

$$b = \frac{2}{27}\left\{-\beta_e^{6}(1-\omega_o^{2})^{3}+\beta_o^{6}\,\frac{(1-\Omega^{2}-\omega_o^{2})\left[(1-\Omega^{2}-\omega_o^{2})^{2}-9\Omega^{2}\omega_o^{4}\right]}{(1-\Omega^{2})^{3}}\right.$$

$$-3\beta_o^{4}\beta_e^{2}\,\frac{(1-\omega_o^{2})\left[(1-\Omega^{2}-\omega_o^{2})^{2}-3\Omega^{2}\omega_o^{4}\right]}{(1-\Omega^{2})^{2}}$$

$$\left.+3\beta_o^{2}\beta_e^{4}\,\frac{(1-\omega_o^{2})^{2}(1-\Omega^{2}-\omega_o^{2})}{(1-\Omega^{2})}\right\}$$

$$\sqrt{\frac{b^{2}}{4}+\frac{a^{3}}{27}} = \frac{3\sqrt{-3}}{27}\left\{\frac{\beta_o^{6}\Omega\omega_o^{2}}{(1-\Omega^{2})^{3}}\left[(1-\Omega^{2}-\omega_o^{2})^{2}-\Omega^{2}\omega_o^{4}\right]\right.$$

$$-2\beta_o^{4}\beta_e^{2}\,\frac{\Omega\omega_o^{2}(1-\omega_o^{2})(1-\Omega^{2}-\omega_o^{2})}{(1-\Omega^{2})^{2}}$$

$$\left.+\beta_o^{2}\beta_e^{4}\,\frac{\Omega\omega_o^{2}(1-\omega_o^{2})^{2}}{(1-\Omega^{2})}\right\}$$

$$A = \frac{1}{3}\left[\beta_e^{2}(1-\omega_o^{2})-\beta_o^{2}\,\frac{1-\Omega^{2}-\omega_o^{2}}{1-\Omega^{2}}\right]-\frac{\beta_o^{2}\Omega\omega_o^{2}}{\sqrt{-3}(1-\Omega^{2})}$$

$$B = \frac{1}{3}\left[\beta_e^{2}(1-\omega_o^{2})-\beta_o^{2}\,\frac{1-\Omega^{2}-\omega_o^{2}}{1-\Omega^{2}}\right]+\frac{\beta_o^{2}\Omega\omega_o^{2}}{\sqrt{-3}(1-\Omega^{2})}\quad .$$

Thus, the three propagation constants for the three waves are given by

$$s_I^{2} = A+B-\frac{p}{3} = \beta_e^{2}(1-\omega_o^{2}) \tag{3.48}$$

$$s_{II}^{2} = -\frac{A+B}{2}+\frac{A-B}{2}\sqrt{-3}-\frac{p}{3} = \beta_o^{2}\left(\frac{1-\Omega-\omega_o^{2}}{1-\Omega}\right) \tag{3.49}$$

$$s_{III}^{2} = -\frac{A+B}{2}-\frac{A-B}{2}\sqrt{-3}-\frac{p}{3} = \beta_o^{2}\left(\frac{1+\Omega-\omega_o^{2}}{1+\Omega}\right) \tag{3.50}$$

The wave characterized by Eq. (3.48) is called the plasma wave as shown before, and since Eq. (3.48) is obtained by applying the same formula, $(A+B-\frac{p}{3})$, which has also been used to define k_1^2 in Eq. (3.39), the wave with propagation constant equal to k_1 will be called the "modified plasma wave". The remaining propagation constant k_3, then, should be identified as the propagation constant for the "modified extraordinary wave". The close relation between the wave characterized by Eq. (3.39) and the plasma wave can also be shown by making $\Omega \rightarrow 0$ in Eq. (3.45). This equation then reduces to the dispersion relation of the plasma wave as given by Eq. (3.48).

3.4 Characteristics of Waves

In Part A some characteristics of two types of waves involved in magnetoionic theory will be reviewed briefly which will be helpful in discussing the general case later. Magnetoionic theory has been investigated quite thoroughly by various authors (Ratcliffe[2], Budden[5]), and liberal use will be made of their results. In Part B the characteristics of the plasma wave have also been analyzed carefully, which should be helpful in understanding the modified plasma wave which is closely related to the plasma wave. In Part C a detailed analysis of the propagation constants as a function of the normalized plasma frequency will be given and the results presented in graphical form. The purpose of this analysis is to determine the conditions for which the various waves propagate and also to assist in clarifying the terminology developed in Section 3.3. In Part D the surfaces showing the variations of propagation constants with respect to propagation direction will be sought. These surfaces will be used to obtain the asymptotic solution for the radiation fields in the next chapter.

[A] Electromagnetic Waves in a Cold Plasma

The ordinary wave as characterized by Eq. (3.31) does not depend on the static magnetic field because it is linearly polarized with its electric field parallel to the static magnetic field. Hence, the electrons are forced to move only parallel to the static magnetic field, and the wave behaves as if the field were absent. The extraordinary wave as given by Eq. (3.32) is affected by the static magnetic field. For intermediate inclinations of the static magnetic field with respect to the propagation direction, θ is different from 0 or $\frac{\pi}{2}$, and the variations of the propagation constant as given by Eqs. (3.29) and (3.30) with ω_o^2 are shown by Fig. 1 and Fig. 2 (Budden[5]). The dotted lines show the limiting positions for $\theta = 0$ and $\theta = \frac{\pi}{2}$, and the thick lines are typical curves which always lie in the shaded regions bounded by the dotted lines. The dotted lines are the curves for Eqs. (3.31), (3.32), (3.33) and (3.34), and the line $\omega_o^2 = 1$. The thick curve marked 0 would deform continuously into the straight line for ordinary wave at $\theta = \frac{\pi}{2}$, and the thick curve marked x would deform into extradordinary wave at $\theta = \frac{\pi}{2}$. One value of s^2 is infinite when

$$\omega_o^2 = \frac{1-\Omega^2}{1-\Omega^2 \cos^2\theta} \tag{3.51}$$

Physically, both ω_o and s must be real, thus graphically, the region of interest to wave propagation is confined to the first quadrant.

[B] Plasma Wave

The restoring force of the plasma wave is electrostatic, and the limiting case of very low electron temperature was studied by Tonks and Langmuir[42]. They have derived the plasma frequency, ω_p, with which the electrons oscillate

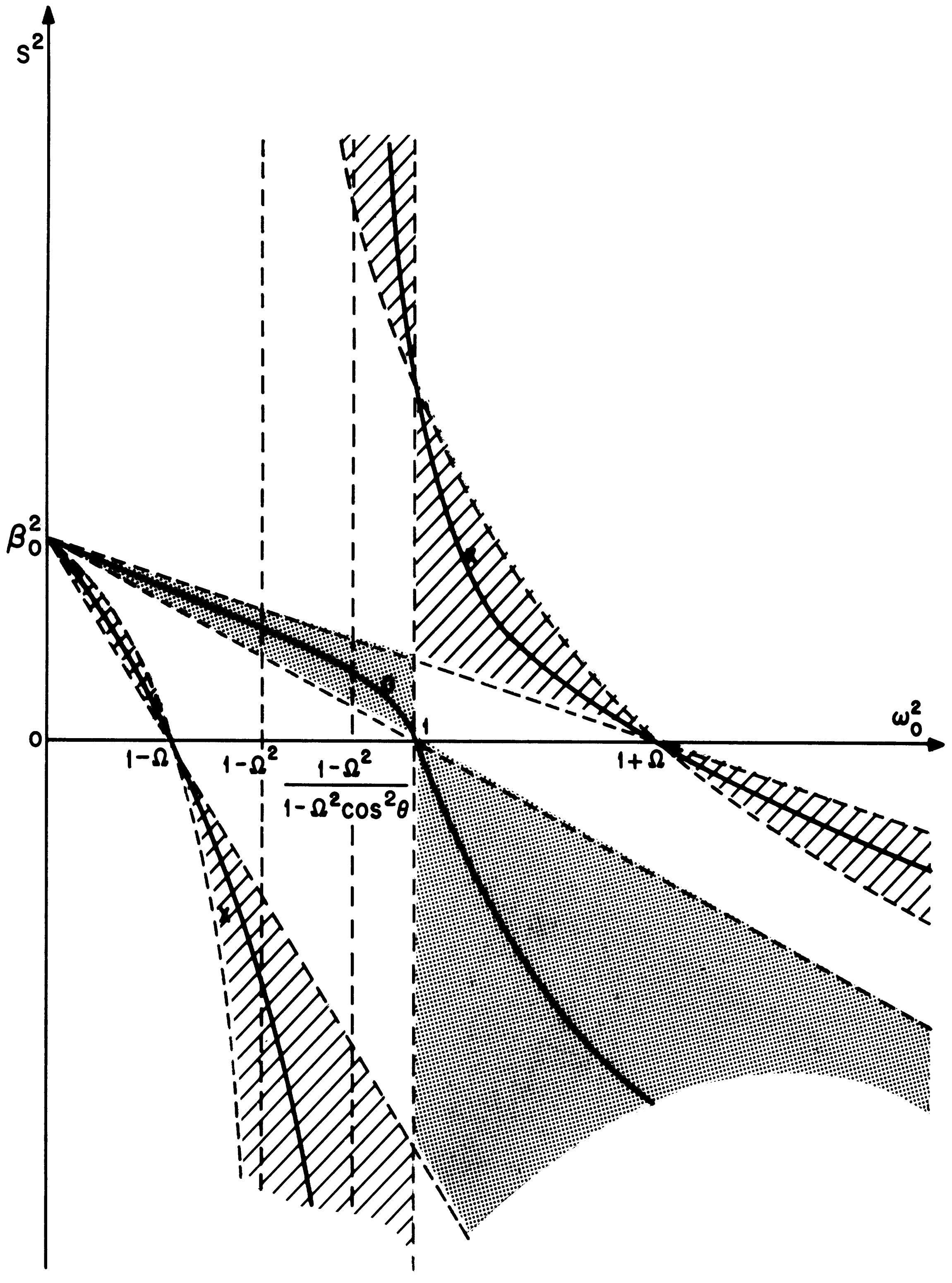


FIG. 1: (PROPAGATION CONSTANT)^2 VS. ω_p^2/ω^2

$\Omega < 1$, $0 < \theta < \pi/2$, $U \rightarrow 0$

Fig. 1.

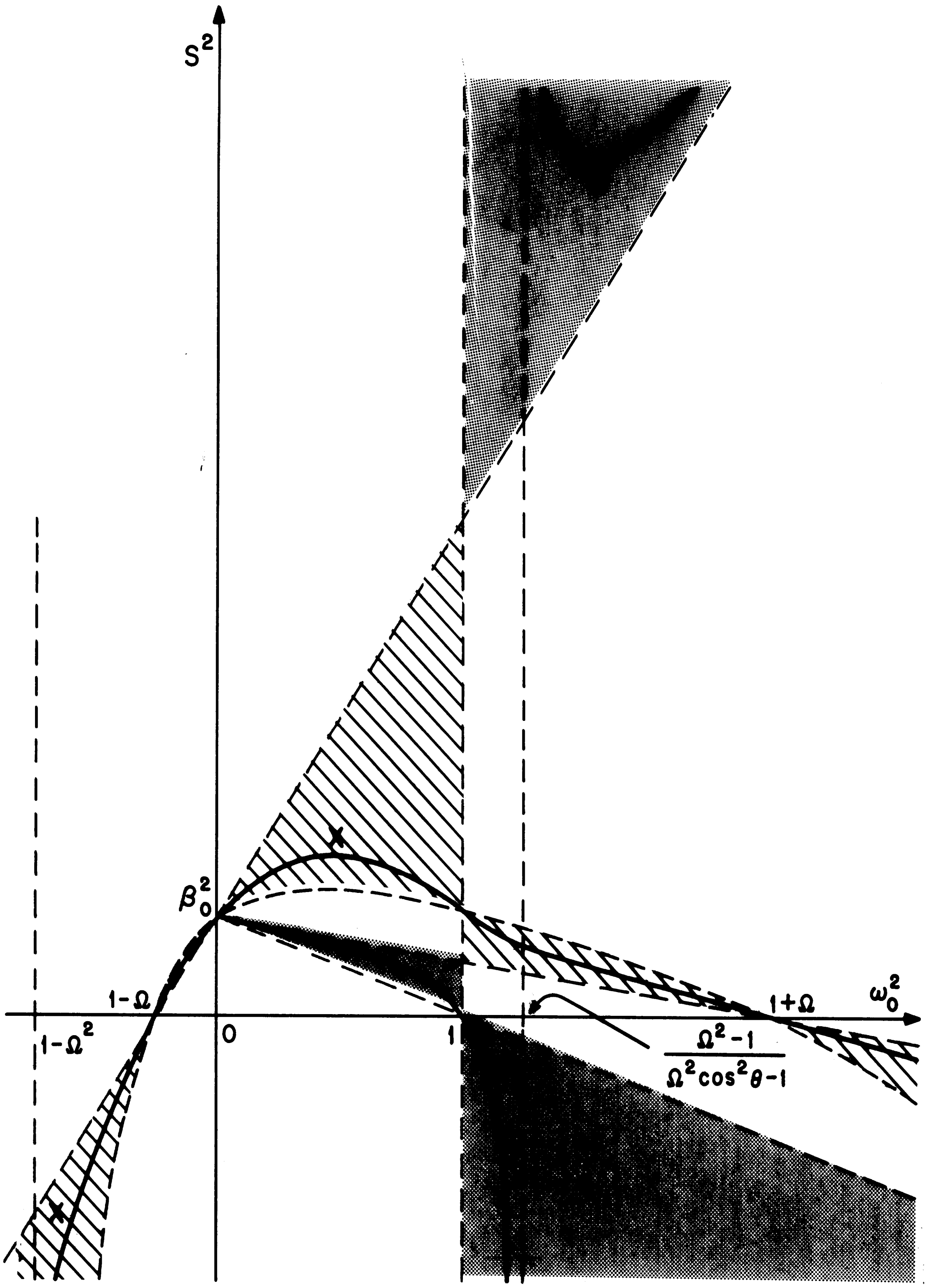


FIG. 2: (PROPAGATION CONSTANT)^2 VS. ω_p^2/ω^2

$\Omega > 1, \quad 0 < \theta < \pi/2, \quad U \rightarrow 0$

regardless of wavelength. Bohm and Gross[43] have given the microscopic analysis of plasma oscillation and obtained the dispersion relation for the plane plasma wave as given by Eq. (3.24). Or, more exactly, the sound velocity, U, in Eq. (3.24) is given by the expression $[\gamma k_B T/m]^{1/2}$, where γ is the ratio of the specific heat at constant pressure to that at constant volume ($\gamma = 3$ for electron gas), and k_B is Boltzmann constant. The plasma wave is a longitudinal wave, in which $\bar{E}$ and $\bar{V}$ are parallel to the direction of propagation. It resembles the sound waves that propagate in a neutral gas, but there exists a fundamental difference between the two waves (Denisse and Delcroix[8]). The former is supported by short range incoherent collisions, while the latter is supported by the coupling between the charged particles provided by the electrostatic field. The range of the forces due to the electrostatic field is limited only by the Debye length.

Since the phase velocity v_p of the plasma wave is given by

$$v_p = \frac{\omega}{s} = U/(1-\omega_o^2)^{1/2} \; , \tag{3.52}$$

and the group velocity v_g is given by

$$v_g = \frac{d\omega}{ds} = U(1-\omega_o^2)^{1/2} \; , \tag{3.53}$$

their variations as a function of ω_o^2 are given by Fig. 3. Both velocities will approach the sound velocity in the electron gas, U, when the frequency ω approaches infinity. Also, the two velocities have the order of magnitude of the sound wave in the electron gas, and are related by

$$v_p v_g = U^2 \tag{3.54}$$

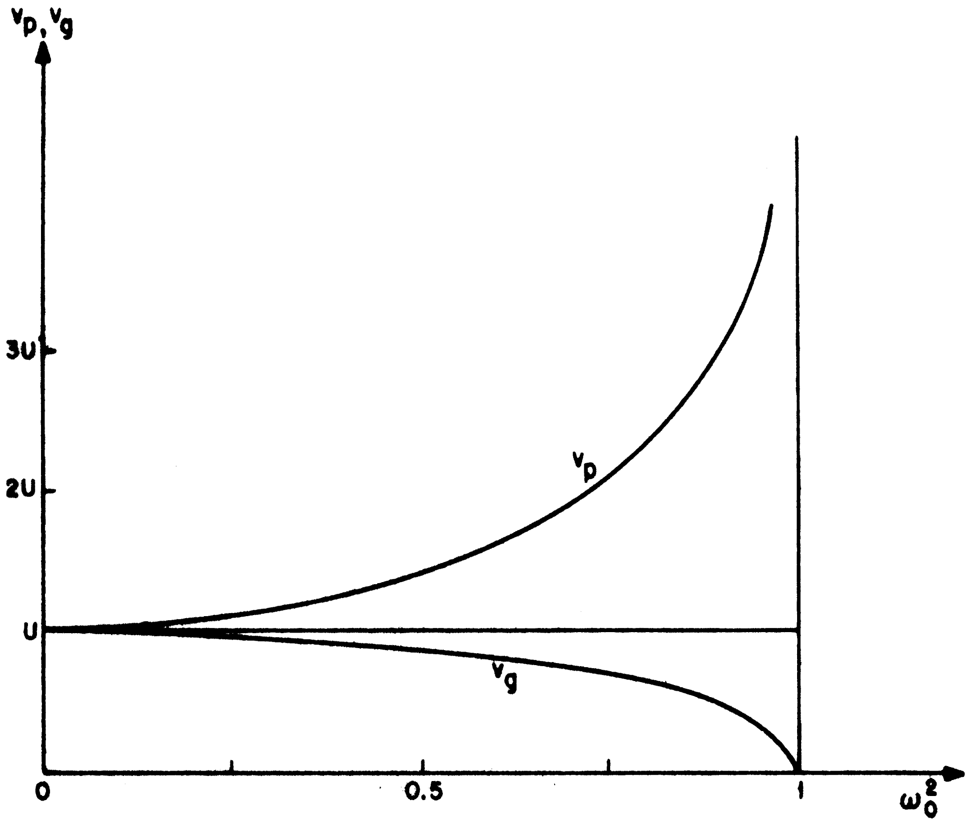


FIG. 3: PHASE VELOCITY AND GROUP VELOCITY VS. ω_o^2

[c] **Propagation Constants vs. $\omega_o^2 = \omega_p^2/\omega^2$**

In this part the dispersion relation will be analyzed as a function of the normalized plasma frequency squared for various values of the normalized gyro-frequency. $\Omega=0$ is considered first which corresponds to the situation where there is no magnetic field. $\Omega < 1$ and $\Omega > 1$ are considered next which corresponds respectively to the case when the gyro-frequency is less than and larger than the angular frequency of the waves. Finally, $\Omega = 1$ is considered which corresponds to the case when the angular frequency is equal to the gyro-frequency.

The results of this analysis are presented in graphical form in Fig. 4 through Fig. 13. In these graphs the square of the propagation constants are displayed as functions of ω_o^2. As explained with respect to Fig. 1 and Fig. 2, we are interested in only the first quadrant in these graphs since only those waves, whose propagation constants are in this region can propagate, and physically there is no negative ω_o^2. We will plot and analyze k_1^2, k_2^2 and k_3^2 in a manner similar to that used to obtain Fig. 1 and Fig. 2. Thus, the two limiting cases, $\theta = 0$ and $\theta = \frac{\pi}{2}$, corresponding respectively to longitudinal and transverse propagation, will be analyzed first. The curves for these two limiting cases provide the boundary lines for the shaded areas where k_1^2, k_2^2 and k_3^2 always lie.

a) $\Omega = 0$:

Physically, this is the limiting case with no static magnetic field which was discussed in Sec. 3.3, and the dispersion relation is given by Eq. (3.22) as

$$\left[s^2 - \beta_o^2(1-\omega_o^2)\right]^2 \left[s^2 - \beta_e^2(1-\omega_o^2)\right] = 0 .$$

Variations of s^2 with respect to $\omega_o^2 = \frac{\omega_p^2}{\omega^2}$ can be seen from Fig. 4.

Practically, the acoustic velocity is, of course, much less than the velocity of light, thus

$$\beta_e^2 >> \beta_o^2 . \tag{3.55}$$

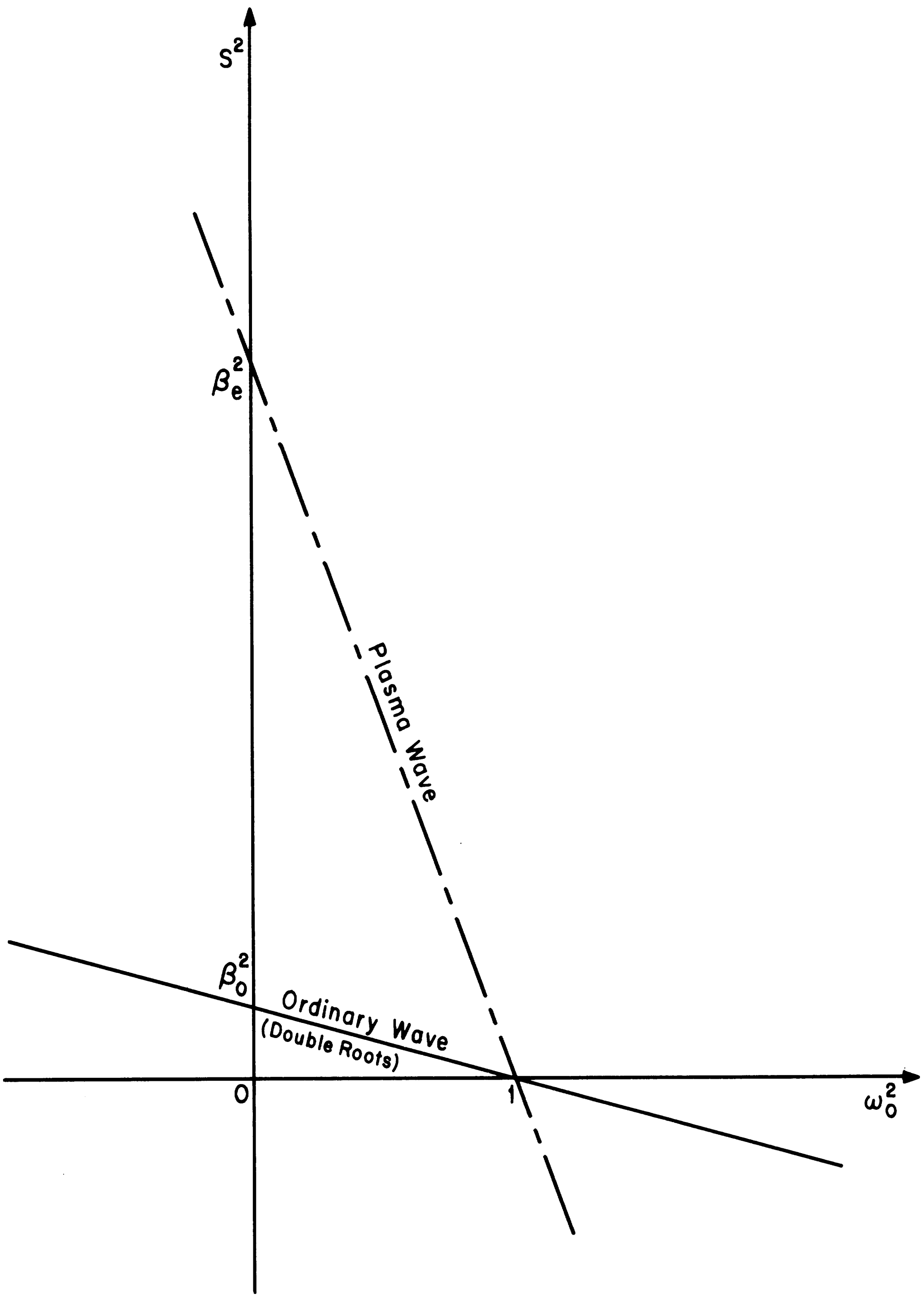


FIG. 4: (PROPAGATION CONSTANT)^2 VS. ω_p^2/ω^2, $\Omega = 0$

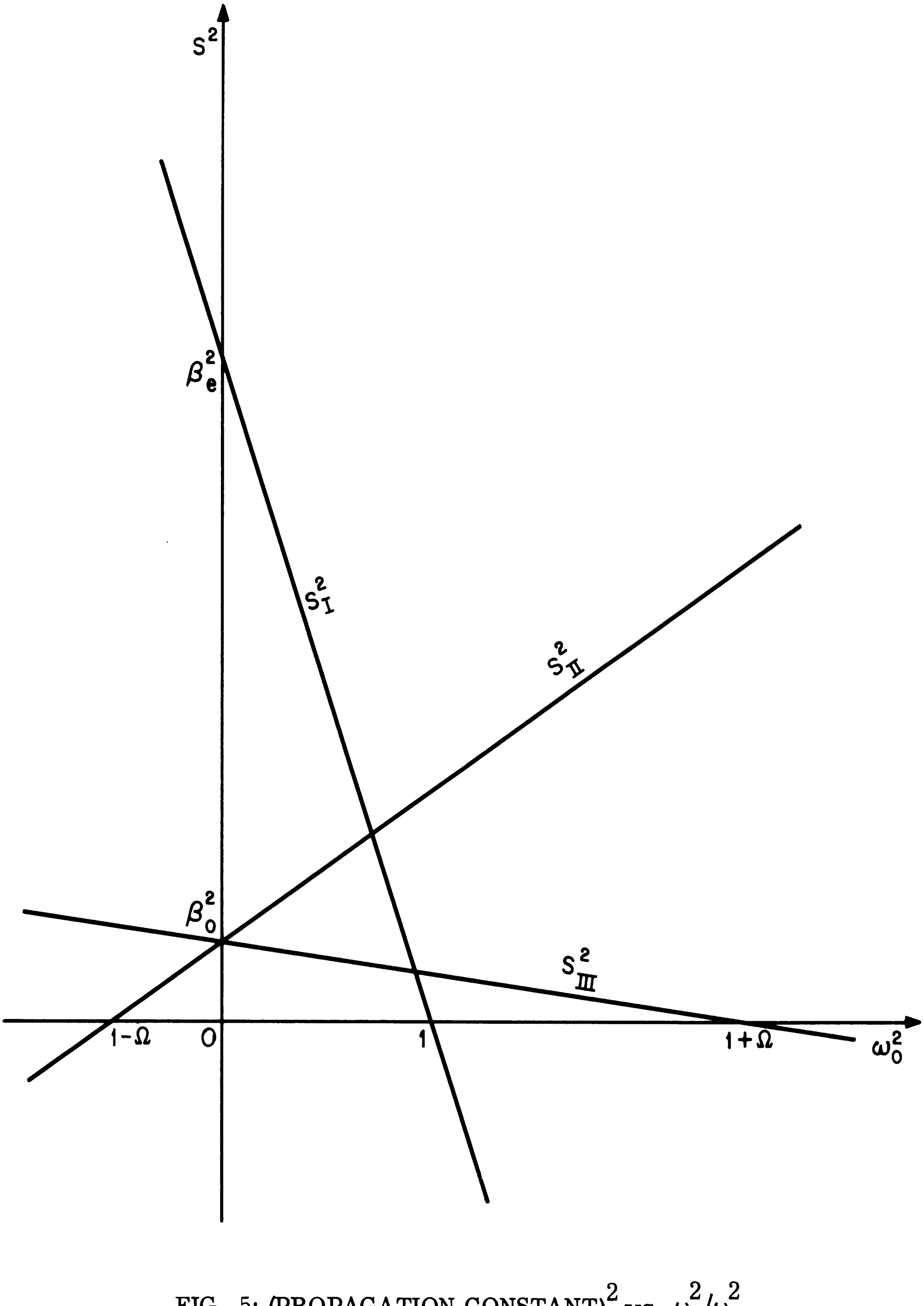


FIG. 5: (PROPAGATION CONSTANT)^2 VS. ω_p^2/ω^2

$\Omega > 1, \quad \theta = 0$

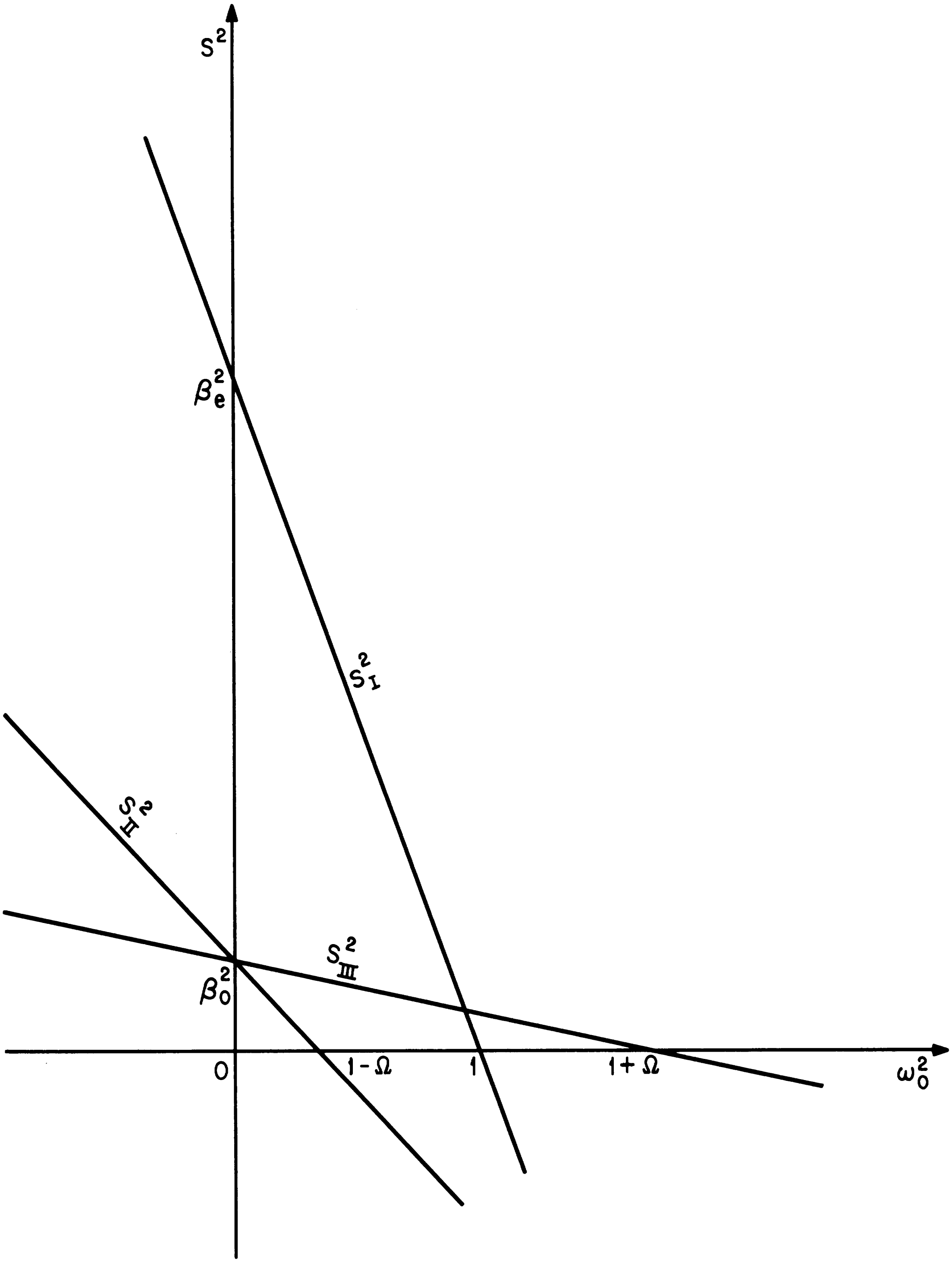


FIG. 6: (PROPAGATION CONSTANT)^2 VS. ω_p^2/ω^2

$\Omega < 1, \quad \theta = 0$

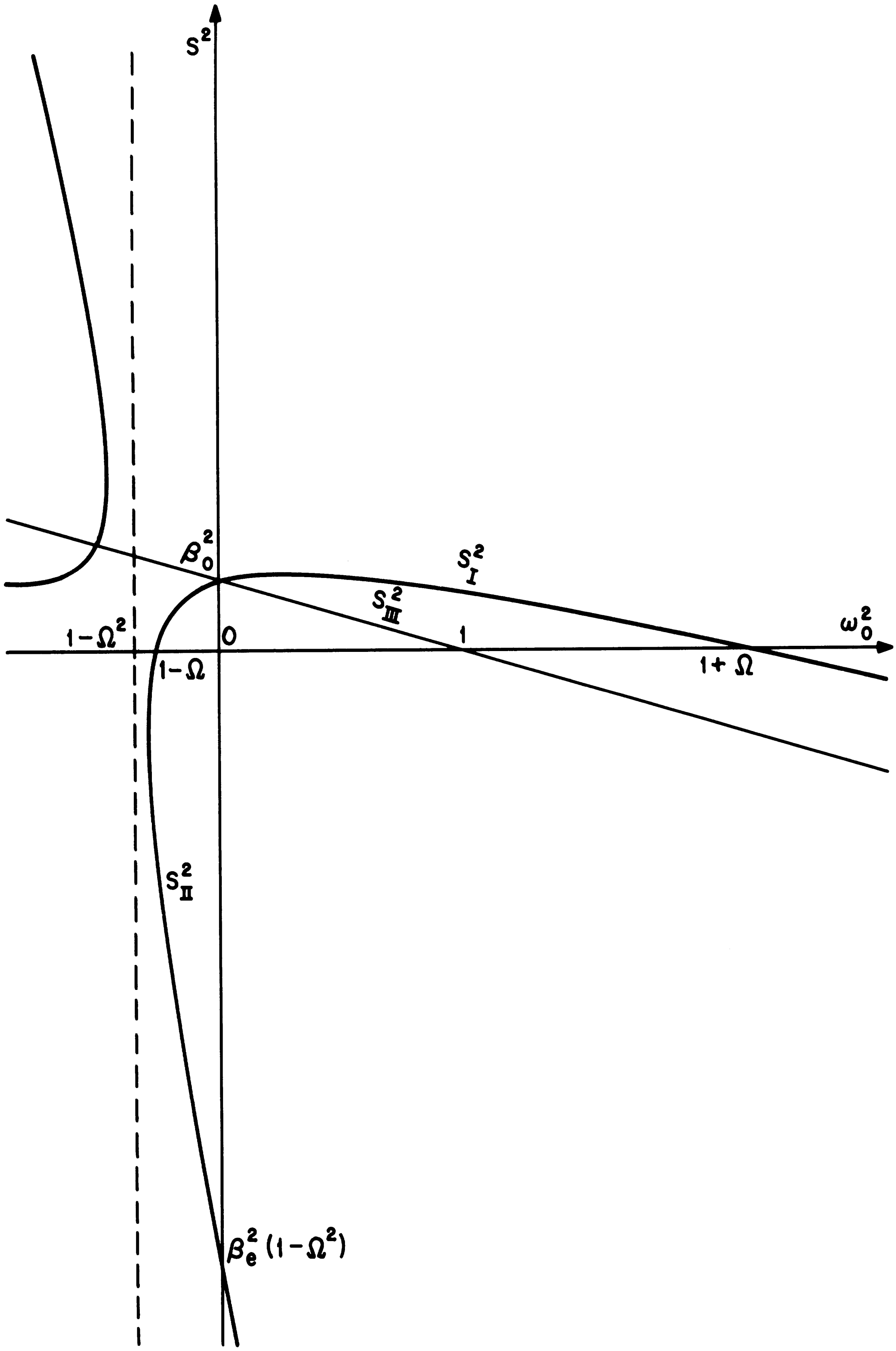


FIG. 7: (PROPAGATION CONSTANT)^2 VS. ω_p^2/ω^2 $\Omega > 1$, $\theta = \pi/2$

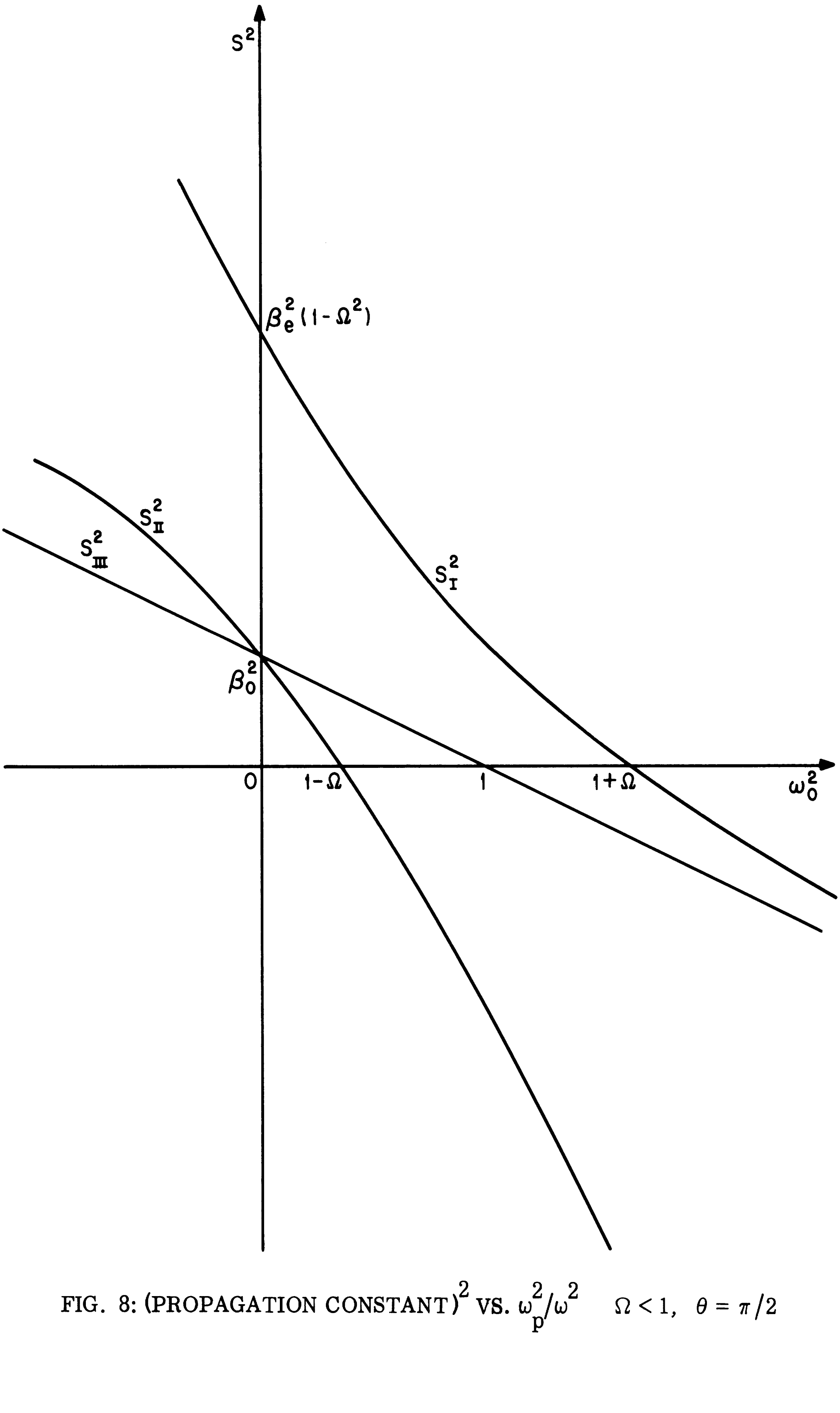


FIG. 8: (PROPAGATION CONSTANT)^2 VS. ω_p^2/ω^2 $\quad \Omega < 1, \quad \theta = \pi/2$

Fig. 8

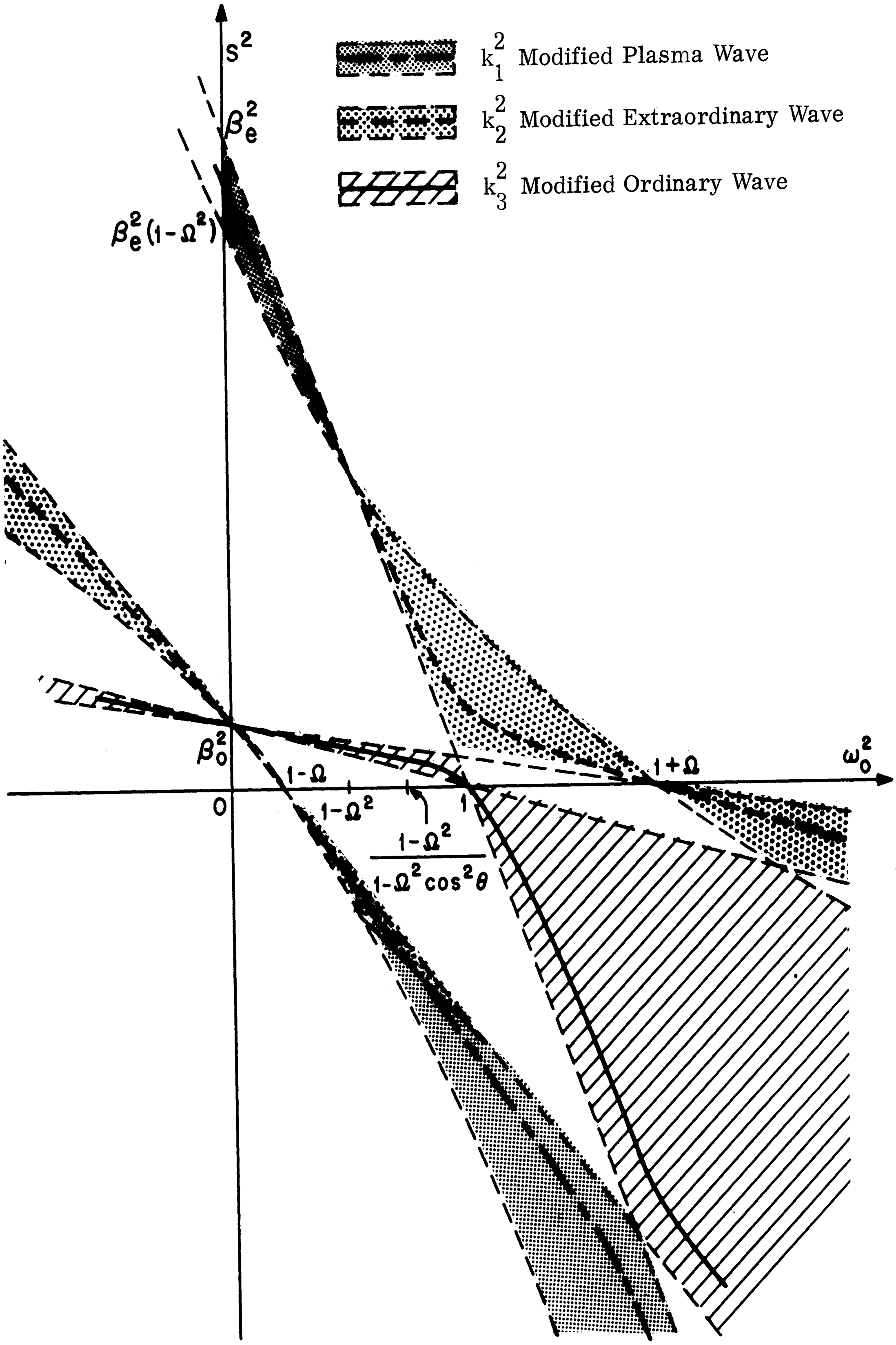


FIG. 9: (PROPAGATION CONSTANT)^2 VS. ω_p^2/ω^2

$\Omega < 1, \quad 0 < \theta < \pi/2$

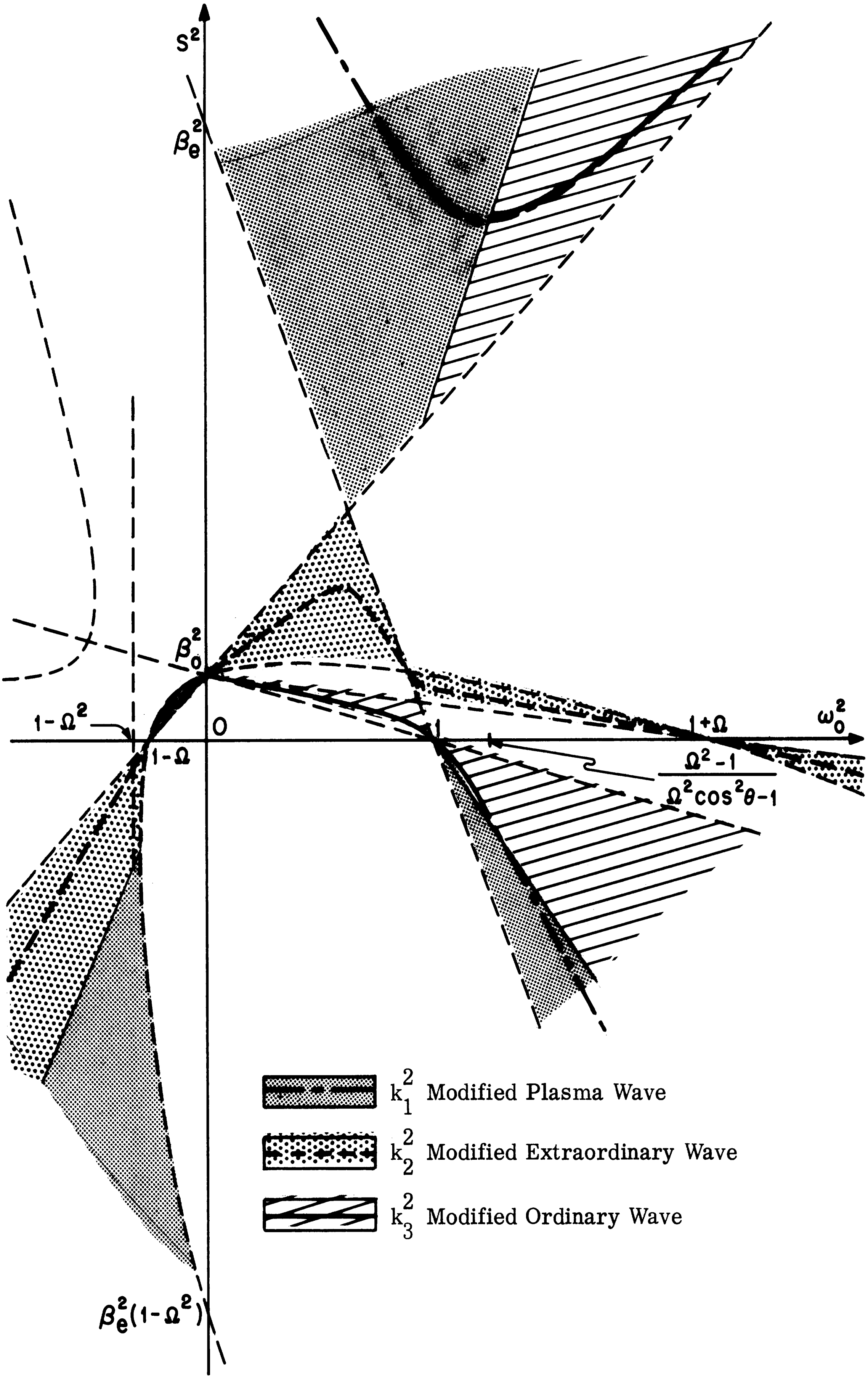


FIG. 10: (PROPAGATION CONSTANT)^2 VS. ω_p^2/ω^2

$\Omega > 1, \quad 0 < \theta < \pi/2$

Fig 10

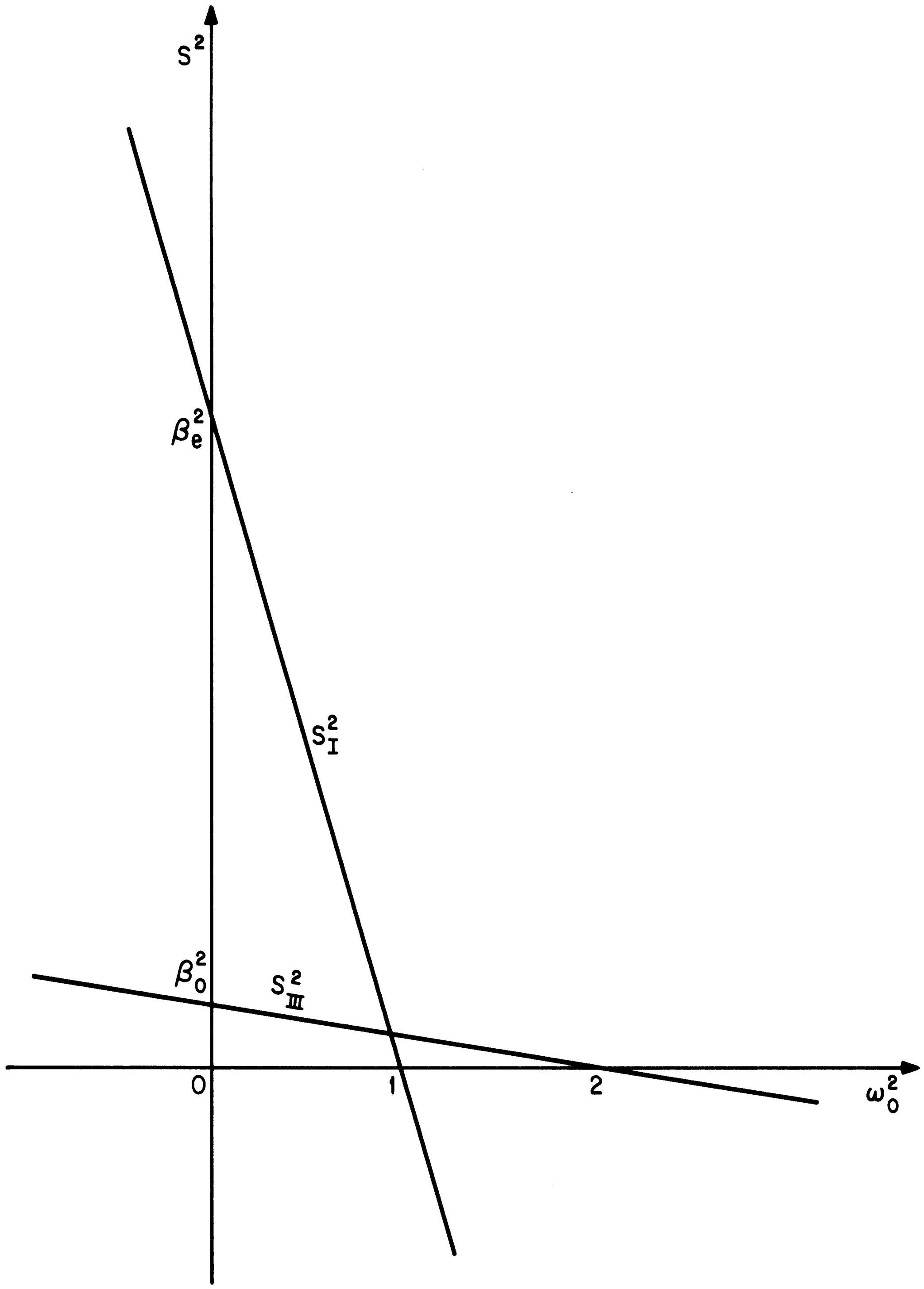


FIG. 11: (PROPAGATION CONSTANT)^2 VS. ω_p^2/ω^2

$\Omega = 1, \quad \theta = 0$

Fig. 11.

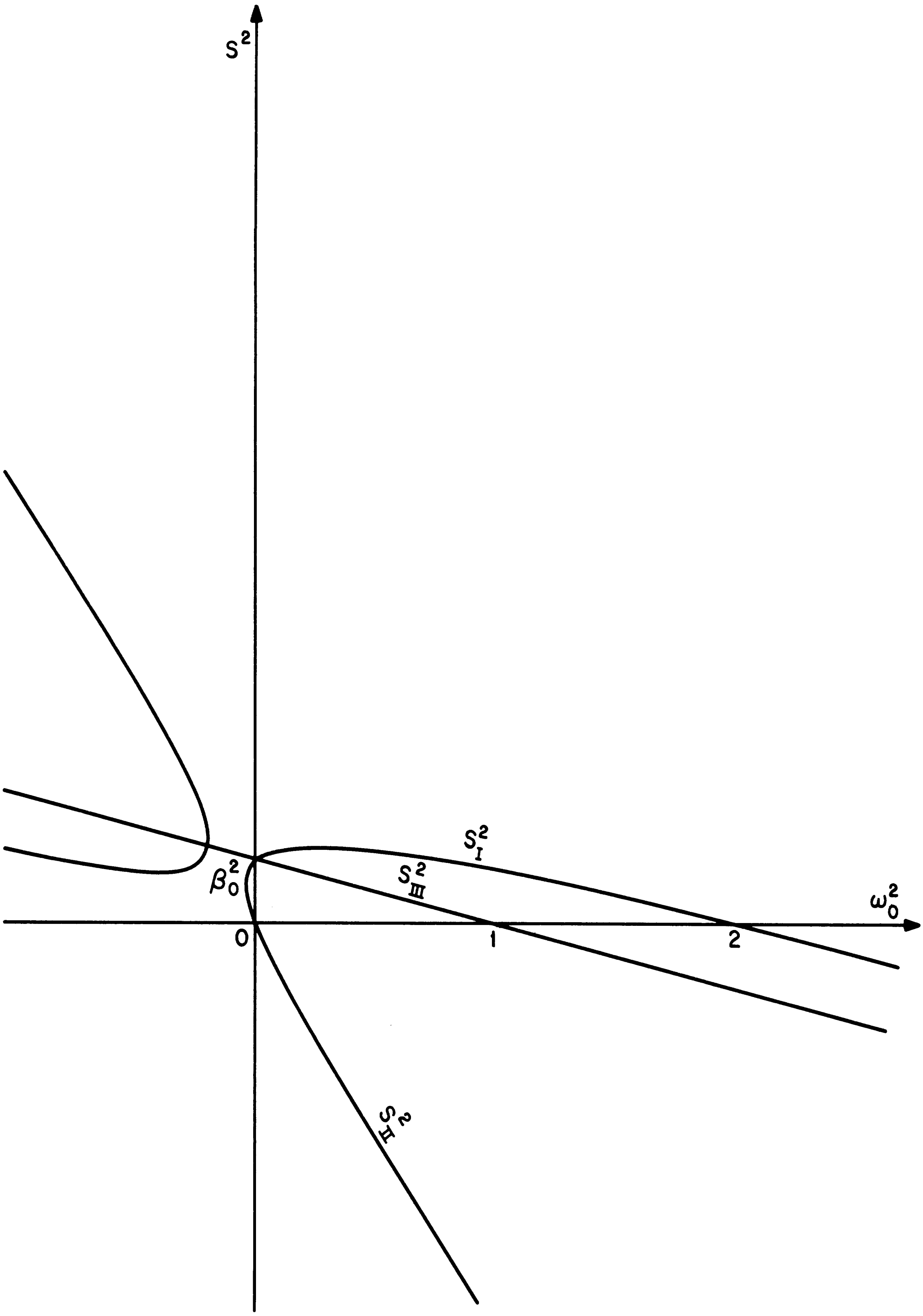


FIG. 12: (PROPAGATION CONSTANT)^2 VS. ω_p^2/ω^2

$\Omega = 1, \quad \theta = \pi/2$

Fig 12

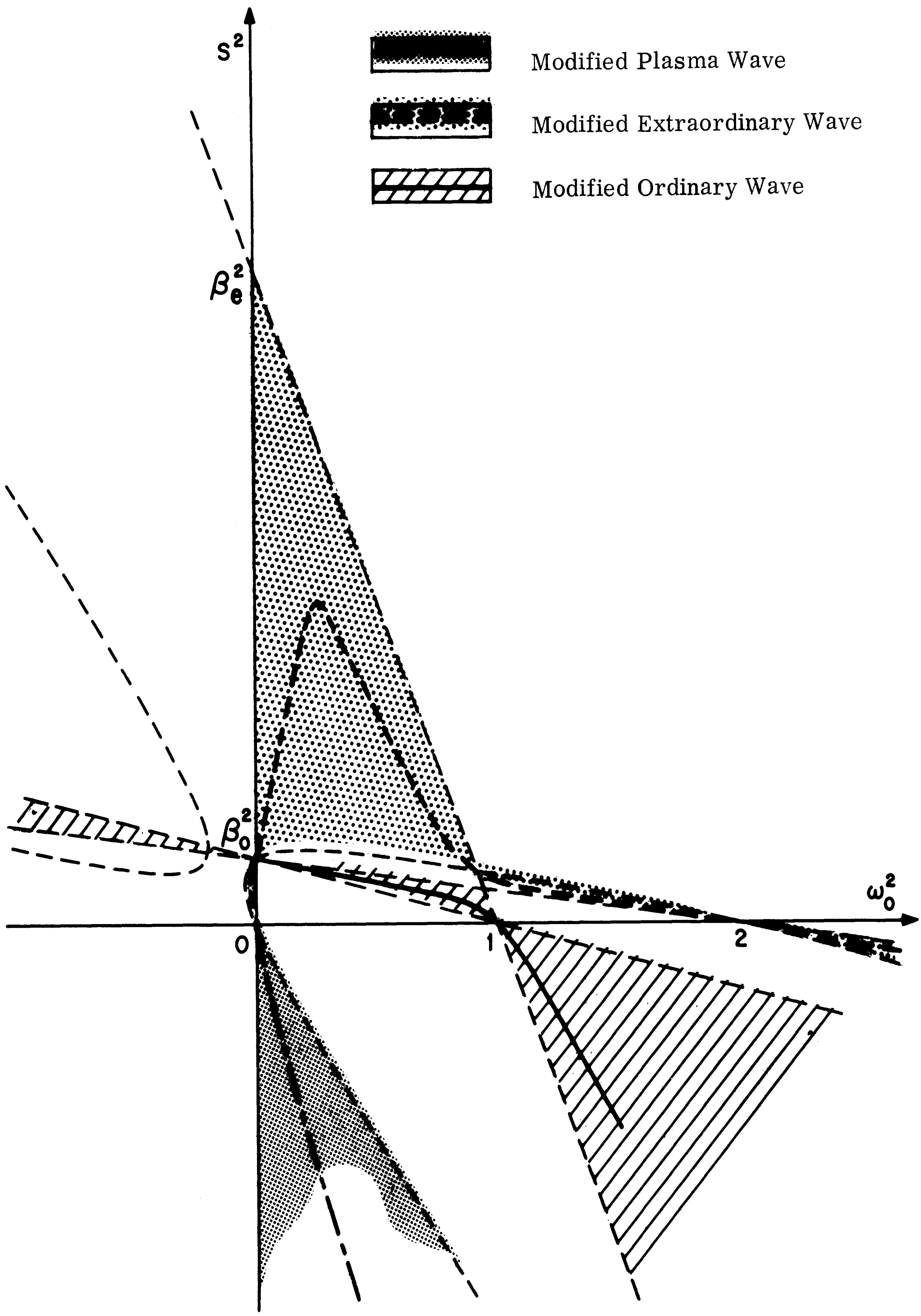


FIG. 13: (PROPAGATION CONSTANT)^2 VS. ω_p^2/ω^2

$\Omega = 1, \quad 0 < \theta < \pi/2$

Fig. 13

However, in order to illustrate the important features of the analysis of the waves propagating in an electron gas on the graphs the value of β_e^2 will be chosen such that

$$\beta_e^2 = 10\beta_o^2 \quad .$$

b) $\Omega > 1$ and $\Omega < 1$:

The two most general situations are for $\Omega > 1$ and $\Omega < 1$. The three propagation constants for the modified plasma wave, modified ordinary wave and modified extraordinary wave for these two situation are shown in Fig. 9 and Fig. 10 respectively.

Two special cases will be discussed first. The third case is the general case.

Case I: Propagation Along the Magnetic Field ($\theta = 0$)

Three propagation constants are given by Eqs. (3.48), (3.49) and (3.50). Their graphs are plotted in Fig. 5 and Fig. 6. s_I is the propagation constant for the plasma wave and s_{II} and s_{III} are the propagation constants for the two electromagnetic waves. As mentioned in Sec. 3.3, the distinction between the ordinary and extraordinary wave is not clear in this case, but in keeping with the definition used in this work (Denisse and Delcroix[8]) s_{II} is the propagation constant for the modified extraordinary wave and s_{III} the propagation constant for the modified ordinary wave.

Case II: Propagation Across the Magnetic Field ($\theta = \frac{\pi}{2}$)

The three propagation constants in this case are given by Eqs. (3.45), (3.46) and (3.47). The two roots s_I^2 and s_{II}^2 given by Eqs. (3.45) and (3.46) can be combined to yield an equation of second degree in s^2 and ω_o^2 which represents a conical section. The equation is

$$(s^2)^2 + (\beta_o^2+\beta_e^2)s^2\omega_o^2 + \beta_o^2\beta_e^2(\omega_o^2)^2 + s^2\left[\beta_e^2(\Omega^2-1) - \beta_o^2\right]$$

$$-2\omega_o^2\beta_o^2\beta_e^2 + \beta_o^2\beta_e^2(1-\Omega^2) = 0 \ . \tag{3.56}$$

Equation (3.56) can be analyzed by considering it as the general equation of second degree

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \quad , \tag{3.57}$$

where $x = \omega_o^2$ and $y = s^2$. Since the following inequality

$$B^2 - 4AC = (\beta_o^2 - \beta_e^2)^2 > 0$$

is satisfied, this conical section is a hyperbola.

When $\omega_o^2 = 0$,

$$s_{II}^2 = \begin{cases} \beta_o^2 & \text{when } \Omega^2 < 1 - \dfrac{\beta_o^2}{\beta_e^2} \approx 1 \\[2ex] \beta_e^2(1-\Omega^2) & \text{when } \Omega^2 > 1 - \dfrac{\beta_o^2}{\beta_e^2} \approx 1 \end{cases}$$

$$s_I^2 = \begin{cases} \beta_e^2(1-\Omega^2) & \text{when } \Omega^2 < 1 - \dfrac{\beta_o^2}{\beta_e^2} \approx 1 \\[2ex] \beta_o^2 & \text{when } \Omega^2 > 1 - \dfrac{\beta_o^2}{\beta_e^2} \approx 1 \end{cases}$$

Two points of intersection with the ω_o^2 - axis are obtained at $\omega_o^2 = 1 \pm \Omega$. If

$1 - \omega_o^2 = -\Omega$,

$$\frac{\beta_o^4\beta_e^4}{4}\left[-\Omega\left(\frac{1}{\beta_o^2}+\frac{1}{\beta_e^2}\right)-\frac{\Omega^2}{\beta_o^2}\right]^2 = \frac{\beta_o^2\beta_e^2}{2}\left[\Omega\left(\frac{1}{\beta_o^2}+\frac{1}{\beta_e^2}\right)+\frac{\Omega^2}{\beta_o^2}\right]$$

and so $s_I^2 = 0$ at this point. At another point $1-\omega_o^2 = \Omega$

$$(1-\omega_o^2)\left(\frac{1}{\beta_o^2}+\frac{1}{\beta_e^2}\right)-\frac{\Omega^2}{\beta_o^2} = \frac{\Omega}{\beta_o^2}\left[\frac{\beta_o^2}{\beta_e^2}+(1-\Omega)\right]$$

and we can see that

$$\begin{cases} s_{II}^2 = 0 \quad \text{when}\ \Omega < 1 \\ s_I^2 = 0 \quad \text{when}\ \Omega > 1 + \dfrac{\beta_o^2}{\beta_e^2} \approx 1 \end{cases}$$

The detailed analysis of the slope of the hyperbola, and the coordinate transformation applied to Eq. (3.56) in order to reduce it to the standard hyperbolic equation, is given in Appendix D.

The standard hyperbolic equation expressed in the new coordinate system $(\omega_o'^2 - s'^2)$ is

$$\frac{(1+\beta_o^2\beta_e^2)\sqrt{1+\beta_o^4\beta_e^4+\beta_o^4+\beta_e^4}+(1+\beta_o^4\beta_e^4+\beta_o^4+\beta_e^4)}{2\sqrt{1+\beta_o^4\beta_e^4+\beta_o^4+\beta_e^4}}\left\{(\omega_o'^2)+\frac{\left[\beta_e^2(\Omega^2-1)-\beta_o^2\right]\sqrt{1-\beta_o^2\beta_e^2+\sqrt{1+\beta_o^4\beta_e^4+\beta_o^4+\beta_e^4}}-2\beta_o^2\beta_e^2\sqrt{-1+\beta_o^2\beta_e^2+\sqrt{1+\beta_o^4\beta_e^4+\beta_o^4+\beta_e^4}}}{\left(1+\beta_o^2\beta_e^2+\sqrt{1+\beta_o^4\beta_e^4+\beta_o^4+\beta_e^4}\right)\sqrt{2\sqrt{1+\beta_o^4\beta_e^4+\beta_o^4+\beta_e^4}}}\right\}^2$$

$$-\frac{(1+\beta_o^4\beta_e^4+\beta_o^4+\beta_e^4)-(1+\beta_o^2\beta_e^2)\sqrt{1+\beta_o^4\beta_e^4+\beta_o^4+\beta_e^4}}{2\sqrt{1+\beta_o^4\beta_e^4+\beta_o^4+\beta_e^4}}\left\{(s'^2)-\frac{\left[\beta_e^2(\Omega^2-1)-\beta_o^2\right]\sqrt{-1+\beta_o^2\beta_e^2+\sqrt{1+\beta_o^4\beta_e^4+\beta_o^4+\beta_e^4}}+2\beta_o^2\beta_e^2\sqrt{1-\beta_o^2\beta_e^2+\sqrt{1+\beta_o^4\beta_e^4+\beta_o^4+\beta_e^4}}}{\left[\sqrt{1+\beta_o^4\beta_e^4+\beta_o^4+\beta_e^4}-(1+\beta_o^2\beta_e^2)\right]\sqrt{2\sqrt{1+\beta_o^4\beta_e^4+\beta_o^4+\beta_e^4}}}\right\}^2$$

$$=\frac{\beta_o^2\beta_e^6\Omega^2}{(\beta_o^2-\beta_e^2)^2}\left\{\left(1-\frac{\beta_o^2}{\beta_e^2}\right)^2-\Omega^2\right\} \qquad (3.58)$$

The transverse axis of the hyperbola is parallel to s'^2 - axis when

$$\Omega > 1 - \frac{\beta_o^2}{\beta_e^2} \approx 1$$

and the transverse axis is parallel to $\omega_o'^2$ - axis when

$$\Omega < 1 - \frac{\beta_o^2}{\beta_e^2} \approx 1 \ .$$

Additional descriptions of the hyperbola which can be seen from Eq. (3.58) are itemized as follows:

(i) when $\Omega^2 < 1 + \frac{\beta_o^2}{\beta_e^2} \approx 1$, the center of the hyperbola translates to positive $\omega_o'^2$-axis.

(ii) When $\Omega^2 > 1 + \frac{\beta_o^2}{\beta_e^2} \approx 1$, the center of the hyperbola translates to positive s'^2 - axis.

(iii) The slopes of the asymptotes in $\omega_o'^2 - s'^2$ plane have the absolute value

$$\left[\frac{(1+\beta_o^2\beta_e^2) + \sqrt{1+\beta_o^4\beta_e^4+\beta_o^4+\beta_e^4}}{-(1+\beta_o^2\beta_e^2) + \sqrt{1+\beta_o^4\beta_e^4+\beta_o^4+\beta_e^4}}\right]^{1/2} > 1$$

The hyperbola constructed from the two propagation constants s_I^2 and s_{II}^2 and the propagation constant for the ordinary wave s_{III}^2 are plotted in Fig. 7 and Fig. 8.

Case III: Intermediate Inclination $\left(0 < \theta < \frac{\pi}{2}\right)$

The numerical values of three propagation constants given by Eqs. (3.39), (3.40) and (3.41) could, of course, be obtained directly from these equations. However, such a procedure is tedious and the important features for this case can be obtained by an approximate analysis using the well known results for the two types of waves in cold plasma as shown in Fig. 1 and Fig. 2. The main difficulty in this analysis is to determine the points of discontinuity for three propagation constants, where the transition of modified plasma wave to modified extraordinary wave takes place in Fig. 9 and the transition of modified plasma wave to modified ordinary wave takes place in Fig. 10. These points can be found approximately from the fact that the appearance of modified plasma wave is associated with the disappearance of the discontinuities in the graphs of Fig. 1 and Fig. 2 (Ginzburg[4]). Thus, these transition take place at

$$\omega_o^2 = \frac{1-\Omega^2}{1-\Omega^2\cos^2\theta} \quad .$$

This result can also be found approximately from the original Eq. (3.20). Since it is true that in most of the range of ω_o^2

$$k_1^2 >> k_2^2$$

$$k_1^2 >> k_3^2$$

and so k_1^2 can be obtained approximately by first two terms of Eq. (3.20) as

$$k_1^2 \simeq \frac{(1-\Omega^2-\omega_o^2+\Omega^2\omega_o^2\cos^2\theta)\,\beta_e^2}{(1-\Omega^2\cos^2\theta)} \tag{3.59}$$

(the coefficient of s^4 is simplified by using the relation $\beta_e^2 >> \beta_o^2$). The expression in Eq. (3.59) assumes the minimum number in the neighborhood of

$$1-\Omega^2-\omega_o^2+\Omega^2\omega_o^2\cos^2\theta = 0$$

or

$$\omega_o^2 = \frac{1-\Omega^2}{1-\Omega^2\cos^2\theta} \quad .$$

The bounding curves for the shaded areas in Fig. 9 and Fig. 10 are obtained from Figs. 5, 6, 7 and 8. The curves for the three propagation constants, k_1^2, k_2^2 and k_3^2 always lie in these shaded areas.

c) $\Omega = 1$:

This condition is satisfied when the angular frequency of the wave is equal to the electron gyrofrequency, and the Eq. (3.20) reduces to

$$s^6\sin^2\theta+s^4(2\omega_o^2\beta_o^2+\omega_o^2\beta_e^2\sin^2\theta-2\beta_o^2\sin^2\theta)+s^2\beta_o^2\Big[(1-\omega_o^2)^2(2\beta_e^2+\beta_o^2)$$
$$-(2\beta_e^2+\beta_o^2\cos^2\theta-\beta_e^2\omega_o^2\cos^2\theta-\beta_e^2\omega_o^2)\Big]+\beta_e^2\beta_o^4\omega_o^2(\omega_o^2-1)(\omega_o^2-2) = 0 \tag{3.60}$$

One propagation constant becomes zero at $\omega_o^2 = 0$, 1 and 2, and also one propagation constant becomes infinite at $\theta = 0$.

Case I: Propagation Along the Magnetic Field ($\theta = 0$)

In this case Eq. (3.60) can be simplified to

$$2s^4+s^2\Big[\omega_o^2(2\beta_e^2+\beta_o^2)-2(\beta_e^2+\beta_o^2)\Big]+\beta_e^2\beta_o^2(\omega_o^2-1)(\omega_o^2-2) = 0 \tag{3.61}$$

and the two roots of Eq. (3.61) are given by

$$s_I^2 = \beta_e^2(1-\omega_o^2) \ , \tag{3.62}$$

and

$$s_{III}^2 = \frac{\beta_o^2}{2}(2-\omega_o^2) \quad . \tag{3.63}$$

Equation (3.62) gives the propagation constant for the plasma wave, and Eq. (3.63) characterizes one of the electromagnetic waves which can be reduced from Eq. (3.50). s_I^2 and s_{III}^2 versus ω_o^2 are plotted in Fig. 11.

Case II: Propagation Across the Magnetic Field ($\theta = 90^o$)

Equation (3.60) will give

$$s^6 + s^4\left[2\beta_o^2(\omega_o^2-1)+\omega_o^2\beta_e^2\right] + s^2\beta_o^2\left[(1-\omega_o^2)^2(2\beta_e^2+\beta_o^2)-(2-\omega_o^2)\beta_e^2\right]$$

$$+\beta_e^2\beta_o^4\omega_o^2(\omega_o^2-1)(\omega_o^2-2) = 0 \tag{3.64}$$

and

$$a = -\frac{1}{3}\left\{\left[\beta_o^2(1-\omega_o^2)+\omega_o^2\beta_e^2\right]^2 + 3\omega_o^2\beta_e^2\beta_o^2\right\}$$

$$b = \frac{1}{27}\left\{2\left[\beta_o^2(1-\omega_o^2)+\omega_o^2\beta_e^2\right]^3 + 9\omega_o^2(1-\omega_o^2)\beta_o^4\beta_e^2 + 9\omega_o^4\beta_o^2\beta_e^4\right\}$$

$$\frac{b^2}{4}+\frac{a^3}{27} = \frac{-\omega_o^4\beta_o^4\beta_e^4}{4\times 27}\left\{\left[\beta_o^2(1-\omega_o^2)+\omega_o^2\beta_e^2\right]^2 + 4\omega_o^2\beta_o^2\beta_e^2\right\}$$

Thus we have $\frac{b^2}{4}+\frac{a^3}{27} \leq 0$ for all ω_o^2, and so we have three real and unequal roots for all ω_o^2 except at two negative points;

$$\omega_o^2 = \frac{-\beta_o^2\left\{(3\beta_e^2-\beta_o^2) \pm 2\beta_e\sqrt{2\beta_e^2-\beta_o^2}\right\}}{(\beta_o^2-\beta_e^2)^2}$$

and also at $\omega_o^2 = 0$, where we have $\frac{b^2}{4}+\frac{a^3}{27} = 0$ which means at least two real roots are equal.

Actually, we can separate the ordinary electromagnetic wave from Eq. (3.64) and obtain

$$\left[s^2-\beta_o^2(1-\omega_o^2)\right]\left\{s^4+s^2\left[\beta_o^2(\omega_o^2-1)+\omega_o^2\beta_e^2\right]+\beta_o^2\beta_e^2\omega_o^2(\omega_o^2-2)\right\}=0 \tag{3.65}$$

which gives the propagation constants of the three waves as

$$s_I^2=-\frac{1}{2}\left[\beta_o^2(\omega_o^2-1)+\omega_o^2\beta_e^2\right]+\sqrt{\frac{1}{4}\left[\beta_o^2(\omega_o^2-1)-\omega_o^2\beta_e^2\right]^2+\beta_o^2\beta_e^2\omega_o^2} \tag{3.66}$$

$$s_{II}^2=-\frac{1}{2}\left[\beta_o^2(\omega_o^2-1)+\omega_o^2\beta_e^2\right]-\sqrt{\frac{1}{4}\left[\beta_o^2(\omega_o^2-1)-\omega_o^2\beta_e^2\right]^2+\beta_o^2\beta_e^2\omega_o^2} \tag{3.67}$$

$$s_{III}^2=\beta_o^2(1-\omega_o^2) \tag{3.68}$$

Eq. (3.66) and Eq. (3.67) together give

$$s^4+s^2\omega_o^2(\beta_o^2+\beta_e^2)+\omega_o^4\beta_o^2\beta_e^2-s^2\beta_o^2-2\beta_o^2\beta_e^2\omega_o^2=0 \tag{3.69}$$

which represents a hyperbola. Equation (3.69) can also be deduced from Eq. (3.56), and an analysis similar to that applied to Eq. (3.56) can be used for the analysis of Eq. (3.69). Some of this analysis is given in Appendix D.

The standard form of the hyperbolic equation obtained by application of coordinate transformation to Eq. (3.69) is

$$
\begin{aligned}
&\Big[(1+\beta_o^2\beta_e^2)(1+\beta_o^4\beta_e^4+\beta_o^4+\beta_e^4)^{1/2} + (1+\beta_o^4\beta_e^4+\beta_o^4+\beta_e^4)\Big] \times \\
&\Big[2(1+\beta_o^4\beta_e^4+\beta_o^4+\beta_e^4)^{1/2}\Big]^{-1} \Big\{ \omega_o'^2 - \Big[\beta_o^2\Big(1-\beta_o^2\beta_e^2+\sqrt{1+\beta_o^4\beta_e^4+\beta_o^4+\beta_e^4}\Big)^{1/2} \\
&+2\beta_o^2\beta_e^2\Big(-1+\beta_o^2\beta_e^2+\sqrt{1+\beta_o^4\beta_e^4+\beta_o^4+\beta_e^4}\Big)^{1/2}\Big] \Big(1+\beta_o^2\beta_e^2+\sqrt{1+\beta_o^4\beta_e^4+\beta_o^4+\beta_e^4}\Big)^{-1} \times \\
&\Big(2\sqrt{1+\beta_o^4\beta_e^4+\beta_o^4+\beta_e^4}\Big)^{-1/2} \Big\}^2 - \Big[(1+\beta_o^4\beta_e^4+\beta_o^4+\beta_e^4) \\
&-(1+\beta_o^2\beta_e^2)(1+\beta_o^4\beta_e^4+\beta_o^4+\beta_e^4)^{1/2}\Big]\Big[2(1+\beta_o^4\beta_e^4+\beta_o^4+\beta_e^4)^{1/2}\Big]^{-1}\Big\{ s'^2 - \\
&\Big[-\beta_o^2\Big(-1+\beta_o^2\beta_e^2+\sqrt{1+\beta_o^4\beta_e^4+\beta_o^4+\beta_e^4}\Big)^{1/2} + 2\beta_o^2\beta_e^2\Big(1-\beta_o^2\beta_e^2+\sqrt{1+\beta_o^4\beta_e^4+\beta_o^4+\beta_e^4}\Big)^{1/2}\Big] \times \\
&\Big(\sqrt{1+\beta_o^4\beta_e^4+\beta_o^4+\beta_e^4}-1-\beta_o^2\beta_e^2\Big)^{-1}\Big(2\sqrt{1+\beta_o^4\beta_e^4+\beta_o^4+\beta_e^4}\Big)^{-1/2}\Big\}^2 \\
&= \beta_o^2\beta_e^2\Big[1-\beta_e^4(\beta_o^2-\beta_e^2)^{-2}\Big] \qquad (3.70)
\end{aligned}
$$

The transverse axis of the hyperbola is always parallel to the s'^2-axis. The center of the hyperbola translates to the positive $\omega_o'^2$-axis and s'^2-axis. It can be shown that the absolute value of the slopes for the asymptotes is greater than one. This hyperbola intersects the ω_o^2-axis at $\omega_o^2 = 0$ and 2, and the s^2-axis at $s^2 = \beta_o^2$ and 0. Figure 12 shows the graphs for Eqs. (3.66), (3.67) and (3.68).

Case III: Intermediate Inclination $(0 < \theta < \frac{\pi}{2})$

This case is similar to the case given in Fig. 10. The three propagation constants obtained from Eq. (3.60) are very tedious to plot directly, so some singular points are discussed first and the boundary lines for their variations are introduced from Fig. 11 and Fig. 12.

At $\omega_o^2 = 0$,

$$\begin{cases} s^2 = 0 \\ (s^2-\beta_o^2)^2 = 0 \end{cases} \tag{3.71}$$

At $\omega_o^2 = 1$,

$$\begin{cases} s^4\sin^2\theta + s^2(2\beta_o^2\cos^2\theta + \beta_e^2\sin^2\theta) - \beta_o^2(\beta_o^2\cos^2\theta + \beta_e^2\sin^2\theta) = 0 \\ s^2 = 0 \end{cases} \tag{3.72}$$

which gives $s^2 = \beta_o^2/2$ and $s^2 = \infty$ at $\theta = 0$, and

$$s^2 = \frac{-\beta_e^2 \pm \sqrt{\beta_e^2(\beta_e^2+4\beta_o^2)}}{2} \qquad \text{at } \theta = \frac{\pi}{2} \quad .$$

At $\omega_o^2 = 2$,

$$\begin{cases} s^4\sin^2\theta + 2s^2(\beta_o^2+\beta_o^2\cos^2\theta + \beta_e^2\sin^2\theta) \\ \qquad + \beta_o^2(\beta_o^2\sin^2\theta + 2\beta_e^2\cos^2\theta + 2\beta_e^2) = 0, \\ s^2 = 0. \end{cases} \tag{3.73}$$

Rearrangement of (3.73) gives

$$\tan^2\theta = \frac{-4\beta_o^2(s^2+\beta_e^2)}{s^4+2(\beta_o^2+\beta_e^2)s^2+\beta_o^2(\beta_o^2+2\beta_e^2)} \quad . \tag{3.74}$$

Thus at

$$\theta = 0: \begin{cases} s^2 = -\beta_e^2 \\ s^2 = \infty \end{cases},$$

and at

$$\theta = \frac{\pi}{2}: \begin{cases} s^2 = -2\beta_e^2 - \beta_o^2 \\ s^2 = -\beta_o^2 \end{cases}.$$

Three propagation constants are plotted in Fig. 13. This is a limiting case and the identifications as modified plasma waves, modified extraordinary waves and modified ordinary waves can not be seen clearly from the graph. As it stands, we do not know how the thick dotted curve for the modified extraordinary wave changes to the straight line for the plasma wave. More elaborate analysis is necessary in this case.

[D] Effect of Direction

In order to discuss the dependence of the propagation constant on direction, $\tan^2\theta$ is derived from Eq. (3.20) as

$$\begin{aligned}\tan^2\theta = \Big\{ & s^6(\Omega^2-1) + s^4\left[(1-\omega_o^2)(\beta_e^2+2\beta_o^2) - \Omega^2(\beta_e^2+2\beta_o^2-\beta_e^2\omega_o^2)\right] \\ & + s^2\beta_o^2\left[-(1-\omega_o^2)^2(2\beta_e^2+\beta_o^2) + \Omega^2(2\beta_e^2+\beta_o^2-2\beta_e^2\omega_o^2)\right] \\ & + \beta_e^2\beta_o^4(1-\omega_o^2)\left[(1-\omega_o^2)^2 - \Omega^2\right] \Big\} \Big\{ s^6 + s^4\left[\Omega^2\beta_e^2 - (1-\omega_o^2)(\beta_e^2+2\beta_o^2)\right] \\ & + s^2\beta_o^2\left[(1-\omega_o^2)^2(2\beta_e^2+\beta_o^2) - \Omega^2(2\beta_e^2-\beta_e^2\omega_o^2)\right] \\ & - \beta_e^2\beta_o^4(1-\omega_o^2)\left[(1-\omega_o^2)^2 - \Omega^2\right] \Big\}^{-1} \end{aligned} \tag{3.75}$$

Equation (3.75) can be factored as

$$\tan^2\theta = (\Omega^2-1)\left[s^2-\beta_e^2(1-\omega_o^2)\right]\left[s^2-\beta_o^2\left(1-\frac{\omega_o^2}{1-\Omega}\right)\right]\left[s^2-\beta_o^2\left(1-\frac{\omega_o^2}{1+\Omega}\right)\right] \times$$

$$\left[s^2-\beta_o^2(1-\omega_o^2)\right]^{-1}\left\{s^2-\frac{\beta_o^2\beta_e^2}{2}\left[(1-\omega_o^2)\left(\frac{1}{\beta_o^2}+\frac{1}{\beta_e^2}\right)-\frac{\Omega^2}{\beta_o^2}\right]\right.$$

$$\left.-\sqrt{\frac{\beta_o^4\beta_e^4}{4}\left[(1-\omega_o^2)\left(\frac{1}{\beta_o^2}+\frac{1}{\beta_e^2}\right)-\frac{\Omega^2}{\beta_o^2}\right]^2-\beta_o^2\beta_e^2\left[(1-\omega_o^2)^2-\Omega^2\right]}\right\}^{-1}$$

$$\times\left\{s^2-\frac{\beta_o^2\beta_e^2}{2}\left[(1-\omega_o^2)\left(\frac{1}{\beta_o^2}+\frac{1}{\beta_e^2}\right)-\frac{\Omega^2}{\beta_o^2}\right]\right.$$

$$\left.+\sqrt{\frac{\beta_o^4\beta_e^4}{4}\left[(1-\omega_o^2)\left(\frac{1}{\beta_o^2}+\frac{1}{\beta_e^2}\right)-\frac{\Omega^2}{\beta_o^2}\right]^2-\beta_o^2\beta_e^2\left[(1-\omega_o^2)^2-\Omega^2\right]}\right\}^{-1} \qquad (3.76)$$

So, along the direction of magnetic field there are three waves as given by Eqs. (3.48), (3.49) and (3.50), and at right angles to the magnetic field there are three waves as given by Eqs. (3.45), (3.46) and (3.47).

Now we want to investigate which values of the parameters ω_o^2, Ω^2 and θ give propagation, and which values give attenuation. The boundaries of these regions are the lines along which $s^2 = \infty$ and $s^2 = 0$, which are called "resonance" and "cutoff" respectively by Allis[44].

The principal cutoffs are given by

$$(1-\omega_o^2) = 0 \qquad (3.77)$$

$$\left(1-\frac{\omega_o^2}{1-\Omega}\right) = 0 \qquad (3.78)$$

$$(1 - \frac{\omega_o^2}{1+\Omega}) = 0 \qquad (3.79)$$

The principal resonances are given by

$$\Omega = \pm 1 \qquad (3.80)$$

Resonance occurs also at the angle θ which satisfies the condition

$$\tan^2\theta = (\Omega^2 - 1) \qquad (3.81)$$

This can happen only for $\Omega \geq 1$ and is obtained from the original dispersion relation (3.20) by setting the first coefficient equal to zero, i.e., $\Omega^2 \cos^2\theta - 1 = 0$. There is one more boundary line existing at

$$\omega_o^2 + \Omega^2 = 1 \quad . \qquad (3.82)$$

In the case of a cold electron plasma this condition gives resonance for the extra-ordinary wave as can be seen from the following expression:

$$s^2 = \beta_o^2 \frac{(1-\omega_o^2)^2 - \Omega^2}{1-\Omega^2-\omega_o^2} \quad . \qquad (3.83)$$

This is also quite apparent from Fig. 9, where the transition between the modified plasma wave and the modified extraordinary wave takes place at the angle satisfying

$$\omega_o^2 = \frac{1-\Omega^2}{1-\Omega^2\cos^2\theta} \qquad (3.84)$$

Equation (3.82) is the lower boundary of (3.84) corresponding to $\theta = \frac{\pi}{2}$. Equation (3.84) can be rewritten as

$$\cos^2\theta = \frac{1-(\Omega^2+\omega_o^2)}{-\omega_o^2\Omega^2} \qquad (3.85)$$

and so this transition angle exists only for $\Omega^2 + \omega_o^2 > 1$.

In Fig. 14 sample plots of $(s\text{-}\theta)$ curves are given for eight regions in the $(\omega_o^2\text{-}\Omega^2)$ plane. These curves will be called dispersion curves. The surface of revolution obtained by the rotation of the dispersion curve around the s_2 axis is known as the Fresnel phase surface. The boundaries of the eight regions in Fig. 14 are given by Eqs. (3.77), (3.78), (3.79), (3.80) and (3.82). The direction of magnetic field is assumed to be in the Ω^2-axis direction. The propagation constant of light in free space, $\beta_o = \frac{\omega}{c}$, is given by the dotted circle in the figure as a reference. These dispersion curves are deduced from some wave normal surfaces calculated by Allis, Buchsbaum and Bers[40], and also confirmed by our numerical results in Chapter V. The wave normal surface shows the variation of the phase velocity v_p with respect to the direction in space, thus it has the inverse relation with the Fresnel phase surface, which can be seen from the definition of the propagation constant, i.e.,

$$s \equiv \frac{\omega}{v_p} .$$

In region 1, corresponding to $0 < \omega_o^2 < 1 - \Omega$ of Fig. 9, three distinct dispersion curves exist. In region 2, corresponding to $1 - \Omega < \omega_o^2 < 1 - \Omega^2$ of Fig. 9, only a modified ordinary wave and a modified plasma wave exist. In region 3, corresponding to $1 - \Omega^2 < \omega_o^2 < 1$ of Fig. 9, the transition takes place between the modified plasma wave and the modified extraordinary wave at the angle θ_t satisfying Eq. (3.85). In region 4, corresponding to $1 < \omega_o^2 < 1 + \Omega$ of Fig. 9, only a modified extraordinary wave exists. In region 5, corresponding to $\omega_o^2 > 1 + \Omega$ of Fig. 9, none of these waves exist.

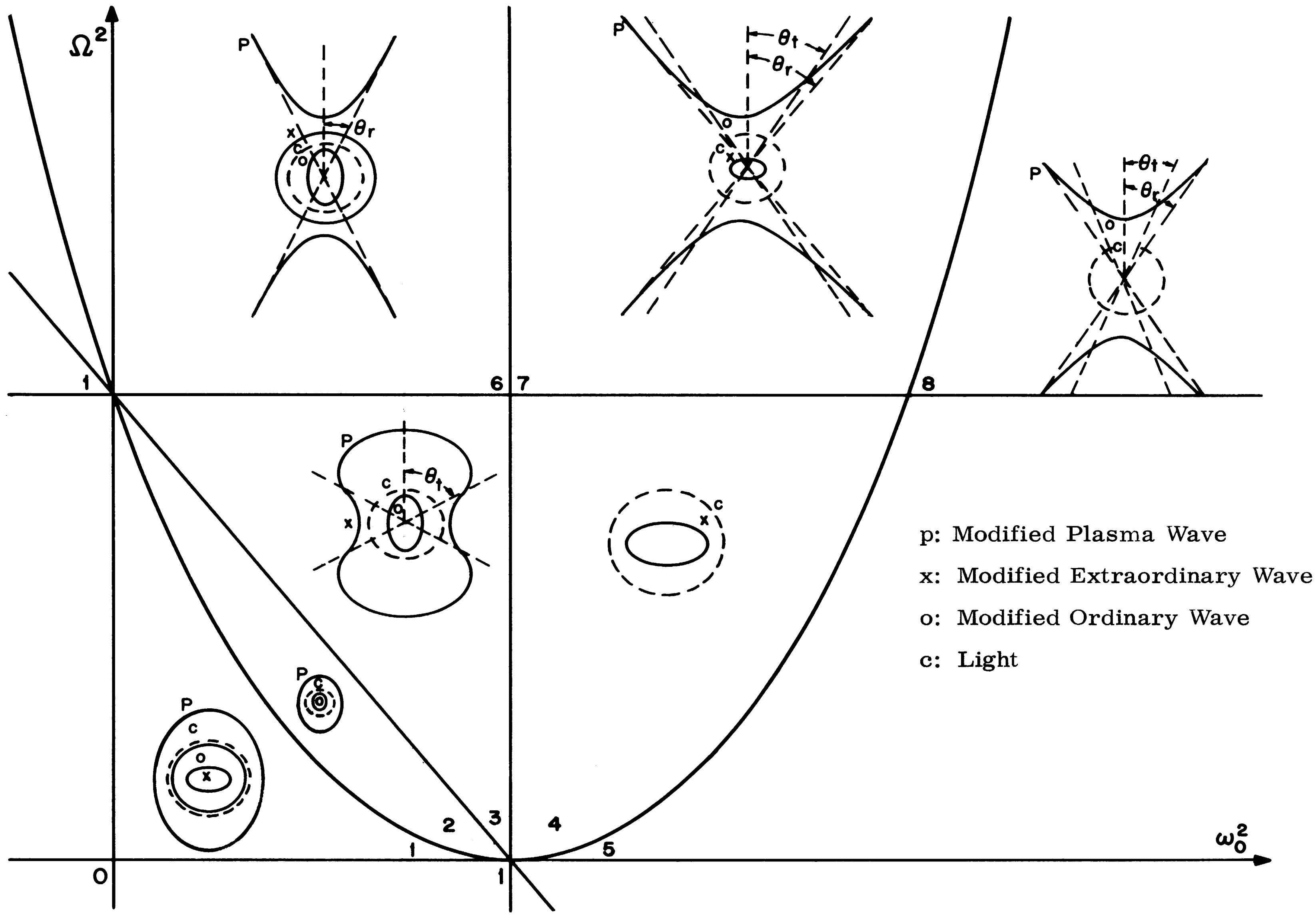


FIG. 14: DISPERSION CURVES FOR THREE WAVES

The dispersion curves in regions 6, 7 and 8 can be explained in conjunction with Fig. 10. In region 6, corresponding to $0 < \omega_o^2 < 1$ of Fig. 10, there are three distinct dispersion curves. The resonance angle θ_r for the modified plasma wave is given by Eq. (3.81). In region 7, corresponding to the range $1 < \omega_o^2 < 1 + \Omega$ in Fig 10, the transition between the modified ordinary wave and the modified plasma wave takes place at the angle θ_t satisfying Eq. (3.85), and the resonance angle θ_r for the modified plasma wave is given by Eq. (3.81). Between θ_t and θ_r there is the relation

$$\theta_r = \tan^{-1}\sqrt{\Omega^2 - 1} > \theta_t = \tan^{-1}\sqrt{(\Omega^2 - 1)\left(1 - \frac{\Omega^2}{\Omega^2 + \omega_o^2 - 1}\right)} \qquad (3.86)$$

Region 8, corresponding to $\omega_o^2 > 1 + \Omega$ in Fig. 10, has the same features as region 7 except that the modified extraordinary wave does not exist.

In the limiting case of $\Omega = 1$, it can be seen from Fig. 13 that there are two waves which exist in the range between $0 < \omega_o^2 < 1$ with closed dispersion curves, while there is only one wave in the range $1 < \omega_o^2 < 2$ which also has a closed dispersion curve. No wave is possible for $\omega_o^2 > 2$.

When $\Omega = 0$, only the plasma wave and the ordinary wave exist. Their dispersion curves are circles.

In general, these three wave constants do not become equal except at the transition angle θ_t in regions 3, 7 and 8.

CHAPTER IV

WAVE EXCITATION IN ONE-FLUID PLASMA

4.1 Introduction

In this chapter the general excitation problem in a homogeneous electron fluid plasma, which is compressible and uniformly impressed by a constant magnetic field, will be treated by applying the formal solution obtained in Chapter II. The dispersion relations analyzed in Chapter III will be utilized directly in calculating the excited fields. This type of problem, as far as is known, has never been treated in the literature.

The excitation problems discussed in the literature so far maybe divided into three categories:

(1) Cold plasma problems with a uniformly impressed constant magnetic field. In this type of work the longitudinal plasma wave does not come into picture. Typical examples of this type of problem are the works of Arbel[13], and Arbel and Felsen[20]. They start their formulation with "ordinary" and "extraordinary" modes.

(2) Compressible plasma without an externally applied constant magnetic field. With these assumptions Cohen [30] has shown that the field can be separated into two types of modes; one mode is transverse in nature and has all the fluctuating magnetic field, and another mode is longitudinal in nature and has all the fluctuating density field. The radiation of this acoustic-type of wave has been investigated by Hessel and Shmoys[23], Whale[22], Chen[45] and Wait[46].

(3) Two-dimensional problems in a compressible plasma with externally impressed constant magnetic field. Seshadri [24] has investigated the radiation

characteristics of a line magnetic current source in a homogeneous compressible plasma of infinite extent with an externally impressed uniform magnetic field.

The problems studied in this chapter include and extend the problems of the third category. First, the unified and systematic formulation developed in Chapter II will be applied to two-dimensional problems, and Seshadri's solution will emerge as a special case. Next, general three-dimensional problems will be treated where Lighthill's [38] method will be used to obtain the asymptotic solutions for the excited fields. The equivalence relations obtained between the various types of sources, by means of which the fields excited by one type of source can be expressed in terms of the fields excited by another type of source, are some of the highlights of the unified operator transform method.

4.2 Two-Dimensional Problems

Here we consider those excitation problems where the fields are not varying in the direction of the constant magnetic field, which was assumed to be in the y-direction.

[A] Field Solution in Transform Space for All Types of Sources

The transforms of the magnetic field and the density fluctuation field are given by Eq. (2.86), namely

$$\begin{bmatrix} I_t(s) \\ V_e(s) \end{bmatrix} = \left[1 - N(s)\right]^{-1} F(s)$$

To find the explicit expressions for $I_t(s)$ and $V_e(s)$ we have to find the inverse of the matrix $\left[1-N(s)\right]$, and the Fourier transform of the general source function

$$F(s) = \mathscr{F}_1(s) + \oint \mathscr{W}_{12}(s,v)\mathscr{F}_2(v) = \begin{bmatrix} J_t(s) + \dfrac{s^*}{\omega\mu_o} W_t(s) \\ \\ W_e(s) + \dfrac{N_o}{\omega} s' J_e(s) \end{bmatrix} . \tag{4.1}$$

The inverse matrix for the two-dimensional problem is given by

$$\left[1-N(s)\right]^{-1} = \frac{1}{\det.\left[1-N(s)\right]} \begin{bmatrix} M_{11} & 0 & M_{31} & 0 \\ 0 & M_{22} & 0 & M_{42} \\ M_{13} & 0 & M_{33} & 0 \\ 0 & M_{24} & 0 & M_{44} \end{bmatrix} \qquad (4.2)$$

where det. $\left[1-N(s)\right]$ can be easily obtained from Eq. (3.19) by setting $s_2 = 0$ and is

$$\det.\left[1-N(s)\right] = (\omega^2-\omega_p^2-s^2c^2)\left[(\omega^2-\omega_p^2-s^2c^2)(\omega^2-\omega_p^2-s^2U^2)-\omega_c^2(\omega^2-s^2c^2)\right]$$

$$\times\, (\omega^2-\omega_p^2)^{-1}\left[(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2\right]^{-1}$$

and

$$M_{11} = \left(1-\frac{s_1^2c^2}{\omega^2-\omega_p^2}\right)\left[1-\frac{s^2c^2(\omega^2-\omega_p^2-\omega_c^2)+s^2U^2(\omega^2-\omega_p^2-s^2c^2)}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\right] \qquad (4.3)$$

$$M_{13} = -\frac{s_1s_3c^2}{\omega^2-\omega_p^2}\left[1-\frac{s^2c^2(\omega^2-\omega_p^2-\omega_c^2)+s^2U^2(\omega^2-\omega_p^2-s^2c^2)}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\right] \qquad (4.4)$$

$$M_{22} = \left[1-\frac{s^2U^2(\omega^2-\omega_p^2)}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\right]\left[1-\frac{s^2c^2}{\omega^2-\omega_p^2}\right] \qquad (4.5)$$

$$M_{24} = -\frac{\omega_c\omega_p^2s^2/e}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\left[1-\frac{s^2c^2}{\omega^2-\omega_p^2}\right] \qquad (4.6)$$

$$M_{31} = M_{13} \qquad (4.7)$$

$$M_{33} = \left[1 - \frac{s_3^2 c^2}{\omega^2 - \omega_p^2}\right]\left[1 - \frac{s^2 c^2(\omega^2 - \omega_p^2 - \omega_c^2) + s^2 U^2(\omega^2 - \omega_p^2 - s^2 c^2)}{(\omega^2 - \omega_p^2)^2 - \omega_c^2 \omega^2}\right] \tag{4.8}$$

$$M_{42} = -\frac{ec^2 U^2 \omega_c s^2}{(\omega^2 - \omega_p^2)^2 - \omega_c^2 \omega^2}\left[1 - \frac{s^2 c^2}{\omega^2 - \omega_p^2}\right] \tag{4.9}$$

$$M_{44} = \left[1 - \frac{s^2 c^2(\omega^2 - \omega_p^2 - \omega_c^2)}{(\omega^2 - \omega_p^2)^2 - \omega_c^2 \omega^2}\right]\left[1 - \frac{s^2 c^2}{\omega^2 - \omega_p^2}\right] \tag{4.10}$$

Here $s^2 \equiv s_1^2 + s_3^2$.

The four components of the four-vector general source function are found from

Eq. (4.1) to be

$$F_1(s) = J_{tx}(s) + \frac{s_3 ec^2}{\omega m(\omega^2 - \omega_p^2)} f_y(s)$$

$$+ \frac{is_2 c^2}{(\omega^2 - \omega_p^2)^2 - \omega_c^2 \omega^2}\left\{\frac{e}{m}\left[\omega_c f_x(s) + i\frac{(\omega^2 - \omega_p^2)}{\omega} f_z(s)\right] + i\frac{\omega_c \omega_p^2}{\omega} j_x(s)\right\}$$

$$+ \frac{is_3 c^2}{\omega^2 - \omega_p^2} j_y(s) - \frac{is_2 c^2(\omega^2 - \omega_p^2 - \omega_c^2)}{(\omega^2 - \omega_p^2)^2 - \omega_c^2 \omega^2} j_z(s) \tag{4.11}$$

$$F_2(s) = J_{ty}(s) + \frac{ic^2}{(\omega^2 - \omega_p^2)^2 - \omega_c^2 \omega^2}\left\{-\frac{e}{m}\left[\omega_c\left[s_3 f_z(s) + s_1 f_x(s)\right] + \frac{(\omega^2 - \omega_p^2)}{i\omega}\left[s_3 f_x(s) - s_1 f_z(s)\right]\right]\right.$$

$$\left. - \frac{i\omega_c \omega_p^2}{\omega}\left[s_3 j_z(s) + s_1 j_x(s)\right] - (\omega^2 - \omega_p^2 - \omega_c^2)\left[s_3 j_x(s) - s_1 j_z(s)\right]\right\} \tag{4.12}$$

$$
\begin{aligned}
F_3(s) = {} & J_{tz}(s) - \frac{s_1 e c^2}{\omega m(\omega^2-\omega_p^2)} f_y(s) \\
& + \frac{i s_2 c^2}{(\omega^2-\omega_p^2)^2 - \omega_c^2\omega^2} \left\{ \frac{e}{m}\left[\omega_c f_z(s) + \frac{(\omega^2-\omega_p^2)}{i\omega} f_x(s)\right] + \frac{i\omega_c\omega_p^2}{\omega} j_z(s) \right. \\
& \left. + (\omega^2-\omega_p^2-\omega_c^2) j_x(s) \right\} - \frac{i s_1 c^2}{\omega^2-\omega_p^2} j_y(s)
\end{aligned}
\tag{4.13}
$$

$$
\begin{aligned}
F_4(s) = {} & W_e(s) - \frac{s_2}{\omega^2-\omega_p^2}\left[-\frac{i}{m} f_y(s) + \frac{\omega_p^2}{e\omega} j_y(s)\right] \\
& - \frac{\omega/e}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\left\{ \frac{e s_1}{m}\left[\omega_c f_z(s) + \frac{\omega^2-\omega_p^2}{i\omega} f_x(s)\right] \right. \\
& + s_1\omega_p^2\left[i\frac{\omega_c}{\omega} j_z(s) + \frac{\omega^2-\omega_p^2}{\omega^2} j_x(s)\right] + \frac{e s_3}{m}\left[-\omega_c f_x(s) + \frac{\omega^2-\omega_p^2}{i\omega} f_z(s)\right] \\
& \left. + s_3\omega_p^2\left[-i\frac{\omega_c}{\omega} j_x(s) + \frac{\omega^2-\omega_p^2}{\omega^2} j_z(s)\right]\right\} .
\end{aligned}
\tag{4.14}
$$

The transform of the electric current source, $j(s) \equiv \oint d(s, r)\bar{J}$, and the transform of the mechanical body source, $f(s) \equiv \oint d(s, r)\bar{F}$, have been used in Eqs. (4.11) through (4.14). In deriving Eqs. (4.11) through (4.14), without loss of generality, the constant magnetic field is assumed to be directed along the positive y-axis, thus these equations may be used in three-dimensional problems.

After setting $s_2 = 0$ in Eqs. (4.11), (4.12), (4.13) and (4.14) and using the result in Eq. (2.86), the three components of the transform of the magnetic field and the transform of the density fluctuation field may be expressed as follows:

$$I_{tx}(s) = \frac{1}{\omega^2-\omega_p^2-s^2c^2}\left\{(\omega^2-\omega_p^2)\left[J_{tx}(s) + \frac{s_3ec^2}{\omega m(\omega^2-\omega_p^2)}\, f_y(s) + \frac{is_3c^2}{\omega^2-\omega_p^2}\, j_y(s)\right] - s_1^2c^2J_{tx}(s) - s_1s_3c^2J_{tz}(s)\right\} \tag{4.15}$$

$$\begin{aligned} I_{ty}(s) = {} & \frac{1}{\Delta}\left[(\omega^2-\omega_p^2)(\omega^2-\omega_p^2-s^2U_e^2) - \omega_c^2\omega^2\right] F_2(s) \\ & - \frac{ec^2U^2\omega_c s^2}{\Delta}\left\{W_e(s) - \frac{\omega/e}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\left[\frac{es_1}{m}\left(\omega_c f_z(s) + \frac{\omega^2-\omega_p^2}{i\omega}\, f_x(s)\right)\right.\right. \\ & + s_1\omega_p^2\left(i\frac{\omega_c}{\omega}\, j_z(s) + \frac{\omega^2-\omega_p^2}{\omega^2}\, j_x(s)\right) + \frac{es_3}{m}\left(-\omega_c f_x(s) + \frac{\omega^2-\omega_p^2}{i\omega}\, f_z(s)\right) \\ & \left.\left. + s_3\omega_p^2\left(-i\frac{\omega_c}{\omega}\, j_x(s) + \frac{\omega^2-\omega_p^2}{\omega^2}\, j_z(s)\right)\right]\right\} \end{aligned} \tag{4.16}$$

$$I_{tz}(s) = \frac{1}{\omega^2-\omega_p^2-s^2c^2}\left\{-\frac{s_1ec^2}{\omega m}\, f_y(s) - is_1c^2j_y(s) - s_1s_3c^2J_{tx}(s) + (\omega^2-\omega_p^2-s_3^2c^2)J_{tz}(s)\right\} \tag{4.17}$$

$$\begin{aligned} V_e(s) = {} & \frac{-1}{\Delta}\left(\frac{\omega_c\omega_p^2s^2}{e}\right)F_2(s) + \frac{\left[(\omega^2-\omega_p^2)(\omega^2-\omega_p^2-s^2c^2) - \omega_c^2(\omega^2-s^2c^2)\right]}{\Delta}\left\{W_e(s)\right. \\ & - \frac{\omega/e}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\left[\frac{es_1}{m}\left(\omega_c f_z(s) + \frac{\omega^2-\omega_p^2}{i\omega}\, f_x(s)\right) + s_1\omega_p^2\left(i\frac{\omega_c}{\omega}\, j_z(s) + \frac{\omega^2-\omega_p^2}{\omega^2}\, j_x(s)\right)\right. \\ & \left.\left. + \frac{es_3}{m}\left(-\omega_c f_x(s) + \frac{\omega^2-\omega_p^2}{i\omega}\, f_z(s)\right) + s_3\omega_p^2\left(-i\frac{\omega_c}{\omega}\, j_x(s) + \frac{\omega^2-\omega_p^2}{\omega^2}\, j_z(s)\right)\right]\right\} \end{aligned} \tag{4.18}$$

where $\Delta \equiv (\omega^2-\omega_p^2-s^2c^2)(\omega^2-\omega_p^2-s^2U^2) - \omega_c^2(\omega^2-s^2c^2)$

B Physical Interpretation

The types of waves excited by the different types of sources can be determined from the original Maxwell-Euler's Eqs. (2.27) through (2.30), or from the explicit solutions in transform space, Eqs. (4.15) through (4.18).

Although it can be shown directly from Eq. (2.27) through (2.30) that for the magnetic field in the y-direction and no field variation in the y-direction the fields can be separated into two independent sets of components, such a separation can also be seen clearly by examination of Eqs. (4.15) through (4.18). In particular, since $\bar{h}$ and n are the inverse transforms of $I_t(s)$ and $V_e(s)$ respectively, their characteristics depend upon the two types of poles of $I_t(s)$ and $V_e(s)$. The poles of $I_{ty}(s)$ and $V_e(s)$ are determined by

$$\Delta = (\omega^2 - \omega_p^2 - s^2 c^2)(\omega^2 - \omega_p^2 - s^2 U^2) - \omega_c^2(\omega^2 - s^2 c^2) = 0 \tag{4.19}$$

which is the dispersion relation for the coupled extraordinary wave and plasma wave, or the modified extraordinary wave and the modified plasma wave. The poles of $I_{tx}(s)$ and $I_{tz}(s)$ are given by

$$\omega^2 - \omega_p^2 - c^2 s^2 = 0 \tag{4.20}$$

which is the dispersion relation for the ordinary wave.

Thus the propagation constants determined by (4.19) are given by Eq. (3.45) and Eq. (3.46) and the corresponding field components are E_x, E_z, h_y, V_x, V_z and n. Similarly, the propagation constant determined by (4.20) is given by Eq. (3.47) and the corresponding field components are E_y, h_x, h_z and V_y.

Explicitly, the results can be summarized as follows:

(a) A line magnetic current source $\bar{K} = \hat{y}K_o\delta(x)\delta(z)$ will excite the modified extraordinary wave and the modified plasma wave, but not an ordinary wave. This can be seen from the fact that the y-component of the transform of the magnetic current source, $J_{ty}(s)$, exists only in $F_2(s)$, which will produce $I_{ty}(s)$ and $V_e(s)$ as given by Eqs. (4.16) and (4.18).

(b) A line electric current source $\bar{J} = \hat{y}J_o\delta(x)\delta(z)$ will excite only ordinary waves, since only $I_{tx}(s)$ and $I_{tz}(s)$ are different from zero as can be seen from Eqs. (4.15) and (4.17).

(c) The transverse components of the electric current source will excite the modified extraordinary wave and the modified plasma wave, but not the ordinary wave, since $j_x(s)$ and $j_z(s)$ exist only in the Eqs. (4.16) and (4.18).

(d) The transverse components of the magnetic current source $\bar{K}$ will excite only the ordinary wave, since $J_{tx}(s)$ and $J_{tz}(s)$ exist only in the Eqs. (4.15) and (4.17).

(e) The transverse components of the mechanical body source $\bar{F}$ will excite the modified extraordinary wave and the modified plasma wave, and the longitudinal component of $\bar{F}$ will excite only the ordinary wave. This can be seen from the existence of $f_x(s)$ and $f_z(s)$ in the Eqs. (4.16) and (4.18), and the existence of $f_y(s)$ in the Eqs. (4.15) and (4.17).

(f) The electron fluid flux source Q will excite only the modified extraordinary wave and the modified plasma wave. Since $W_e(s)$ exists only in the Eqs. (4.16) and (4.18).

[C] Comparison of the Excitation Effects of Different Types of Sources

Applying the Eq. (2.69), and the transform of the magnetic field given by Eqs. (4.15) through (4.17), the transform of the density fluctuation field given by Eq. (4.18), and also the expressions given by Eqs. (2.75) through (2.78), the solutions to four typical, simple excitation problems will be presented in this section. The source terms are chosen such that only the modified extraordinary wave and the modified plasma wave are excited. If necessary, we can always superpose the ordinary wave which is decoupled from the other two waves.

1. Electron Fluid Flux Source

Let $Q = Q_o \delta(x)\delta(z)(2\pi)^2$, then we can obtain its transform

$$W_e(s) = \oint d(s, r)\, \frac{iQ}{\omega} = \frac{iQ_o}{\omega} \tag{4.21}$$

From Eqs. (4.15), (4.16), (4.17) and (4.18) we have

$$I_{tx}(s) = 0 \tag{4.22}$$

$$I_{ty}(s) = \frac{-ec^2U^2\omega_c W_e(s)(s_1^2+s_3^2)}{\Delta} \tag{4.23}$$

$$I_{tz}(s) = 0 \tag{4.24}$$

$$V_e(s) = \frac{\left[-s^2c^2(\omega^2-\omega_p^2-\omega_c^2)-\omega^2\omega_c^2+(\omega^2-\omega_p^2)^2\right]W_e(s)}{\Delta} \tag{4.25}$$

and then from Eq. (2.69) we can obtain

$$\begin{cases} V_{tx}(s) = \dfrac{ie}{\epsilon_o} U^2 \dfrac{\left[s_1(\omega^2 - \omega_p^2 - c^2 s^2) + i\omega\omega_c s_3\right] W_e(s)}{\Delta} \\ V_{ty}(s) = 0 \\ V_{tz}(s) = \dfrac{ie}{\epsilon_o} U^2 \dfrac{\left[s_3(\omega^2 - \omega_p^2 - c^2 s^2) - i\omega\omega_c s_1\right] W_e(s)}{\Delta} \end{cases} \tag{4.26}$$

$$\begin{cases} I_{ex}(s) = \dfrac{-\omega U^2}{N_o} W_e(s) \dfrac{\left[\frac{c^2}{\omega^2} s_3 i\omega\omega_c s^2 - s_1(\omega^2 - \omega_p^2 - c^2 s^2) - i\omega\omega_c s_3\right]}{\Delta} \\ I_{ey}(s) = 0 \\ I_{ez}(s) = \dfrac{-\omega U^2}{N_o} W_e(s) \dfrac{\left[-\frac{c^2}{\omega^2} s_1 i\omega\omega_c s^2 - s_3(\omega^2 - \omega_p^2 - c^2 s^2) + i\omega\omega_c s_1\right]}{\Delta} \end{cases} \tag{4.27}$$

In order to obtain the expressions for the fields in real space, the inverse of two different functions must be known. The details of the evaluation of the inversion integrals are presented in Appendix B and the results given in Eqs. (4.28) and (4.29), respectively.

$$\mathcal{J}_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{i(s_1 x + s_3 z)}}{\Delta} \, ds_1 ds_3$$

$$= \frac{i\pi^2}{c^2 U^2 (s_{II}^2 - s_I^2)} \left[H_o^{(1)}(s_{II} r) - H_o^{(1)}(s_I r)\right] \tag{4.28}$$

$$\mathcal{J}_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(s_1^2 + s_3^2) e^{i(s_1 x + s_3 z)}}{\Delta} \, ds_1 ds_3$$

$$= \frac{i\pi^2}{c^2 U^2 (s_{II}^2 - s_I^2)} \left[s_{II}^2 H_o^{(1)}(s_{II} r) - s_I^2 H_o^{(1)}(s_I r)\right] \tag{4.29}$$

Application of Eqs. (4.28) and (4.29) to Eqs. (4.22) through (4.27) yield the following field solutions:

$$h_y = \frac{\pi^2 e Q_o}{(s_{II}^2 - s_I^2)} \left(\frac{\omega_c}{\omega}\right) \left[s_{II}^2 H_o^{(1)}(s_{II} r) - s_I^2 H_o^{(1)}(s_I r) \right] \tag{4.30}$$

$$n = \frac{\pi^2 Q_o (\omega^2 - \omega_p^2 - \omega_c^2)}{\omega U^2 (s_{II}^2 - s_I^2)} \left[\left(s_{II}^2 - \frac{(\omega^2 - \omega_p^2)^2 - \omega^2 \omega_c^2}{c^2 (\omega^2 - \omega_p^2 - \omega_c^2)} \right) H_o^{(1)}(s_{II} r) \right.$$

$$\left. - \left(s_I^2 - \frac{(\omega^2 - \omega_p^2)^2 - \omega^2 \omega_c^2}{c^2 (\omega^2 - \omega_p^2 - \omega_c^2)} \right) H_o^{(1)}(s_I r) \right] \tag{4.31}$$

$$E_x = \frac{-eU^2 Q_o}{\epsilon_o \omega} \left[-i(\omega^2 - \omega_p^2) \frac{\partial}{\partial x} \mathcal{J}_1 + \omega \omega_c \frac{\partial}{\partial z} \mathcal{J}_1 + ic^2 \frac{\partial}{\partial x} \mathcal{J}_2 \right] \tag{4.32}$$

$$E_z = \frac{-eU^2 Q_o}{\epsilon_o \omega} \left[-i(\omega^2 - \omega_p^2) \frac{\partial}{\partial z} \mathcal{J}_1 - \omega \omega_c \frac{\partial}{\partial x} \mathcal{J}_1 + ic^2 \frac{\partial}{\partial z} \mathcal{J}_2 \right] \tag{4.33}$$

$$V_x = \frac{iQ_o U^2}{N_o} \left[-i(\omega^2 - \omega_p^2) \frac{\partial}{\partial x} \mathcal{J}_1 + \omega \omega_c \frac{\partial}{\partial z} \mathcal{J}_1 - c^2 \frac{\omega_c}{\omega} \frac{\partial}{\partial z} \mathcal{J}_2 + ic^2 \frac{\partial}{\partial x} \mathcal{J}_2 \right] \tag{4.34}$$

$$V_z = \frac{iQ_o U^2}{N_o} \left[-i(\omega^2 - \omega_p^2) \frac{\partial}{\partial z} \mathcal{J}_1 - \omega \omega_c \frac{\partial}{\partial x} \mathcal{J}_1 + c^2 \frac{\omega_c}{\omega} \frac{\partial}{\partial x} \mathcal{J}_2 + ic^2 \frac{\partial}{\partial z} \mathcal{J}_2 \right] \tag{4.35}$$

where s_I and s_{II} are the propagation constants for the modified plasma wave and the modified extraordinary wave as given by Eqs. (3.45) and (3.46), and $H_o^{(1)}$ is the conventional Hankel function of first kind of order zero. Also it is clear that $h_x = h_z = E_y = V_y = 0$.

2. Mechanical Body Source

Let $\bar{F} = \hat{x} F_o \delta(x) \delta(z) (2\pi)^2$, then its transform is

$$f(s) = \oint d(s, r)\, \bar{F} = \hat{x} F_o \tag{4.36}$$

Applying Eqs. (4.15) through (4.18), the following expressions are obtained:

$$\left\{\begin{aligned} I_{ty}(s) &= \left(\frac{ec^2 F_o}{m\omega^2}\right) \frac{-\omega s_3(\omega^2-\omega_p^2) - i\omega_c \omega^2 s_1 + \omega U^2 s_3(s_1^2+s_3^2)}{\Delta} \\ I_{tx}(s) &= I_{tz}(s) = 0 \end{aligned}\right. \tag{4.37}$$

$$V_e(s) = \left(-\frac{iF_o}{m\omega}\right) \frac{-s_1\omega(\omega^2-\omega_p^2) + \omega c^2 s_1 s^2 + i\omega_c s_3(\omega^2 - c^2 s^2)}{\Delta} \tag{4.38}$$

The transforms of the electric field and the fluid velocity field can then be obtained from Eq. (2.69) as

$$\left\{\begin{aligned} V_{tx}(s) &= -\left(\frac{eF_o}{\epsilon_o m\omega}\right) \frac{\omega(\omega^2 - c^2 s_1^2) + i\omega_c c^2 s_1 s_3 - \omega(\omega_p^2 + U^2 s_3^2)}{\Delta} \\ V_{ty}(s) &= 0 \\ V_{tz}(s) &= \left(\frac{eF_o}{\epsilon_o m\omega}\right) \frac{\omega c^2 s_1 s_3 + i\omega_c(\omega^2 - c^2 s_3^2) - \omega U^2 s_1 s_3}{\Delta} \end{aligned}\right. \tag{4.39}$$

$$\left\{\begin{aligned} I_{ex}(s) &= \left(\frac{c^2}{\omega^2}\right)\left(\frac{iF_o}{mN_o}\right) \frac{-\omega^3 s^2 + \omega\omega_p^2 s_3^2 + \frac{\omega^3}{\omega_c^2}(\omega^2-\omega_p^2) + \omega U^2 s_3^2\left(s^2 - \frac{\omega^2}{c^2}\right)}{\Delta} \\ I_{ey}(s) &= 0 \\ I_{ez}(s) &= \left(-\frac{c^2}{\omega^2}\right)\left(\frac{iF_o}{mN_o}\right) \frac{\omega\omega_p^2 s_1 s_3 - i\omega_c \omega s^2 + i\omega_c \frac{\omega^4}{c^2} + \omega U^2 s_1 s_3\left(s^2 - \frac{\omega^2}{c^2}\right)}{\Delta} \end{aligned}\right. \tag{4.40}$$

By using the relations given by Eqs. (4.28) and (4.29), all the inverse transforms for Eqs. (4.37), (4.38), (4.39) and (4.40) can be obtained as follows:

$$\begin{cases} E_x = -\left(\dfrac{eN_o}{\omega\epsilon_o}\right)\left(\dfrac{F_o}{mN_o}\right)\left[\omega(\omega^2-\omega_p^2)\vartheta_1 - i\omega_c c^2 \dfrac{\partial^2}{\partial x\partial z}\vartheta_1 + \omega c^2 \dfrac{\partial^2}{\partial x^2}\vartheta_1 + \omega U^2 \dfrac{\partial^2}{\partial z^2}\vartheta_1\right] \\ E_y = 0 \\ E_z = \left(\dfrac{eN_o}{\omega\epsilon_o}\right)\left(\dfrac{F_o}{mN_o}\right)\left[i\omega_c\omega^2\vartheta_1 - \omega(c^2-U^2)\dfrac{\partial^2}{\partial x\partial z}\vartheta_1 + i\omega_c c^2\dfrac{\partial^2}{\partial z^2}\vartheta_1\right] \end{cases} \tag{4.41}$$

$$\begin{cases} h_y = \dfrac{c^2 eF_o}{\omega^2 m}\left[\dfrac{-\omega(\omega^2-\omega_p^2)}{i}\dfrac{\partial}{\partial z}\vartheta_1 - \omega_c\omega^2\dfrac{\partial}{\partial x}\vartheta_1 + \dfrac{\omega U^2}{i}\dfrac{\partial}{\partial z}\vartheta_2\right] \\ h_x = h_z = 0 \end{cases} \tag{4.42}$$

$$\begin{cases} V_x = i\left(\dfrac{c^2}{\omega^2}\right)\left(\dfrac{F_o}{mN_o}\right)\left[\dfrac{\omega^3}{c^2}(\omega^2-\omega_p^2)\vartheta_1 - \omega^3\vartheta_2 - \omega\left(\omega_p^2 - \dfrac{U^2\omega^2}{c^2}\right)\dfrac{\partial^2}{\partial z^2}\vartheta_1 - \omega U^2\dfrac{\partial^2}{\partial z^2}\vartheta_2\right] \\ V_y = 0 \\ V_z = \left(\dfrac{-ic^2}{\omega^2}\right)\left(\dfrac{F_o}{mN_o}\right)\left[i\omega_c\dfrac{\omega^4}{c^2}\vartheta_1 - i\omega_c\omega^2\vartheta_2 - \omega\left(\omega_p^2 - \dfrac{U^2\omega^2}{c^2}\right)\dfrac{\partial^2}{\partial x\partial z}\vartheta_1 - \omega U^2\dfrac{\partial^2}{\partial x\partial z}\vartheta_2\right] \end{cases} \tag{4.43}$$

$$n = \left(\dfrac{-iF_o}{\omega m}\right)\left[\omega_c\omega^2\dfrac{\partial}{\partial z}\vartheta_1 - \dfrac{\omega(\omega^2-\omega_p^2)}{i}\dfrac{\partial}{\partial x}\vartheta_1 + \dfrac{\omega c^2}{i}\dfrac{\partial}{\partial x}\vartheta_2 - \omega_c c^2\dfrac{\partial}{\partial z}\vartheta_2\right] \tag{4.44}$$

3. A Line Magnetic Current Source

A line magnetic current source can be expressed as

$$\overline{K} = \hat{y}K_o\,\delta(x)\,\delta(z)\,(2\pi)^2 \tag{4.45}$$

then its transform is

$$J_{ty}(s) = \oint d(s, r) \frac{-iK_o \delta(x)\delta(z)(2\pi)^2}{\omega\mu_o} = \frac{-iK_o}{\omega\mu_o} \tag{4.46}$$

As before, the application of Eqs. (4.15) through (4.18) gives

$$\begin{cases} I_{ty}(s) = \left(\dfrac{-iK_o}{\omega\mu_o}\right) \dfrac{(\omega^2-\omega_p^2)(\omega^2-\omega_p^2-s^2U^2)-\omega^2\omega_c^2}{\Delta} \\ I_{tx}(s) = I_{tz}(s) = 0 \end{cases} \tag{4.47}$$

$$V_e(s) = \left(\frac{iK_o\omega_c\omega_p^2}{e\omega\mu_o}\right)\frac{s^2}{\Delta} \tag{4.48}$$

and then, the application of Eq. (2.69) gives

$$\begin{cases} V_{tx}(s) = \left(\dfrac{ic^2K_o}{\omega^2}\right) \dfrac{-s_3\omega^2(\omega^2-\omega_c^2-\omega_p^2)+i\omega\omega_c\omega_p^2 s_1+\omega^2U^2s_3s^2}{\Delta} \\ V_{ty}(s) = 0 \\ V_{tz}(s) = \left(\dfrac{ic^2K_o}{\omega^2}\right) \dfrac{s_1\omega^2(\omega^2-\omega_c^2-\omega_p^2)+i\omega\omega_c\omega_p^2 s_3-\omega^2U^2s_1s^2}{\Delta} \end{cases} \tag{4.49}$$

$$\begin{cases} I_{ex}(s) = \dfrac{-1}{eN_o} s_3 I_{ty}(s) - i\dfrac{\omega\epsilon_o}{eN_o} V_{tx}(s) \\ I_{ey}(s) = 0 \\ I_{ez}(s) = \dfrac{1}{eN_o} s_1 I_{ty}(s) - i\dfrac{\omega\epsilon_o}{eN_o} V_{tz}(s) \end{cases} \tag{4.50}$$

The field solutions in the real space are obtained from Eqs. (4.47) through (4.50) by applying the inverse transformation and using Eqs. (4.28) and (4.29) as

$$\begin{cases} h_y = \left(\dfrac{iK_o}{\omega\mu_o}\right)\left\{(\omega^2-\omega_p^2)U^2\vartheta_2 - \left[(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2\right]\vartheta_1\right\} \\ h_x = h_z = 0 \end{cases} \tag{4.51}$$

$$n = \left(\frac{iK_o\omega_c\omega_p^2}{e\mu_o\omega}\right)\vartheta_2 \tag{4.52}$$

$$\begin{cases} E_x = ic^2K_o\left\{\dfrac{\omega_c\omega_p^2}{\omega}\dfrac{\partial}{\partial x}\vartheta_1 + i(\omega^2-\omega_p^2-\omega_c^2)\dfrac{\partial}{\partial z}\vartheta_1 - iU^2\dfrac{\partial}{\partial z}\vartheta_2\right\} \\ E_y = 0 \\ E_z = ic^2K_o\left\{\dfrac{\omega_c\omega_p^2}{\omega}\dfrac{\partial}{\partial z}\vartheta_1 - i(\omega^2-\omega_p^2-\omega_c^2)\dfrac{\partial}{\partial x}\vartheta_1 + iU^2\dfrac{\partial}{\partial x}\vartheta_2\right\} \end{cases} \tag{4.53}$$

$$\begin{cases} V_x = \dfrac{K_o\omega_p^2}{e\mu_oN_o\omega}\left[\omega\omega_c\dfrac{\partial}{\partial x}\vartheta_1 + i(\omega^2-\omega_p^2)\dfrac{\partial}{\partial z}\vartheta_1 - iU^2\dfrac{\partial}{\partial z}\vartheta_2\right] \\ V_y = 0 \\ V_z = \dfrac{K_o\omega_p^2}{e\mu_oN_o\omega}\left[\omega\omega_c\dfrac{\partial}{\partial z}\vartheta_1 - i(\omega^2-\omega_p^2)\dfrac{\partial}{\partial x}\vartheta_1 + iU^2\dfrac{\partial}{\partial x}\vartheta_2\right] \end{cases} \tag{4.54}$$

4. Transverse Electric Current Source

Assume that the electric current source is given by

$$\bar{J} = \hat{x}J_o\,\delta(x)\,\delta(z)\,(2\pi)^2, \tag{4.55}$$

then, its transform is given by

$$j_x(s) = \oint d(s,r)\,J_o\,\delta(x)\,\delta(z)\,(2\pi)^2 = J_o\ . \tag{4.56}$$

The solution for the electric current source can be obtained by the same methods used in the previous examples. The results in the transform space are

$$\left\{\begin{aligned} I_{ty}(s) &= \frac{-iJ_o c^2}{\omega^2 \Delta} \; s_1 i\omega\omega_c\omega_p^2 + s_3\omega^2(\omega^2-\omega_c^2-\omega_p^2) - s_3 s^2 U^2\omega^2 \\ I_{tx}(s) &= I_{tz}(s) = 0 \end{aligned}\right. \tag{4.57}$$

$$V_e(s) = \frac{J_o}{e\omega\Delta}\left[s_1\omega_p^2(\omega_p^2-\omega^2) + i\omega\omega_c\omega_p^2 s_3 + s_1 s^2 c^2\omega_p^2\right] \tag{4.58}$$

$$\left\{\begin{aligned} V_{tx}(s) &= \frac{-iJ_o}{\epsilon_o\omega\Delta}\left\{\omega^2(\omega^2-\omega_c^2-\omega_p^2) - s_1^2\left[c^2(\omega^2-\omega_c^2) - U^2\omega_p^2\right] - U^2\omega^2 s^2 + c^2U^2 s_1^2 s^2\right\} \\ V_{ty}(s) &= 0 \\ V_{tz}(s) &= \frac{iJ_o}{\epsilon_o\omega\Delta}\left\{i\omega\omega_c\omega_p^2 + s_1 s_3\left[c^2(\omega^2-\omega_c^2) - U^2\omega_p^2\right] - c^2U^2 s_1 s_3 s^2\right\} \end{aligned}\right. \tag{4.59}$$

$$\left\{\begin{aligned} I_{ex}(s) &= \frac{J_o}{eN_o\Delta}\left[\omega_p^2(\omega_p^2-\omega^2) + ic^2\left(\frac{\omega_c}{\omega}\right)\omega_p^2 s_1 s_3 + c^2\omega_p^2 s_1^2 + U^2\omega_p^2 s_3^2\right] \\ I_{ey}(s) &= 0 \\ I_{ez}(s) &= \frac{J_o}{eN_o\Delta}\left[i\omega\omega_c\omega_p^2 + \omega_p^2(c^2-U^2)s_1 s_3 - ic^2\left(\frac{\omega_c}{\omega}\right)\omega_p^2 s_1^2\right] \end{aligned}\right. \tag{4.60}$$

and in the real space

$$\left\{\begin{aligned} h_y &= \frac{-iJ_o c^2}{\omega^2}\left\{\omega\omega_c\omega_p^2\frac{\partial\vartheta_1}{\partial x} + \frac{\omega^2(\omega^2-\omega_c^2-\omega_p^2)}{i}\frac{\partial\vartheta_1}{\partial z} - \frac{U^2\omega^2}{i}\frac{\partial\vartheta_2}{\partial z}\right\} \\ h_x &= h_z = 0 \end{aligned}\right. \tag{4.61}$$

$$n = \frac{J_o}{e\omega}\left\{\frac{\omega_p^2(\omega_p^2-\omega^2)}{i}\frac{\partial\vartheta_1}{\partial x} + \omega\omega_c\omega_p^2\frac{\partial\vartheta_1}{\partial z} + \frac{c^2\omega_p^2}{i}\frac{\partial\vartheta_2}{\partial x}\right\} \tag{4.62}$$

$$\left\{\begin{aligned}
E_x &= -\frac{iJ_o}{\omega\epsilon_o}\left\{\omega^2(\omega^2-\omega_c^2-\omega_p^2)\vartheta_1 + \left[c^2(\omega^2-\omega_c^2) - U^2\omega_p^2\right]\frac{\partial^2\vartheta_1}{\partial x^2}\right.\\
&\qquad\left. - U^2\omega^2\vartheta_2 - c^2U^2\frac{\partial^2\vartheta_2}{\partial x^2}\right\}\\
E_y &= 0\\
E_z &= \frac{iJ_o}{\omega\epsilon_o}\left\{i\omega\omega_c\omega_p^2\vartheta_1 - \left[c^2(\omega^2-\omega_c^2) - U^2\omega_p^2\right]\frac{\partial^2\vartheta_1}{\partial x\partial z} + c^2U^2\frac{\partial^2\vartheta_2}{\partial x\partial z}\right\}
\end{aligned}\right. \tag{4.63}$$

$$\left\{\begin{aligned}
V_x &= \frac{J_o}{eN_o}\left\{\omega_p^2(\omega_p^2-\omega^2)\vartheta_1 - ic^2\left(\frac{\omega_c}{\omega}\right)\omega_p^2\frac{\partial^2\vartheta_1}{\partial x\partial z} - c^2\omega_p^2\frac{\partial^2\vartheta_1}{\partial x^2} - U^2\omega_p^2\frac{\partial^2\vartheta_1}{\partial z^2}\right\}\\
V_y &= 0\\
V_z &= \frac{J_o}{eN_o}\left\{i\omega\omega_c\omega_p^2\vartheta_1 - \omega_p^2(c^2-U^2)\frac{\partial^2\vartheta_1}{\partial x\partial z} + ic^2\left(\frac{\omega_c}{\omega}\right)\omega_p^2\frac{\partial^2\vartheta_1}{\partial x^2}\right\}
\end{aligned}\right. \tag{4.64}$$

[D] Equivalence Relations

Equivalence relations between different types of sources in real space are obtained from the equivalence relations in transform space, which are derived from Eqs. (4.11), (4.12), (4.13) and (4.14). These relations can be used to obtain the excited fields due to one type of source from the solutions obtained for another type of source. In deriving Eqs. (4.11) through (4.14) for the four components of the source function F(s), the assumption of $\frac{\partial}{\partial y} = 0$, was not used so the equivalence relations obtained from these equations can be applied to the three-dimensional problems in an electron fluid plasma as well as to the two-dimensional problem.

First Relations: In the transform space s, the magnetic current source $J_t(s)$ and the mechanical body source f(s) are related by the following equivalence relations:

$$J_{tx}(s) = \frac{s_3 e c^2}{\omega m(\omega^2-\omega_p^2)} f_y(s) + \frac{i s_2 c^2 \frac{e}{m}}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\left[\omega_c f_x(s) + i\,\frac{\omega^2-\omega_p^2}{\omega} f_z(s)\right] \tag{4.65}$$

$$J_{ty}(s) = \frac{-i c^2 \frac{e}{m}}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\left\{\omega_c\left[s_3 f_z(s)+s_1 f_x(s)\right] + \frac{(\omega^2-\omega_p^2)}{i\omega}\left[s_3 f_x(s) - s_1 f_z(s)\right]\right\} \tag{4.66}$$

$$J_{tz}(s) = \frac{-s_1 e c^2}{\omega m(\omega^2-\omega_p^2)} f_y(s) + \frac{i s_2 c^2 \frac{e}{m}}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\left[\omega_c f_z(s) + \frac{(\omega^2-\omega_p^2)}{i\omega} f_x(s)\right] \tag{4.67}$$

Second Relations: The following equivalence relations exist between the magnetic current source $J_t(s)$ and the electric current source j(s) in the transform space:

$$J_{tx}(s) = \frac{i s_2 c^2}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\left[i\,\frac{\omega_c\omega_p^2}{\omega} j_x(s) - (\omega^2-\omega_p^2-\omega_c^2) j_z(s)\right] + \frac{i s_3 c^2}{\omega^2-\omega_p^2} j_y(s) \tag{4.68}$$

$$J_{ty}(s) = \frac{-ic^2}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\left\{ i\frac{\omega_c\omega_p^2}{\omega}\left[s_3 j_z(s)+s_1 j_x(s)\right] + (\omega^2-\omega_p^2-\omega_c^2)\left[s_3 j_x(s)-s_1 j_z(s)\right]\right\} \tag{4.69}$$

$$J_{tz}(s) = \frac{is_2c^2}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\left[i\frac{\omega_c\omega_p^2}{\omega} j_z(s) + (\omega^2-\omega_p^2-\omega_c^2) j_x(s)\right] - \frac{is_1c^2}{\omega^2-\omega_p^2} j_y(s) \tag{4.70}$$

Third Relation: In the transform space, the following equivalence relation exists between the electron fluid flux source $W_e(s)$ and the mechanical body source f(s):

$$W_e(s) = \frac{is_2}{m(\omega^2-\omega_p^2)} f_y(s) - \frac{\frac{\omega}{m}}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\left\{ s_1\left[\omega_c f_z(s) + \frac{\omega^2-\omega_p^2}{i\omega} f_x(s)\right] + s_3\left[-\omega_c f_x(s) + \frac{\omega^2-\omega_p^2}{i\omega} f_z(s)\right]\right\} \tag{4.71}$$

Fourth Relation: In the transform space, the following equivalence relation exists between the electron fluid flux source $W_e(s)$ and the electric current source j(s):

$$W_e(s) = -\frac{s_2\omega_p^2}{e\omega(\omega^2-\omega_p^2)} j_y(s) - \frac{\frac{\omega_p^2}{e}}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\left\{ s_1\left[i\omega_c j_z(s) + \frac{\omega^2-\omega_p^2}{\omega} j_x(s)\right] + s_3\left[-i\omega_c j_x(s) + \frac{\omega^2-\omega_p^2}{\omega} j_z(s)\right]\right\} \tag{4.72}$$

Equivalence relations in the real space can now be obtained easily from Eqs. (4.65) through (4.72) by applying the inverse transformation.

First Relation:

$$-\frac{i}{\mu_o}\bar{K} = \frac{ec^2}{m\left[(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2\right]}\left\{i(\omega^2-\omega_p^2)\nabla\times\bar{F}+\frac{i\omega_c^2\omega^2}{(\omega^2-\omega_p^2)}\hat{y}\times\nabla F_y - \omega\omega_c\hat{y}\nabla\cdot\bar{F}+\omega\omega_c\frac{\partial\bar{F}}{\partial y}\right\} \quad (4.73)$$

Second Relation:

$$-\frac{i}{\mu_o}\bar{K} = \frac{c^2}{\left[(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2\right]}\left\{-\omega(\omega^2-\omega_p^2-\omega_c^2)\nabla\times\bar{J}-\frac{\omega\omega_c^2\omega_p^2}{(\omega^2-\omega_p^2)}\hat{y}\times\nabla J_y - i\omega_c\omega_p^2\hat{y}\nabla\cdot\bar{J}+i\omega_c\omega_p^2\frac{\partial\bar{J}}{\partial y}\right\} \quad (4.74)$$

Third Relation:

$$\frac{iQ}{\omega} = \frac{1}{m(\omega^2-\omega_p^2)}\frac{\partial F_y}{\partial y}+\frac{\frac{\omega}{m}}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\left\{i\omega_c\frac{\partial F_z}{\partial x}+\frac{\omega^2-\omega_p^2}{\omega}\frac{\partial F_x}{\partial x} - i\omega_c\frac{\partial F_x}{\partial z}+\frac{\omega^2-\omega_p^2}{\omega}\frac{\partial F_z}{\partial z}\right\} \quad (4.75)$$

Fourth Relation:

$$\frac{iQ}{\omega} = \frac{i\omega_p^2}{e\omega(\omega^2-\omega_p^2)}\frac{\partial J_y}{\partial y}-\frac{\frac{\omega_p^2}{e}}{(\omega^2-\omega_p^2)^2-\omega_c^2\omega^2}\left\{\omega_c\frac{\partial J_z}{\partial x}-i\frac{\omega^2-\omega_p^2}{\omega}\frac{\partial J_x}{\partial x} - \omega_c\frac{\partial J_x}{\partial z}-i\frac{\omega^2-\omega_p^2}{\omega}\frac{\partial J_z}{\partial z}\right\} \quad (4.76)$$

4.3 Three-Dimensional Problems

As can be seen from the two-dimensional problems treated in the last section, any type of source excitation problem can be solved with equal ease by using the formal solution derived in Chapter II. In order to show the salient features of this technique when applied to a three-dimensional problem, the radiation fields due to a point current source will be obtained.

[A] Basic Derivation and Analysis

The point electric current source can be expressed as

$$\bar{J} = \hat{y}(2\pi)^3 J_o \,\delta(x)\delta(y)\delta(z) \tag{4.77}$$

and its transform given by

$$j_y(s) = \oint d(s,r)(2\pi)^3 J_o \,\delta(x)\delta(y)\delta(z) = J_o \tag{4.78}$$

From Eqs. (4.11), (4.12), (4.13) and (4.14), the source function can be expressed as

$$\left\{\begin{aligned} F_1(s) &= \frac{is_3 c^2}{\omega^2-\omega_p^2} J_o \\ F_2(s) &= 0 \\ F_3(s) &= -\frac{is_1 c^2}{\omega^2-\omega_p^2} J_o \\ F_4(s) &= -\frac{s_2\omega_p^2}{e\omega(\omega^2-\omega_p^2)} J_o \end{aligned}\right. \tag{4.79}$$

Next, the inverse matrix $\left[1-N(s)\right]^{-1}$ is obtained by applying Eqs. (3.2) through (3.19). The transforms of the magnetic field and the density fluctuation field can now be obtained from Eq. (2.86), and the transforms of the electric field and the fluid velocity field obtained from Eq. (2.69). In order to obtain solution in real space from the resulting expression it is necessary to evaluate the following integral

$$\mathcal{J} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{L(s)\, e^{i(s_1 x + s_2 y + s_3 z)}}{G(s)}\, ds_1 ds_2 ds_3 \tag{4.80}$$

The asymptotic solution to the integral (4.80) can be obtained by applying the principle of stationary phase. For the present problem it is convenient to use the theorem developed by Lighthill[38]. The solution of (4.80) satisfying the radiation condition is asymptotically given by

$$\mathcal{J} = \frac{4\pi^2}{r} \sum \frac{CLe^{i(s_1 x + s_2 y + s_3 z)}}{|\nabla G| \sqrt{|K|}} + 0\left(\frac{1}{r^2}\right) \tag{4.81}$$

as $r \longrightarrow \infty$. The summation is over all points (s_1, s_2, s_3) of the surface $G(s) = \det. \left[1 - N(s)\right] = 0$ where the normal to the surface is parallel to the direction of observation and $(\bar{r} \cdot \nabla G)/(\partial G / \partial \omega) < 0$. At each of these summation points the Gaussian curvature K can not be zero. C is $\mp i$ where $K < 0$ and ∇G is in the direction of $\pm r$, and ± 1 where $K > 0$ and the surface is convex to the direction of $\pm \nabla G$.

Usually, threefold Fourier integrals with axial symmetry are treated by conversion into Hankel transforms, but such a conversion complicates the asymptotic evaluation and also loses sight of the close relation existing between the radiation fields and the dispersion relation. Thus, we will use the asymptotic solution given by Eq. (4.81), which is directly expressed in terms of the characteristics of the phase surfaces.

In the present problem the axis of symmetry is in the y-direction and so the dispersion relation is a function of s_2^2 and $(s_1^2+s_3^2)$, i.e.

$$\det.\left[1-N(s)\right] = f(s_2^2,\ s_1^2+s_3^2)\ .$$

Using the simple notations $b \equiv s_2^2$ and $c \equiv s_1^2+s_3^2$ the Gaussian curvature reduces to

$$K = \frac{f_c\left\{2bc(f_{bb}f_c^2-2f_{bc}f_bf_c+f_{cc}f_b^2)+f_bf_c(bf_b+cf_c)\right\}}{(bf_b^2+cf_c^2)^2} \tag{4.82}$$

and

$$\left|\nabla G\right| = 2\sqrt{bf_b^2+cf_c^2}\ . \tag{4.83}$$

If the surface of revolution can be expressed by

$$s_2 = \varphi(\sqrt{s_1^2+s_3^2}) \tag{4.84}$$

the Gaussian curvature has the simple form

$$K = \frac{\varphi'\ \varphi''}{\sqrt{s_1^2+s_3^2}\,(1+\varphi'^2)^2} \tag{4.85}$$

Because of the axial symmetry, it is only necessary to find the field variation in a plane containing the y-axis. Stationary points will be given by the equation

$$\varphi' = -\left|\tan\phi\right| \tag{4.86}$$

or equivalently by the equations

$$\frac{\partial f}{\partial s_\rho} - \left|\tan\phi\right|\frac{\partial f}{\partial s_2} = 0, \qquad f = 0 \tag{4.87}$$

or

$$s_\rho f_c - \left|\tan\phi\right| s_2 f_b = 0, \qquad f = 0 \tag{4.88}$$

where $\quad s_\rho^2 \equiv s_1^2 + s_3^2 \equiv c.$

At an inflexion point of the plane curve given by (4.84) the Gaussian curvature is zero and Eq. (4.81) can not be used. Instead of (4.81), the following expression for ϑ must be used:

$$\vartheta = \frac{(2\pi)^{\frac{3}{2}} i(\frac{1}{3})! \sqrt{3}}{r^{\frac{5}{6}}} \sum \frac{c' L}{|\nabla G| (\lambda)^{\frac{1}{3}} | K_p |^{\frac{1}{2}}} e^{i\left[\bar{s} \cdot \bar{r} + \frac{1}{4} \pi \operatorname{sgn} K_p\right]} \tag{4.89}$$

where $c' = \pm 1$ depending on whether ∇G is in the direction of $\pm \bar{r}$, and λ is proportional to $\frac{\partial K_m}{\partial s_\rho}$. K_p is the principal curvature for the parallel section and K_m is the principal curvature for the meridian section.

On the surface of revolution the principal directions are on meridians and parallels, and the principal curvatures are given by

$$K_p = \frac{2\varphi_c}{\left[1+4(s_1^2+s_3^2)\varphi_c^2\right]^{\frac{1}{2}}} = \frac{\varphi'}{s_\rho(1+\varphi'^2)^{\frac{1}{2}}} \tag{4.90}$$

$$K_m = \frac{2\left[\varphi_c + 2\varphi_{cc}(s_1^2+s_3^2)\right]}{\left[1+4(s_1^2+s_3^2)\varphi_c^2\right]^{\frac{3}{2}}} = \frac{\varphi''}{(1+\varphi'^2)^{\frac{3}{2}}}, \tag{4.91}$$

where φ is given by (4.84). Equations (4.90) and (4.91) are obtained as inverse of the two roots of Eq. (4.92) solved for the radius of curvature, R;

$$\left[4\varphi_c^2 + 8\varphi_c\varphi_{cc}(s_1^2+s_3^2)\right]R^2 - \left[4\varphi_c + 8\varphi_c^3(s_1^2+s_3^2) + 4\varphi_{cc}(s_1^2+s_3^2)\right] \times$$

$$\sqrt{1+4(s_1^2+s_3^2)\varphi_c^2}\, R + \left[1+4\varphi_c^2(s_1^2+s_3^2)\right]^2 = 0 \tag{4.92}$$

The product of (4.90) and (4.91) gives the Gaussian curvature (4.85). As for λ, it is shown in Appendix E that

$$\lambda = \frac{1}{6} \frac{\partial K_m}{\partial s_\rho} \cos\left(\tan^{-1}|\varphi'|\right) \tag{4.93}$$

[B] Actual Calculation

For the actual calculation of the stationary points it is very complicated to solve either Eq. (4.87) or (4.88) directly. Instead, we will find the radiation direction ϕ corresponding to each point on the dispersion curves in the following way. First, the form given by Eq. (3.20) is used for the dispersion relation, det. $\left[1-N(s)\right] = 0$. Then, for each dispersion surface

$$f(s, \theta) = s - F(\theta) = 0 , \tag{4.94}$$

there are relations of the form

$$\begin{cases} \dfrac{\partial f}{\partial s_\rho} = \sin\theta - \dfrac{\cos\theta}{s}\dfrac{ds}{d\theta} \\[2ex] \dfrac{\partial f}{\partial s_2} = \cos\theta + \dfrac{\sin\theta}{s}\dfrac{ds}{d\theta} \end{cases} . \tag{4.95}$$

Substitution of (4.95) into (4.87) gives

$$\tan(\theta - \phi) = \frac{1}{s}\frac{ds}{d\theta} . \tag{4.96}$$

From Eq. (3.20) the right hand side of the above equation can be expressed as

$$\frac{1}{s}\frac{ds}{d\theta} = \frac{\Omega^2 \sin 2\theta \left[s^4 - s^2(2\beta_o^2 - \beta_e^2\omega_o^2) + \beta_o^2(\beta_o^2 - \beta_e^2\omega_o^2)\right]}{6s^4 A' + 4s^2 B' + 2C'} \tag{4.97}$$

where Eq. (3.20) is written as

$$A's^6 + B's^4 + C's^2 + D' = 0 \tag{4.98}$$

with

$$A' \equiv \Omega^2 \cos^2\theta - 1$$

$$B' \equiv (1-\omega_o^2)(\beta_e^2 + 2\beta_o^2) - \Omega^2(\beta_e^2 + 2\beta_o^2\cos^2\theta - \beta_e^2\omega_o^2\cos^2\theta)$$

$$C' \equiv \beta_o^2\left[-(1-\omega_o^2)^2(2\beta_e^2 + \beta_o^2) + \Omega^2(2\beta_e^2 + \beta_o^2\cos^2\theta - \beta_e^2\omega_o^2\cos^2\theta - \beta_e^2\omega_o^2)\right]$$

$$D' \equiv \beta_e^2\beta_o^4(1-\omega_o^2)\left[(1-\omega_o^2)^2 - \Omega^2\right]$$

Then for each angle θ, corresponding to each point on the dispersion curves, we can calculate ϕ from Eqs. (4.96) and (4.97).

Next, the Gaussian curvature must be evaluated at each stationary phase point from Eq. (4.85). The following two expressions will be used for this purpose:

$$\varphi' = \frac{ds_2}{ds_\rho} = \frac{-\frac{\partial f}{\partial s_\rho}}{\frac{\partial f}{\partial s_2}} = \frac{-\sin\theta + \frac{\cos\theta}{s}\frac{ds}{d\theta}}{\cos\theta + \frac{\sin\theta}{s}\frac{ds}{d\theta}}, \tag{4.99}$$

and

$$\varphi'' = \frac{d\varphi'}{ds_\rho} = \frac{\left(\varphi'\frac{\sin\theta}{s} - \frac{\cos\theta}{s}\right)\left[1 + 2\frac{1}{s^2}\left(\frac{ds}{d\theta}\right)^2 - \frac{1}{s}\frac{d^2s}{d\theta^2}\right]}{\left(\cos\theta + \frac{\sin\theta}{s}\frac{ds}{d\theta}\right)^2} \tag{4.100}$$

Equation (4.99) is obtained from (4.95). Substitution of (4.99) and (4.100) into (4.85) gives the following expression for the Gaussian curvature:

$$K = \frac{\left(1 - \frac{\cot\theta}{s}\frac{ds}{d\theta}\right)\left[1 + 2\frac{1}{s^2}\left(\frac{ds}{d\theta}\right)^2 - \frac{1}{s}\frac{d^2s}{d\theta^2}\right]}{\left[1 + \frac{1}{s^2}\left(\frac{ds}{d\theta}\right)^2\right]^2 s^2} \tag{4.101}$$

where

$$\frac{d^2s}{d\theta^2} = \Bigg\{ (6s^4A' + 4s^2B' + 2C')\Big\{\Omega^2 \sin 2\theta \frac{ds}{d\theta}\left[5s^4 - 3s^2(2\beta_o^2 - \beta_e^2\omega_o^2) + \beta_o^2(\beta_o^2 - \beta_e^2\omega_o^2)\right]$$

$$+ 2s\Omega^2 \cos 2\theta \left[s^4 - s^2(2\beta_o^2 - \beta_e^2\omega_o^2) + \beta_o^2(\beta_o^2 - \beta_e^2\omega_o^2)\right]\Big\} - 2s\Omega^2 \sin 2\theta \quad \times$$

$$\left[s^4 - s^2(2\beta_o^2 - \beta_e^2\omega_o^2) + \beta_o^2(\beta_o^2 - \beta_e^2\omega_o^2)\right] \Big\{(12s^3A' + 4sB')\frac{ds}{d\theta} - \Omega^2 \sin 2\theta \quad \times$$

$$\left[3s^4 - 2s^2(2\beta_o^2 - \beta_e^2\omega_o^2) + \beta_o^2(\beta_o^2 - \beta_e^2\omega_o^2)\right]\Big\}\Bigg\}(6s^4A' + 4s^2B' + 2C')^{-2} \quad . \qquad (4.102)$$

At the limiting angle $\theta = 0$, the Gaussian curvature is given by

$$K = (\varphi'')^2 = \frac{1}{s^2}\left(1 - \frac{1}{s}\frac{d^2s}{d\theta^2}\right)^2 \qquad (4.103)$$

At an inflexion point where $\varphi'' = 0$, K_p as given by (4.90) and λ as given by (4.93) must be calculated. For this purpose it is necessary to know

$$\frac{\partial K_m}{\partial s_\rho} = \frac{\varphi'''(1 + \varphi'^2) - 3\varphi'(\varphi'')^2}{(1 + \varphi'^2)^{\frac{5}{2}}} \quad . \qquad (4.104)$$

φ''' is obtained from (4.100) as

$$\varphi''' = \frac{\frac{d\theta}{ds_\rho}}{\left(\cos\theta + \frac{\sin\theta}{s}\frac{ds}{d\theta}\right)^2} \left\{ \frac{d\theta}{ds_\rho} \left[\frac{1}{s}\frac{d^3 s}{d\theta^3} - \frac{5}{s^2}\frac{ds}{d\theta}\frac{d^2 s}{d\theta^2} + \frac{4}{s^3}\left(\frac{ds}{d\theta}\right)^3 \right] \right.$$

$$+ \left[1 + 2\frac{1}{s^2}\left(\frac{ds}{d\theta}\right)^2 - \frac{1}{s}\frac{d^2 s}{d\theta^2}\right] \left[\frac{\varphi''}{\frac{d\theta}{ds_\rho}} \left(\frac{\sin\theta}{s}\right) + \varphi' \frac{(s\cos\theta - \frac{ds}{d\theta}\sin\theta)}{s^2} \right.$$

$$\left.\left. + \frac{(s\sin\theta + \frac{ds}{d\theta}\cos\theta)}{s^2} \right] \right\} + \frac{2\left(\frac{d\theta}{ds_\rho}\right)^2 \left[1 + 2\frac{1}{s^2}\left(\frac{ds}{d\theta}\right)^2 - \frac{1}{s}\frac{d^2 s}{d\theta^2}\right]}{\left(\cos\theta + \frac{\sin\theta}{s}\frac{ds}{d\theta}\right)^3} \times$$

$$\left\{ -\sin\theta + \frac{\sin\theta}{s}\frac{d^2 s}{d\theta^2} + \frac{ds}{d\theta}\frac{(s\cos\theta - \frac{ds}{d\theta}\sin\theta)}{s^2} \right\} \tag{4.105}$$

where

$$\frac{d\theta}{ds_\rho} = \frac{\cos\theta - \varphi'\sin\theta}{s} .$$

Also, for the actual calculation of the absolute value of ∇G, use is made of the following form instead of Eq. (4.82):

$$|\nabla G| = \sqrt{1 + \left(\frac{1}{s}\frac{ds}{d\theta}\right)^2} \tag{4.106}$$

Asymptotic solutions in the form of Eq. (4.81) can now be calculated for the electric field. After some manipulations, the inversion integrals for the electric field are shown to be

$$E_x = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\frac{J_o}{i\omega\epsilon_o}}{\det.\left[1 - N(s)\right]} \left\{ -\frac{s_2 U^2}{\omega^5(\omega^2 - \omega_c^2)} \left\{ s_1 \omega (c^2 s^2 + \omega_p^2 - \omega^2)(c^2 s_\rho^2 + \omega_p^2) \right.\right.$$

$$\left. - i\omega_c \omega_p^2 \omega^2 s_3 + c^2 \omega s_1 s_2^2 \omega_p^2 \right\} - \left(1 - \frac{U^2}{\omega^2} s_2^2\right)\left(1 - \frac{U^2}{\omega^2} s^2\right) \frac{c^2}{\omega^2} s_1 s_2 + \frac{c^2 \omega_p^2 s_2^2}{\omega^3(\omega^2 - \omega_c^2)} \times$$

$$\left. \left\{ \omega s_1 - i\omega_c s_3 \right\} \right\} e^{i(s_1 x + s_2 y + s_3 z)}\, ds_1\, ds_2\, ds_3 \tag{4.107}$$

$$E_z = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\frac{J_o}{i\omega\epsilon_o}}{\det.\left[1-N(s)\right]} \left\{ -\frac{s_2 U^2}{\omega^5(\omega^2-\omega_c^2)} \left\{ s_3\omega(c^2s^2+\omega_p^2-\omega^2)(c^2s_\rho^2+\omega_p^2) \right.\right.$$

$$\left. + i\omega_c\omega_p^2\omega^2 s_1 + c^2\omega s_3 s_2^2\omega_p^2 \right\} - (1-\frac{U^2}{\omega^2}s_2^2)(1-\frac{c^2}{\omega^2}s^2)\frac{c^2}{\omega^2}s_2 s_3 + \frac{c^2\omega_p^2 s_2}{\omega^3(\omega^2-\omega_c^2)} \quad x$$

$$\left. \left\{\omega s_3 + i\omega_c s_1\right\} \right\} e^{i(s_1x+s_2y+s_3z)}\, ds_1\, ds_2\, ds_3 \qquad (4.108)$$

$$E_y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\frac{J_o}{i\omega\epsilon_o}}{\det.\left[1-N(s)\right]} \left\{ \frac{s_2^2 U^2}{\omega^4(\omega^2-\omega_c^2)} \left[\omega_p^2(2\omega^2-\omega_p^2-2c^2s^2)\right.\right.$$

$$\left. -c^2 s_\rho^2(c^2s^2-\omega^2)\right] - \frac{1}{\omega^2(\omega^2-\omega_c^2)}\left[\omega_p^2(2\omega^2-\omega_p^2-c^2s^2-c^2s_2^2) - U^2 s_\rho^2(c^2s^2+\omega_p^2-\omega^2)\right]$$

$$\left. + (1-\frac{U^2}{\omega^2}s_2^2)\left[1-\frac{c^2}{\omega^2}(s^2+s_2^2)+\frac{c^4}{\omega^4}s_2^2 s^2\right] \right\} \qquad (4.109)$$

The sum of $\hat{x}E_x$ and $\hat{z}E_z$ will be expressed in terms of two vectors in the direction of $\hat{\rho}$ and $\hat{\phi}$ by using the fact that

$$\hat{x}s_1 + \hat{z}s_3 = \bar{s}_\rho = |s_\rho|\,\hat{\rho}\ ,$$

and

$$-\hat{x}s_3 + \hat{z}s_1 = \bar{s}_\rho x\hat{y} = |s_\rho|\,\hat{\phi}\ . \qquad (4.110)$$

These two vector components are

$$E_\phi = \frac{4\pi^2}{r}\sum \frac{Ce^{i\bar{s}\cdot\bar{r}}\,\omega^4(\omega^2-\omega_c^2)}{|\nabla G|\sqrt{|K|}\,U^2c^4}\left(\frac{J_o}{i\omega\epsilon_o}\right) s_\rho \left\{\frac{i\omega_c\omega_p^2c^2s_2}{\omega^3(\omega^2-\omega_c^2)} - \frac{i\omega_c\omega_p^2U^2s_2}{\omega^3(\omega^2-\omega_c^2)}\right\}$$

$$= \frac{4\pi^2}{r}\sum \frac{Ce^{i\bar{s}\cdot\bar{r}}J_o\omega_c\omega_p^2}{|\nabla G|\sqrt{|K|}\,U^2c^4\epsilon_o} s_2 s_\rho(c^2-U^2) \qquad (4.111)$$

and

$$E_{\rho} = \frac{4\pi^2}{r} \sum \frac{C e^{i\bar{s}\cdot\bar{r}} \omega^4(\omega^2-\omega_c^2)}{|\nabla G| \sqrt{|K|}\; U^2 c^4} \left(\frac{J_o}{i\omega\epsilon_o}\right) s_{\rho} \left\{ \frac{c^2 \omega_p^2 s_2^2}{\omega^2(\omega^2-\omega_c^2)} - \frac{s_2^2 U^2}{\omega^4(\omega^2-\omega_c^2)} \times \right.$$

$$\left. \left[(c^2 s^2+\omega_p^2-\omega^2)(c^2 s_{\rho}^2+\omega_p^2)+ c^2\omega_p^2 s_2^2\right] - \left(1-\frac{U^2}{\omega^2} s_2^2\right)\left(1-\frac{c^2}{\omega^2} s^2\right) \frac{c^2}{\omega^2} s_2^2 \right\} \quad (4.112)$$

$\hat{\rho}$ and $\hat{\phi}$ are unit vectors in the cylindrical coordinate system with the y-direction taken as the direction of the axis. The asymptotic solution for the component of electric field in the y-direction is obtained from (4.109) as

$$E_y = \frac{4\pi^2}{r} \sum \frac{C e^{i\bar{s}\cdot\bar{r}} \omega^4(\omega^2-\omega_c^2)}{|\nabla G| \sqrt{|K|}\; U^2 c^4} \left(\frac{J_o}{i\omega\epsilon_o}\right) \left\{ \frac{s_2^2 U^2}{\omega^4(\omega^2-\omega_c^2)} \left[\omega_p^2(2\omega^2-\omega_p^2-2c^2 s^2) \right.\right.$$

$$\left. -c^2 s_{\rho}^2(c^2 s^2-\omega^2)\right] - \frac{1}{\omega^2(\omega^2-\omega_c^2)} \left[\omega_p^2(2\omega^2-\omega_p^2-c^2 s^2-c^2 s_2^2)\right.$$

$$\left.\left. -U^2 s_{\rho}^2(c^2 s^2+\omega_p^2-\omega^2)\right] + \left(1-\frac{U^2}{\omega^2} s_2^2\right) \left[1-\frac{c^2}{\omega^2}(s^2+s_2^2)+\frac{c^4}{\omega^4} s_2^2 s^2\right] \right\} \quad (4.113)$$

Finally, it will be shown that the radiation condition expressed by the requirement $(\bar{r}\cdot \nabla G)/(\partial G/\partial\omega) < 0$ is not essential to the calculation of amplitude variations. First of all, it is easy to see from Eq. (4.88) that if the point (s_2, s_{ρ}) satisfies this equation, its symmetrical point $(-s_2, -s_{\rho})$ will also satisfy this equation. $\bar{r}\cdot \nabla G$ of these two stationary points have different signs, and the radiation requirement selects the one which has a sign opposite to that of $\partial G/\partial\omega$. But, the amplitude of each stationary point contribution in Eqs. (4.111), (4.112) and (4.113) is the function of s_2 and s_{ρ} which appear only in the form of their product or squares, thus it is not necessary to select one of two stationary points, by calculating ∇G and $\partial G/\partial\omega$, in order to plot the amplitude variations.

CHAPTER V

NUMERICAL RESULTS IN THE IONOSPHERIC PLASMA

5.1 Ionospheric Model

Because of the great variations in ionospheric properties depending upon time and geographic location, attempt is made to use those data corresponding to day-time, mid-latitude and late 1962 ionosphere. For electron density profiles use is made of those ionograms utilized by Stone, Bird and Balser[47], and some topside sounding ionograms analyzed by Bauer and Blumle[48]. From these profiles it seems convenient to divide the ionosphere into four regions: (i) Above F-peak region; (ii) Around F-peak region; (iii) E region; and (iv) D region.

As for the electron temperature use is made of the United States Standard Atmosphere, 1962, and it is assumed that thermal equilibrium condition prevails throughout the ionosphere. Thermal non-equilibrium seems to be ascertained in certain altitude regions by Spencer, Brace and Carignan[49], Nagy, Brace, Carignan and Kanal[50], and others, but for our linearized treatment the assumption of thermal equilibrium should be able to represent general characteristics.

The variation of the magnitude of the earth's magnetic field is rather small and thus, it will be assumed constant with a value of 0.5 Gauss, which will give the electron gyrofrequency as $\omega_c = 8.79 \times 10^6$ and $\omega_c^2 = 7.73 \times 10^{13}$.

The following altitudes are chosen for the four regions:

(i) 400 Km: $T = 1,487^\circ K$, $U^2 = 2.25 \times 10^{10}$

$N_o = 1.7 \times 10^5$ electrons/cc, $\omega_p^2 = 5.4 \times 10^{14}$

(ii) 250 Km: $T = 1,357^\circ K$, $U^2 = 2.05 \times 10^{10}$

$N_o = 6 \times 10^5$ electrons/cc, $\omega_p^2 = 1.9 \times 10^{15}$

(iii) 100 Km: $T = 210^{\circ}K$, $U^2 = 3.18 \times 10^9$

$N_o = 3 \times 10^4$ electrons/cc, $\omega_p^2 = 9.54 \times 10^{13}$

(iv) 70 Km: $T = 220^{\circ}K$, $U^2 = 3.34 \times 10^9$

$N_o = 10^2$ electrons/cc, $\omega_p^2 = 3.18 \times 10^{11}$

5.2 Radiation Fields

The magnitudes of the electric field components excited by a point current source oriented in the direction of the earth's magnetic field are calculated from Eqs. (4.111), (4.112) and (4.113) for each stationary point contribution. $|E_M|$ is the magnitude of the projection of the electric field on the meridian plane, which is obtained as

$$|E_M| = (|E_y|^2 + |E_\rho|^2)^{\frac{1}{2}} \quad . \tag{5.1}$$

Calculations are made for four frequencies, which are $\omega = 3 \times 10^5$, $\omega = 3 \times 10^6$, $\omega = 3 \times 10^7$ and $\omega = 3 \times 10^8$. The dispersion curves and E_ϕ', E_M' versus ϕ are given in Fig. 15 through Fig. 67, where

$$E_\phi' = \frac{36\pi \epsilon_o r \, |E_\phi|}{J_o} \tag{5.2}$$

and

$$E_M' = \frac{36\pi \epsilon_o r \, |E_M|}{J_o} \tag{5.3}$$

Due to the symmetrical nature of the physical system and the source involved, E_ϕ', E_M' vs ϕ are plotted only for the range $\phi = 0^{\circ}$ to $\phi = 90^{\circ}$. The angle $\phi = 0^{\circ}$ and $\theta = 0^{\circ}$ corresponds to the direction of the earth's magnetic field.

According to our numerical results, the dependence of the radiation patterns of the excited fields on the altitude of the ionosphere is not too conspicuous.

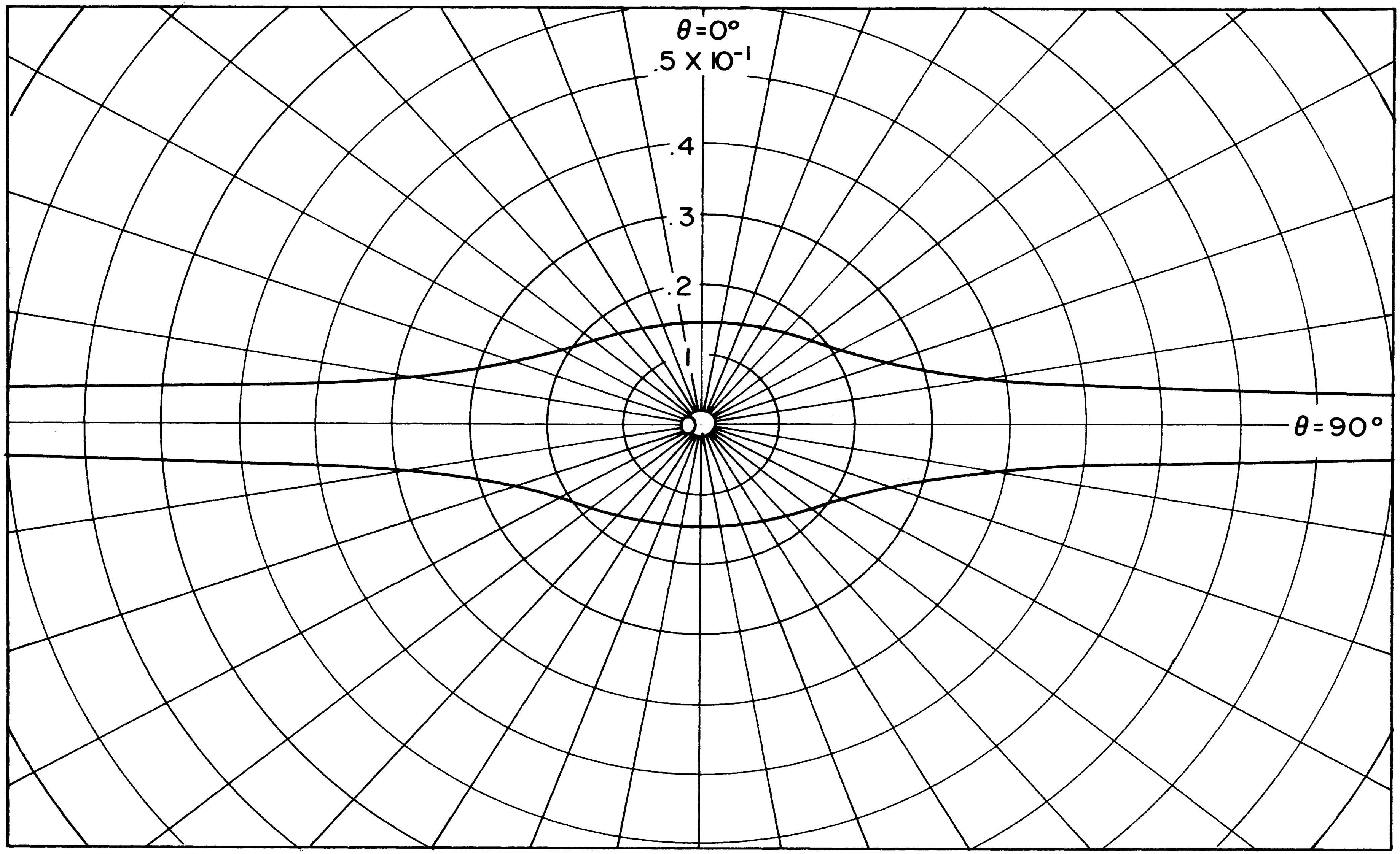


FIG. 15: DISPERSION CURVE
400 KM, $\omega = 3 \times 10^5$

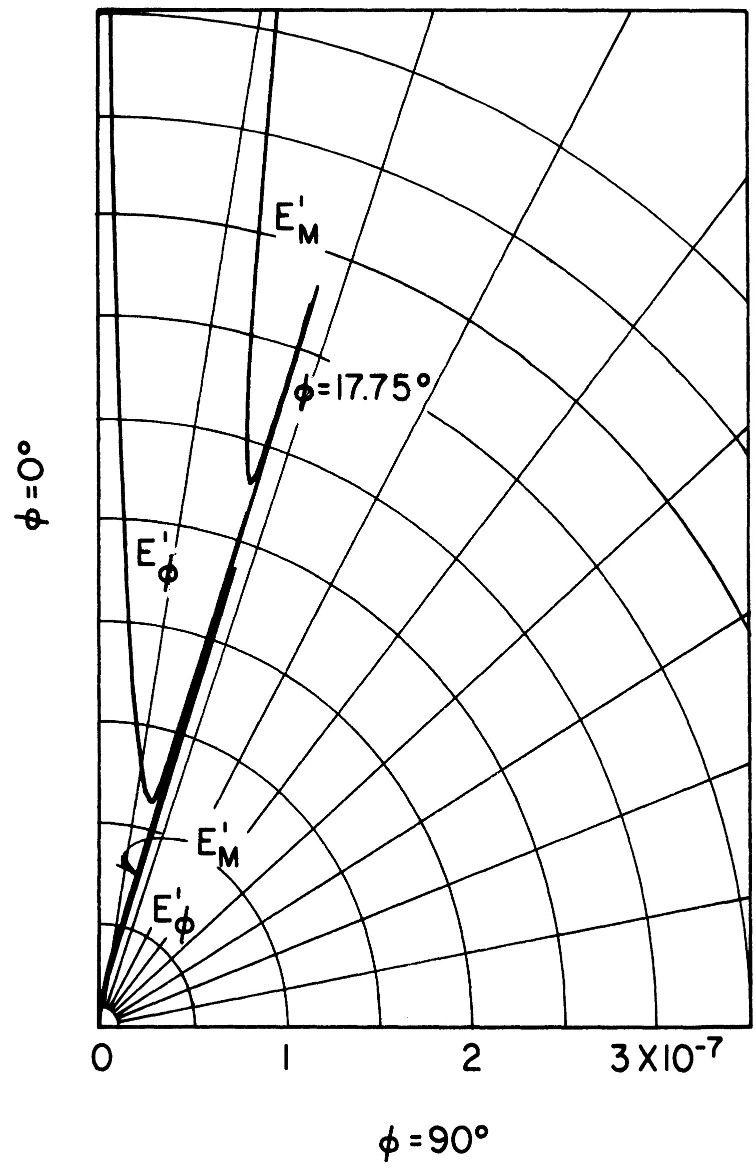


FIG. 16: E'_{ϕ}, E'_{M} VS. ϕ

400 KM, $\omega = 3 \times 10^5$

POINT CURRENT SOURCE ($\phi = 0^{\circ}$)

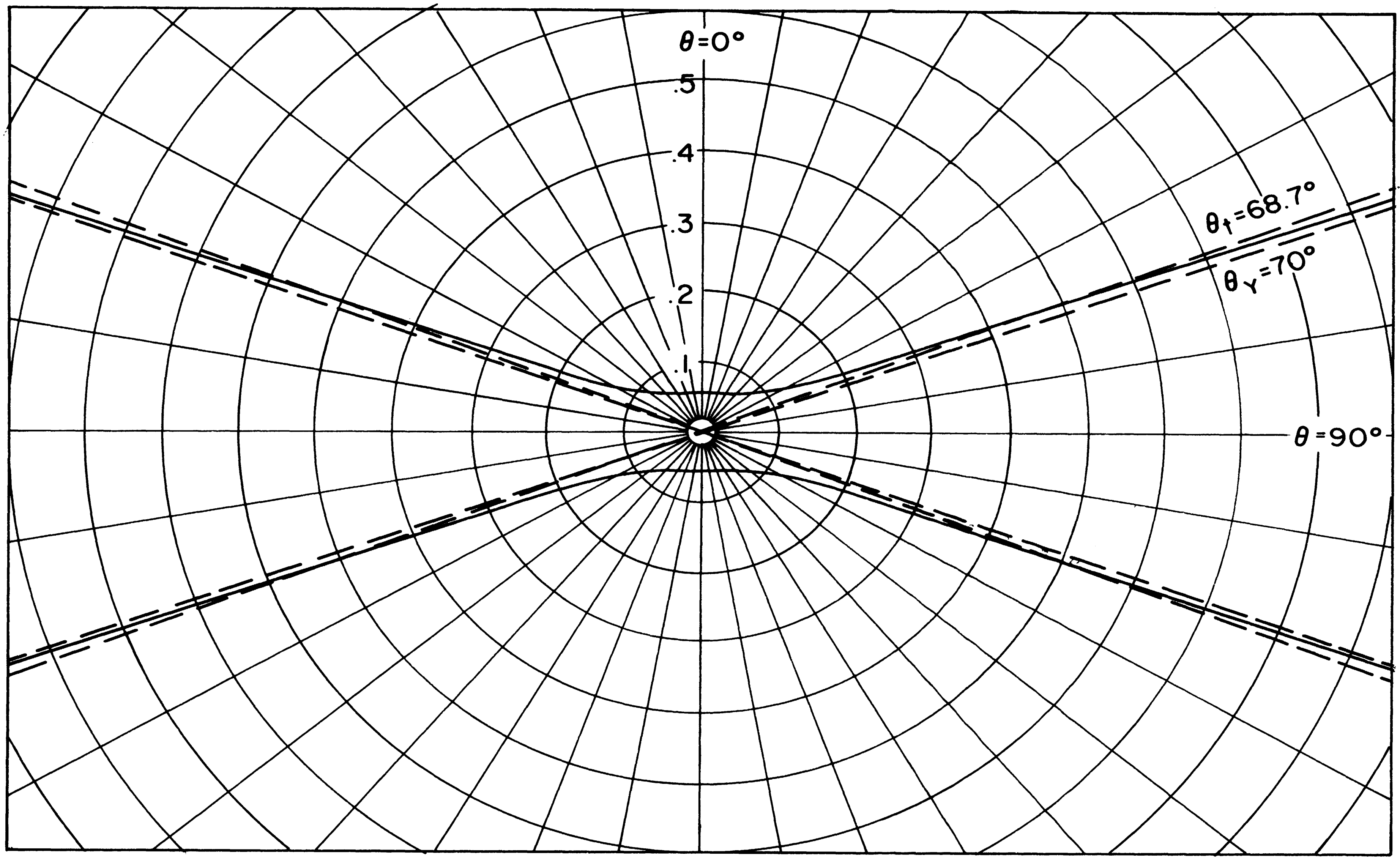


FIG. 17: DISPERSION CURVE

400 KM, $\omega = 3 \times 10^6$

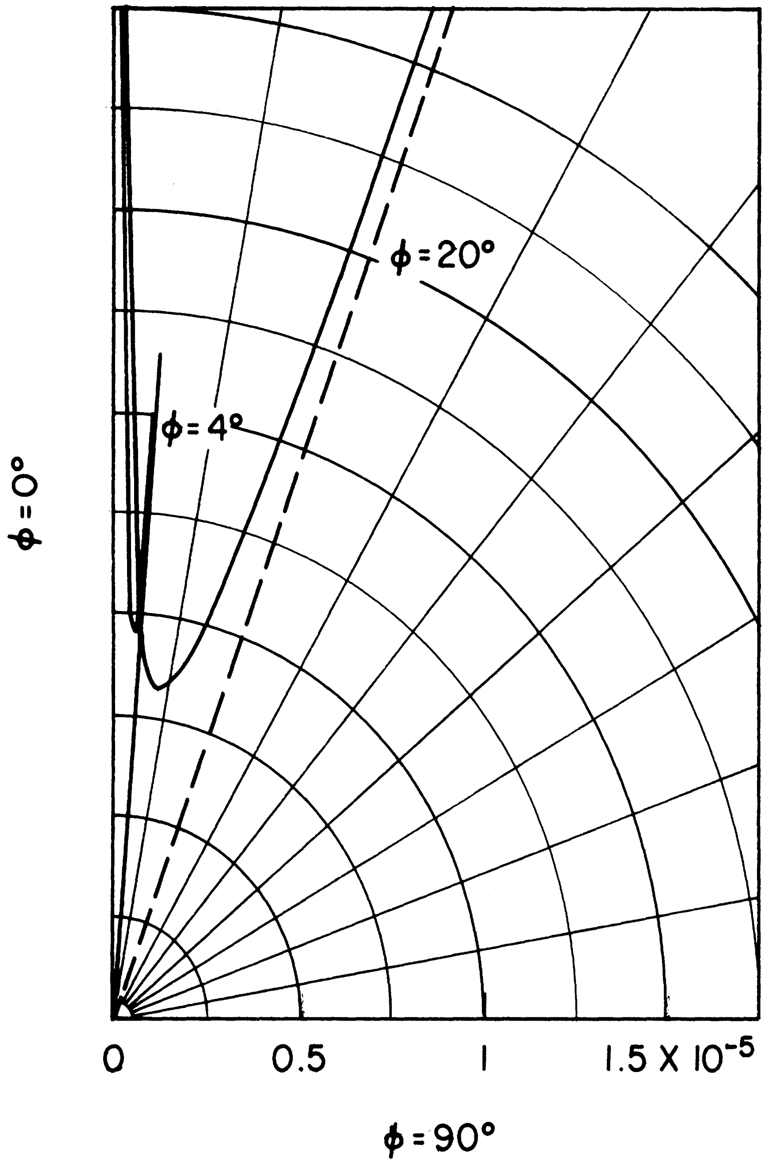


FIG. 18: E'_ϕ VS. ϕ

400 KM, $\omega = 3 \times 10^6$

POINT CURRENT SOURCE ($\phi = 0^0$)

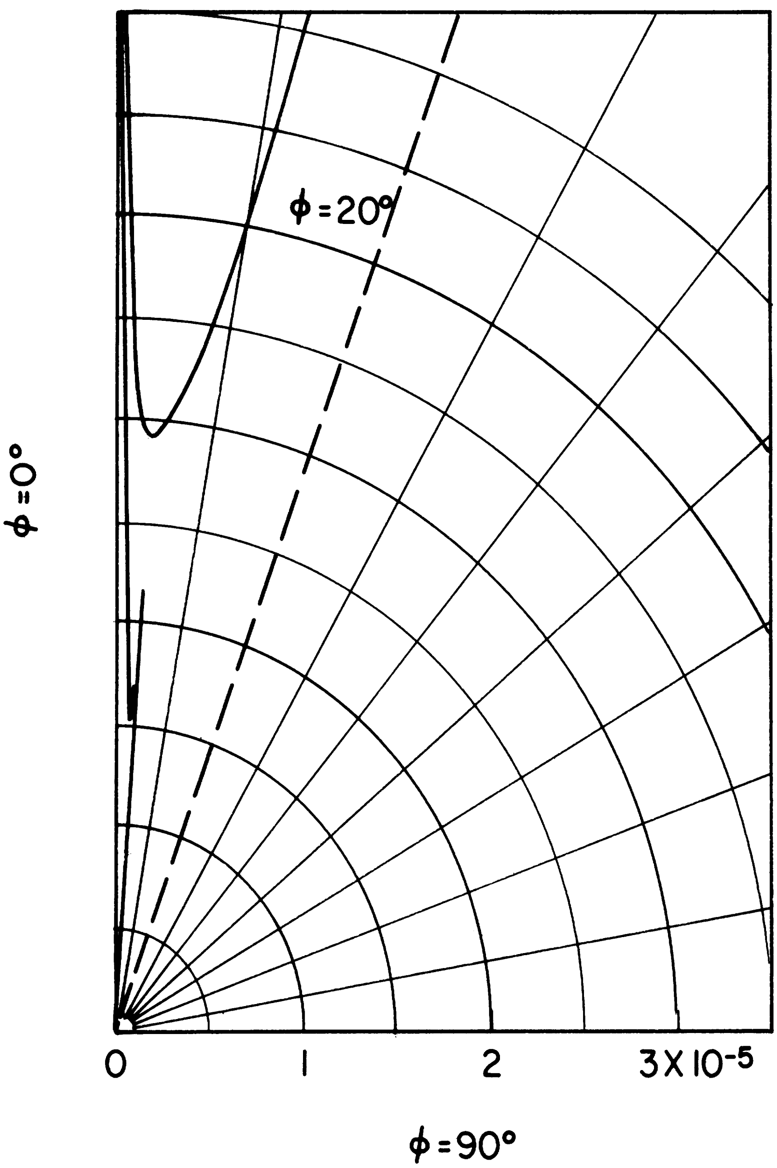


FIG. 19: E'_M VS. ϕ

400 KM, $\omega = 3 \times 10^6$

POINT CURRENT SOURCE (ϕ = 0^o)

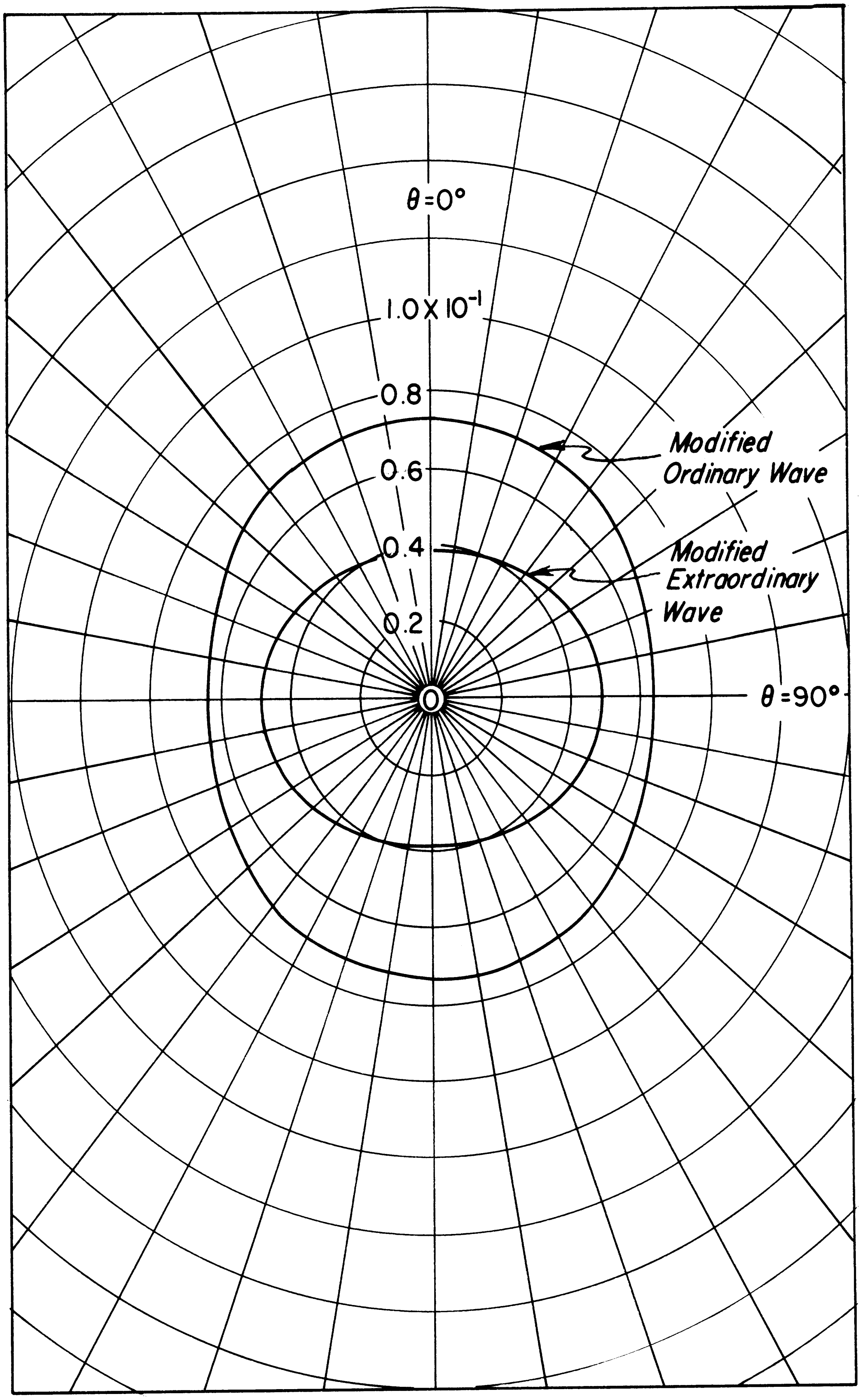


FIG. 20: DISPERSION CURVES FOR MODIFIED ORDINARY AND EXTRAORDINARY WAVES

400 KM, $\omega = 3 \times 10^7$

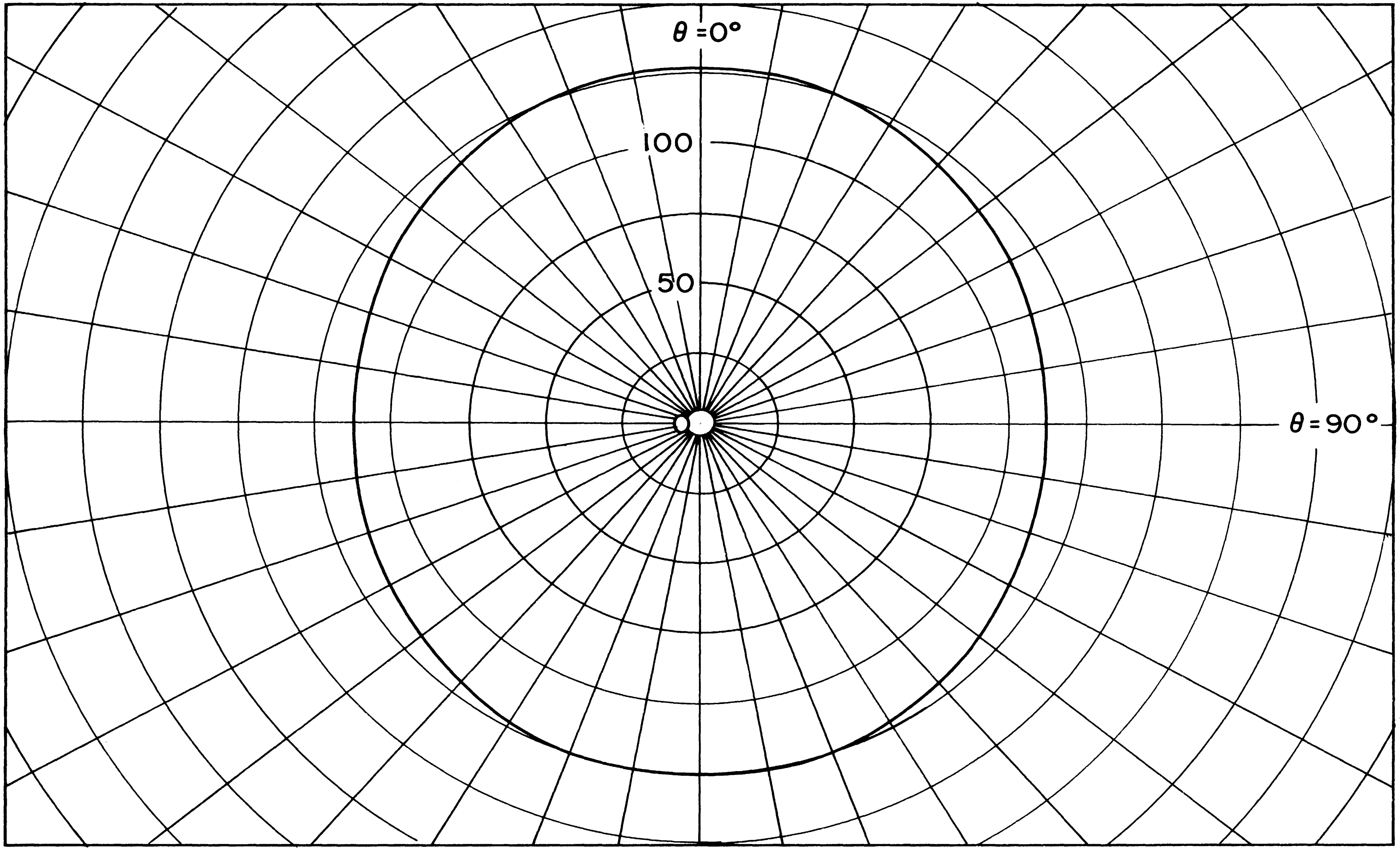


FIG. 21: DISPERSION CURVE FOR MODIFIED PLASMA WAVE

400 KM, $\omega = 3 \times 10^7$

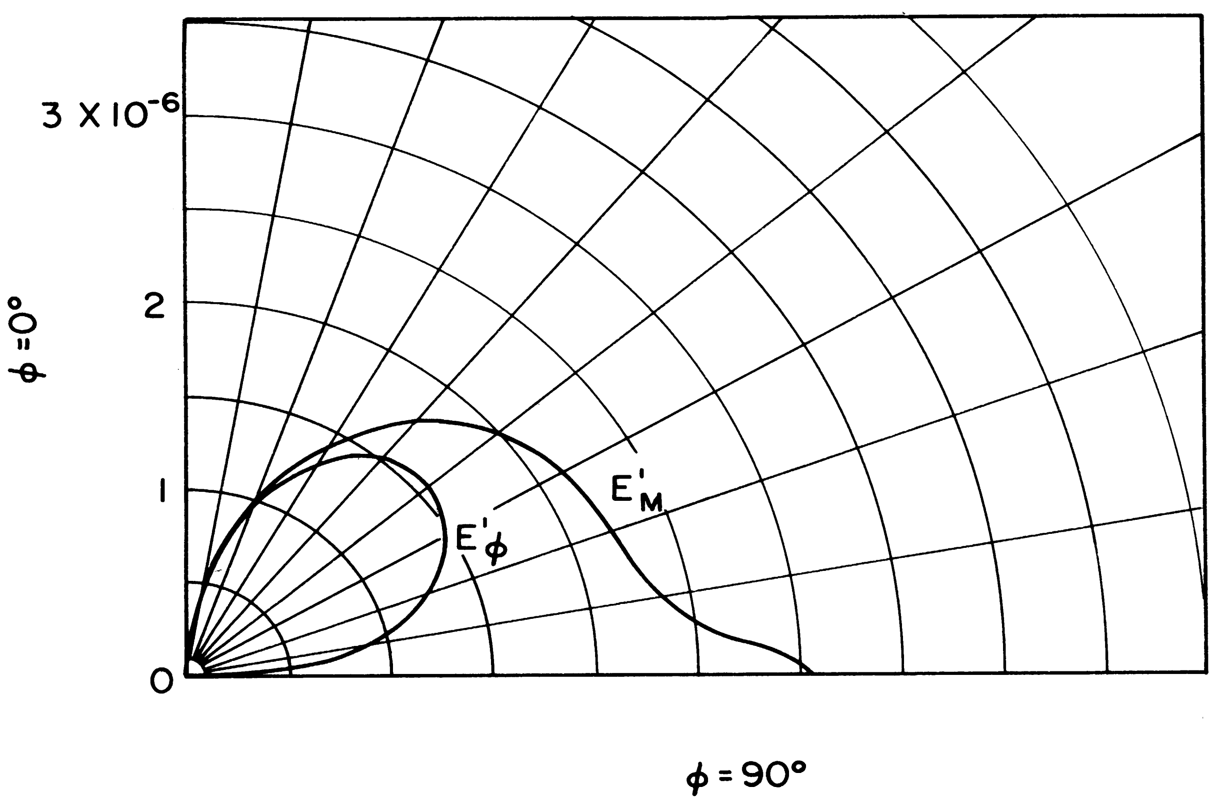


FIG. 22: E'_ϕ, E'_M VS. ϕ FOR MODIFIED ORDINARY WAVE

400 KM, $\omega = 3 \times 10^7$

POINT CURRENT SOURCE ($\phi = 0^o$)

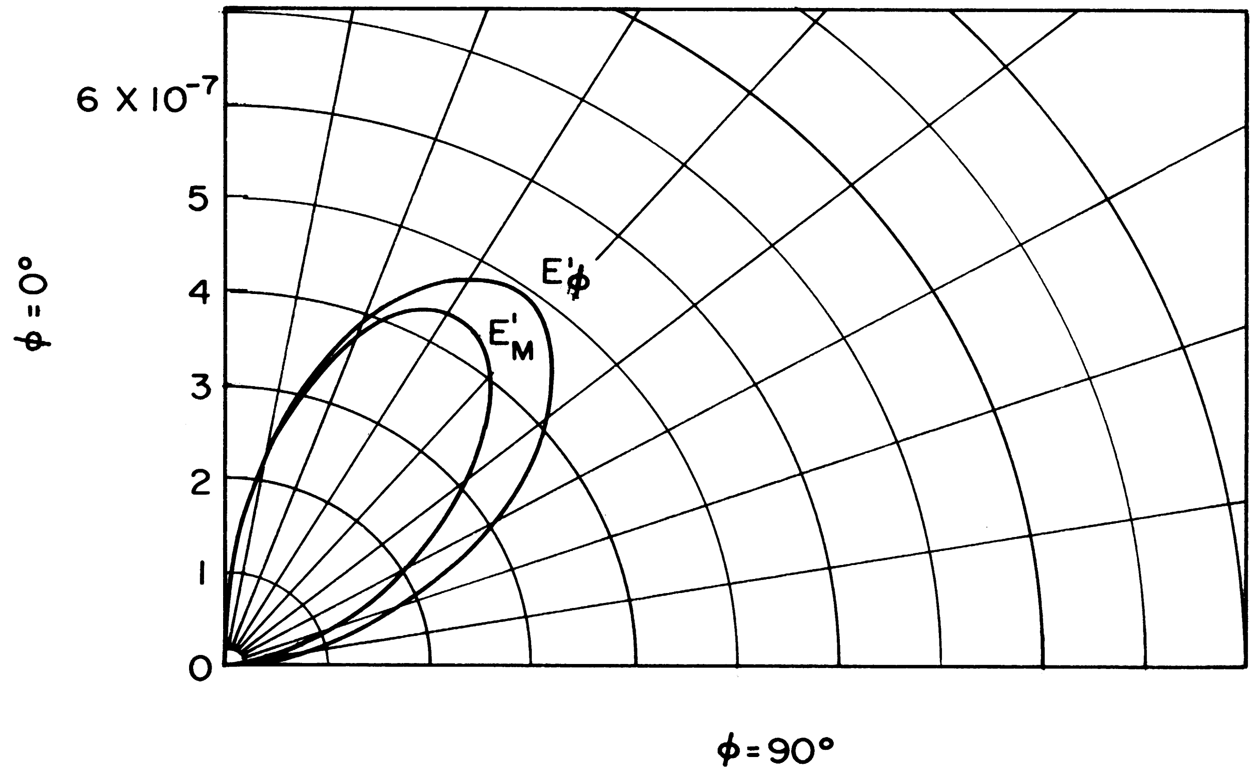


FIG. 23: E'_{ϕ}, E'_M VS. ϕ FOR MODIFIED EXTRAORDINARY WAVE

400 KM, $\omega = 3 \times 10^7$

POINT CURRENT SOURCE ($\phi = 0^{\circ}$)

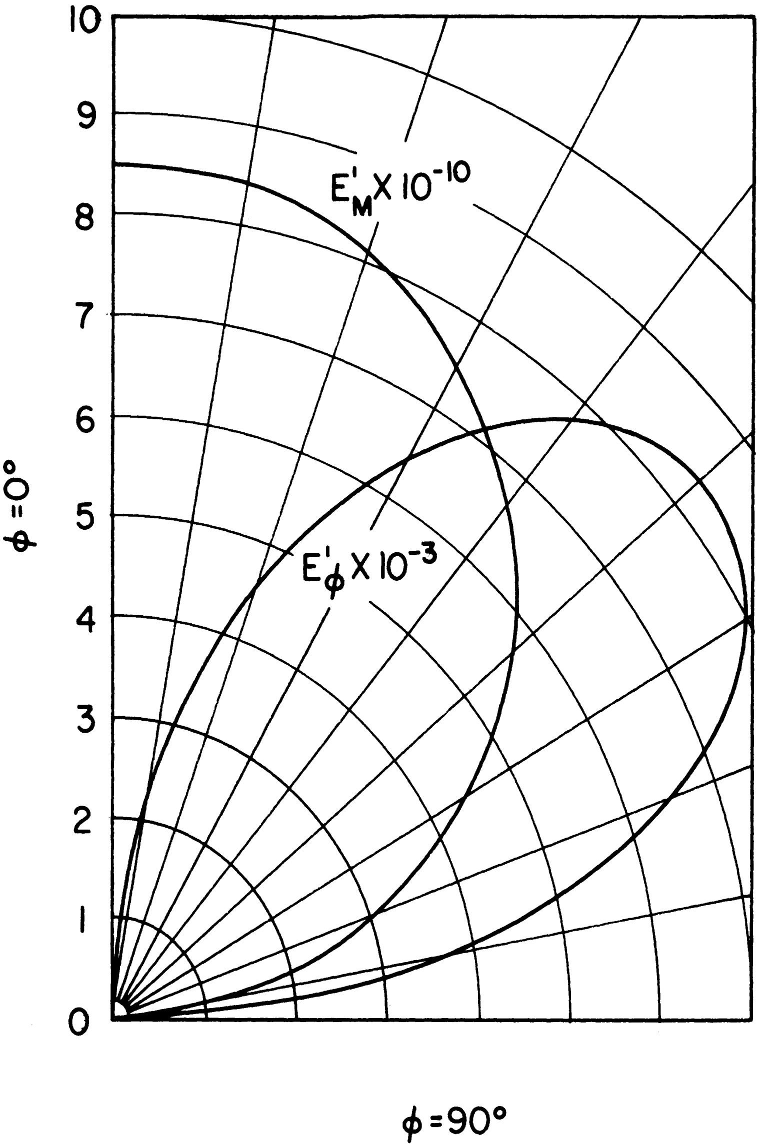


FIG. 24: E'_ϕ, E'_M VS. ϕ FOR MODIFIED PLASMA WAVE

400 KM, $\omega = 3 \times 10^7$

POINT CURRENT SOURCE ($\phi = 0^0$)

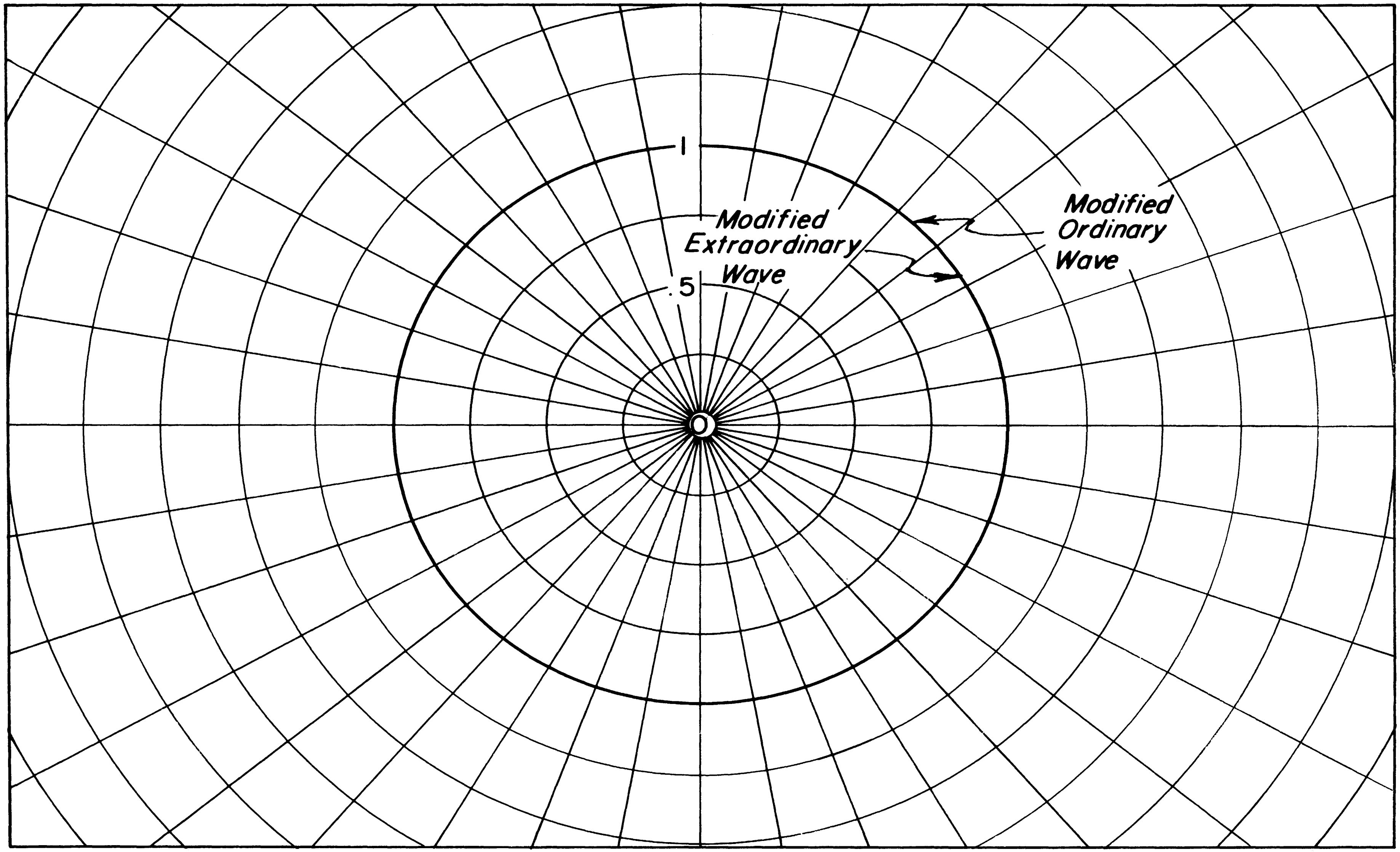


FIG. 25: DISPERSION CURVES FOR MODIFIED ORDINARY AND EXTRAORDINARY WAVES

400 KM, $\omega = 3 \times 10^8$

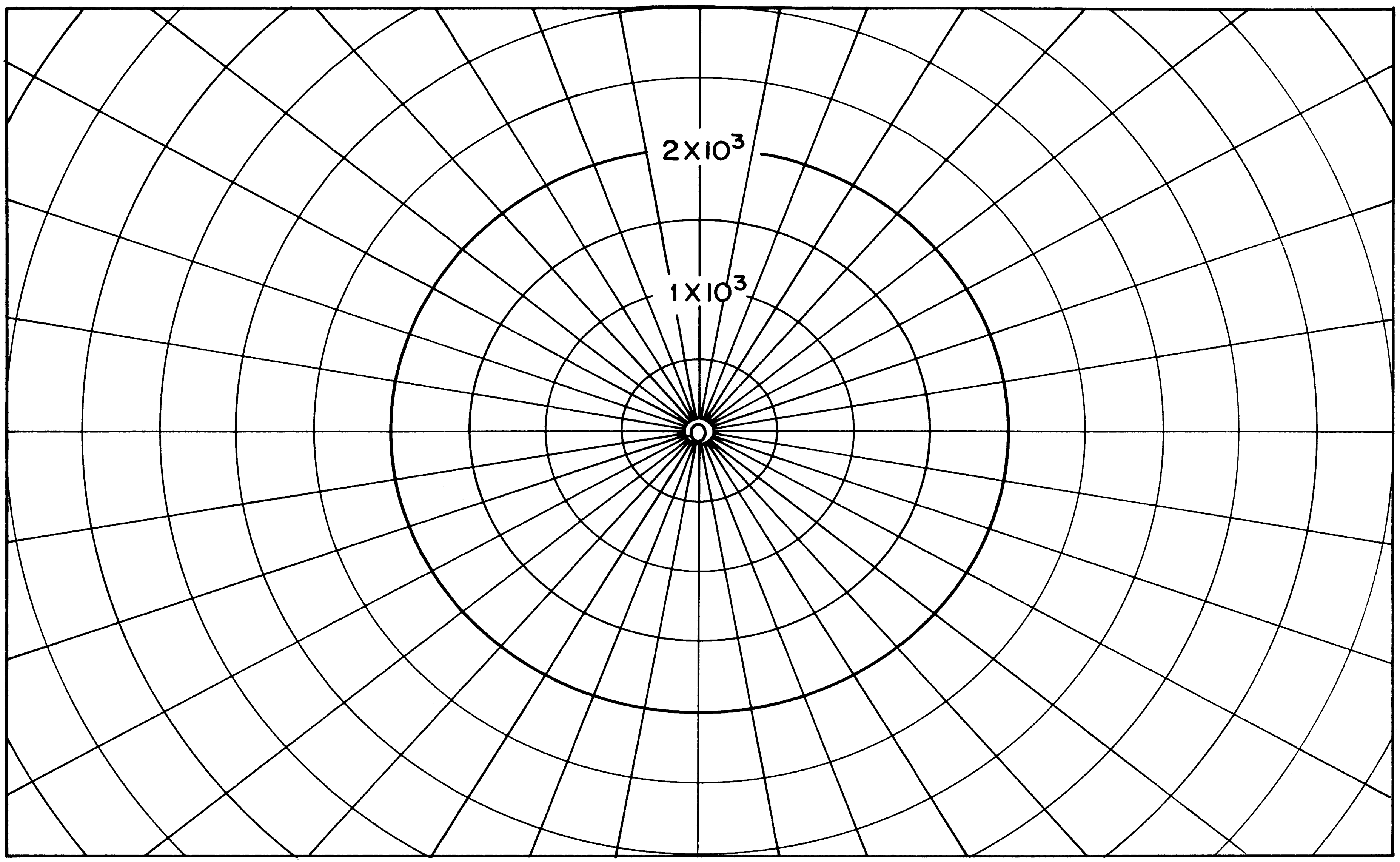


FIG. 26: DISPERSION CURVE FOR MODIFIED PLASMA WAVE

400 KM, $\omega = 3 \times 10^8$

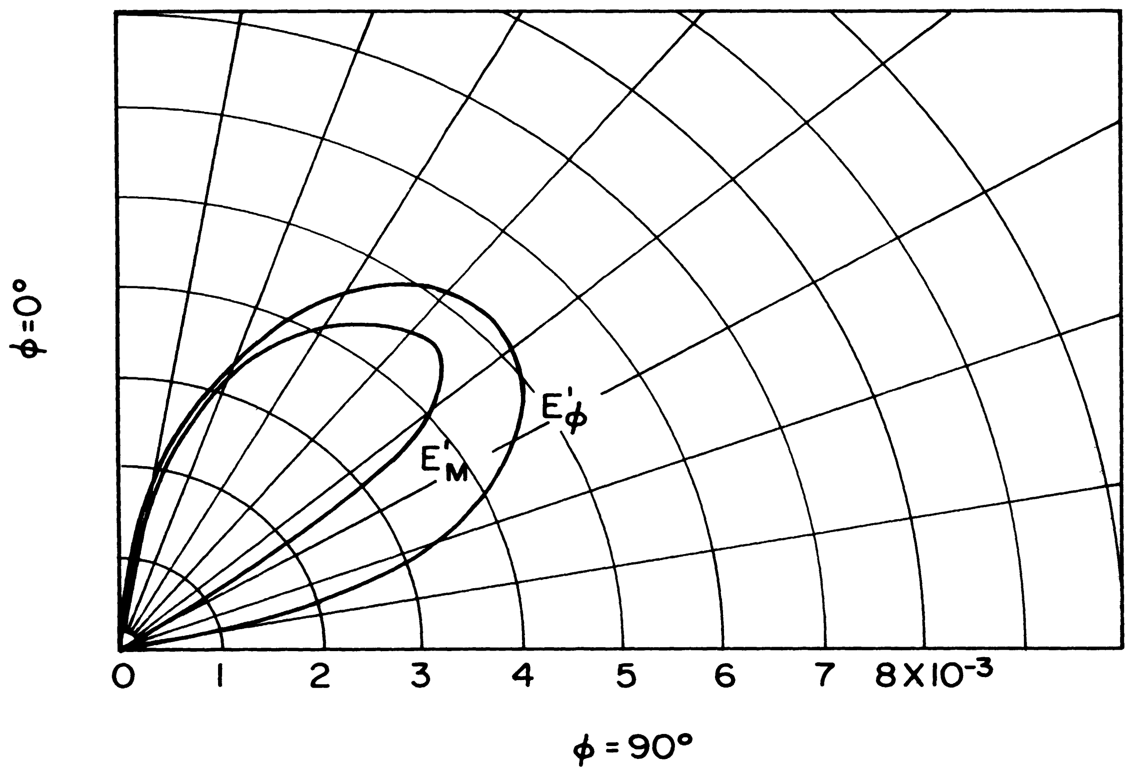


FIG. 27: E'_{ϕ}, E'_M VS. ϕ FOR MODIFIED ORDINARY AND EXTRAORDINARY WAVES

400 KM, $\omega = 3 \times 10^8$

POINT CURRENT SOURCE ($\phi = 0^{\circ}$)

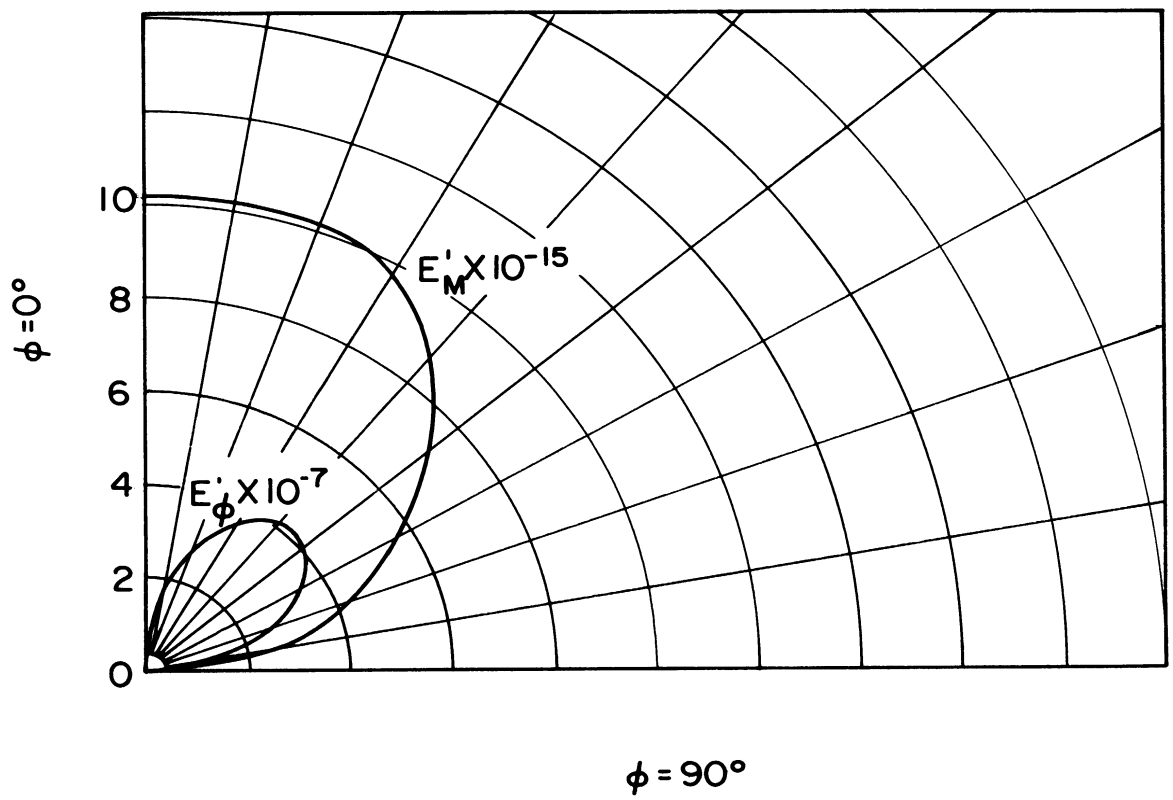


FIG. 28: E'_ϕ, E'_M VS ϕ FOR MODIFIED PLASMA WAVE

400 KM, $\omega = 3 \times 10^8$

POINT CURRENT SOURCE ($\phi = 0^\circ$)

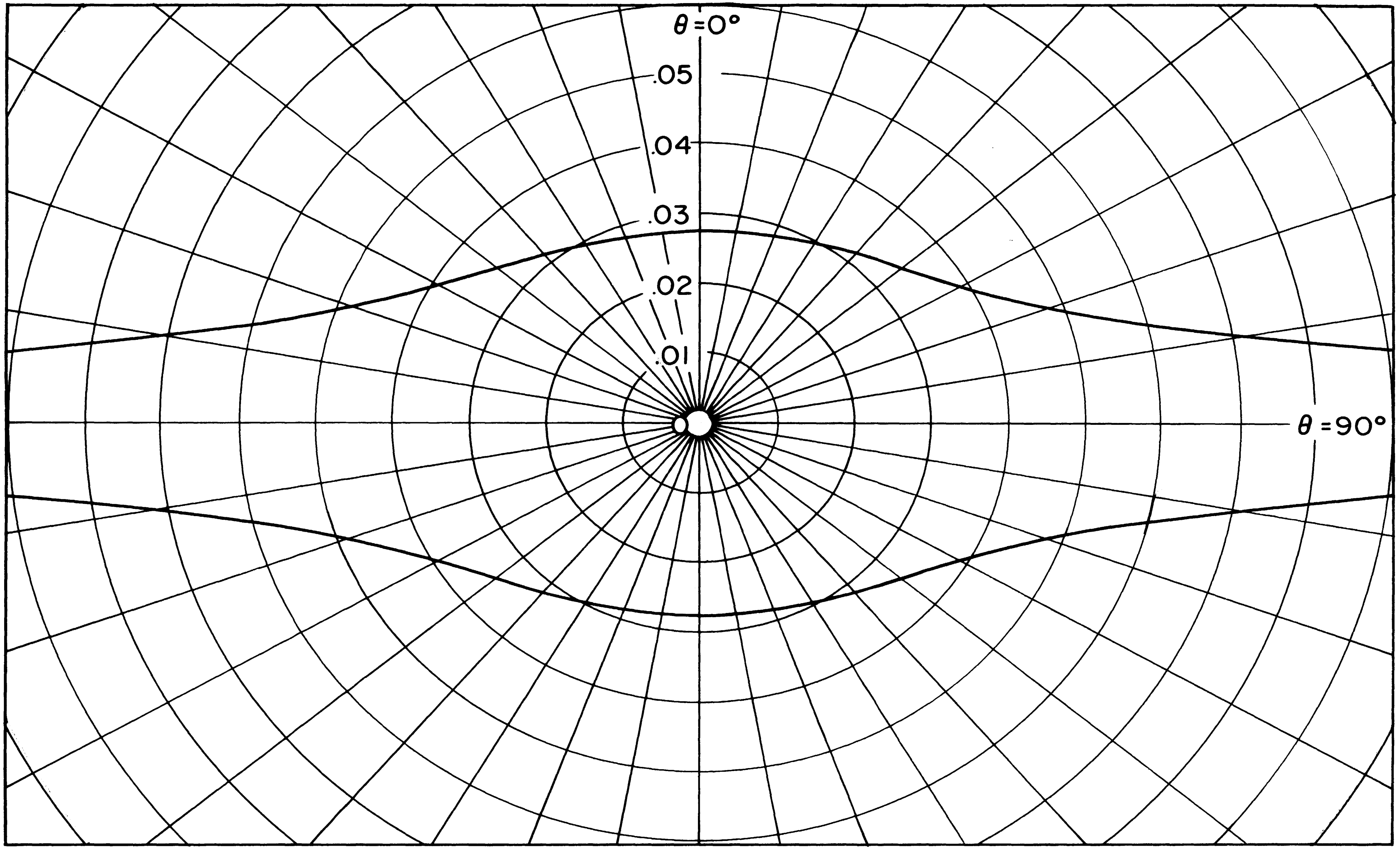


FIG. 29: DISPERSION CURVE

250 KM, $\omega = 3 \times 10^5$

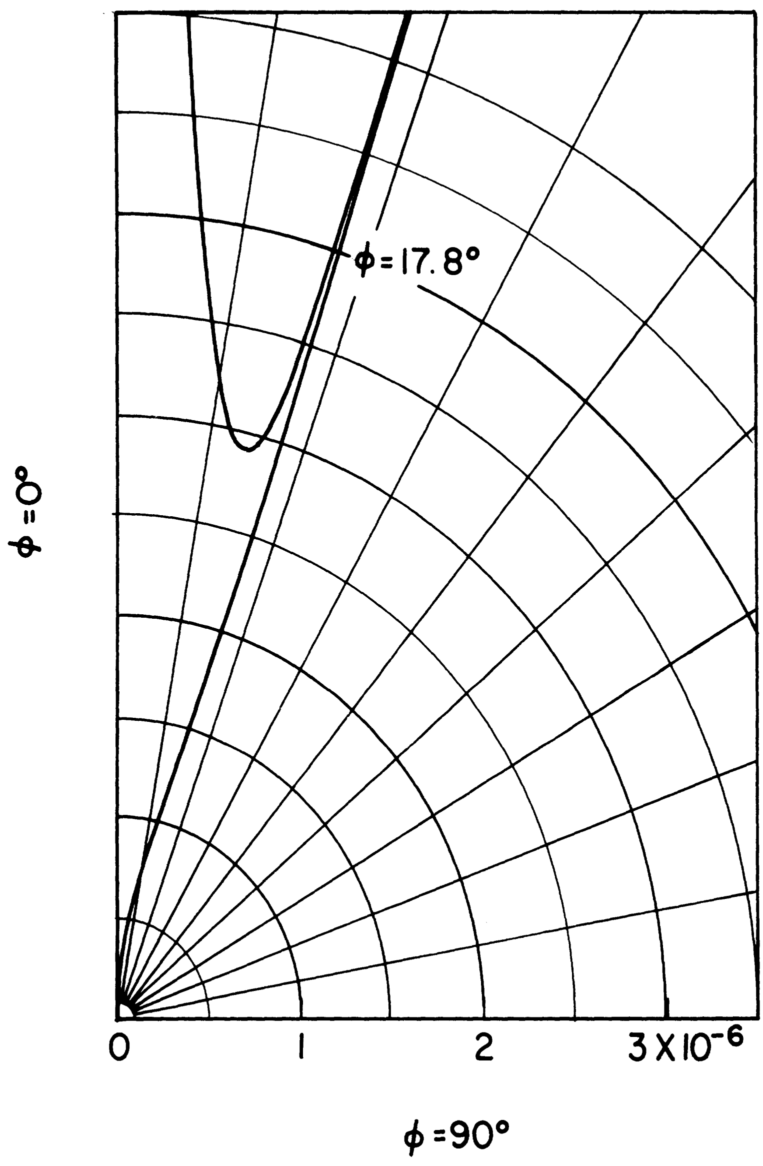


FIG. 30: E'_ϕ VS. ϕ

250 KM, $\omega = 3 \times 10^5$

POINT CURRENT SOURCE ($\phi = 0^0$)

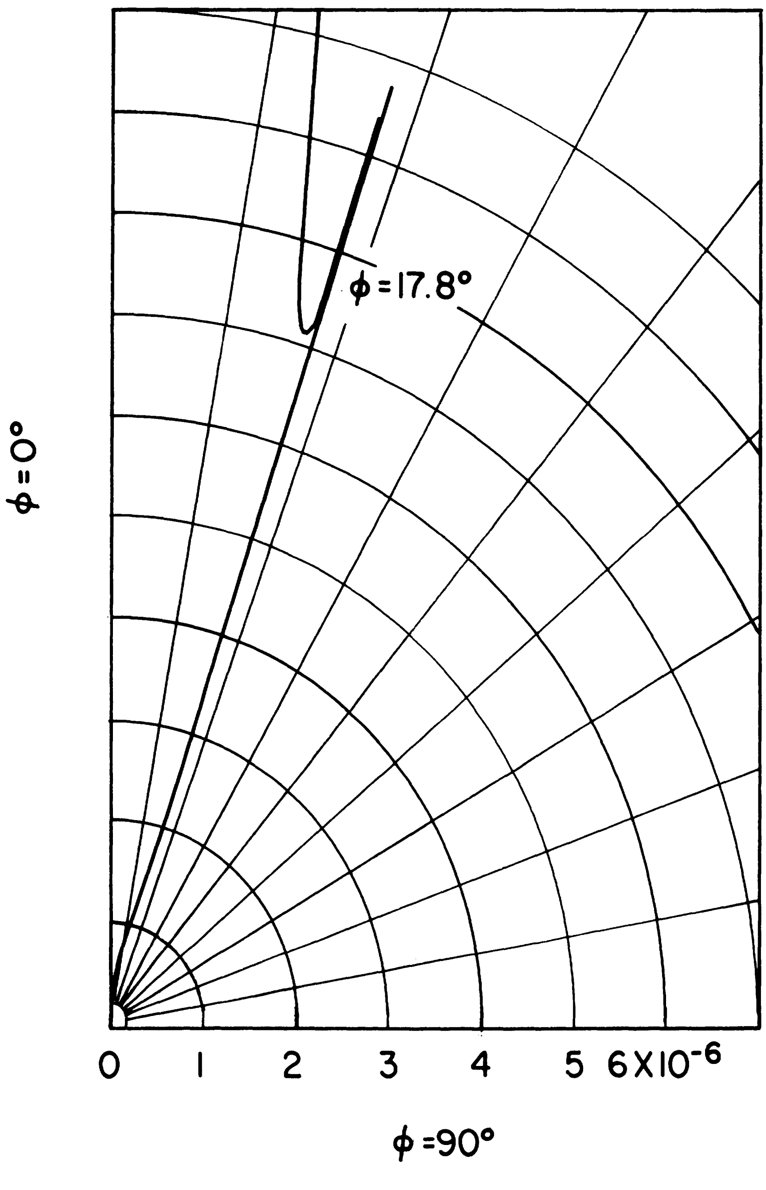


FIG. 31: E'_M VS. ϕ

250 KM, $\omega = 3 \times 10^5$

POINT CURRENT SOURCE (ϕ = 0^o)

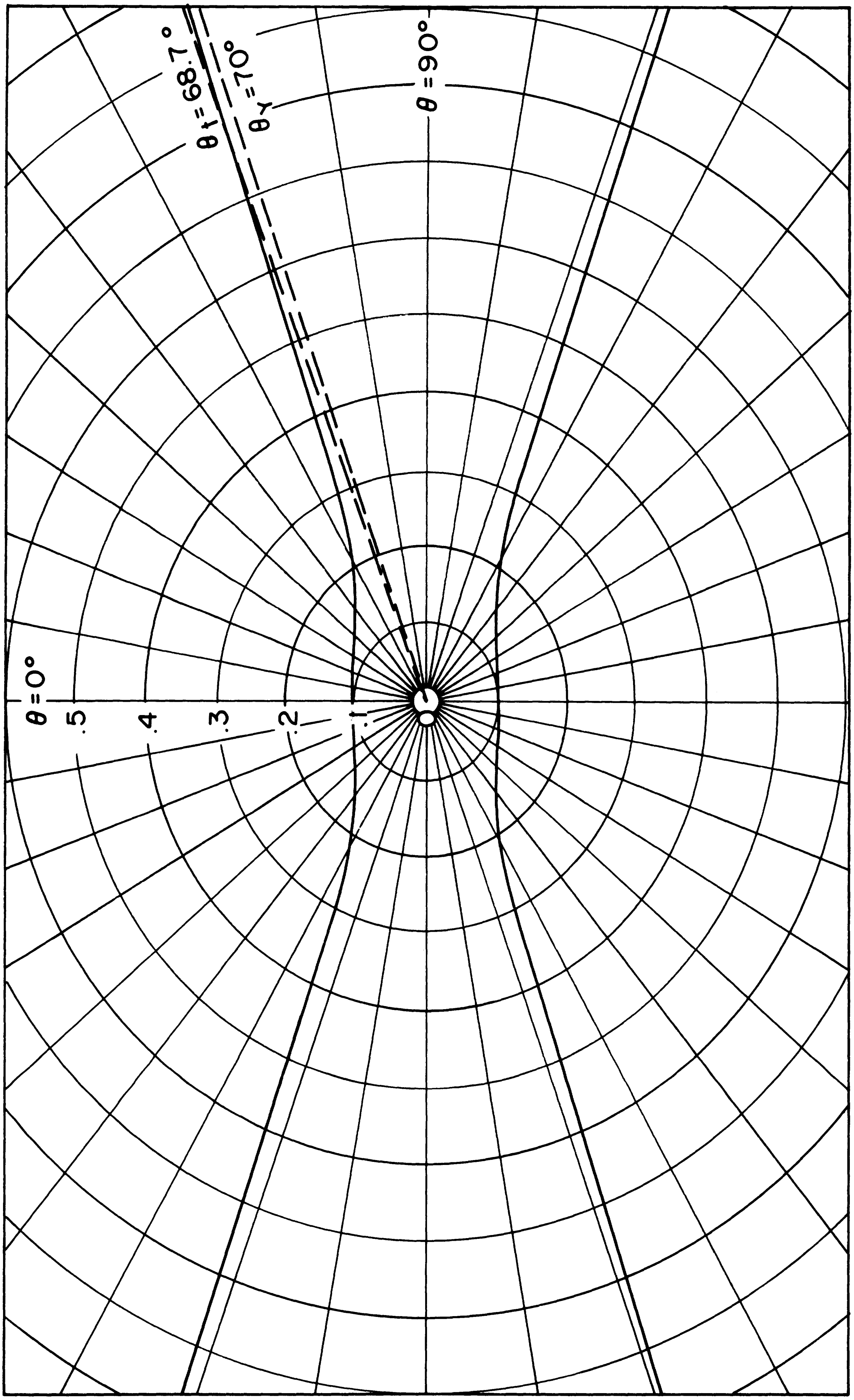


FIG. 32: DISPERSION CURVE
250 KM, $\omega = 3 \times 10^6$

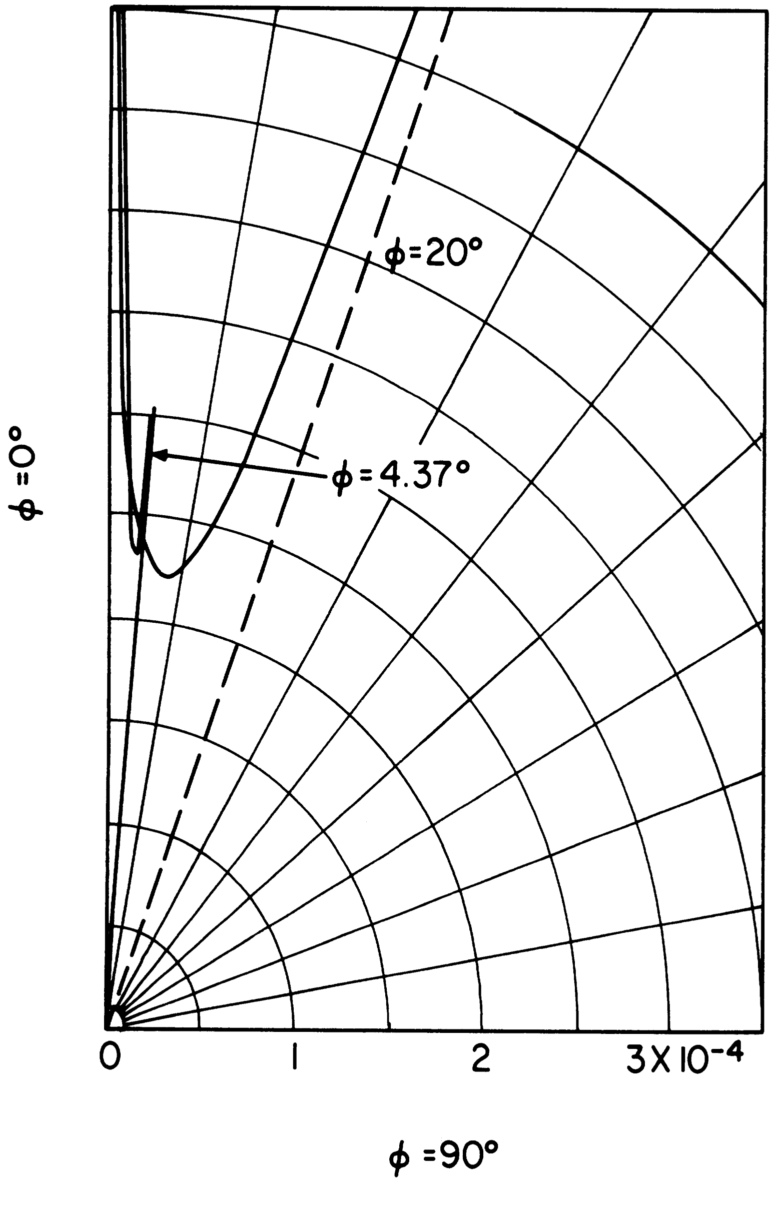


FIG. 33: E'_ϕ VS. ϕ

250 KM, $\omega = 3 \times 10^6$

POINT CURRENT SOURCE ($\phi = 0^o$)

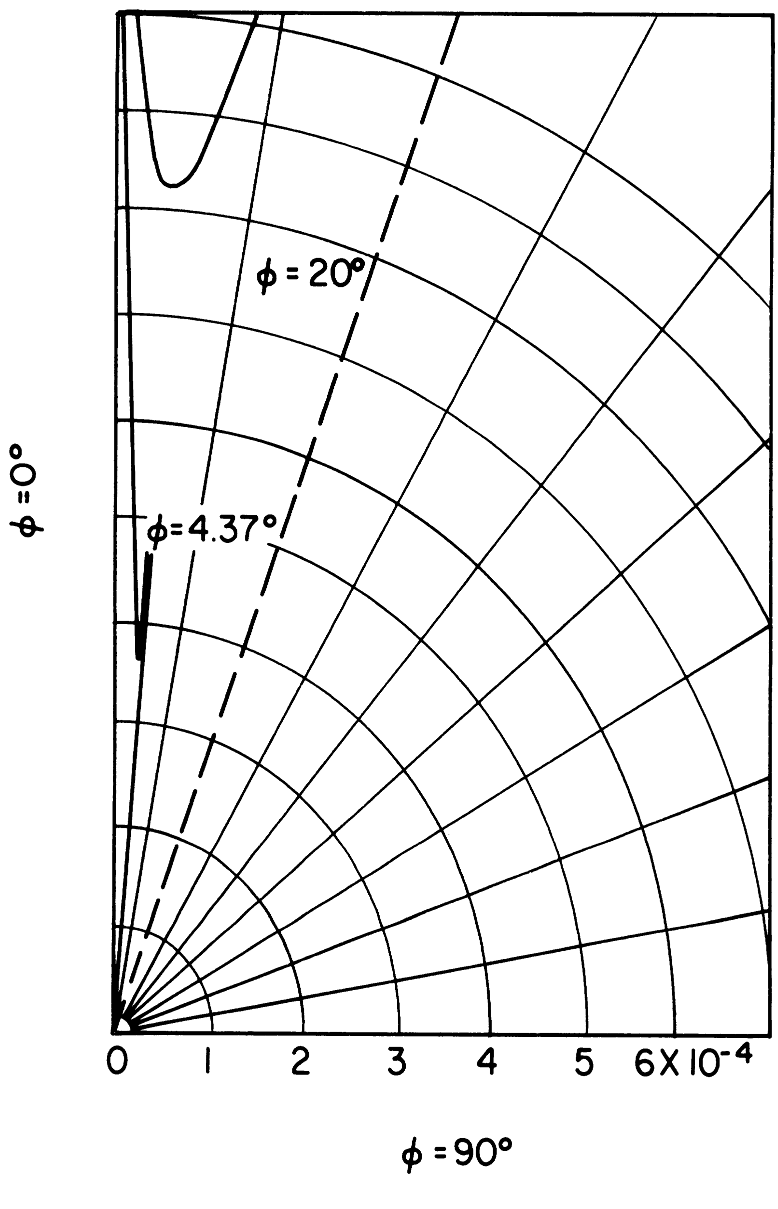


FIG. 34: E'_M VS. ϕ

250 KM, $\omega = 3 \times 10^6$

POINT CURRENT SOURCE ($\phi = 0^0$)

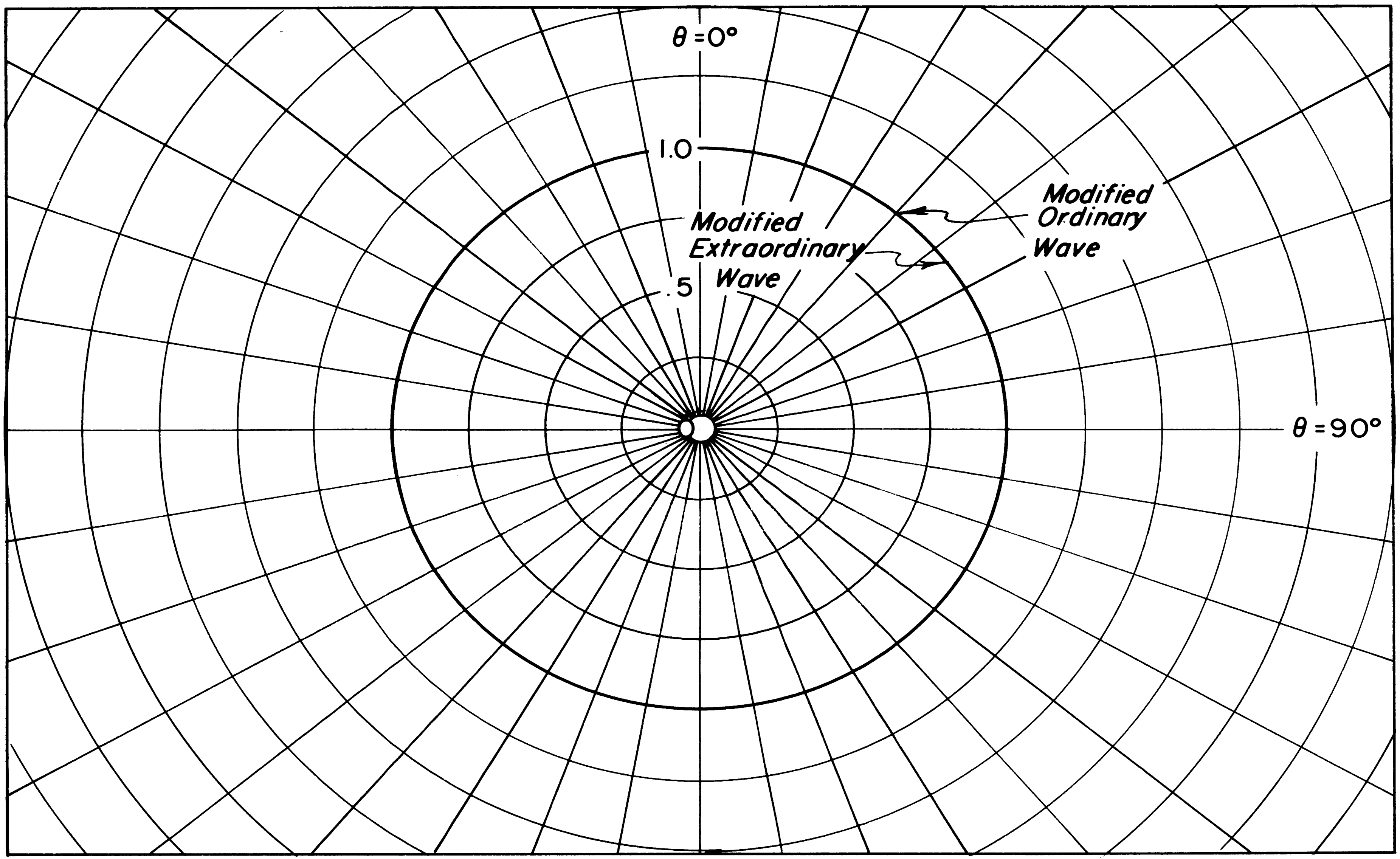


FIG. 35: DISPERSION CURVES FOR MODIFIED ORDINARY AND EXTRAORDINARY WAVES

250 KM, $\omega = 3 \times 10^8$

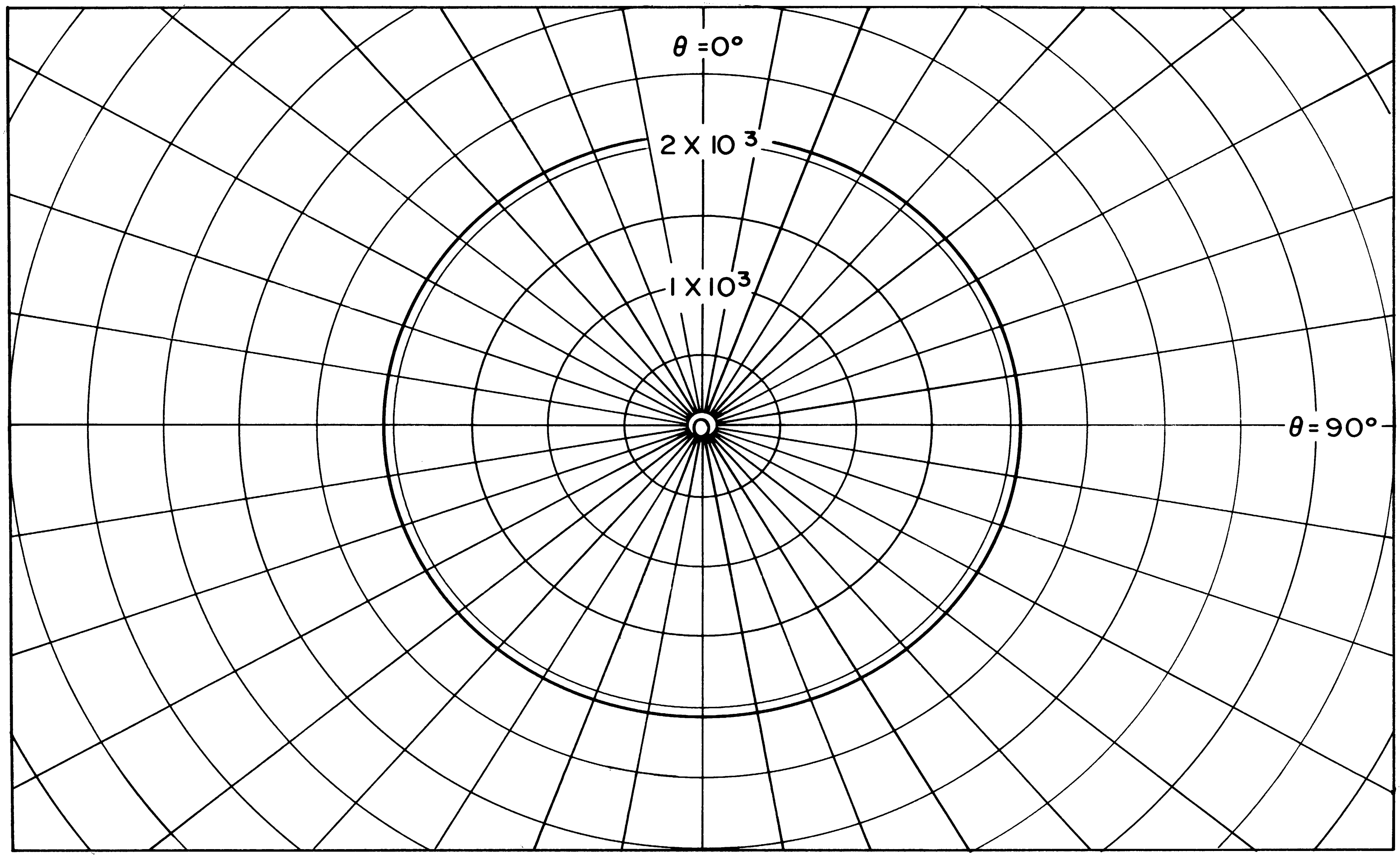


FIG. 36: DISPERSION CURVE FOR MODIFIED PLASMA WAVE

250 KM, $\omega = 3 \times 10^8$

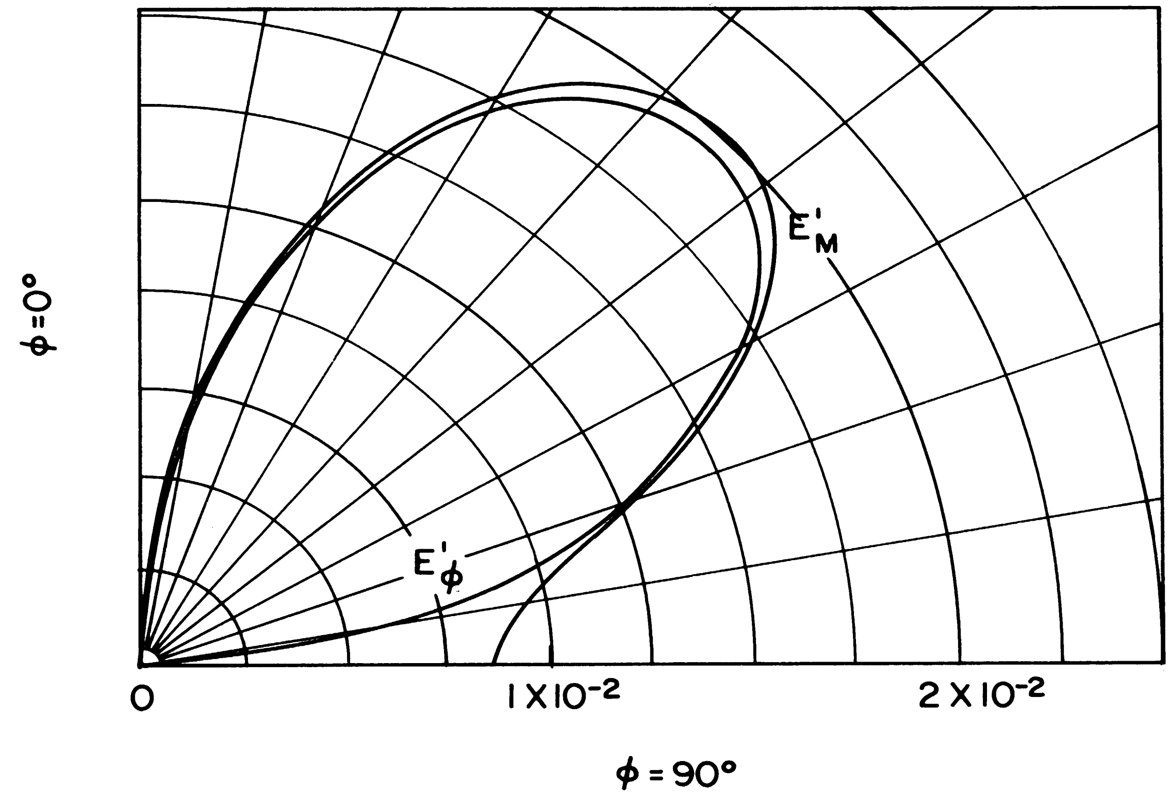


FIG. 37: E'_ϕ, E'_M VS. ϕ FOR MODIFIED ORDINARY WAVE

250 KM, $\omega = 3 \times 10^8$

POINT CURRENT SOURCE ($\phi = 0^{\circ}$)

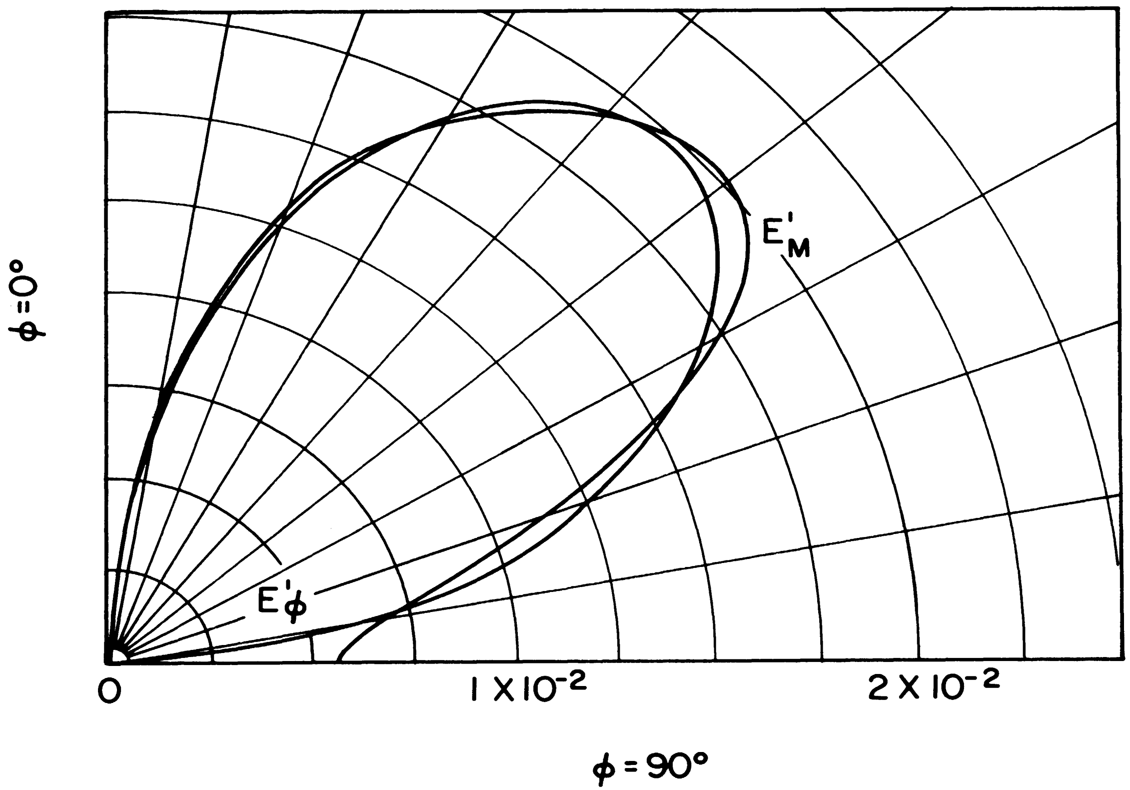


FIG. 38: E'_ϕ, E'_M VS. ϕ FOR MODIFIED EXTRAORDINARY WAVE

250 KM, $\omega = 3 \times 10^8$

POINT CURRENT SOURCE ($\phi = 0^o$)

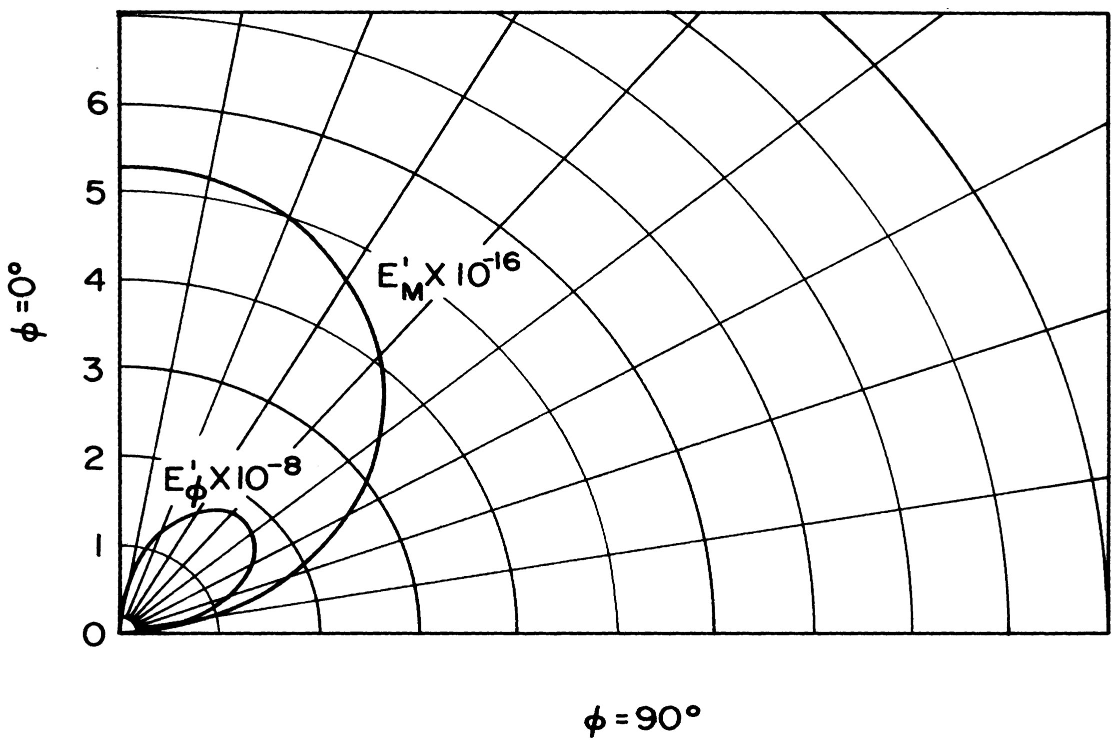


FIG. 39: E'_ϕ, E'_M VS. ϕ FOR MODIFIED PLASMA WAVE
250 KM, $\omega = 3 \times 10^8$
POINT CURRENT SOURCE ($\phi = 0°$)

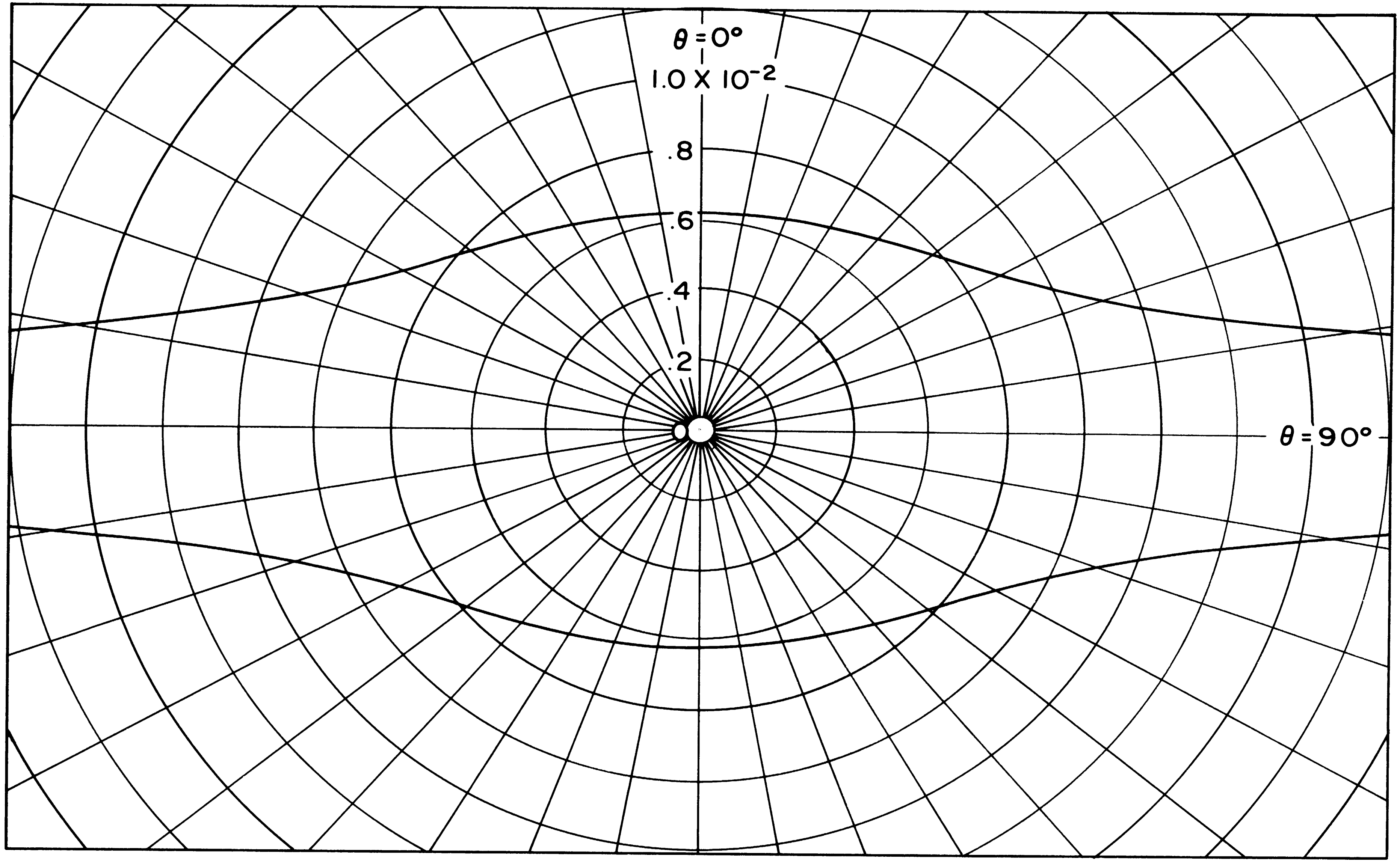


FIG. 40: DISPERSION CURVE
100 KM, $\omega = 3 \times 10^5$

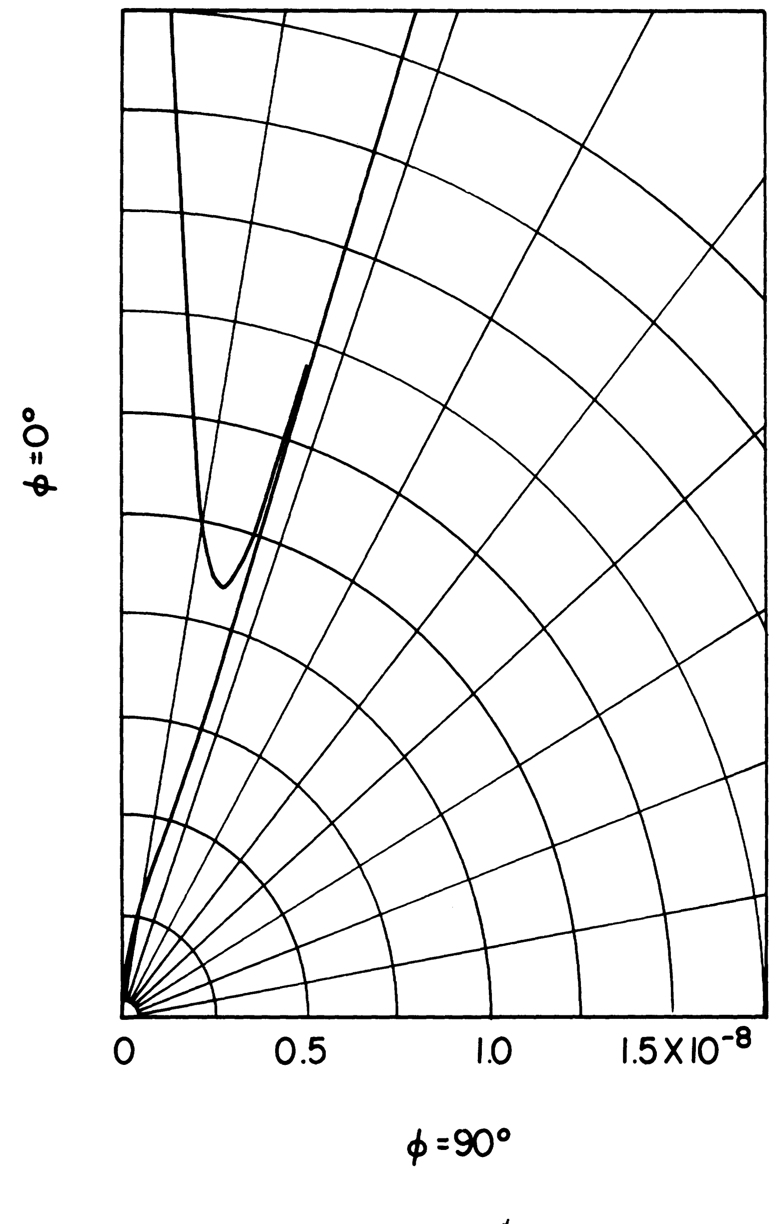


FIG. 41 : E'_{ϕ} VS. ϕ

100 KM, $\omega = 3 \times 10^5$

POINT CURRENT SOURCE ($\phi = 0^{\circ}$)

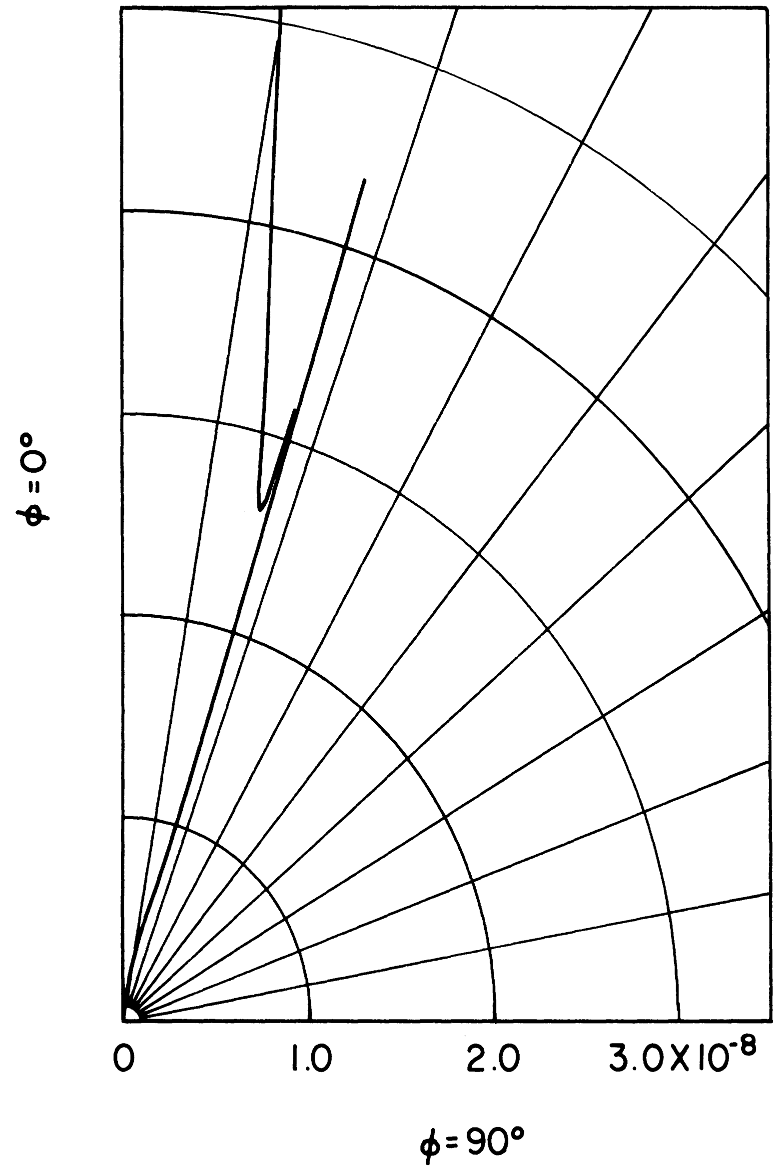


FIG. 42: E'_M VS. ϕ

100 KM, $\omega = 3 \times 10^5$

POINT CURRENT SOURCE ($\phi = 0^\circ$)

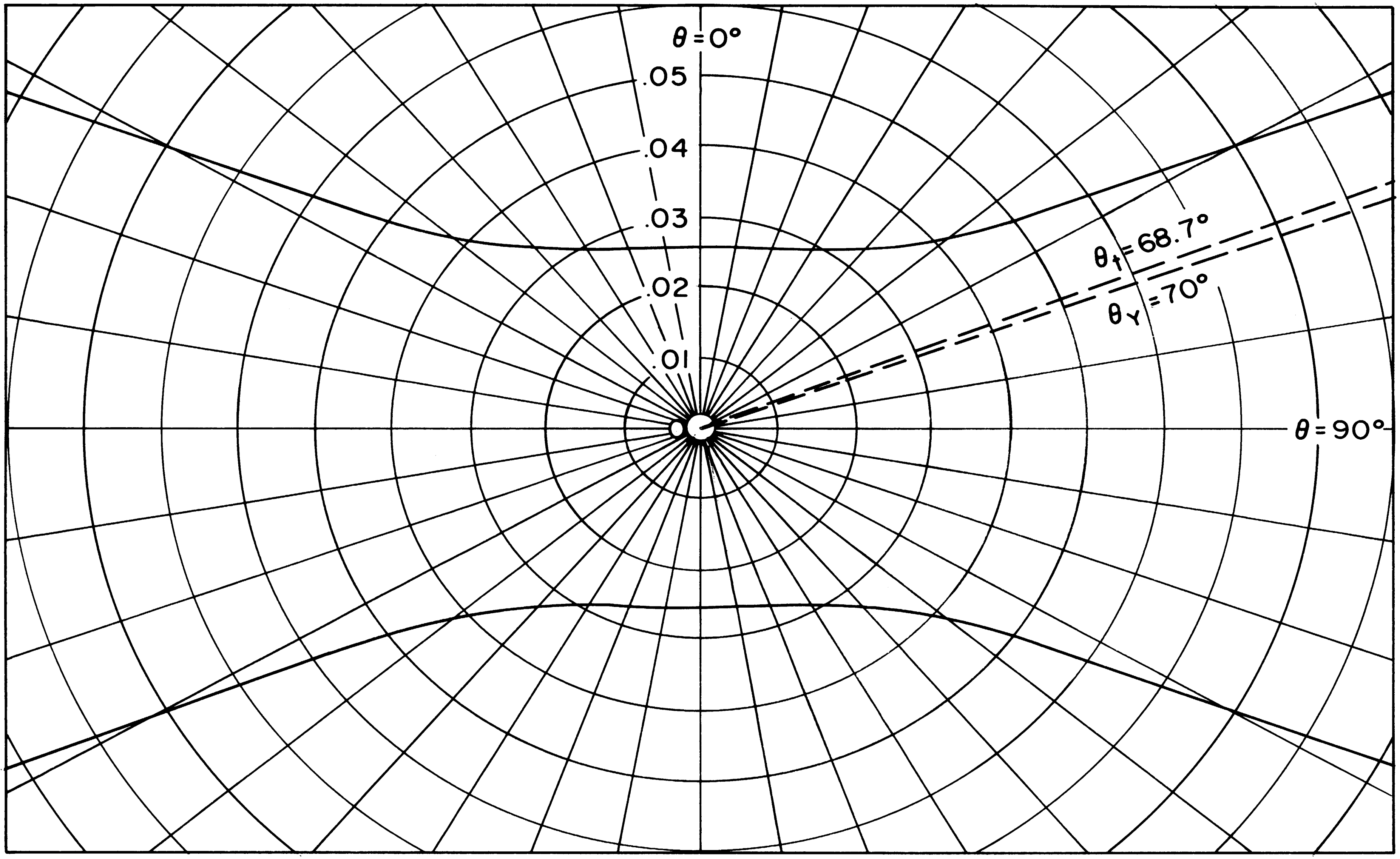


FIG. 43: DISPERSION CURVE
100 KM, $\omega = 3 \times 10^6$

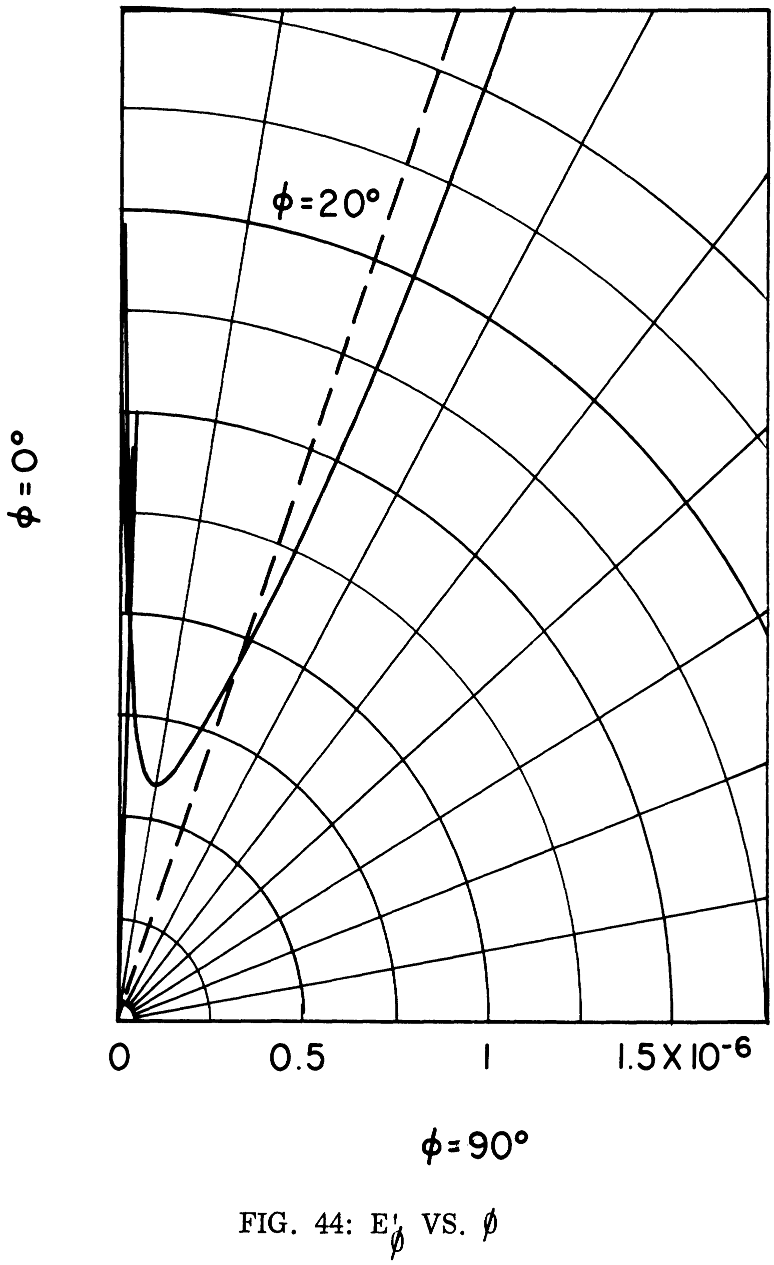


FIG. 44: E'_ϕ VS. ϕ

100 KM, $\omega = 3 \times 10^6$

POINT CURRENT SOURCE ($\phi = 0^0$)

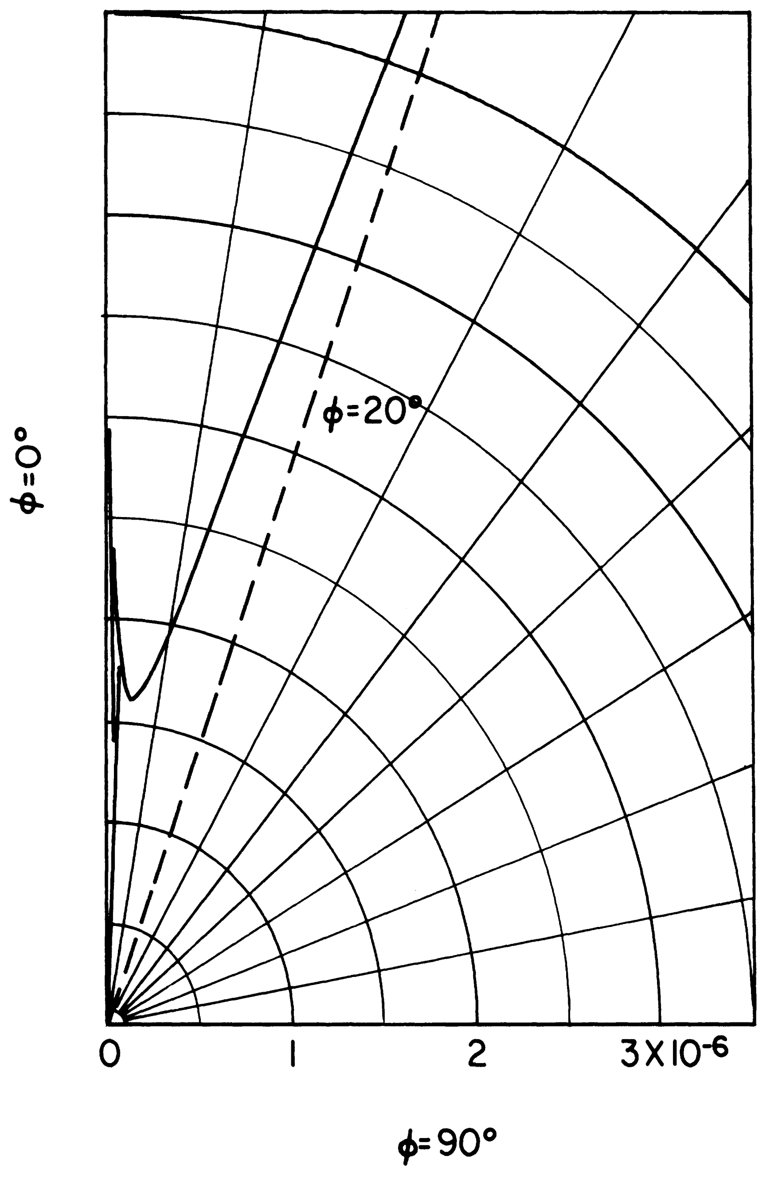


FIG. 45: E'_M VS. ϕ

100 KM, $\omega = 3 \times 10^6$

POINT CURRENT SOURCE (ϕ = 0^0)

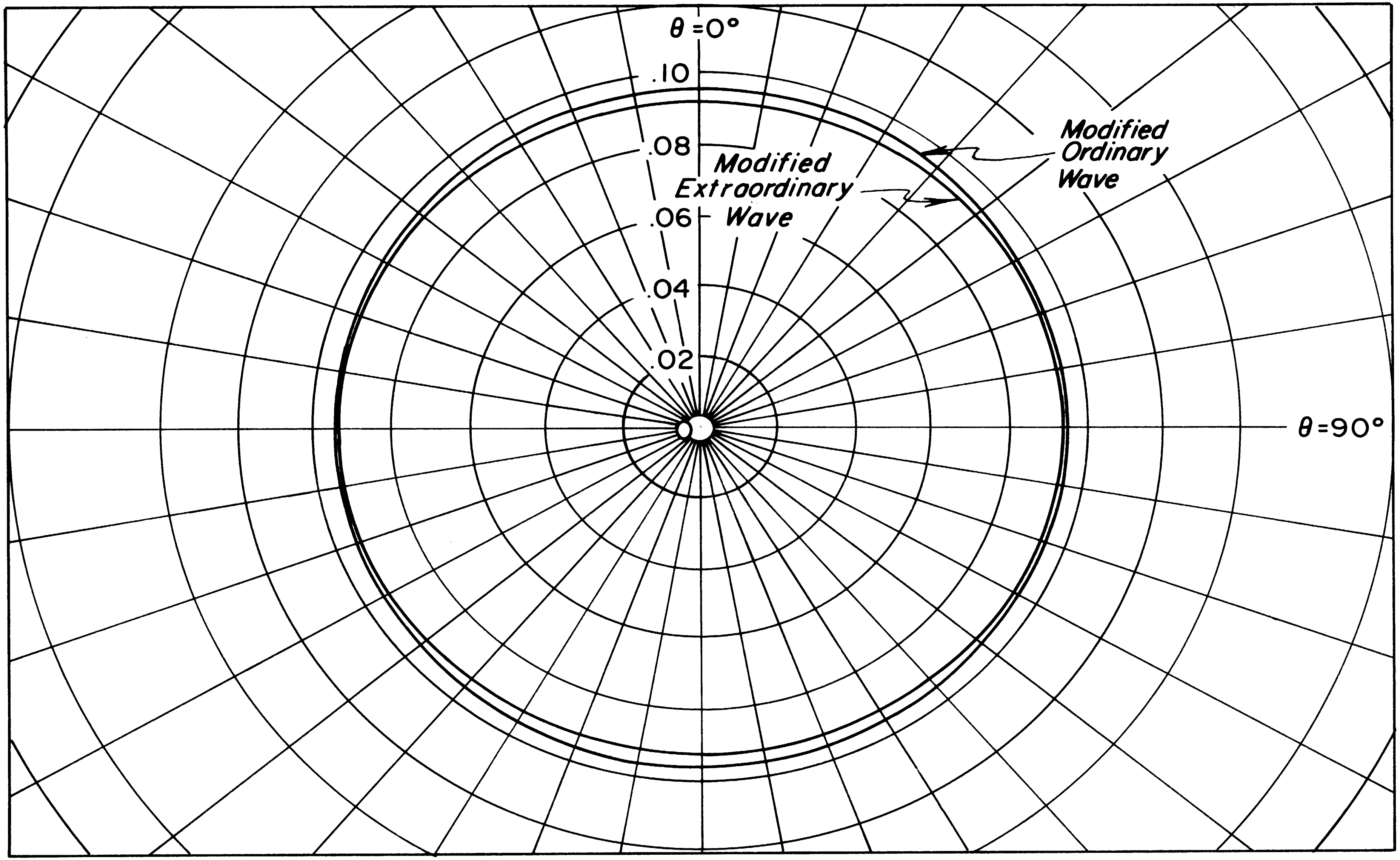


FIG. 46: DISPERSION CURVES FOR MODIFIED ORDINARY AND EXTRAORDINARY WAVES
100 KM, $\omega = 3 \times 10^7$

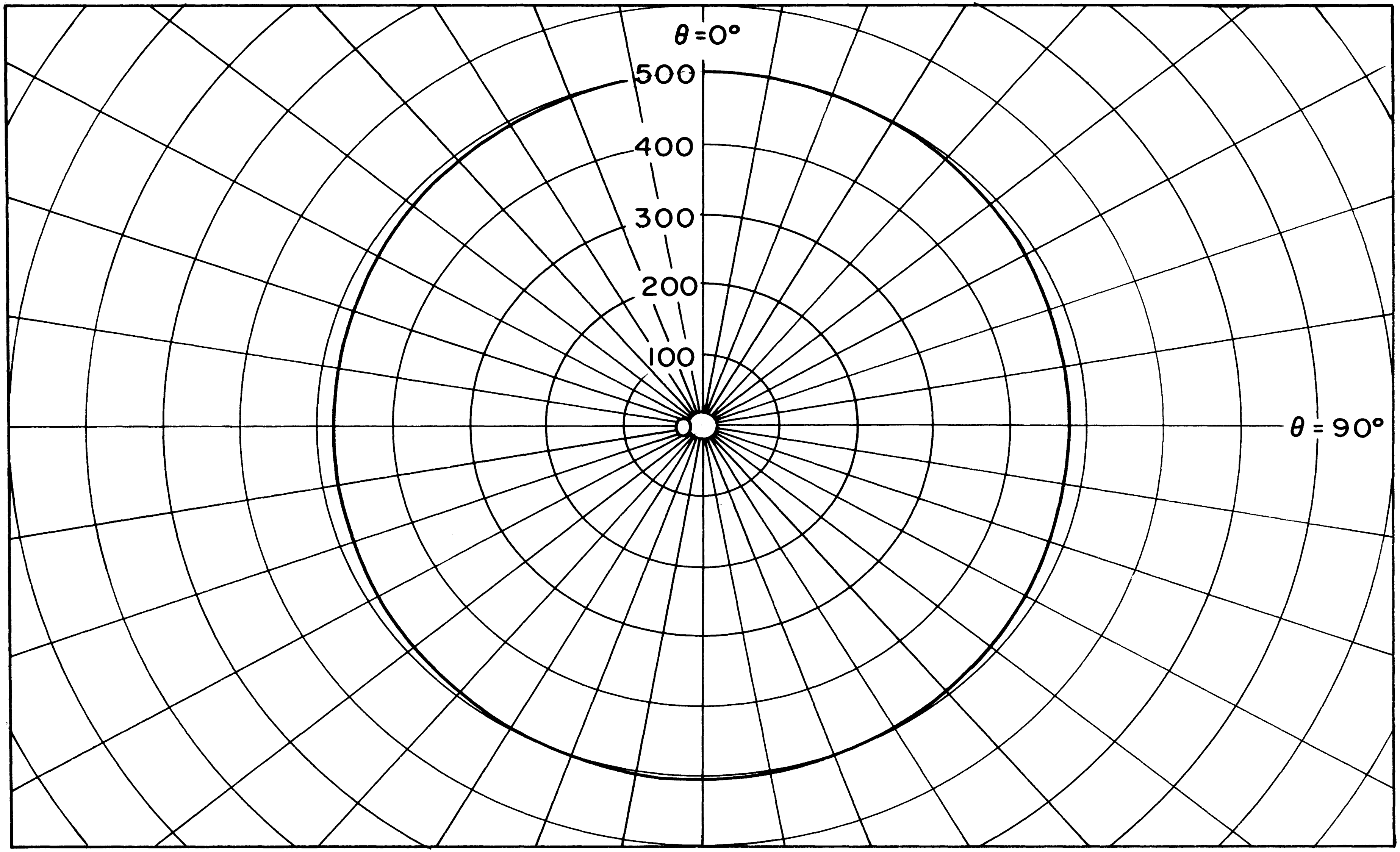


FIG. 47: DISPERSION CURVE FOR MODIFIED PLASMA WAVE
100 KM, $\omega = 3 \times 10^7$

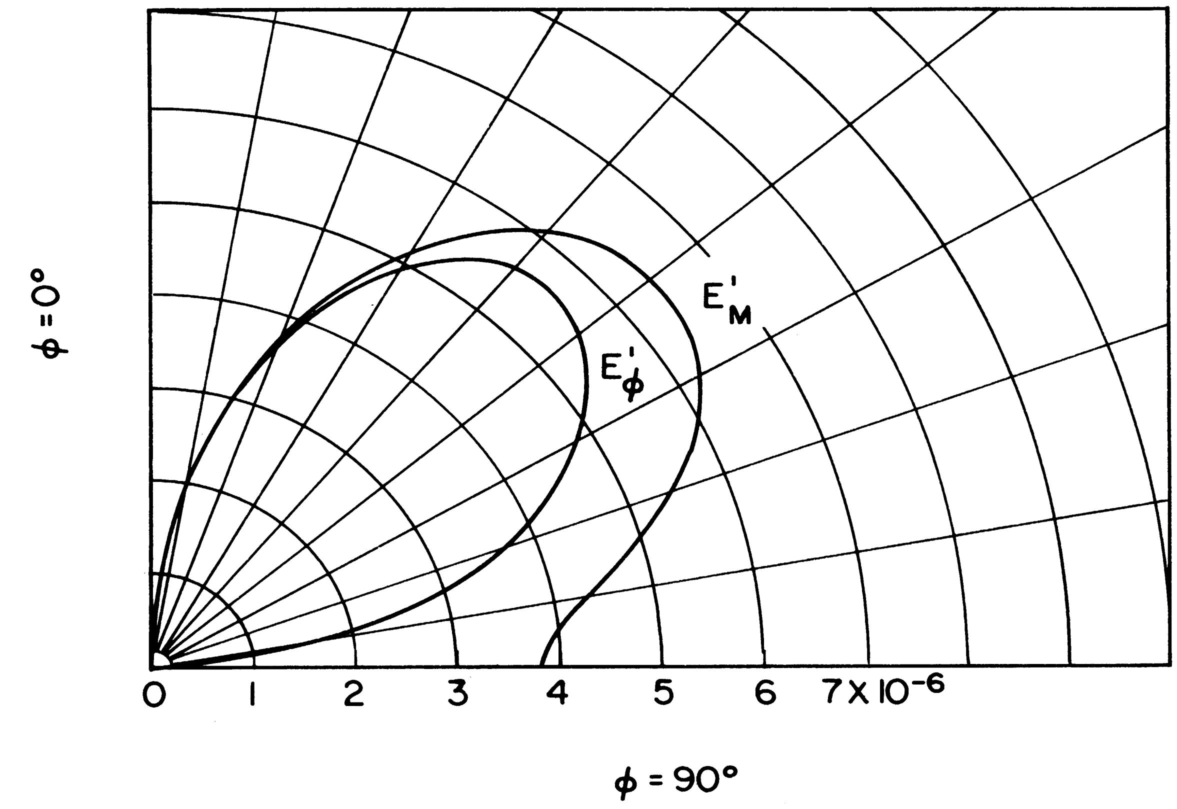


FIG. 48: E'_ϕ, E'_M VS. ϕ FOR MODIFIED ORDINARY WAVE

100 KM, $\omega = 3 \times 10^7$

POINT CURRENT SOURCE ($\phi = 0°$)

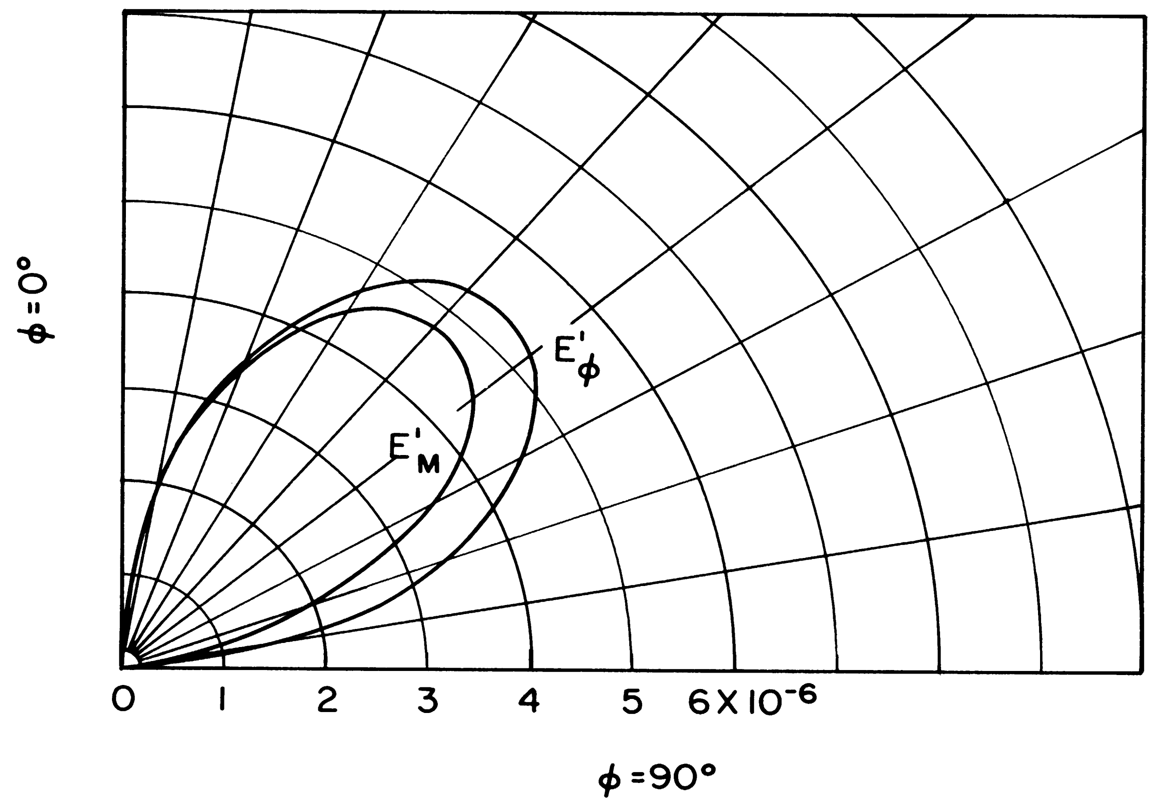


FIG. 49: E'_{ϕ}, E'_{M} VS. ϕ FOR MODIFIED EXTRAORDINARY WAVE

100 KM, $\omega = 3 \times 10^{7}$

POINT CURRENT SOURCE ($\phi = 0^{\circ}$)

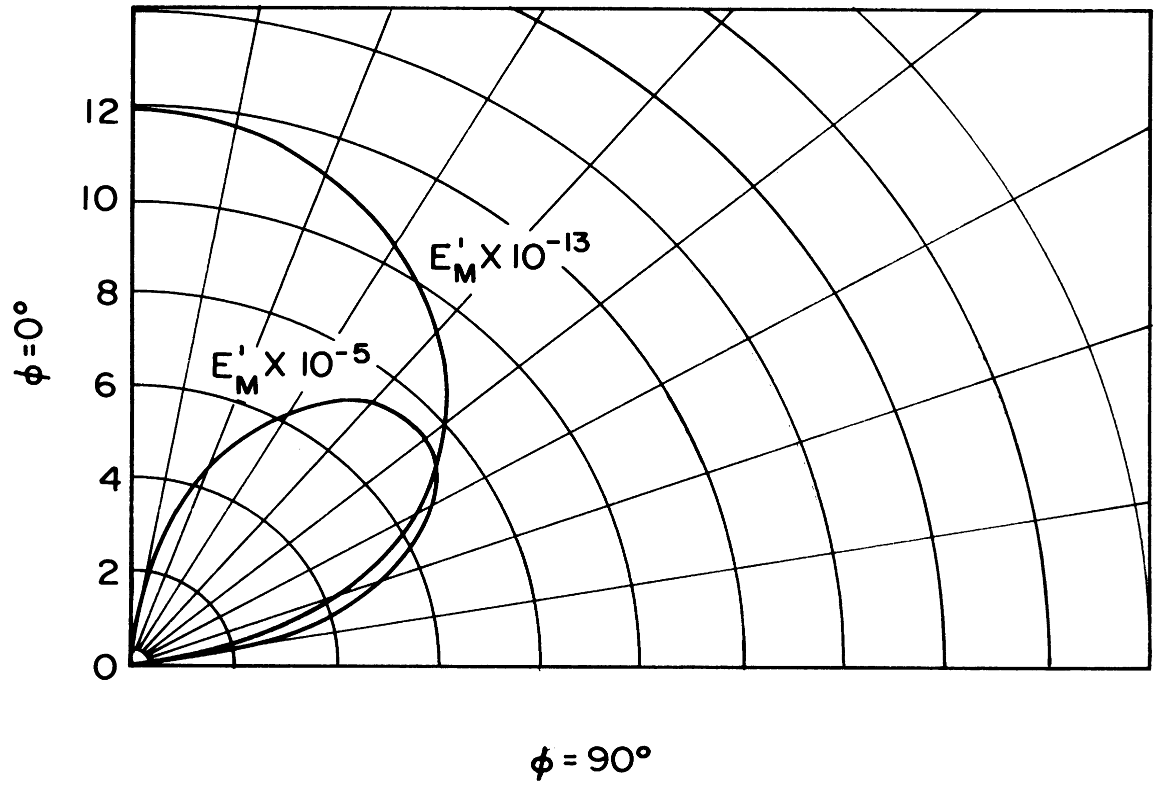


FIG. 50: E'_ϕ, E'_M VS. ϕ FOR MODIFIED PLASMA WAVE

100 KM, $\omega = 3 \times 10^7$

POINT CURRENT SOURCE ($\phi = 0°$)

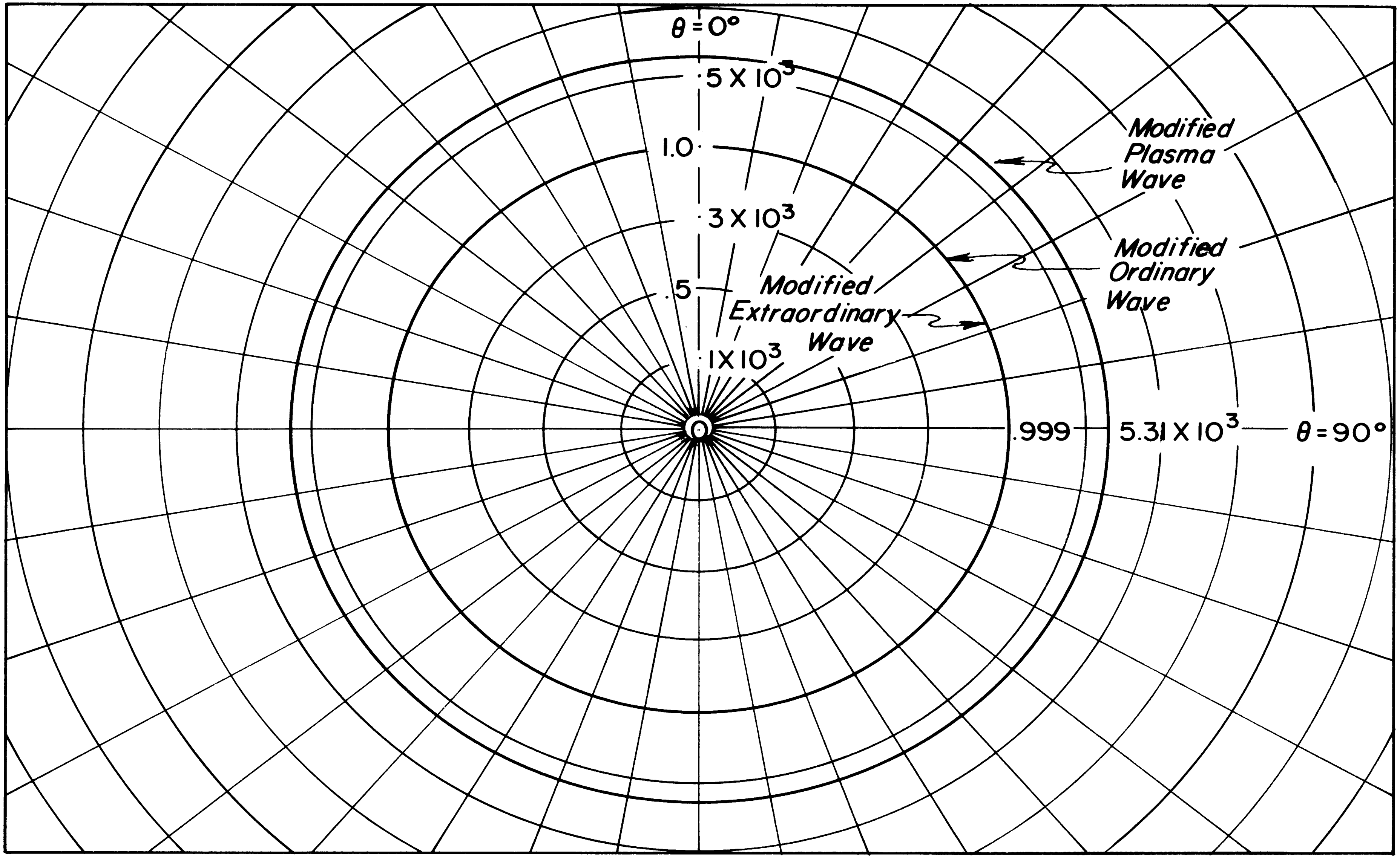


FIG. 51: DISPERSION CURVES FOR MODIFIED ORDINARY, EXTRAORDINARY AND PLASMA WAVES

100 KM, $\omega = 3 \times 10^8$

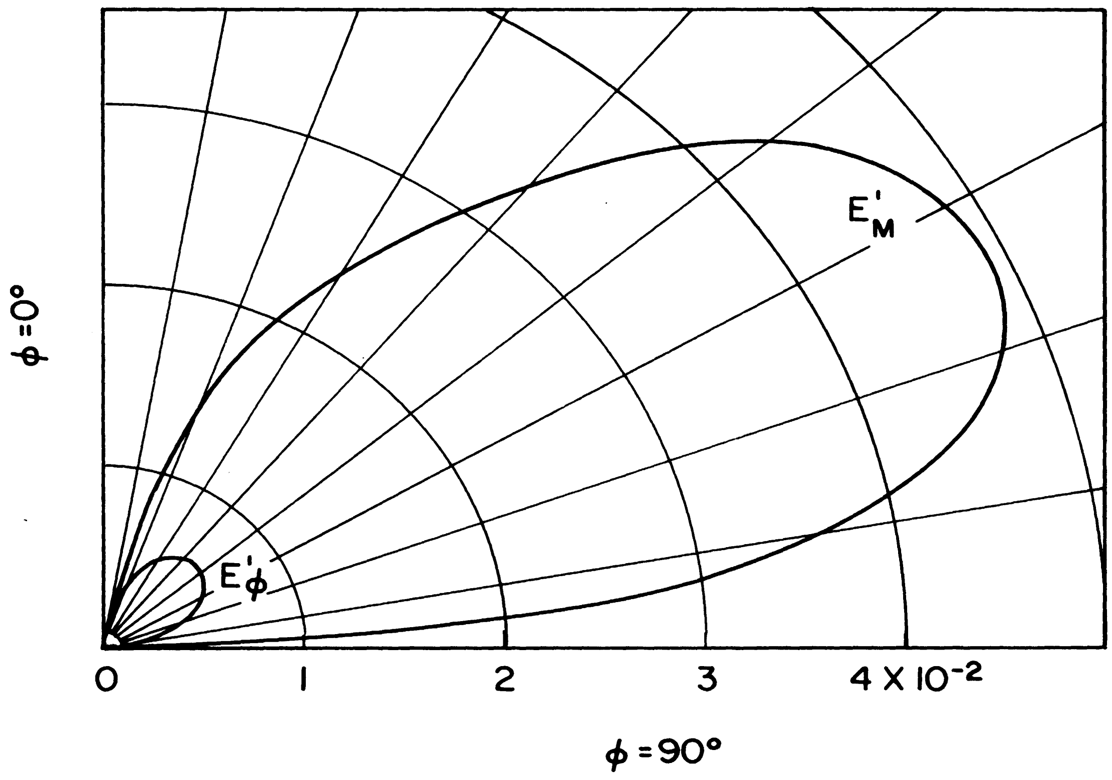


FIG. 52: E'_ϕ, E'_M VS. ϕ FOR MODIFIED ORDINARY AND EXTRAORDINARY WAVES

100 KM, $\omega = 3 \times 10^8$

POINT CURRENT SOURCE ($\phi = 0^\circ$)

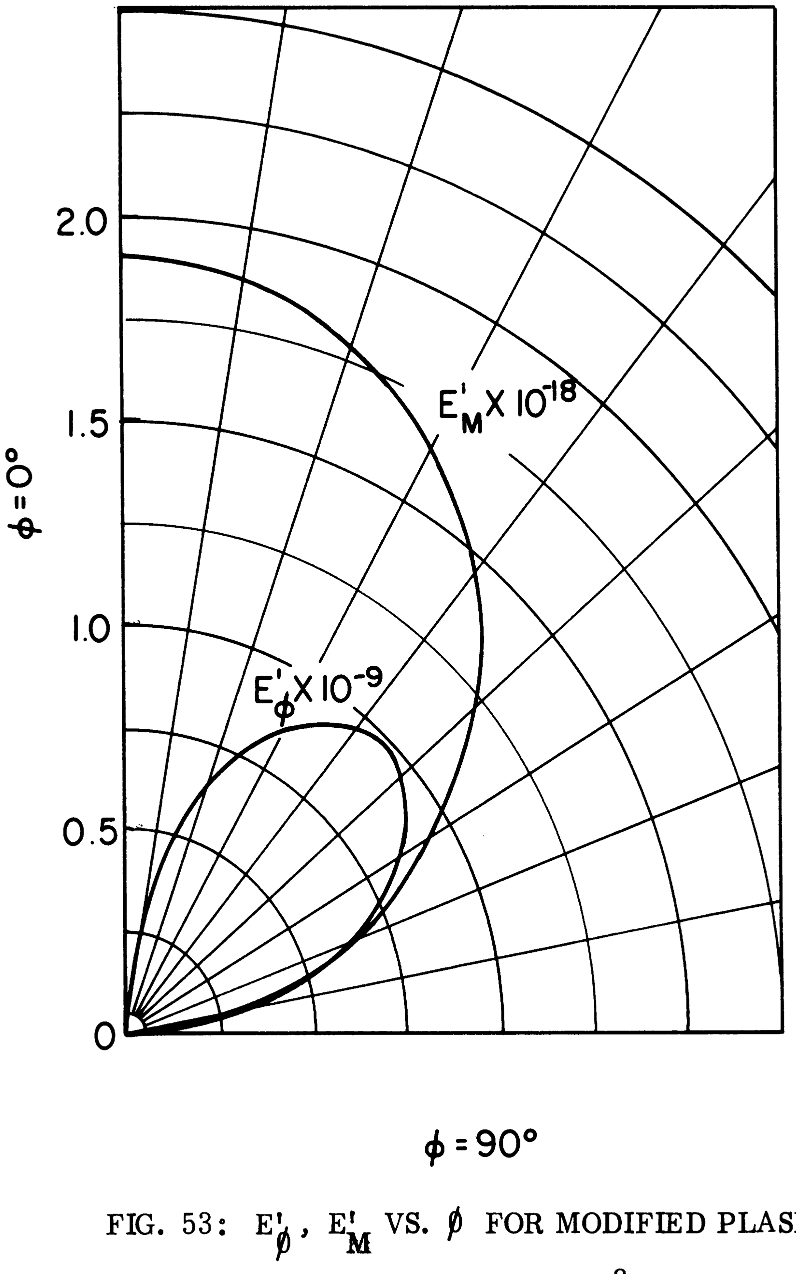


FIG. 53: E'_ϕ, E'_M VS. ϕ FOR MODIFIED PLASMA WAVE

100 KM, $\omega = 3 \times 10^8$

POINT CURRENT SOURCE ($\phi = 0^\circ$)

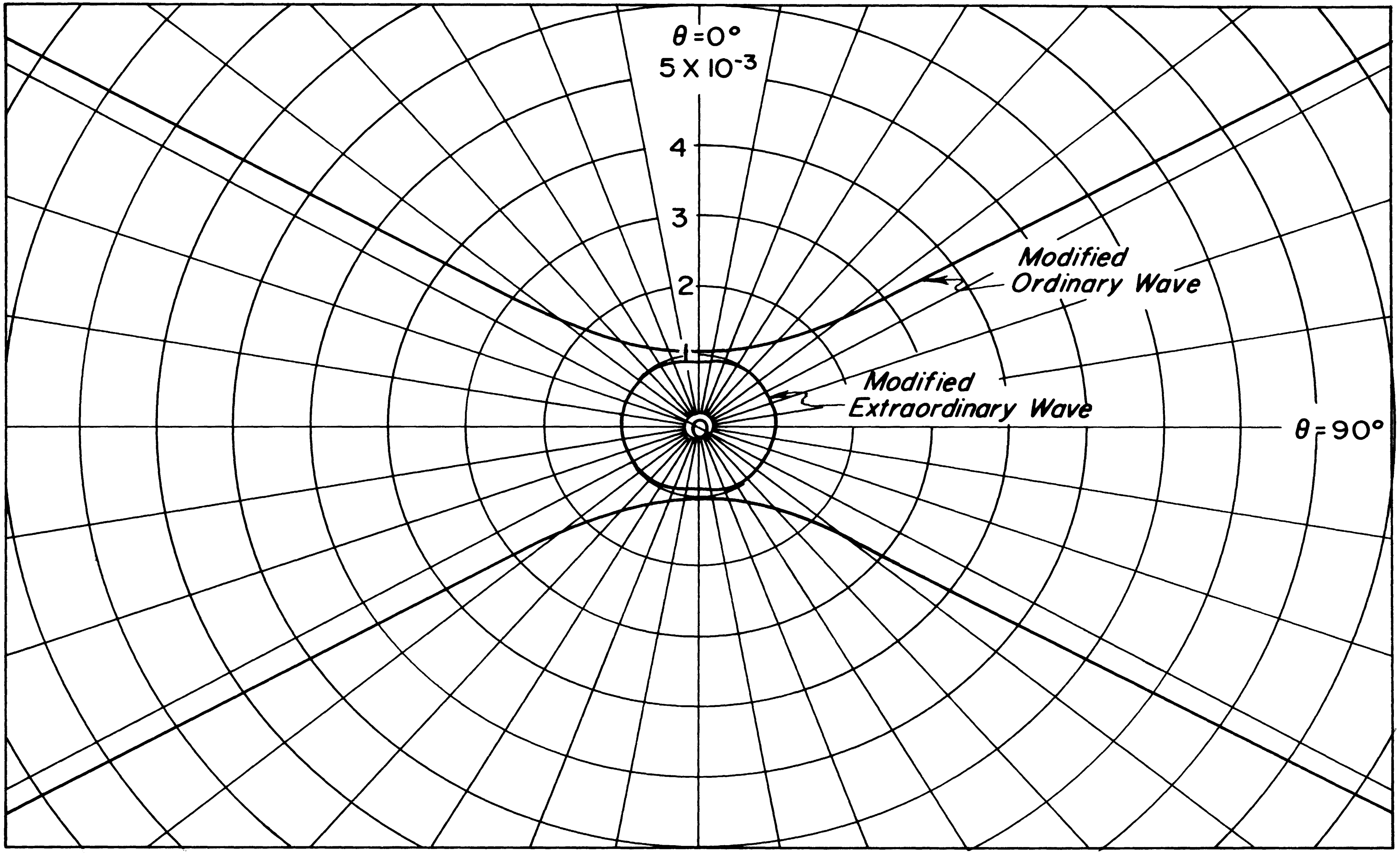


FIG. 54: DISPERSION CURVE

70 KM, $\omega = 3 \times 10^5$

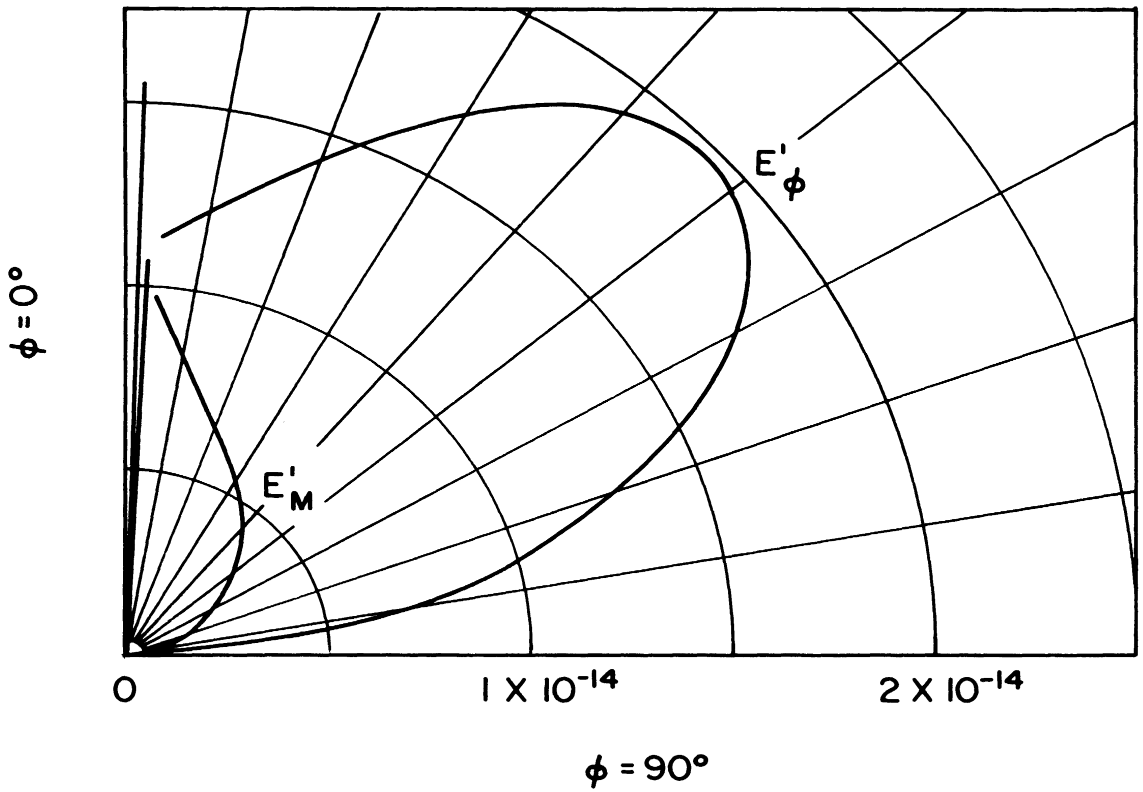


FIG. 55: E'_ϕ, E'_M VS. ϕ FOR MODIFIED EXTRAORDINARY WAVE

70 KM, $\omega = 3 \times 10^5$

POINT CURRENT SOURCE ($\phi = 0^o$)

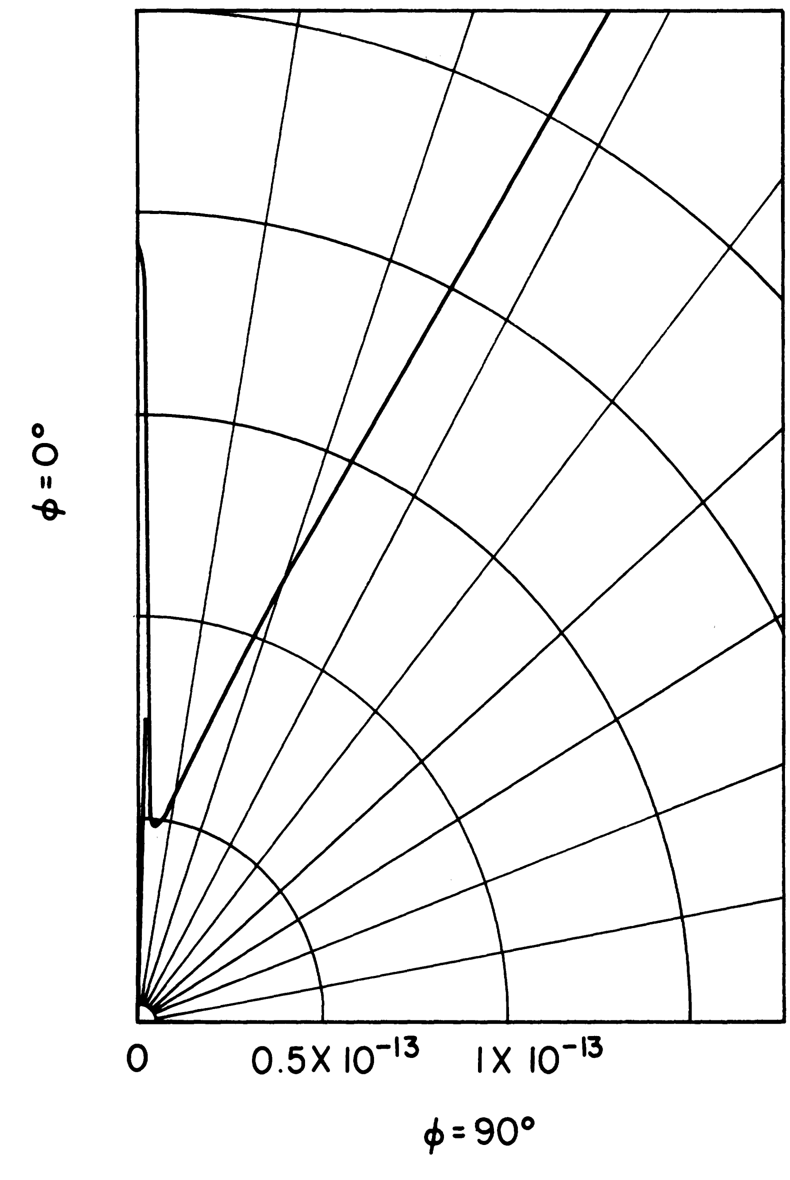


FIG. 56: E'_{ϕ} VS. ϕ FOR MODIFIED ORDINARY WAVE

70 KM, $\omega = 3 \times 10^5$

POINT CURRENT SOURCE ($\phi = 0^{\circ}$)

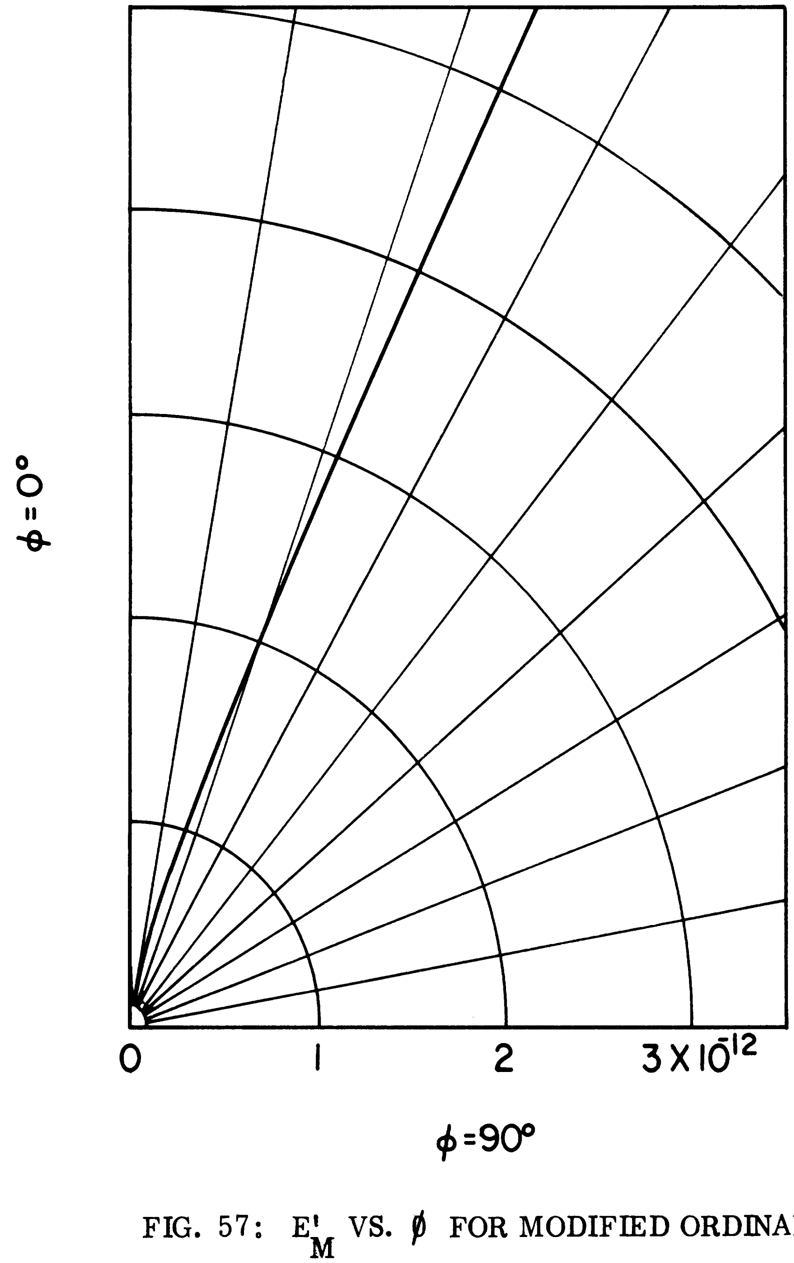


FIG. 57: E'_M VS. ϕ FOR MODIFIED ORDINARY WAVE

70 KM, $\omega = 3 \times 10^5$

POINT CURRENT SOURCE ($\phi = 0^0$)

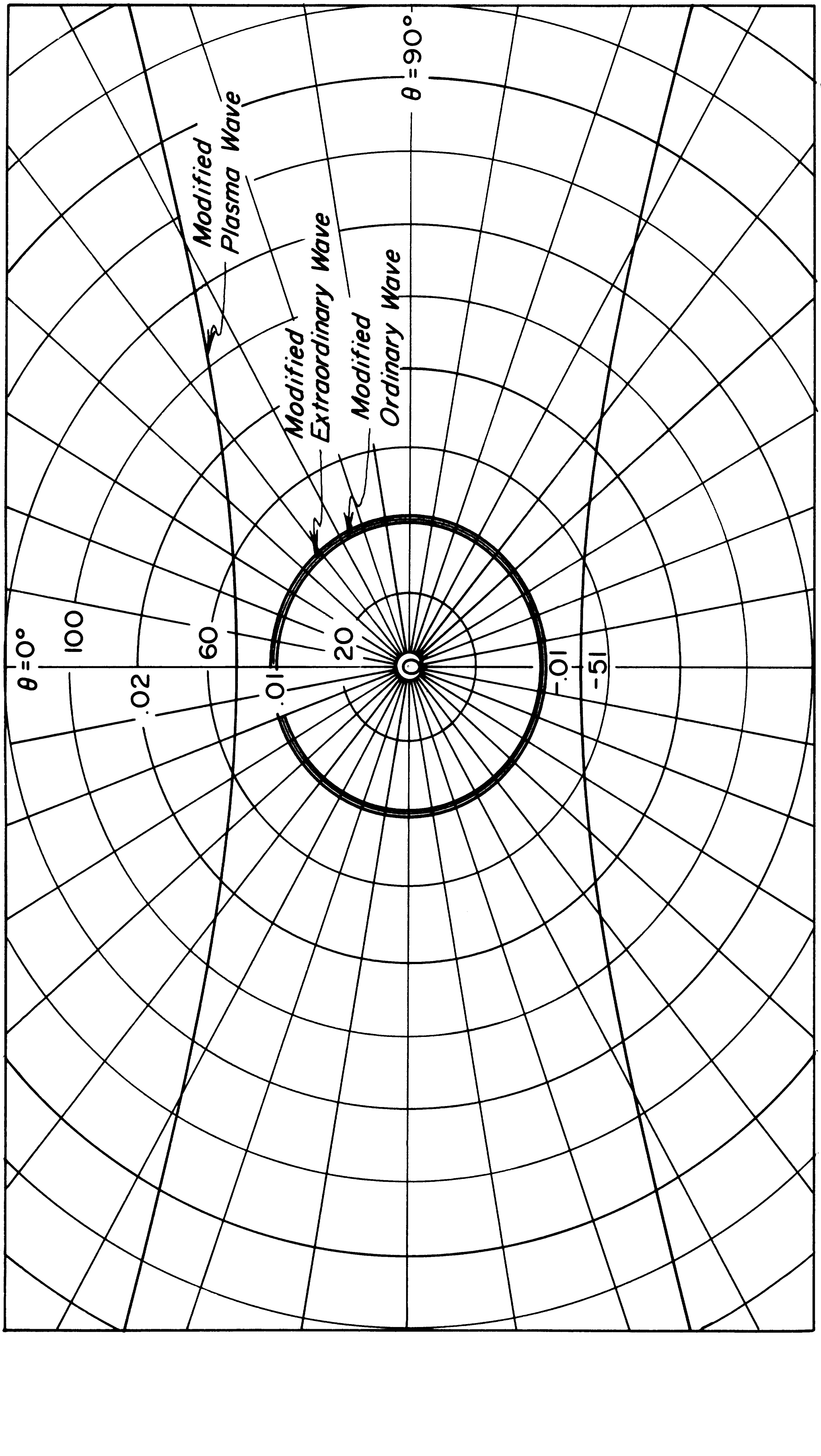


FIG. 58: DISPERSION CURVE
70 KM, $\omega = 3 \times 10^6$

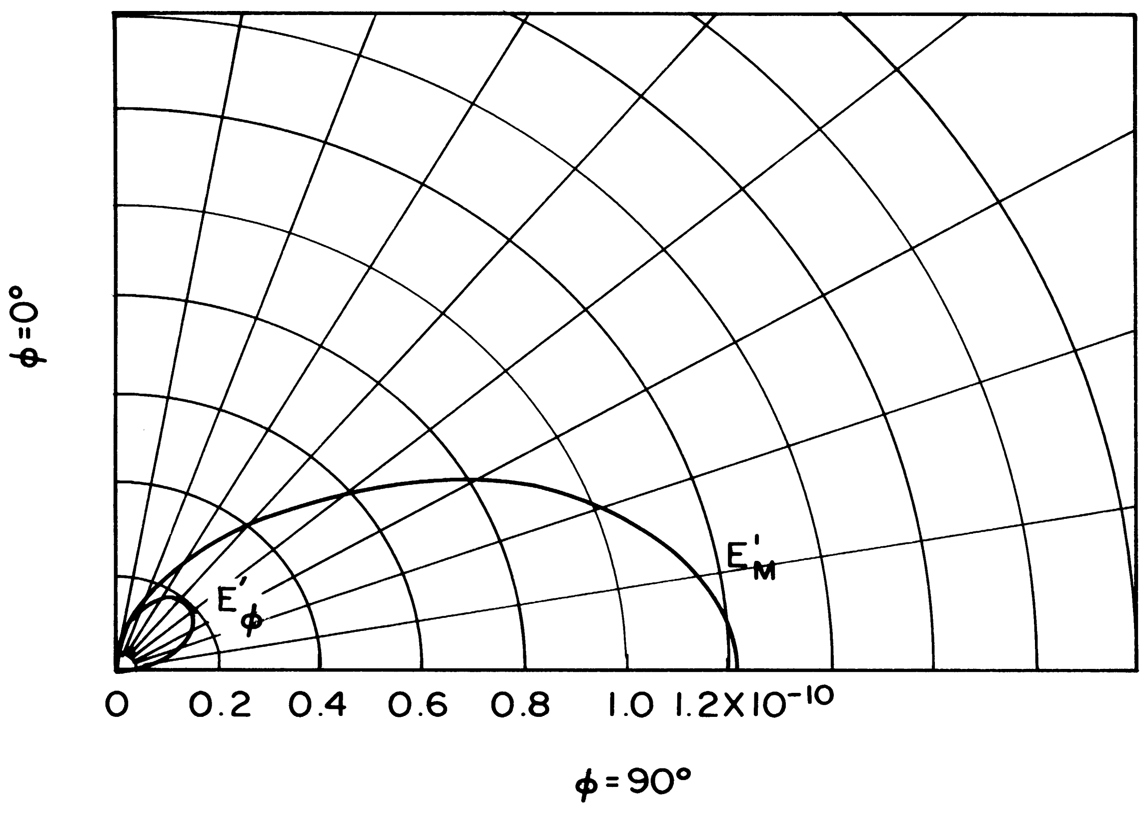


FIG. 59: E'_ϕ, E'_M VS. ϕ FOR MODIFIED ORDINARY WAVE

70 KM, $\omega = 3 \times 10^6$

POINT CURRENT SOURCE ($\phi = 0^o$)

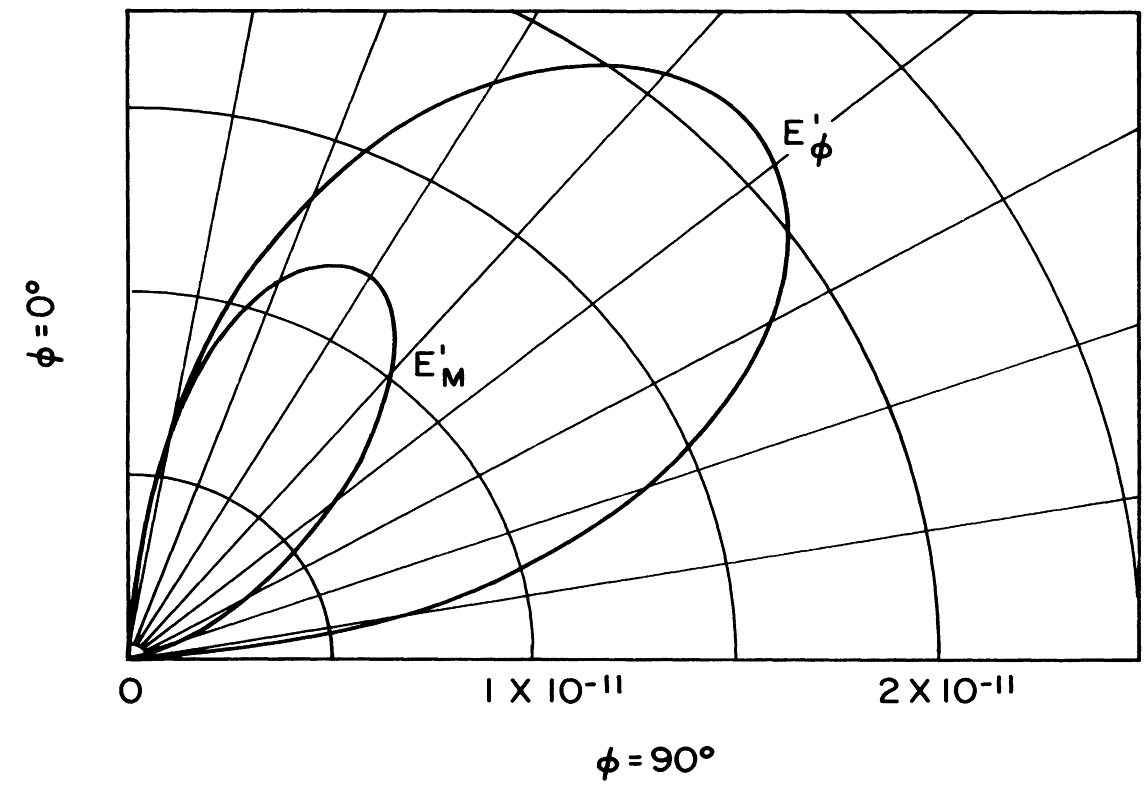


FIG. 60: E'_{ϕ}, E'_M VS. ϕ FOR MODIFIED EXTRAORDINARY WAVE

70 KM, $\omega = 3 \times 10^6$

POINT CURRENT SOURCE ($\phi = 0^{\circ}$)

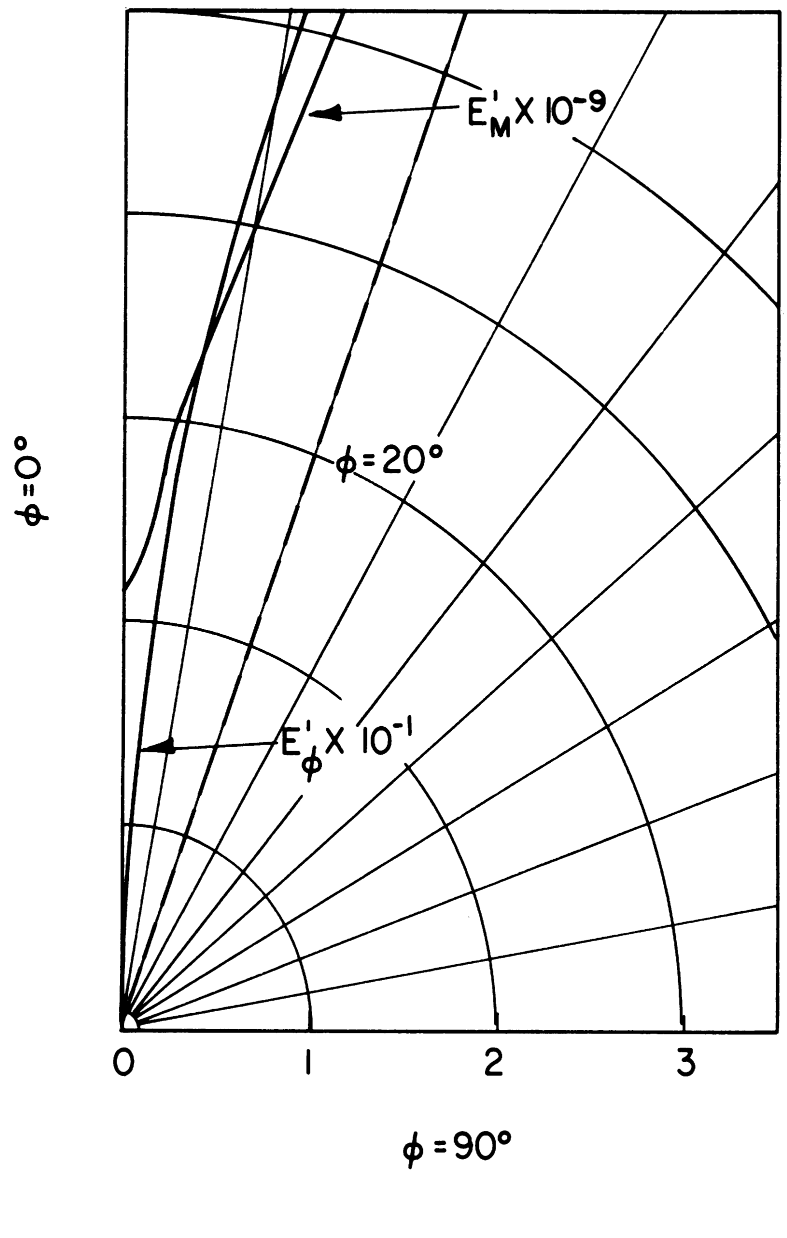


FIG. 61: E'_{ϕ}, E'_M VS. ϕ FOR MODIFIED PLASMA WAVE

70 KM, $\omega = 3 \times 10^6$

POINT CURRENT SOURCE ($\phi = 0^{\circ}$)

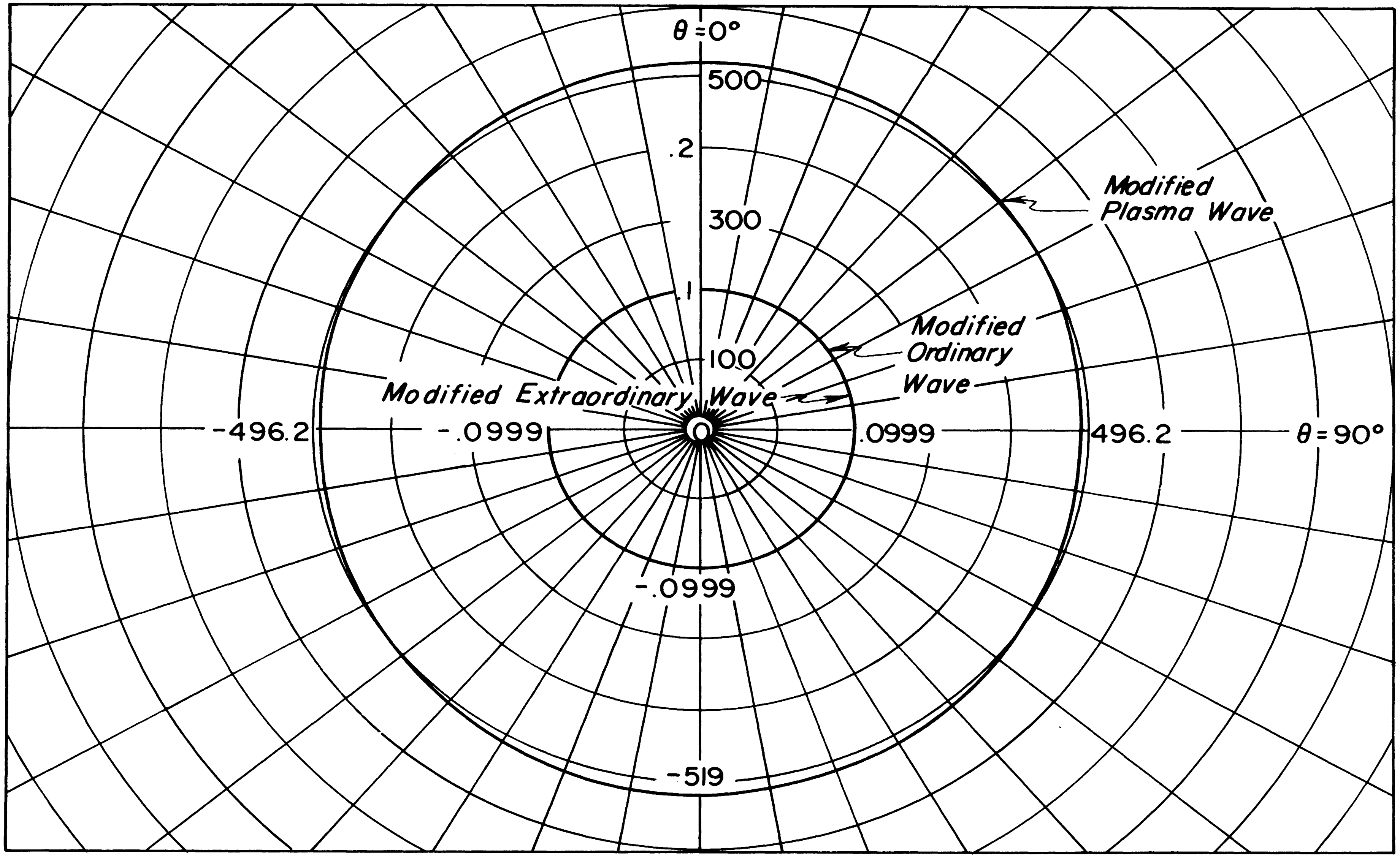


FIG. 62: DISPERSION CURVE

70 KM, $\omega = 3 \times 10^7$

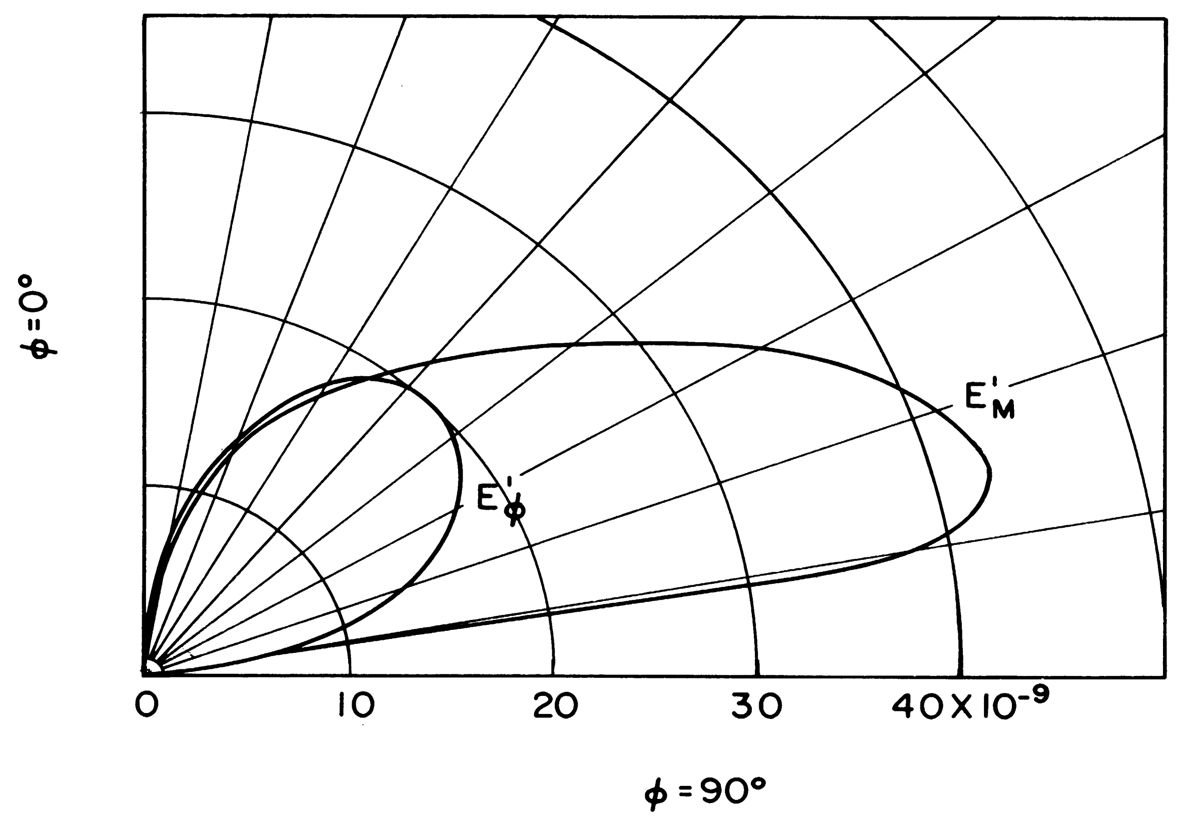


FIG. 63: E'_ϕ, E'_M VS. ϕ FOR MODIFIED ORDINARY AND EXTRAORDINARY WAVES

70 KM, $\omega = 3 \times 10^7$

POINT CURRENT SOURCE ($\phi = 0^o$)

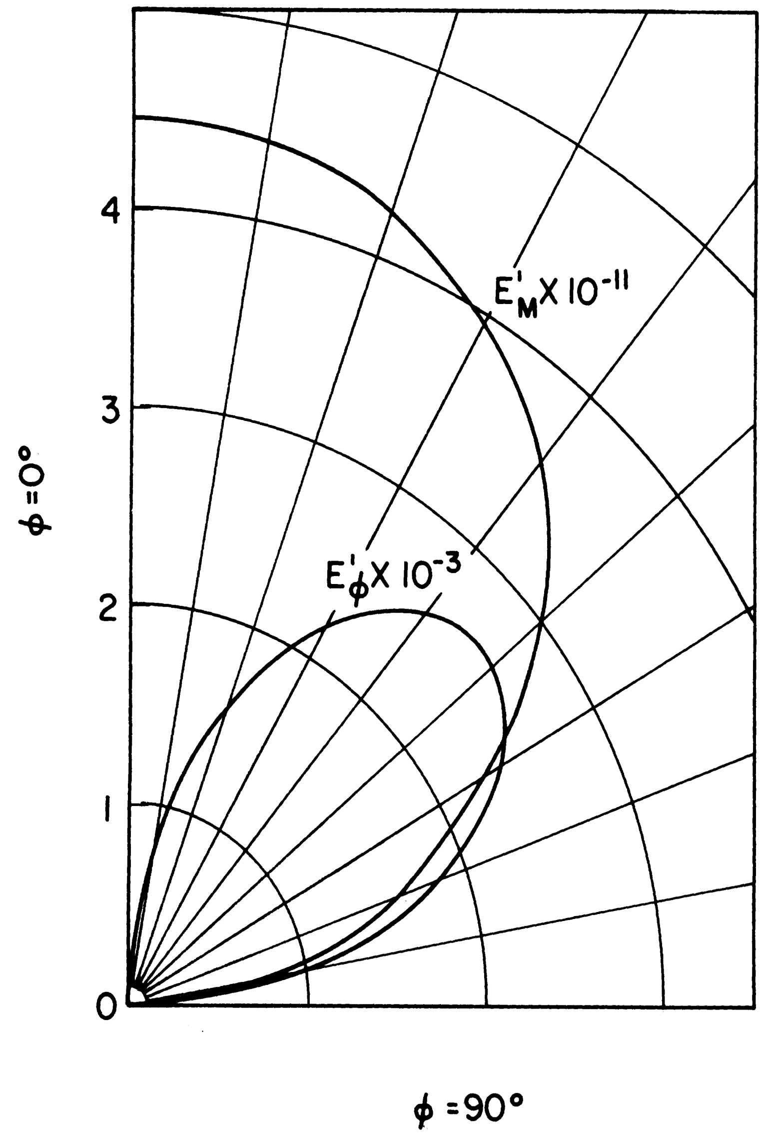


FIG. 64: E'_ϕ, E'_M VS. ϕ FOR MODIFIED PLASMA WAVE

70 KM, $\omega = 3 \times 10^7$

POINT CURRENT SOURCE ($\phi = 0^0$)

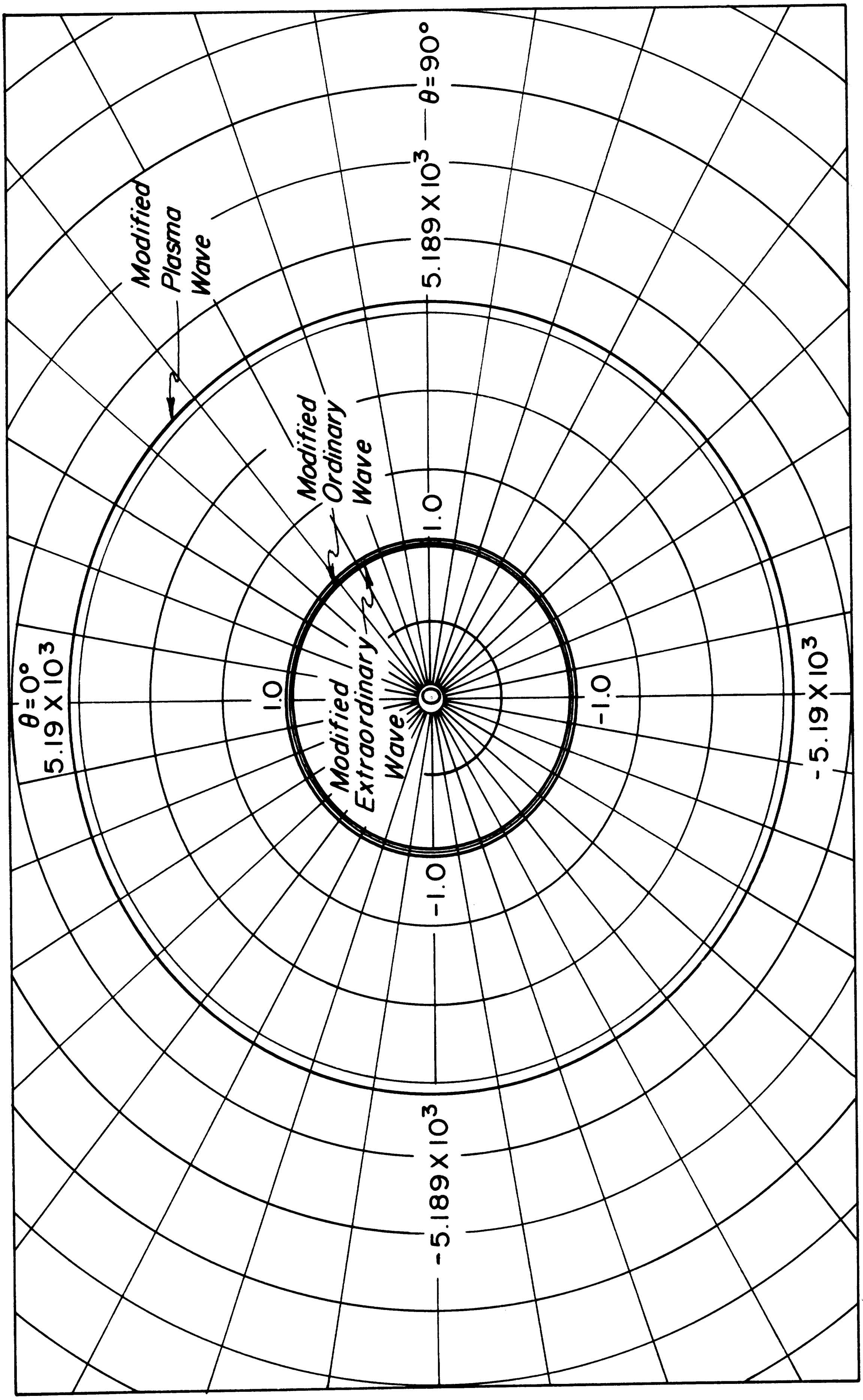


FIG. 65: DISPERSION CURVE
70 KM, $\omega = 3 \times 10^8$

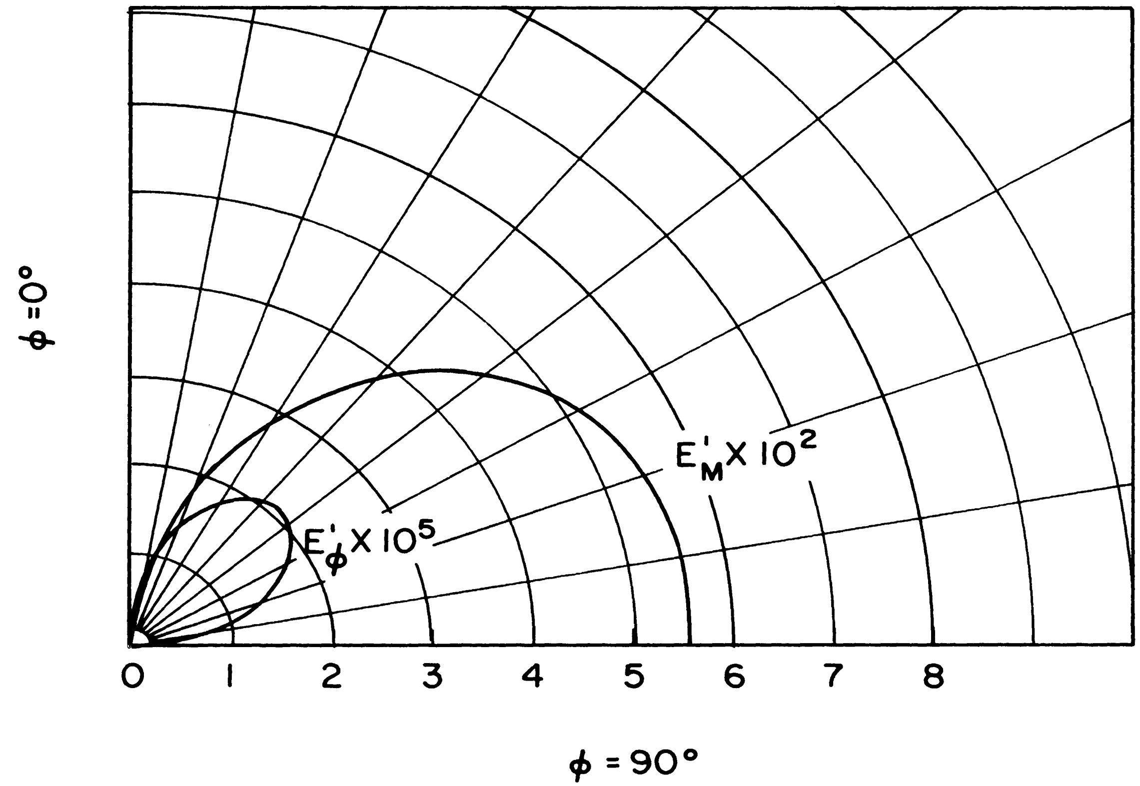


FIG. 66: E'_{ϕ}, E'_M VS. ϕ. FOR MODIFIED ORDINARY AND EXTRAORDINARY WAVES

70 KM, $\omega = 3 \times 10^8$

POINT CURRENT SOURCE ($\phi = 0^{\circ}$)

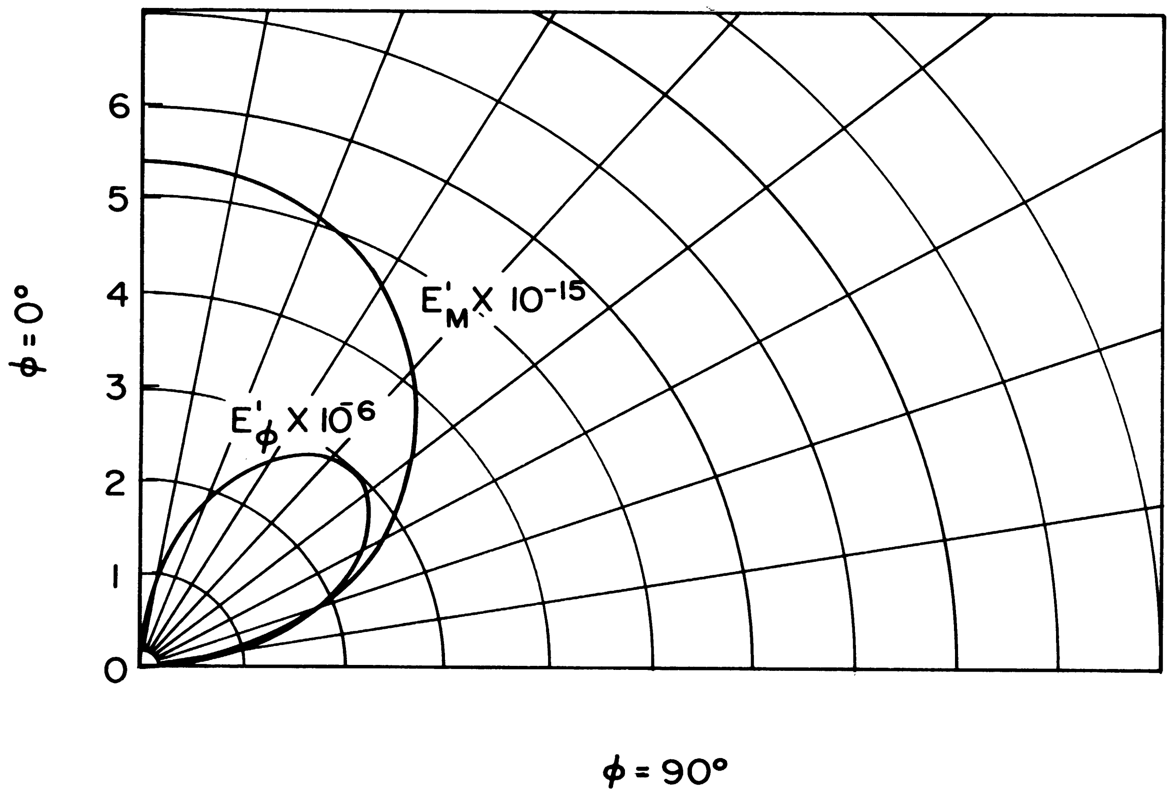


FIG. 67: E'_ϕ, E'_M VS. ϕ FOR MODIFIED PLASMA WAVE

70 KM, $\omega = 3 \times 10^8$

POINT CURRENT SOURCE ($\phi = 0^\circ$)

However, there is quite a variation in the magnitude of the excited fields. Modified plasma waves are essentially composed by E_M components, as can be expected from their longitudinal nature.

The relatively large magnitude of the modified plasma wave compared with the modified electromagnetic waves should be comparable with the result of Hessel and Shmoys[23]. They consider the excitation by a point current source without static magnetic field. Wait[46] has studied the radiation from a slotted-sphere antenna immersed in a compressible plasma, without static magnetic field, and concludes that the relative power in the acoustic type of wave is increased as the dimension of the antenna is reduced. The assumption of an infinitesimal point source might have given an unrealistically large contribution of modified plasma wave, but comparatively strong excitation of this type of wave seems to be possible, because these fields are basically decided by the dispersion relation. Other interpretations of the figures will be given in the following.

[A] At 400 Kilometers

When $\omega = 3 \times 10^5$ the dispersion curve belongs to the region 8 of Fig. 14. The transition angle θ_t for modified ordinary wave to modified plasma wave is equal to $\tan^{-1}(29.18)$. The resonance angle θ_r for the modified plasma wave is $\tan^{-1}(29.3)$. Due to a turning point in Fig. 15, there are two rays existing inside the cone $\phi \leq 17.75^o$ as shown in Fig. 16. This terminology "ray" has been used by Arbel[13] for each stationary phase point contribution. Large contribution due to modified plasma wave is restricted to a very narrow region near the axis $\phi = 0^o$.

When $\omega = 3 \times 10^6$ the dispersion curve as given by Fig. 17 still belongs to the region 8 of Fig. 14, but now $\theta_t = 68.7^o$ and $\theta_r = 70^o$. Due to a turning point, there are three rays existing inside the cone $\phi \leq 4^o$ as indicated in Fig. 18. At $\phi = 20^o$ corresponding to $\theta_r = 70^o$ the stationary point goes to infinity and the asymptotic solution can not be applied. There are two rays existing in the regions $4^o \leq \phi \leq 20^o$. A large modified plasma wave contribution exists near and inside the boundary cone $\phi = 20^o$.

When $\omega = 3 \times 10^7$ the dispersion curves given in Fig. 20 and Fig. 21 belong to the region 1 of Fig. 14. The patterns of $E'_\phi - \phi$ are similar to that of $\hat{F}_i - \theta$ of Arbel(13) for modified ordinary and extraordinary waves, but $E'_M - \phi$ patterns are somewhat different from $\hat{G}_i - \theta$ of Arbel(13). Actually, $\hat{G}_i$ is not the true magnitude of the projection on the meridian plane. The radiation field of the modified plasma wave as given in Fig. 24 is much greater than those of other two waves. The modified plasma wave is elliptically polarized with a small E_ϕ component, except in the direction $\phi = 0^o$ where the wave is linearly polarized in the direction of earth's magnetic field.

When $\omega = 3 \times 10^8$ the dispersion curves are given by Fig. 25 and Fig. 26 and E'_ϕ, E'_M vs ϕ given by Fig. 27 and Fig. 28. Everything is the same as in the case $\omega = 3 \times 10^7$ except for the similarity of the propagation constants of modified ordinary and extraordinary waves, which are nearly equal to the propagation constant of light in free space.

[B] At 250 Kilometers

Most of the features are the same as in the case of [A] at 400 Km. Dispersion curves and E'_ϕ, E'_M vs ϕ patterns are given by Fig. 29 through Fig. 39.

When $\omega = 3 \times 10^7$, the criterion corresponds to region 5 of Fig. 14 and there is no radiation field existing. At $\omega = 3 \times 10^8$, the modified ordinary and extraordinary waves have almost equal magnitude of E_M and E_ϕ everywhere except in the direction of earth's magnetic field where there are no waves of these types existing, and also in the direction $\phi = 90^o$ where these waves are linearly polarized in y-direction. These features can be seen from Fig. 37 and Fig. 38.

[C] At 100 Kilometers

All features are the same as in the case of altitude 400 Km, except for the appearance of second turning point near $\theta = 60^o$ at $\omega = 3 \times 10^6$. The dispersion curves and E'_ϕ, E'_M vs ϕ patterns are given by Fig. 40 through Fig. 53.

At $\omega = 3 \times 10^6$ the second turning point on the dispersion curve causes one ray to appear outside the cone $\phi = 20^o$. Actually, there is another ray existing outside the cone $\phi = 20^o$ with a large intensity which can not be seen in Fig. 44.

[D] At 70 Kilometers

When $\omega = 3 \times 10^5$ the dispersion curves given by Fig. 54 belong to region 7 of Fig. 14. The patterns near $\phi = 0^o$ in Fig. 55 are not indicated because of the turning point very close to $\theta = 0^o$. As for the modified ordinary wave, there are two turning points, one near $\phi = 0^o$ and another one near $\phi = 29.5^o$ as shown in Fig. 56 and Fig. 57. θ_r and θ_t are still the same numbers given for the case of 400 Km and $\omega = 3 \times 10^5$.

When $\omega = 3 \times 10^6$ the dispersion curves given by Fig. 58 belong to region 6 of Fig. 14. There are no turning points for any dispersion curves. E'_ϕ, E'_M vs ϕ patterns are given by Figs. 59, 60 and 61.

The main features at $\omega = 3 \times 10^7$ and $\omega = 3 \times 10^8$ as given by Fig. 62 through Fig. 67 are the same as in the case at 400 Km.

APPENDIX A

GENERAL FORMULATION FOR THREE-FLUID PLASMA

A.1 Operator Form

In order to be able to obtain a proper operator form of the basic Eqs. (2.1) through (2.8), we will express $\bar{E}$, $\bar{V}_e$, $\bar{V}_i$ and $\bar{V}_n$ in terms of $\bar{h}$, n_e, n_i, n_n, $\bar{J}$, $\bar{F}_e$, $\bar{F}_i$ and $\bar{F}_n$ by employing Eqs. (2.2), (2.3), (2.5) and (2.7). The procedure is as follows.

First, $\bar{E}$ is eliminated between Eqs. (2.2) and (2.3) to obtain an equation involving $\bar{V}_e$, $\bar{V}_i$ and $\bar{V}_n$. Next, $\bar{E}$ is eliminated between Eqs. (2.2) and (2.5) to obtain another equation involving $\bar{V}_e$, $\bar{V}_i$ and $\bar{V}_n$. Then, $\bar{V}_n$ will be eliminated from these two new equations and (2.7) to get the following pair of equations:

$$A\bar{V}_e + B\bar{V}_i + \omega_{ce}\bar{V}_e x\hat{b} = \bar{S}_e \qquad (A.1)$$

$$C\bar{V}_e + D\bar{V}_i - \omega_{ci}\bar{V}_i x\hat{b} = \bar{S}_i \qquad (A.2)$$

where

$$A \equiv -i\omega + i\frac{\omega_{pe}^2}{\omega} + \nu_{ei} + \nu_{en} - \frac{\nu_{en}\nu_{ne}}{-i\omega+\nu_{ni}+\nu_{ne}} \qquad (A.3)$$

$$B \equiv -i\frac{\omega_{pe}^2}{\omega} - \nu_{ei} - \frac{\nu_{en}\nu_{ni}}{-i\omega+\nu_{ni}+\nu_{ne}} \qquad (A.4)$$

$$C \equiv -i\frac{\omega_{pi}^2}{\omega} - \nu_{ie} - \frac{\nu_{in}\nu_{ne}}{-i\omega+\nu_{ni}+\nu_{ne}} \qquad (A.5)$$

$$D \equiv -i\omega + i\frac{\omega_{pi}^2}{\omega} + \nu_{ie} + \nu_{in} - \frac{\nu_{in}\nu_{ni}}{-i\omega+\nu_{ni}+\nu_{ne}} \qquad (A.6)$$

$$\bar{S}_e \equiv \frac{e\nabla x\bar{h}}{i\omega\epsilon_o m_e} - \frac{U_e^2 \nabla n_e}{N_o} - \frac{U_n^2 \nu_{en} \nabla n_n}{N_1(\nu_{ni}+\nu_{ne}-i\omega)}$$

$$- \frac{e\bar{J}}{i\omega\epsilon_o m_e} + \frac{\bar{F}_e}{N_o m_e} + \frac{\nu_{en}\bar{F}_n}{N_1 m_n(\nu_{ni}+\nu_{ne}-i\omega)} \tag{A.7}$$

$$\bar{S}_i \equiv - \frac{e\nabla x\bar{h}}{i\omega\epsilon_o m_i} - \frac{U_i^2 \nabla n_i}{N_o} - \frac{U_n^2 \nu_{in} \nabla n_n}{N_1(\nu_{ni}+\nu_{ne}-i\omega)}$$

$$+ \frac{e\bar{J}}{i\omega\epsilon_o m_i} + \frac{\bar{F}_i}{N_o m_i} + \frac{\nu_{in}\bar{F}_n}{N_1 m_n(\nu_{ni}+\nu_{ne}-i\omega)} \tag{A.8}$$

ω_{ce} and ω_{ci} in Eqs. (A.1) and (A.2) are the electron and ion cyclotron frequencies given by $\omega_{ce} = \frac{eB_o}{m_e}$ and $\omega_{ci} = \frac{eB_o}{m_i}$, respectively, and ω_{pe} and ω_{pi} in Eqs. (A.3) through (A.6) are the electron and ion plasma frequencies given by $\omega_{pe}^2 = \frac{e^2 N_o}{m_e \epsilon_o}$ and $\omega_{pi}^2 = \frac{e^2 N_o}{m_i \epsilon_o}$, respectively.

The pair of Eqs. (A.1) and (A.2) can be easily solved for $\bar{V}_e$ and $\bar{V}_i$ in terms of $\bar{S}_e$, $\bar{S}_i$, $\hat{b}\hat{b}\cdot\bar{S}_e$, $\hat{b}\hat{b}\cdot\bar{S}_i$, $\hat{b}x\bar{S}_e$ and $\hat{b}x\bar{S}_i$. Then $\bar{E}$ can be found from Eq. (2.2), and $\bar{V}_n$ can be found from Eq. (2.7). Utilization of the expression of $\bar{S}_e$ and $\bar{S}_i$ as given by Eqs. (A.7) and (A.8) will, then, give the solutions for $\bar{E}$, $\bar{V}_e$, $\bar{V}_i$ and $\bar{V}_n$ in terms of $\bar{h}$, n_e, n_i, n_n, $\bar{J}$, $\bar{F}_e$, $\bar{F}_i$ and $\bar{F}_n$. These four equations, each of them involving $\bar{E}$, $\bar{V}_e$, $\bar{V}_i$ and $\bar{V}_n$ separately, together with the original Eqs. (2.1), (2.4), (2.6) and (2.8) can now be put into the following desirable matrix form.

$$
\begin{bmatrix}
\bar{\bar{I}} & 0 & 0 & 0 & i\dfrac{\nabla \times \bar{\bar{I}}}{\omega\mu_o} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & \dfrac{iN_o}{\omega}(\nabla\cdot \bar{\bar{I}})' + \dfrac{i}{\omega}\nabla N_o \cdot \bar{\bar{I}} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & \dfrac{iN_o}{\omega}(\nabla\cdot \bar{\bar{I}})' + \dfrac{i}{\omega}\nabla N_o \cdot \bar{\bar{I}} & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & \dfrac{iN_1}{\omega}(\nabla\cdot \bar{\bar{I}})' + \dfrac{i}{\omega}\nabla N_1 \cdot \bar{\bar{I}} \\
A_{11} & A_{12} & A_{13} & A_{14} & \bar{\bar{I}} & 0 & 0 & 0 \\
A_{21} & A_{22} & A_{23} & A_{24} & 0 & \bar{\bar{I}} & 0 & 0 \\
A_{31} & A_{32} & A_{33} & A_{34} & 0 & 0 & \bar{\bar{I}} & 0 \\
A_{41} & A_{42} & A_{43} & A_{44} & 0 & 0 & 0 & \bar{\bar{I}}
\end{bmatrix}
\begin{bmatrix}
\bar{h} \\ n_e \\ n_i \\ n_n \\ \bar{E} \\ \bar{V}_e \\ \bar{V}_i \\ \bar{V}_n
\end{bmatrix}
=
\begin{bmatrix}
-\dfrac{i\bar{K}}{\omega\mu_o} \\ \dfrac{iQ_e}{\omega} \\ \dfrac{iQ_i}{\omega} \\ \dfrac{iQ_n}{\omega} \\ \bar{S}_1 \\ \bar{S}_2 \\ \bar{S}_3 \\ \bar{S}_4
\end{bmatrix}
\qquad (A.9)
$$

where the matrix components A_{ij} and the source vectors $\bar{S}_1$, $\bar{S}_2$, $\bar{S}_3$ and $\bar{S}_4$ are given by

$$A_{11} = \frac{eN_o}{i\epsilon_o\omega}(A_{31}-A_{21}) - \frac{i}{\epsilon_o\omega}\nabla \times \bar{\bar{I}} \tag{A.10}$$

$$A_{12} = \frac{eN_o}{i\epsilon_o\omega}(A_{32}-A_{22}) \tag{A.11}$$

$$A_{13} = \frac{eN_o}{i\epsilon_o\omega}(A_{33}-A_{23}) \tag{A.12}$$

$$A_{14} = \frac{eN_o}{i\epsilon_o\omega}(A_{34}-A_{24}) \tag{A.13}$$

$$A_{21} = \frac{1}{\Delta_1}\Bigg[\frac{1}{\omega_{ce}^2}\left(\frac{D}{\omega_{ci}^2}\Delta_2+A\right)+\frac{1}{\Delta_2}\left(\frac{ABC}{\omega_{ce}^2}-\frac{2BCD}{\omega_{ce}\omega_{ci}}+\frac{D^3}{\omega_{ci}^2}+D\right)\hat{b}\hat{b}\cdot$$

$$+\frac{1}{\omega_{ce}}\left(\frac{D^2}{\omega_{ci}^2}-\frac{BC}{\omega_{ce}\omega_{ci}}+1\right)\hat{b}\times\Bigg]\frac{-e\nabla\times\bar{\bar{I}}}{i\omega\epsilon_o m_e}+\frac{B}{\Delta_1}\Bigg[\frac{-1}{\omega_{ce}\omega_{ci}}\left(\frac{1}{\omega_{ce}\omega_{ci}}\Delta_2+1\right)$$

$$+\frac{1}{\Delta_2}\left(\frac{BC+AD}{\omega_{ce}\omega_{ci}}-\frac{D^2}{\omega_{ci}^2}-\frac{A^2}{\omega_{ce}^2}-1\right)\hat{b}\hat{b}\cdot$$

$$+\frac{1}{\omega_{ce}\omega_{ci}}\left(\frac{A}{\omega_{ce}}-\frac{D}{\omega_{ci}}\right)\hat{b}\times\Bigg]\frac{e\nabla\times\bar{\bar{I}}}{i\omega\epsilon_o m_i} \tag{A.14}$$

$$A_{22} = \frac{1}{\Delta_1}\Bigg[\frac{1}{\omega_{ce}^2}\left(\frac{D}{\omega_{ci}^2}\Delta_2+A\right)+\frac{1}{\Delta_2}\left(\frac{ABC}{\omega_{ce}^2}-\frac{2BCD}{\omega_{ce}\omega_{ci}}+\frac{D^3}{\omega_{ci}^2}+D\right)\hat{b}\hat{b}\cdot$$

$$+\frac{1}{\omega_{ce}}\left(\frac{D^2}{\omega_{ci}^2}-\frac{BC}{\omega_{ce}\omega_{ci}}+1\right)\hat{b}\times\Bigg]\frac{U_e^2\nabla\cdot\bar{\bar{I}}}{N_o} \tag{A.15}$$

$$A_{23}=\frac{B}{\Delta_1}\left[\frac{-1}{\omega_{ce}\omega_{ci}}\left(\frac{1}{\omega_{ce}\omega_{ci}}\Delta_2+1\right)+\frac{1}{\Delta_2}\left(\frac{BC+AD}{\omega_{ce}\omega_{ci}}-\frac{D^2}{\omega_{ci}^2}-\frac{A^2}{\omega_{ce}^2}-1\right)\hat{b}\hat{b}\cdot\right.$$

$$\left.+\frac{1}{\omega_{ce}\omega_{ci}}\left(\frac{A}{\omega_{ce}}-\frac{D}{\omega_{ci}}\right)\hat{b}x\right]\frac{U_i^2\,\nabla\cdot\bar{\bar{I}}}{N_e} \qquad (A.16)$$

$$A_{24}=\frac{U_n^2}{N_1(\nu_{ni}+\nu_{ne}-i\omega)}\left\{\frac{1}{\Delta_1}\left[\frac{1}{\omega_{ce}^2}\left(\frac{D}{\omega_{ci}^2}\Delta_2+A\right)+\frac{1}{\Delta_2}\left(\frac{ABC}{\omega_{ce}^2}-\frac{2BCD}{\omega_{ce}\omega_{ci}}+\frac{D^3}{\omega_{ci}^2}+D\right)\hat{b}\hat{b}\cdot\right.\right.$$

$$\left.+\frac{1}{\omega_{ce}}\left(\frac{D^2}{\omega_{ci}^2}-\frac{BC}{\omega_{ce}\omega_{ci}}+1\right)\hat{b}x\right]\nu_{en}\nabla\cdot\bar{\bar{I}}+\frac{B}{\Delta_1}\left[\frac{-1}{\omega_{ce}\omega_{ci}}\left(\frac{\Delta_2}{\omega_{ce}\omega_{ci}}+1\right)\right.$$

$$\left.\left.+\frac{1}{\Delta_2}\left(\frac{BC+AD}{\omega_{ce}\omega_{ci}}-\frac{D^2}{\omega_{ci}^2}-\frac{A^2}{\omega_{ce}^2}-1\right)\hat{b}\hat{b}\cdot+\frac{1}{\omega_{ce}\omega_{ci}}\left(\frac{A}{\omega_{ce}}-\frac{D}{\omega_{ci}}\right)\hat{b}x\right]\nu_{in}\nabla\cdot\bar{\bar{I}}\right\} \qquad (A.17)$$

$$A_{31}=\frac{C}{\Delta_1}\left[\frac{-1}{\omega_{ce}\omega_{ci}}\left(\frac{\Delta_2}{\omega_{ce}\omega_{ci}}+1\right)+\frac{1}{\Delta_2}\left(\frac{BC+AD}{\omega_{ce}\omega_{ci}}-\frac{D^2}{\omega_{ci}^2}-\frac{A^2}{\omega_{ce}^2}-1\right)\hat{b}\hat{b}\cdot\right.$$

$$\left.+\frac{1}{\omega_{ce}\omega_{ci}}\left(\frac{A}{\omega_{ce}}-\frac{D}{\omega_{ci}}\right)\hat{b}x\right]\frac{-e\nabla x\bar{\bar{I}}}{i\omega\epsilon_o m_e}$$

$$+\frac{1}{\Delta_1}\left[\frac{1}{\omega_{ci}^2}\left(\frac{A}{\omega_{ce}^2}\Delta_2+D\right)+\frac{1}{\Delta_2}\left(\frac{BCD}{\omega_{ci}^2}-\frac{2ABC}{\omega_{ce}\omega_{ci}}+\frac{A^3}{\omega_{ce}^2}+A\right)\hat{b}\hat{b}\cdot\right.$$

$$\left.-\frac{1}{\omega_{ci}}\left(\frac{A^2}{\omega_{ce}^2}-\frac{BC}{\omega_{ce}\omega_{ci}}+1\right)\hat{b}x\right]\frac{e\nabla x\bar{\bar{I}}}{i\omega\epsilon_o m_i} \qquad (A.18)$$

$$A_{32}=\frac{C}{\Delta_1}\left[\frac{-1}{\omega_{ce}\omega_{ci}}\left(\frac{\Delta_2}{\omega_{ce}\omega_{ci}}+1\right)+\frac{1}{\Delta_2}\left(\frac{BC+AD}{\omega_{ce}\omega_{ci}}-\frac{D^2}{\omega_{ci}^2}-\frac{A^2}{\omega_{ce}^2}-1\right)\hat{b}\hat{b}\cdot\right.$$

$$\left.+\frac{1}{\omega_{ce}\omega_{ci}}\left(\frac{A}{\omega_{ce}}-\frac{D}{\omega_{ci}}\right)\hat{b}x\right]\frac{U_e^2}{N_o}\nabla\cdot\bar{\bar{I}} \qquad (A.19)$$

$$A_{33} = \frac{1}{\Delta_1}\left[\frac{1}{\omega_{ci}^2}\left(\frac{A}{\omega_{ce}^2}\Delta_2 + D\right) + \frac{1}{\Delta_2}\left(\frac{BCD}{\omega_{ci}^2} - \frac{2ABC}{\omega_{ce}\omega_{ci}} + \frac{A^3}{\omega_{ce}^2} + A\right)\hat{b}\hat{b}\cdot - \frac{1}{\omega_{ci}}\left(\frac{A^2}{\omega_{ce}^2} - \frac{BC}{\omega_{ce}\omega_{ci}} + 1\right)\hat{b}\times\right]\frac{U_i^2}{N_o}\nabla\cdot\bar{\bar{I}} \qquad (A.20)$$

$$A_{34} = \frac{U_n^2}{N_i(\nu_{ni} + \nu_{ne} - i\omega)}\left\{\frac{C}{\Delta_1}\left[\frac{-1}{\omega_{ce}\omega_{ci}}\left(\frac{\Delta_2}{\omega_{ce}\omega_{ci}} + 1\right) + \frac{1}{\Delta_2}\left(\frac{BC+AD}{\omega_{ce}\omega_{ci}} - \frac{D^2}{\omega_{ci}^2} - \frac{A^2}{\omega_{ce}^2} - 1\right)\hat{b}\hat{b}\cdot + \frac{1}{\omega_{ce}\omega_{ci}}\left(\frac{A}{\omega_{ce}} - \frac{D}{\omega_{ci}}\right)\hat{b}\times\right]\right.$$

$$\nu_{en}\nabla\cdot\bar{\bar{I}} + \frac{1}{\Delta_1}\left[\frac{1}{\omega_{ci}^2}\left(\frac{A}{\omega_{ce}^2}\Delta_2 + D\right) + \frac{1}{\Delta_2}\left(\frac{BCD}{\omega_{ci}^2} - \frac{2ABC}{\omega_{ce}\omega_{ci}} + \frac{A^3}{\omega_{ce}^2} + A\right)\hat{b}\hat{b}\cdot\right.$$

$$\left.\left. - \frac{1}{\omega_{ci}}\left(\frac{A^2}{\omega_{ce}^2} - \frac{BC}{\omega_{ce}\omega_{ci}} + 1\right)\hat{b}\times\right]\nu_{in}\nabla\cdot\bar{\bar{I}}\right\} \qquad (A.21)$$

$$A_{41} = \frac{(\nu_{ne}A_{21} + \nu_{ni}A_{31})}{(\nu_{ne} + \nu_{ni} - i\omega)} \qquad (A.22)$$

$$A_{42} = \frac{(\nu_{ne}A_{22} + \nu_{ni}A_{32})}{(\nu_{ne} + \nu_{ni} - i\omega)} \qquad (A.23)$$

$$A_{43} = \frac{(\nu_{ne}A_{23} + \nu_{ni}A_{33})}{(\nu_{ne} + \nu_{ni} - i\omega)} \qquad (A.24)$$

$$A_{44} = \frac{\left[\nu_{ne}A_{24} + \nu_{ni}A_{34} + \frac{U_n^2\nabla\cdot\bar{\bar{I}}}{N_1}\right]}{(\nu_{ne} + \nu_{ni} - i\omega)} \qquad (A.25)$$

$$\bar{S}_1 = \frac{eN_o}{i\epsilon_o\omega}\left(\bar{S}_3 - \bar{S}_2\right) - \frac{i}{\epsilon_o\omega}\bar{J} \tag{A.26}$$

$$\bar{S}_4 = \frac{\left[\nu_{ne}\bar{S}_2 + \nu_{ni}\bar{S}_3 + \dfrac{\bar{F}_n}{N_1 m_n}\right]}{(\nu_{ne} + \nu_{ni} - i\omega)} \tag{A.27}$$

$$\bar{S}_2 = \frac{1}{\Delta_1}\left[\frac{1}{\omega_{ce}^2}\left(\frac{D}{\omega_{ci}^2}\Delta_2 + A\right) + \frac{1}{\Delta_2}\left(\frac{ABC}{\omega_{ce}^2} - \frac{2BCD}{\omega_{ce}\omega_{ci}} + \frac{D^3}{\omega_{ci}^2} + D\right)\hat{b}\hat{b}\cdot\right.$$

$$\left. + \frac{1}{\omega_{ce}}\left(\frac{D^2}{\omega_{ci}^2} - \frac{BC}{\omega_{ce}\omega_{ci}} + 1\right)\hat{b}\times\right]\left(\frac{-e\bar{J}}{i\omega\epsilon_o m_e} + \frac{\bar{F}_e}{N_o m_e} + \frac{\nu_{en}\bar{F}_n}{N_1 m_n(\nu_{ni} + \nu_{ne} - i\omega)}\right)$$

$$+ \frac{B}{\Delta_1}\left[\frac{-1}{\omega_{ce}\omega_{ci}}\left(\frac{\Delta_2}{\omega_{ce}\omega_{ci}} + 1\right) + \frac{1}{\Delta_2}\left(\frac{BC + AD}{\omega_{ce}\omega_{ci}} - \frac{D^2}{\omega_{ci}^2} - \frac{A^2}{\omega_{ce}^2} - 1\right)\hat{b}\hat{b}\cdot\right.$$

$$\left. + \frac{1}{\omega_{ce}\omega_{ci}}\left(\frac{A}{\omega_{ce}} - \frac{D}{\omega_{ci}}\right)\hat{b}\times\right]\left(\frac{e\bar{J}}{i\omega\epsilon_o m_i} + \frac{\bar{F}_i}{N_o m_i} + \frac{\nu_{in}\bar{F}_n}{N_1 m_n(\nu_{ni} + \nu_{ne} - i\omega)}\right)$$

(A.28)

$$\bar{S}_3 = \frac{C}{\Delta_1}\left[\frac{-1}{\omega_{ce}\omega_{ci}}\left(\frac{\Delta_2}{\omega_{ce}\omega_{ci}} + 1\right) + \frac{1}{\Delta_2}\left(\frac{BC + AD}{\omega_{ce}\omega_{ci}} - \frac{D^2}{\omega_{ci}^2} - \frac{A^2}{\omega_{ce}^2} - 1\right)\hat{b}\hat{b}\cdot\right.$$

$$\left. + \frac{1}{\omega_{ce}\omega_{ci}}\left(\frac{A}{\omega_{ce}} - \frac{D}{\omega_{ci}}\right)\hat{b}\times\right]\left(\frac{-e\bar{J}}{i\omega\epsilon_o m_e} + \frac{\bar{F}_e}{N_o m_e} + \frac{\nu_{en}\bar{F}_n}{N_1 m_n(\nu_{ni} + \nu_{ne} - i\omega)}\right)$$

$$+ \frac{1}{\Delta_1}\left[\frac{1}{\omega_{ci}^2}\left(\frac{A}{\omega_{ce}^2}\Delta_2 + D\right) + \frac{1}{\Delta_2}\left(\frac{BCD}{\omega_{ci}^2} - \frac{2ABC}{\omega_{ce}\omega_{ci}} + \frac{A^3}{\omega_{ce}^2} + A\right)\hat{b}\hat{b}\cdot\right.$$

$$\left. - \frac{1}{\omega_{ci}}\left(\frac{A^2}{\omega_{ce}^2} - \frac{BC}{\omega_{ce}\omega_{ci}} + 1\right)\hat{b}\times\right]\left(\frac{e\bar{J}}{i\omega\epsilon_o m_i} + \frac{\bar{F}_i}{N_o m_i} + \frac{\nu_{in}\bar{F}_n}{N_1 m_n(\nu_{ni} + \nu_{ne} - i\omega)}\right)$$

(A.29)

In the above expressions

$$\Delta_1 \equiv \left(\frac{\Delta_2}{\omega_{ce}\omega_{ci}}\right)^2 + \frac{A^2}{\omega_{ce}^2} - \frac{2BC}{\omega_{ce}\omega_{ci}} + \frac{D^2}{\omega_{ci}^2} + 1 \tag{A.30}$$

and

$$\Delta_2 \equiv AD - BC. \tag{A.31}$$

The matrix Eq. (A.9) can be put into an operator form as

$$\mathcal{W}\,\psi(r) = \phi(r) \tag{A.32}$$

where

$$\psi(r) \equiv \begin{bmatrix} \bar{h} \\ n_e \\ n_i \\ n_n \\ \bar{E} \\ \bar{V}_e \\ \bar{V}_i \\ \bar{V}_n \end{bmatrix} \qquad \phi(r) \equiv \begin{bmatrix} \frac{-i\bar{K}}{\omega\mu_o} \\ \frac{iQ_e}{\omega} \\ \frac{iQ_i}{\omega} \\ \frac{iQ_n}{\omega} \\ \bar{S}_1 \\ \bar{S}_2 \\ \bar{S}_3 \\ \bar{S}_4 \end{bmatrix}. \tag{A.33}$$

Equation (A.32) can be considered as an abstract relation between the sources and the resultant fields. $\psi(r)$ is the eighteen-vector representing the field quantities, $\phi(r)$ is an eighteen-vector representing the source quantities, and $\mathcal{W}$ is the system matrix differential operator relating the field to the sources.

A.2 Generalized Telegraphist's Equations

Generalized Fourier transform as given by Eqs. (2.10) and (2.11) will be used to transform the operator Eq. (A.32). Then, Eqs. (2.14) and (2.15) give the transform pairs for the field vector and the source vector as given by Eq. (A.33), and the Eq. (2.16) gives the transform for the matrix differential operator, $\mathcal{W}$, which can be obtained from Eq. (A.9).

The resultant integral equation of the first kind as given by Eq. (2.17) is repeated here as follows:

$$\oint \mathcal{W}(u,s)\,\psi(s) = \phi(u) \tag{A.34}$$

Equation (A.34) may be put into the generalized forms of the telegraphist's equation by partitioning the transform of the field vector, $\bar{\psi}(s)$, the transform of the source vector, $\bar{\phi}(s)$, and the transform of the matrix differential operator, $\mathcal{W}(u,s)$, as follows:

$$\bar{\psi}(s) = \oint d(s,r) \begin{bmatrix} \bar{h} \\ n_e \\ n_i \\ n_n \\ \bar{E} \\ \bar{V}_e \\ \bar{V}_i \\ \bar{V}_n \end{bmatrix} = \begin{bmatrix} I_t(s) \\ V_e(s) \\ V_i(s) \\ V_n(s) \\ V_t(s) \\ I_e(s) \\ I_i(s) \\ I_n(s) \end{bmatrix} \tag{A.35}$$

$$\mathcal{J}(s) = \oint d(s,r) \begin{bmatrix} \dfrac{-i\bar{K}}{\omega\mu_o} \\ \dfrac{iQ_e}{\omega} \\ \dfrac{iQ_i}{\omega} \\ \dfrac{iQ_n}{\omega} \\ \bar{S}_1 \\ \bar{S}_2 \\ \bar{S}_3 \\ \bar{S}_4 \end{bmatrix} \equiv \begin{bmatrix} J_t(s) \\ W_e(s) \\ W_i(s) \\ W_n(s) \\ W_t(s) \\ J_e(s) \\ J_i(s) \\ J_n(s) \end{bmatrix} \tag{A.36}$$

where $I_t(s)$, $V_t(s)$, $I_e(s)$, $I_i(s)$, $I_n(s)$, $J_t(s)$, $W_t(s)$, $J_e(s)$, $J_i(s)$ and $J_n(s)$ are three by one column matrices, and $V_e(s)$, $V_i(s)$, $V_n(s)$, $W_e(s)$, $W_i(s)$ and $W_n(s)$ are scalars.

In view of the orthonomality of the transformation kernels, $c(r,s)$ and $d(s,r)$, the eighteen-dyadic kernel $\mathcal{W}(u,s)$ can be partitioned as

$$\mathcal{W}(u,s) = \begin{bmatrix} \bar{\bar{1}}(u,s) & 0 & 0 & 0 & -Y_t(u,s) & 0 & 0 & 0 \\ 0 & 1(u,s) & 0 & 0 & 0 & -Z_e(u,s) & 0 & 0 \\ 0 & 0 & 1(u,s) & 0 & 0 & 0 & -Z_i(u,s) & 0 \\ 0 & 0 & 0 & 1(u,s) & 0 & 0 & 0 & -Z_n(u,s) \\ -Z_t(u,s) & -T_{te}(u,s) & -T_{ti}(u,s) & -T_{tn}(u,s) & \bar{\bar{1}}(u,s) & 0 & 0 & 0 \\ -T_{et}(u,s) & -Y_e(u,s) & -Y_{ei}(u,s) & -Y_{en}(u,s) & 0 & \bar{\bar{1}}(u,s) & 0 & 0 \\ -T_{it}(u,s) & -Y_{ie}(u,s) & -Y_i(u,s) & -Y_{in}(u,s) & 0 & 0 & \bar{\bar{1}}(u,s) & 0 \\ -T_{nt}(u,s) & -Y_{ne}(u,s) & -Y_{ni}(u,s) & -Y_n(u,s) & 0 & 0 & 0 & \bar{\bar{1}}(u,s) \end{bmatrix} \tag{A.37}$$

In Eq. (A.37), $1(u,s)$ is a Dirac or Kronecker delta function, which is the same as the scalar form of the idemfactor $\bar{\bar{1}}(u,s)$. The three-dyadic immittance functions $Y_t(u,s)$ and $Z_t(u,s)$; the three-dyadic transfer functions $T_{et}(u,s)$, $T_{it}(u,s)$ and $T_{nt}(u,s)$; the three-row-vector impedance functions $Z_e(u,s)$, $Z_i(u,s)$ and $Z_n(u,s)$; the three-column-vector admittance functions $Y_e(u,s)$, $Y_{ei}(u,s)$, $Y_{en}(u,s)$, $Y_{ie}(u,s)$, $Y_i(u,s)$, $Y_{in}(u,s)$, $Y_{ne}(u,s)$, $Y_{ni}(u,s)$ and $Y_n(u,s)$; the three-column-vector transfer functions $T_{te}(u,s)$, $T_{ti}(u,s)$ and $T_{tn}(u,s)$ are defined as follows:

$$-Y_t(u,s) \equiv \oint d(u,r)\, i\, \frac{\nabla \times \bar{\bar{1}}}{\omega\mu_o}\, c(r,s) \tag{A.38}$$

$$-Z_e(u,s) \equiv \oint d(u,r) \left[\frac{iN_o}{\omega} (\nabla \cdot \bar{\bar{1}})' + \frac{i}{\omega} \nabla N_o \cdot \bar{\bar{1}} \right] c(r,s) \tag{A.39}$$

$$-Z_i(u,s) \equiv -Z_e(u,s) \tag{A.40}$$

$$-Z_n(u,s) \equiv \oint d(u,r) \left[\frac{iN_1}{\omega} (\nabla \cdot \bar{\bar{1}})' + \frac{i}{\omega} \nabla N_1 \cdot \bar{\bar{1}} \right] c(r,s) \tag{A.41}$$

$$-Z_t(u,s) \equiv \oint d(u,r)\, A_{11}\, c(r,s) \tag{A.42}$$

$$-T_{te}(u,s) \equiv \oint d(u,r)\, A_{12}\, c(r,s) \tag{A.43}$$

$$-T_{ti}(u,s) \equiv \oint d(u,r)\, A_{13}\, c(r,s) \tag{A.44}$$

$$-T_{tn}(u,s) \equiv \oint d(u,r)\, A_{14}\, c(r,s) \tag{A.45}$$

$$-T_{et}(u,s) \equiv \oint d(u,r)\, A_{21}\, c(r,s) \tag{A.46}$$

$$-Y_e(u,s) \equiv \oint d(u,r)A_{22}\, c(r,s) \tag{A.47}$$

$$-Y_{ei}(u,s) \equiv \oint d(u,r)A_{23}\, c(r,s) \tag{A.48}$$

$$-Y_{en}(u,s) \equiv \oint d(u,r)A_{24}\, c(r,s) \tag{A.49}$$

$$-T_{it}(u,s) \equiv \oint d(u,r)A_{31}\, c(r,s) \tag{A.50}$$

$$-Y_{ie}(u,s) \equiv \oint d(u,r)A_{32}\, c(r,s) \tag{A.51}$$

$$-Y_i(u,s) \equiv \oint d(u,r)A_{33}\, c(r,s) \tag{A.52}$$

$$-Y_{in}(u,s) \equiv \oint d(u,r)A_{34}\, c(r,s) \tag{A.53}$$

$$-T_{nt}(u,s) \equiv \oint d(u,r)A_{41}\, c(r,s) \tag{A.54}$$

$$-Y_{ne}(u,s) \equiv \oint d(u,r)A_{42}\, c(r,s) \tag{A.55}$$

$$-Y_{ni}(u,s) \equiv \oint d(u,r)A_{43}\, c(r,s) \tag{A.56}$$

$$-Y_n(u,s) \equiv \oint d(u,r)A_{44}\, c(r,s) \tag{A.57}$$

Although the same notation $d(u, r)$ and $c(r, s)$ have been used for the transformation kernels and the inverse transformation kernels in Eqs. (A.38) through (A.57), it should be clear that they are different; e.g., in Eqs. (A.39) and (A.41) the transformation kernels are scalars and the inverse transformation kernels are three-diagonal-dyadics, and in Eqs. (A.38), (A.42), (A.46), (A.50) and (A.54) both the transformation kernels and inverse transformation kernels are three-diagonal-dyadics.

By substituting Eqs. (A.35), (A.36) and (A.37) into the integral Eqs. (A.34) the following generalized telegraphist's equations can be obtained:

$$I_t(u) = J_t(u) + \oint Y_t(u,s)\,V_t(s) \qquad \text{(A.58)}$$

$$V_t(u) = W_t(u) + \oint Z_t(u,s)\,I_t(s) + \oint T_{te}(u,s)\,V_e(s) + \oint T_{ti}(u,s)\,V_i(s) + \oint T_{tn}(u,s)\,V_n(s) \qquad \text{(A.59)}$$

$$I_e(u) = J_e(u) + \oint Y_e(u,s)\,V_e(s) + \oint T_{et}(u,s)\,I_t(s) + \oint Y_{ei}(u,s)\,V_i(s) + \oint Y_{en}(u,s)\,V_n(s) \qquad \text{(A.60)}$$

$$V_e(u) = W_e(u) + \oint Z_e(u,s)\,I_e(s) \qquad \text{(A.61)}$$

$$I_i(u) = J_i(u) + \oint Y_i(u,s)\,V_i(s) + \oint T_{it}(u,s)\,I_t(s) + \oint Y_{ie}(u,s)\,V_e(s) + \oint Y_{in}(u,s)\,V_n(s) \qquad \text{(A.62)}$$

$$V_i(u) = W_i(u) + \oint Z_i(u,s)\,I_i(s) \qquad \text{(A.63)}$$

$$I_n(u) = J_n(u) + \oint Y_n(u,s)\,V_n(s) + \oint T_{nt}(u,s)\,I_t(s) + \oint Y_{ne}(u,s)\,V_e(s) + \oint Y_{ni}(u,s)\,V_i(s) \qquad \text{(A.64)}$$

$$V_n(u) = W_n(u) + \oint Z_n(u,s)\,I_n(s) \qquad \text{(A.65)}$$

A.3 Fredholm Integral Equation

By properly partitioning $\Psi(s)$, $\Phi(s)$ and $\mathcal{W}(u, s)$ as given by Eqs. (A.35), (A.36) and (A.37), the basic Eqs. (2.1) through (2.8) can be reformulated into the general form of the Fredholm Integral Equation of the second kind.

First of all, the transform of the field vector is partitioned into three column vectors each with six components as follows:

$$\Psi(s) \equiv \begin{bmatrix} \Psi_1(s) \\ \Psi_2(s) \\ \Psi_3(s) \end{bmatrix} \tag{A.66}$$

where

$$\Psi_1(s) \equiv \begin{bmatrix} I_t(s) \\ V_e(s) \\ V_i(s) \\ V_n(s) \end{bmatrix}, \quad \Psi_2(s) \equiv \begin{bmatrix} V_t(s) \\ I_e(s) \end{bmatrix}, \quad \Psi_3(s) \equiv \begin{bmatrix} I_i(s) \\ I_n(s) \end{bmatrix}.$$

Similarly, the transform of the source vector is partitioned as three six-column-vectors

$$\Phi(s) \equiv \begin{bmatrix} \Phi_1(s) \\ \Phi_2(s) \\ \Phi_3(s) \end{bmatrix} \tag{A.67}$$

where

$$\Phi_1(s) \equiv \begin{bmatrix} J_t(s) \\ W_e(s) \\ W_i(s) \\ W_n(s) \end{bmatrix}, \quad \Phi_2(s) \equiv \begin{bmatrix} W_t(s) \\ J_e(s) \end{bmatrix}, \quad \Phi_3(s) \equiv \begin{bmatrix} J_i(s) \\ J_n(s) \end{bmatrix}$$

Next, the transform of the matrix differential operator as given by Eq. (A.37) is partitioned in the following form

$$\mathcal{W}(u,s) \equiv \begin{bmatrix} \bar{I}(u,s) & -\mathcal{W}_{12}(u,s) & -\mathcal{W}_{13}(u,s) \\ -\mathcal{W}_{21}(u,s) & \bar{I}(u,s) & 0 \\ -\mathcal{W}_{31}(u,s) & 0 & \bar{I}(u,s) \end{bmatrix} \tag{A.68}$$

Substitution of Eqs. (A.66), (A.67) and (A.68) into the integral Eq. (A.34) gives three coupled integral equations which are

$$\Psi_1(u) = \Phi_1(u) + \oint \mathcal{W}_{12}(u,s)\Psi_2(s) + \oint \mathcal{W}_{13}(u,s)\Psi_3(s) \tag{A.69}$$

$$\Psi_2(u) = \Phi_2(u) + \oint \mathcal{W}_{21}(u,s)\Psi_1(s) \tag{A.70}$$

$$\Psi_3(u) = \Phi_3(u) + \oint \mathcal{W}_{31}(u,s)\Psi_1(s) \tag{A.71}$$

where

$$\mathcal{W}_{12}(u,s) \equiv \begin{bmatrix} Y_t(u,s) & 0 \\ 0 & Z_e(u,s) \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{A.72}$$

$$\mathcal{W}_{13}(u,s) \equiv \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ Z_i(u,s) & 0 \\ 0 & Z_n(u,s) \end{bmatrix} \tag{A.73}$$

$$\mathcal{W}_{21}(u,s) = \begin{bmatrix} Z_t(u,s) & T_{te}(u,s) & T_{ti}(u,s) & T_{tn}(u,s) \\ T_{et}(u,s) & Y_e(u,s) & Y_{ei}(u,s) & Y_{en}(u,s) \end{bmatrix} \qquad (A.74)$$

$$\mathcal{W}_{31}(u,s) = \begin{bmatrix} T_{it}(u,s) & Y_{ie}(u,s) & Y_i(u,s) & Y_{in}(u,s) \\ T_{nt}(u,s) & Y_{ne}(u,s) & Y_{ni}(u,s) & Y_n(u,s) \end{bmatrix} \qquad (A.75)$$

Finally, the substitution of Eqs. (A.70) and (A.71) into Eq. (A.69) gives the desired Fredholm integral equation of the second kind for the field variable $\Psi_1(s)$ as

$$\Psi_1(u) = F(u) + \oint K(u,s)\Psi_1(s) \qquad (A.76)$$

where we have defined

$$F(u) \equiv \phi_1(u) + \oint \mathcal{W}_{12}(u,s)\phi_2(s) + \oint \mathcal{W}_{13}(u,s)\phi_3(s) \qquad (A.77)$$

and

$$K(u,s) \equiv \oint \mathcal{W}_{12}(u,v)\,\mathcal{W}_{21}(v,s) + \oint \mathcal{W}_{13}(u,v)\,\mathcal{W}_{31}(v,s) \qquad (A.78)$$

which are both known functions.

Thus, we have reduced the order of the matrices to be manipulated from 18 x 18 to 6 x 6, and also, we have all the advantages of solving the integral equation of the second kind.

APPENDIX B

EVALUATION OF INVERSE TRANSFORMATION FOR TWO-DIMENSIONAL PROBLEMS

Basically, the following integral must be evaluated

$$\vartheta_2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{(s_1^2 + s_3^2)\, e^{i(s_1 x + s_3 z)}}{\Delta}\, ds_1\, ds_3 \qquad \text{(B.1)}$$

where

$$\Delta \equiv (\omega^2 - \omega_p^2 - c^2 s^2)(\omega^2 - \omega_p^2 - U^2 s^2) - \omega_c^2 (\omega^2 - c^2 s^2) \; .$$

From $\Delta = 0$, we can obtain

$$s_{II} \equiv \left[\frac{(\omega^2 - \omega_p^2)(c^2 + U^2) - c^2 \omega_c^2}{2c^2 U^2} - \sqrt{\frac{\left[(\omega^2 - \omega_p^2)(c^2 + U^2) - c^2 \omega_c^2\right]^2 - 4c^2 U^2 \left[(\omega^2 - \omega_p^2)^2 - \omega^2 \omega_c^2\right]}{4c^4 U^4}} \right]^{\frac{1}{2}} \qquad \text{(B.2)}$$

and

$$s_{I} \equiv \left[\frac{(\omega^2 - \omega_p^2)(c^2 + U^2) - c^2 \omega_c^2}{2c^2 U^2} + \sqrt{\frac{\left[(\omega^2 - \omega_p^2)(c^2 + U^2) - c^2 \omega_c^2\right]^2 - 4c^2 U^2 \left[(\omega^2 - \omega_p^2)^2 - \omega^2 \omega_c^2\right]}{4c^4 U^4}} \right]^{\frac{1}{2}} \qquad \text{(B.3)}$$

where s_{II} and s_I are the propagation constants for the coupled waves.

It is easy to see that in the limit as $\omega_c \rightarrow 0$

$$s_{II} \rightarrow \sqrt{\frac{\omega^2 - \omega_p^2}{c^2}}$$

which is the propagation constant of the ordinary electromagnetic wave, and

$$s_{I} \rightarrow \sqrt{\frac{\omega^2 - \omega_p^2}{U^2}}$$

which is the propagation constant of the plasma wave.

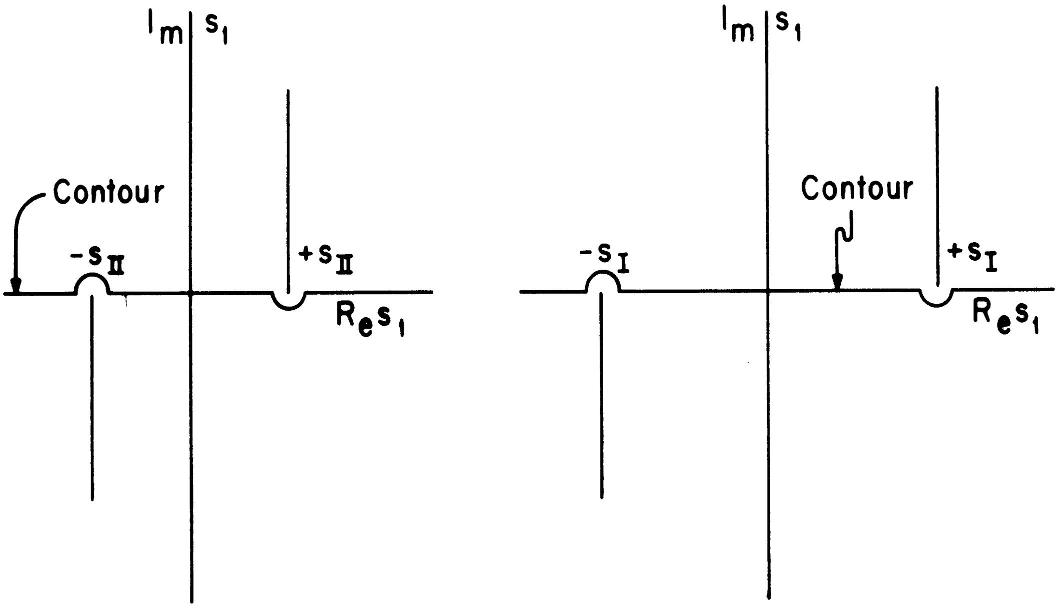
Equation (B.1) will be integrated with respect to the s_3-plane first. There are simple poles at

$$s_3 = \pm\sqrt{s_{II}^2 - s_1^2} \qquad \text{and} \qquad s_3 = \pm\sqrt{s_I^2 - s_1^2}$$

If these poles are on the real axis, the integration path must be indented in such a way so as to produce outgoing waves, i.e., the contour should go above the negative pole and below the positive pole. The contour should be closed in the upper-half plane for $z > 0$ and in the lower-half plane for $z < 0$. The result is

$$\mathcal{J}_2 = \frac{i\pi}{c^2 U^2 (s_{II}^2 - s_I^2)} \int_{-\infty}^{\infty} \left[\frac{s_{II}^2}{\sqrt{s_{II}^2 - s_1^2}} \exp i \left(s_1 x + \sqrt{s_{II}^2 - s_1^2}\, |z| \right) - \frac{s_I^2}{\sqrt{s_I^2 - s_1^2}} \exp i \left(s_1 x + \sqrt{s_I^2 - s_1^2}\, |z| \right) \right] ds_1 \tag{B.4}$$

The integral in (B.4) has branch points at $s_1 = \pm s_{II}$ and $s_1 = \pm s_I$. The choice of branch cuts and contours are shown in the following sketch, which ensures outward traveling phase fronts but not necessarily outward energy propagation.



Branch Cuts

The remaining part of the integration is best performed by the following substitution:

$$s_1 = s_{II} \cos(\theta + \phi) \tag{B.5}$$

for the first part, and

$$s_1 = s_I \cos(\theta + \phi) \tag{B.6}$$

for the second part, where

$$\phi = \tan^{-1} \frac{|z|}{x} \tag{B.7}$$

Only the first part of the integration indicated in (B.4) will be carried out in detail, since the second part is of the same form.

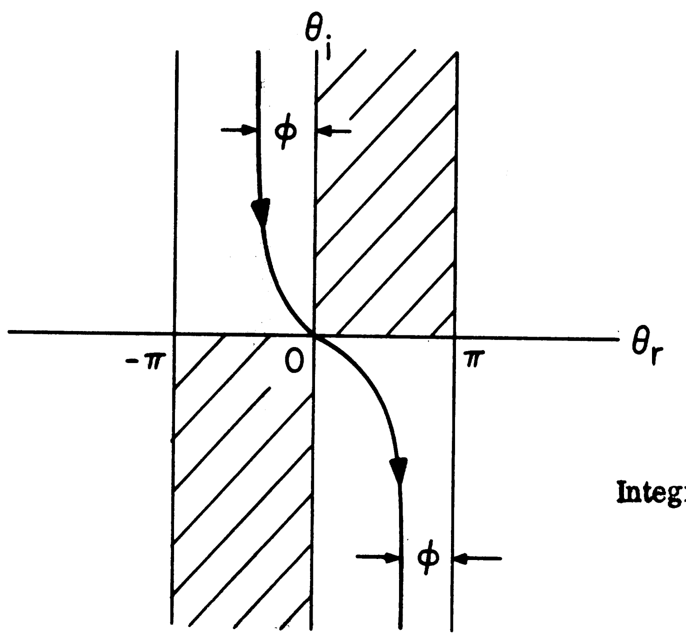
From (B.5)

$$\frac{ds_1}{d\theta} = -s_{II} \sin(\theta + \phi) \quad .$$

The integration limits for the new variable θ can be obtained by expanding $\cos(\theta+\phi)$ as

$$\cos\left[(\theta_r+\phi)+i\theta_i\right] = \cos(\theta_r+\phi)\cosh\theta_i - i\sin(\theta_r+\phi)\sinh\theta_i \quad .$$

For $s_1 = -\infty$ we have $\theta_r+\phi = \pi$ and $\theta_i = -\infty$, and for $s_1 = +\infty$ we have $\theta_r+\phi = 0$ and $\theta_i = +\infty$. The path of integration in the θ -plane is given in the following sketch.



Integration Path

This yields the original definition of the Hankel functions given by Sommerfeld[51].

Thus,

$$\vartheta_2 = \frac{i\pi}{c^2U^2(s_{II}^2-s_I^2)}\left[s_{II}^2\int_{i\infty-\phi}^{-i\infty+\pi-\phi} e^{is_{II}r\cos\theta}\,d\theta - s_I^2\int_{i\infty-\phi}^{-i\infty+\pi-\phi} e^{is_I r\cos\theta}\,d\theta\right]$$

$$= \frac{i\pi^2}{c^2U^2(s_{II}^2-s_I^2)}\left[s_{II}^2 H_o^{(1)}(s_{II}r) - s_I^2 H_o^{(1)}(s_I r)\right] \quad . \tag{B.8}$$

APPENDIX C

THREE ROOTS FOR THE CUBIC EQUATION

To express the three roots k_1^2, k_2^2, k_3^2 as given by Eqs. (3.39), (3.40) and (3.41) in terms of the original coefficients in the dispersion relation, it is only necessary to find the expressions for a, b and $\left[\frac{b^2}{4} + \frac{a^3}{27}\right]$ as follows.

In terms of original quantities

$$a = \frac{1}{3}(3q - p^2)$$

$$= \frac{1}{3}\left\{ - \frac{\left[\beta_e^2(1-\Omega^2) - 2\omega_o^2\beta_o^2\right]^2}{(\Omega^2\cos^2\theta - 1)^2} + \frac{\beta_e^2\beta_o^2\left[(1-\Omega^2)(3\omega_o^2-2) + 2\omega_o^2(3-\omega_o^2)\right] - \beta_o^4\omega_o^2(2+3\omega_o^2) - 2\beta_e^4\omega_o^2(1-\Omega^2)}{(\Omega^2\cos^2\theta - 1)} - \left[(\beta_o^2 - \beta_e^2\omega_o^2)^2 + \beta_e^2\beta_o^2\omega_o^2\right]\right\} \qquad (C.1)$$

$$b = \frac{1}{27}\left(2p^3 - 9pq + 27r\right)$$

$$= \frac{1}{27}\Bigg\{ (\beta_o^{\,2} + \beta_e^{\,2}\omega_o^{\,2})\left[2(\beta_o^{\,2} - \beta_e^{\,2}\omega_o^{\,2})^2 - \beta_o^{\,2}\beta_e^{\,2}\omega_o^{\,2}\right]$$

$$+ \frac{3\beta_e^{\,2}\beta_o^{\,4}\left[(1-\Omega^2)(2-3\omega_o^{\,2}) - 6(1-\omega_o^{\,2})^2\omega_o^{\,2} + 7\omega_o^{\,4}\right] - 6\beta_o^{\,6}\omega_o^{\,2}(3\omega_o^{\,2} - 1) + 3\omega_o^{\,2}\beta_o^{\,2}\beta_e^{\,4}\left[(1-\Omega^2)(1-3\omega_o^{\,2}) - 2\omega_o^{\,2}(3-\omega_o^{\,2})\right] + 6\omega_o^{\,4}(1-\Omega^2)\beta_e^{\,6}}{(\Omega^2\cos^2\theta - 1)}$$

$$+ \frac{\left[\beta_e^{\,2}(1-\Omega^2) - 2\omega_o^{\,2}\beta_o^{\,2}\right]\left\{3\beta_o^{\,4}\omega_o^{\,2}(2+3\omega_o^{\,2}) + 3\beta_e^{\,2}\beta_o^{\,2}\left[(1-\Omega^2)(2-3\omega_o^{\,2}) + 2\omega_o^{\,2}(\omega_o^{\,2}-3)\right] + 6\omega_o^{\,2}(1-\Omega^2)\beta_e^{\,4}\right\}}{(\Omega^2\cos^2\theta - 1)^2}$$

$$+ \frac{2\left[\beta_e^{\,2}(1-\Omega^2) - 2\omega_o^{\,2}\beta_o^{\,2}\right]^3}{(\Omega^2\cos^2\theta - 1)^3}\Bigg\} \qquad \text{(C.2)}$$

$$
\frac{b^2}{4}+\frac{a^3}{27}=\frac{\omega_o^4}{27}\Bigg\{-\frac{1}{4}\beta_o^4\beta_e^4(\beta_o^2-\beta_e^2\omega_o^2)^2+\frac{1}{(\Omega^2\cos^2\theta-1)}\Big\{\beta_o^4\beta_e^8\left[-\frac{\omega_o^2}{2}(1+\omega_o^2)(1-\Omega^2)-\omega_o^4(1-\omega_o^2)^2\right]
$$

$$
+\beta_o^6\beta_e^6\left[-\frac{1}{2}(1-4\omega_o^2)(1-\Omega^2)+\omega_o^2(4-\frac{7}{2}\omega_o^2+\omega_o^4)\right]+\beta_o^8\beta_e^4\left[-\frac{1}{2}(1-\Omega^2)-\frac{1}{2}(2+7\omega_o^2-2\omega_o^4)\right]+\beta_o^{10}\beta_e^2(2+\omega_o^2)-\beta_o^{12}\Big\}
$$

$$
+\frac{1}{(\Omega^2\cos^2\theta-1)^2}\Big\{\beta_o^4\beta_e^8\left[-\frac{1}{4}(1+4\omega_o^2+\omega_o^4)(1-\Omega^2)^2-2\omega_o^2(1-\omega_o^2)^2(1-\Omega^2)-\omega_o^4(1-\omega_o^2)^2\right]
$$

$$
+\beta_o^6\beta_e^6\left[\frac{1}{2}(5\omega_o^2-2)(1-\Omega^2)^2+(-2+\frac{33}{2}\omega_o^2-15\omega_o^4+5\omega_o^6)(1-\Omega^2)+\omega_o^2(10-23\omega_o^2+17\omega_o^4-4\omega_o^6)\right]
$$

$$
+\beta_o^8\beta_e^4\left[-\frac{1}{4}(1-\Omega^2)^2+(4-15\omega_o^2+4\omega_o^4)(1-\Omega^2)-(1+23\omega_o^2-\frac{155}{4}\omega_o^4+20\omega_o^6-2\omega_o^8)\right]
$$

$$
+\beta_o^{10}\beta_e^2\left[(5\omega_o^2-2)(1-\Omega^2)+(2+17\omega_o^2-20\omega_o^4+7\omega_o^6)\right]+\beta_o^{12}(2\omega_o^4-4\omega_o^2-1)\Big\}
$$

(C.3)

Cont'd on next page

$$
\begin{aligned}
&+\frac{1}{(\Omega^2\cos^2\theta-1)^3}\Bigg\{\beta_o^4\beta_e^8(1-\Omega^2)\left[-\frac{1}{2}(1+\omega_o^2)(1-\Omega^2)^2-(1-\omega_o^2)^2(1-\Omega^2)-2\omega_o^2(1-\omega_o^2)^2\right]\\
&+\beta_o^6\beta_e^6\left[\frac{1}{2}(2\omega_o^2-1)(1-\Omega^2)^3+\frac{1}{2}(\omega_o^2+2)^2(1-\Omega^2)^2+(9\omega_o^6-22\omega_o^4+15\omega_o^2-2)(1-\Omega^2)+4\omega_o^2(2-5\omega_o^2+4\omega_o^4-\omega_o^6)\right]\\
&+\beta_o^8\beta_e^4\left[-\frac{1}{2}(2-\omega_o^2+6\omega_o^4)(1-\Omega^2)^2+(4-22\omega_o^2+\frac{31}{2}\omega_o^4-4\omega_o^6)(1-\Omega^2)-2\omega_o^2(\omega_o^6+6\omega_o^4-17\omega_o^2+10)\right]\\
&+\beta_o^{10}\beta_e^2\left[-(2-9\omega_o^2+4\omega_o^4-3\omega_o^6)(1-\Omega^2)+2\omega_o^2(8-6\omega_o^2-\frac{5}{2}\omega_o^4+2\omega_o^6)\right]-\beta_o^{12}\omega_o^2(4+2\omega_o^2-4\omega_o^4+\omega_o^6)\Bigg\}\\
&+\frac{\left[\beta_e^2(1-\Omega^2)-2\omega_o^2\beta_o^2\right]^2}{(\Omega^2\cos^2\theta-1)^4}\Bigg\{\beta_o^4\beta_e^4\left[-\frac{1}{4}(1-\Omega^2)^2-(1-\omega_o^2)^2\right]+\beta_o^6\beta_e^2\left[\frac{-\omega_o^2}{2}(1-\Omega^2)+(2-\omega_o^2)(1-\omega_o^2)\right]-\frac{1}{4}(2-\omega_o^2)^2\beta_o^8\Bigg\}\Bigg\}.
\end{aligned}
\tag{C.3}
$$

APPENDIX D

SOME ANALYSES FOR THE HYPERBOLAS

Some supplementary analyses are given here for the hyperbolas shown in Fig. 7, Fig. 8 and Fig. 12. First, Eqs. (3.56) and (3.69) are transformed to the standard hyperbolic equation by the following coordinate transformation:

$$\begin{cases} \omega_o^2 = \omega_o'^2 \cos\theta - s'^2 \sin\theta \\ s^2 = \omega_o'^2 \sin\theta + s'^2 \cos\theta \end{cases} \tag{D.1}$$

where

$$\tan 2\theta = \frac{\beta_o^2 + \beta_e^2}{\beta_o^2 \beta_e^2 - 1}, \qquad \cos 2\theta = \frac{\beta_o^2 \beta_e^2 - 1}{\sqrt{1 + \beta_o^4 \beta_e^4 + \beta_o^4 + \beta_e^4}}$$

$$\sin\theta = \sqrt{\frac{1 - \cos 2\theta}{2}} = \left[\frac{1 - \beta_o^2 \beta_e^2 + \sqrt{1 + \beta_o^4 \beta_e^4 + \beta_o^4 + \beta_e^4}}{2\sqrt{1 + \beta_o^4 \beta_e^4 + \beta_o^4 + \beta_e^4}} \right]^{\frac{1}{2}}$$

$$\cos\theta = \sqrt{\frac{1 + \cos 2\theta}{2}} = \left[\frac{-1 + \beta_o^2 \beta_e^2 + \sqrt{1 + \beta_o^4 \beta_e^4 + \beta_o^4 + \beta_e^4}}{2\sqrt{1 + \beta_o^4 \beta_e^4 + \beta_o^4 + \beta_e^4}} \right]^{\frac{1}{2}}$$

The results of this transformation are Eq. (3.58) and Eq. (3.70).

Next, the slopes of the hyperbolas in Fig. 7 and Fig. 8 are analyzed as follows. From Eq. (3.45) and Eq. (3.46) we can obtain

$$\frac{ds^2}{d\omega_o^2} = -\frac{\beta_o^2\beta_e^2}{2}\left(\frac{1}{\beta_o^2}+\frac{1}{\beta_e^2}\right)$$

$$\pm\left(\frac{1}{2}\right)\frac{-\frac{\beta_o^4\beta_e^4}{2}\left(\frac{1}{\beta_o^2}+\frac{1}{\beta_e^2}\right)\left[(1-\omega_o^2)\left(\frac{1}{\beta_o^2}+\frac{1}{\beta_e^2}\right)-\frac{\Omega^2}{\beta_o^2}\right]+2\beta_o^2\beta_e^2(1-\omega_o^2)}{\sqrt{\frac{\beta_o^4\beta_e^4}{4}\left[(1-\omega_o^2)\left(\frac{1}{\beta_o^2}+\frac{1}{\beta_e^2}\right)-\frac{\Omega^2}{\beta_o^2}\right]^2-\beta_o^2\beta_e^2\left[(1-\omega_o^2)^2-\Omega^2\right]}} . \tag{D.2}$$

This slope will become infinite when

$$\beta_o^2\beta_e^2\left[(1-\omega_o^2)\left(\frac{1}{\beta_o^2}+\frac{1}{\beta_e^2}\right)-\frac{\Omega^2}{\beta_o^2}\right]^2 = 4\left[(1-\omega_o^2)^2-\Omega^2\right]$$

$$\therefore\ (1-\omega_o^2) = \frac{\frac{\Omega^2}{\beta_o^2}\left(\frac{1}{\beta_o^2}+\frac{1}{\beta_e^2}\right)\pm\frac{2\Omega}{\beta_o^3\beta_e}\sqrt{\Omega^2-\left(1-\frac{\beta_o^2}{\beta_e^2}\right)^2}}{\left(\frac{1}{\beta_o^2}-\frac{1}{\beta_e^2}\right)^2} .$$

Thus, there are two points where $\frac{ds^2}{d\omega_o^2} \longrightarrow \infty$ if $\Omega > 1-\frac{\beta_o^2}{\beta_e^2} \approx 1$. Since $\frac{1}{\beta_o} >> \frac{1}{\beta_e}$ these two points can be given approximately as

$$\omega_o^2 \approx 1-\Omega^2 \tag{D.3}$$

This slope will become zero at

$$\pm\sqrt{\frac{\beta_o^4\beta_e^4}{4}\left[(1-\omega_o^2)\left(\frac{1}{\beta_o^2}+\frac{1}{\beta_e^2}\right)-\frac{\Omega^2}{\beta_o^2}\right]^2-\beta_o^2\beta_e^2\left[(1-\omega_o^2)^2-\Omega^2\right]}$$

$$= -\frac{\beta_o^2\beta_e^2}{2}\left[(1-\omega_o^2)\left(\frac{1}{\beta_o^2}+\frac{1}{\beta_e^2}\right)-\frac{\Omega^2}{\beta_o^2}\right]+\frac{2(1-\omega_o^2)}{\left(\frac{1}{\beta_o^2}+\frac{1}{\beta_e^2}\right)} ,$$

that is

$$(1-\omega_o^2)^2(\beta_o^2-\beta_e^2)^2-2\Omega^2\beta_e^2(\beta_o^2+\beta_e^2)(1-\omega_o^2)+\Omega^2(\beta_o^2+\beta_e^2)^2=0 \quad ,$$

$$\therefore (1-\omega_o^2)=\frac{\Omega^2\beta_e^2(\beta_o^2+\beta_e^2)\pm\Omega\beta_e^2(\beta_o^2+\beta_e^2)\sqrt{\Omega^2-\left(1-\frac{\beta_o^2}{\beta_e^2}\right)^2}}{(\beta_o^2-\beta_e^2)^2}$$

So, again there are two points where $\frac{ds^2}{d\omega_o^2}=0$ if $\Omega > 1-\frac{\beta_o^2}{\beta_e^2}\approx 1$. If the condition $\beta_e^2 >> \beta_o^2$ is applied, these two points are approximately given by

$$\omega_o^2 \approx (1-\Omega^2)\pm\Omega\sqrt{\Omega^2-1} \quad , \tag{D.4}$$

and also if $\Omega^2 >> 1$

$$\omega_o^2 \approxeq \begin{cases} 1 \\ 1-2\Omega^2 \end{cases} \quad . \tag{D.5}$$

The slope of the hyperbola in Fig. 12 is analyzed as follows. From Eqs. (3.66) and (3.67)

$$2\frac{ds^2}{d\omega_o^2}=-(\beta_o^2+\beta_e^2)\pm\frac{\left[\beta_o^2(\omega_o^2-1)-\omega_o^2\beta_e^2\right](\beta_o^2-\beta_e^2)+2\beta_o^2\beta_e^2}{\sqrt{\left[\beta_o^2(\omega_o^2-1)-\omega_o^2\beta_e^2\right]^2+4\beta_o^2\beta_e^2\omega_o^2}} \quad , \tag{D.6}$$

thus

$$\frac{ds^2}{d\omega_o^2}\longrightarrow\infty \qquad \text{at}$$

$$\omega_o^2=\frac{-\beta_o^2(3\beta_e^2-\beta_o^2)\pm 2\beta_o^2\beta_e\sqrt{2\beta_e^2-\beta_o^2}}{(\beta_o^2-\beta_e^2)^2}\approx\frac{\left(-3\beta_e^2\pm 2\sqrt{2}\beta_e^2\right)\beta_o^2}{(\beta_o^2-\beta_e^2)^2}<0\, , \tag{D.7}$$

and

$$\frac{ds^2}{d\omega_o^2} \longrightarrow 0 \qquad \text{at}$$

$$\omega_o^2 = \frac{-\beta_o^2(3\beta_e^2-\beta_o^2) \pm \beta_o(\beta_o^2+\beta_e^2)\sqrt{2\beta_e^2-\beta_o^2}}{(\beta_o^2-\beta_e^2)^2} \approx \frac{-3\beta_o^2\beta_e^2 \pm \sqrt{2}\beta_o\beta_e^3}{(\beta_o^2-\beta_e^2)^2} \approx \pm\sqrt{2}\frac{\beta_o}{\beta_e} .$$

(D. 8)

APPENDIX E

DERIVATION OF λ

The expression for λ given by Eq. (4.93), which is repeated here for convenience, will be proved in the following:

$$\lambda = \frac{1}{6}\frac{\partial K_m}{\partial s_\rho}\cos(\tan^{-1}|\varphi'|)\,. \tag{4.93}$$

First, by choosing tangent plane as the coordinate plane, the phase surface at the point $(s'_{10}, s'_{20}, s'_{30})$ can be expressed as

$$s'_2 = s'_{20} + \frac{1}{2}\left[\frac{\partial^2 s'_2}{\partial s_1'^2}(s'_1 - s'_{10})^2 + 2\frac{\partial^2 s'_2}{\partial s'_1 \partial s'_3}(s'_1 - s'_{10})(s'_3 - s'_{30})\right.$$

$$\left. + \frac{\partial^2 s'_2}{\partial s_3'^2}(s'_3 - s'_{30})^2\right] + \frac{1}{6}\left[(s'_1 - s'_{10})D_{s'_1} + (s'_3 - s'_{30})D_{s'_3}\right]^3 s'_2$$

$$+ \text{higher order terms} \tag{E.1}$$

If we also make the s'_1-axis lie in the parallel plane and the s'_3-axis lie in the meridian plane, the following relations can be obtained.

$$\frac{\partial^2 s'_2}{\partial s'_1 \partial s'_3} = 0$$

$$\frac{\partial^2 s'_2}{\partial s_1'^2} = K_p \tag{E.2}$$

$$\frac{\partial^2 s'_2}{\partial s_3'^2} = K_m$$

and the coefficient of the term with $(s'_3 - s'_{30})^3$ is

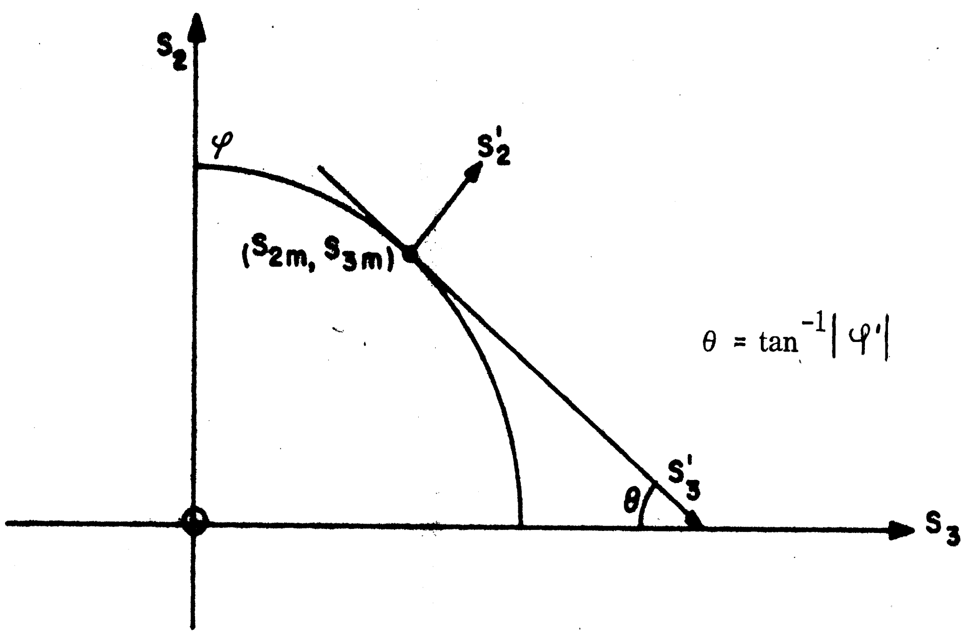
$$\lambda = \frac{1}{6} \frac{\partial^3 s'_2}{\partial s'^3_3} = \frac{1}{6} \frac{\partial K_m}{\partial s'_3} \quad . \tag{E.3}$$

To transform back to the original coordinate system, use is made of the relation (see sketch below)

$$\begin{cases} s_3 = s'_3 \cos\theta + s'_2 \sin\theta + s_{3m} \\ s_2 = -s'_3 \sin\theta + s'_2 \cos\theta + s_{2m} \end{cases} ,$$

and

$$\frac{\partial K_m}{\partial s'_3} = \frac{\partial s_3}{\partial s'_3} \frac{\partial K_m}{\partial s_3} = \cos\theta \frac{\partial K_m}{\partial s_3} \quad . \tag{E.4}$$



Translation of Coordinate in s_2-s_3 Plane

Substitution of (E.4) into (E.3) completes the proof.

APPENDIX F

PROBLEMS OF INHOMOGENEOUS PLASMA

[1] Introduction

The unified operator approach for excitation problems in an ionized medium formulated in this thesis can be, in principle, used for a general linear medium either homogeneous or inhomogeneous.

However, due to the complexity of the resulting equations, direct application of the formulation to an inhomogeneous medium has not been tested. In this appendix, some general thoughts regarding the solution of excitation problems in an inhomogeneous medium are discussed. This phase of work should be a subject of further research.

[2] Perturbation Solution

The integral formulation of the problem is naturally useful in finding approximate solutions for a medium with weak inhomogeneities so that the classical Born approximation may be applied. For example, if a medium is homogeneous except for a small region where the density is somewhat different from the homogeneous region, the kernel of the integral equation to be solved, i.e.

$$\bar{\psi}(u) = F(u) + \oint K(u,s)\,\bar{\psi}(s)\,, \tag{F.1}$$

can be represented by

$$K(u,s) = K_h(u,s) + K_\Delta \tag{F.2}$$

where $K_h(u,s)$ is an ideal kernel corresponding to a homogeneous medium, and K_Δ is a small operator caused by weak inhomogeneity. If $\Psi_o(u)$ is the solution for the homogeneous medium, we may set

$$\Psi(u) = \Psi_o(u) + \Psi_\Delta \ , \tag{F.3}$$

and the substitution of Eqs. (F.2) and (F.3) into Eq. (F.1) gives

$$\Psi_o(u) + \Psi_\Delta = F(u) + \oint \left[K_h(u,s) + K_\Delta\right]\left[\Psi_o(s) + \Psi_\Delta\right] . \tag{F.4}$$

By neglecting higher order small terms, and using the relation

$$\Psi_o(u) = F(u) + \oint K_h(u,s)\,\Psi_o(s) \ , \tag{F.5}$$

the perturbed field can be approximately given by

$$\Psi_\Delta = \oint K_\Delta \Psi_o(s) \ , \tag{F.6}$$

which can be thought of as classical approximation of scattered field applied to both electromagnetic and plasma waves.

For arbitrary inhomogeneity, if the kernel can be approximated by a degenerate form such as

$$K(u,s) = A(u)\,B(s) + K_o(u,s) \ , \tag{F.7}$$

where the small perturbation is expressed by $K_o(u, s)$ whose resolvent kernel $H_o(u, s)$ can be given by a rapidly convergent Neumann series, the resolvent kernel of this integral equation can be expressed exactly as follows. The integral equation is given by Eq. (2.70)

$$\Psi_1(u) = F(u) + \oint K(u,s)\,\Psi_1(s) \ . \tag{2.70}$$

Its solution is given in terms of a resolvent kernel $H(u, s)$ as

$$\Psi_1(u) = F(u) + \oint H(u,s)\,F(s) \ . \tag{F.8}$$

Upon substituting Eq. (F.8) into Eq. (2.70), the following resolvent equation is obtained

$$H(u,s) = K(u,s) + \mathop{\not{S}} K(u,v)\, H(v,s)\ . \tag{F.9}$$

Similarly, the following equation can be obtained

$$H_o(u,s) = K_o(u,s) + \mathop{\not{S}} K_o(u,v)\, H_o(v,s)\ . \tag{F.10}$$

Now, let

$$A_o(u) = A(u) + \mathop{\not{S}} H_o(u,s)\, A(s) \tag{F.11}$$

$$B_o(s) = B(s) + \mathop{\not{S}} B(u)\, H_o(u,s) \tag{F.12}$$

$$R_o = \mathop{\not{S}} B(s)\, A_o(s) = \mathop{\not{S}} B_o(s)\, A(s) \tag{F.13}$$

then, from Eqs. (F.10) and (F.11) it can be seen that

$$A_o(u) = A(u) + \mathop{\not{S}} K_o(u,v)\, A_o(v) \tag{F.14}$$

Also, by assuming

$$H(u,s) = H_o(u,s) + H_x(u,s)\ , \tag{F.15}$$

then, substituting Eqs. (F.7) and (F.15) into Eq. (F.9), and employing the relation given by Eqs. (F.10) and (F.11), we can obtain the following integral equation

$$H_x(u,s) = A(u)B_o(s) + \mathop{\not{S}} K(u,v)\, H_x(v,s)\ . \tag{F.16}$$

Thus, the solution of Eq. (F.16) will completely define the desired resolvent $H(u,s)$. By subtracting and adding $\mathop{\not{S}} A(u)B(s)A_o(s)$ in the right-hand side of Eq. (F.14) we have

$$A_o(u) = A(u)\left[1 - R_o\right] + \mathop{\not{S}} \left[A(u)\,B(v) + K_o(u,v)\right] A_o(v)\ . \tag{F.17}$$

Post multiplication of $[1-R_o]^{-1} B_o(s)$ to Eq. (F.17) gives

$$A_o(u)\,[1-R_o]^{-1} B_o(s) = A(u)\,B_o(s) + \oint K(u,v) A_o(v)\,[1-R_o]^{-1} B_o(s). \qquad \text{(F.18)}$$

Upon comparing Eq. (F.18) with Eq. (F.16), it is seen that

$$H_x(u,s) = A_o(u)\,[1-R_o]^{-1} B_o(s) \qquad \text{(F.19)}$$

and

$$H(u,s) = H_o(u,s) + A_o(u)\,[1-R_o]^{-1} B_o(s). \qquad \text{(F.20)}$$

3] General Numerical Scheme

In principle, the integral equation formulated in this work, i.e.

$$\Psi_1(u) = F(u) + \lambda \oint K(u,s)\,\Psi_1(s)\,, \qquad \text{(F.21)}$$

can be formally solved in terms of the resolvent kernel, where we may reformulate the kernel by making λ as a frequency dependent parameter or use the original kernel with $\lambda = 1$. The general mathematical techniques of obtaining the resolvent kernel in the one-dimensional case have been discussed by Smithies[52] and the formal extension of this method to the case of a dyadic kernel has been discussed by Diament[28]. In principle, if we write the resolvent kernel as

$$H(u,s;\lambda) = \frac{c(u,s;\lambda)}{P(\lambda)} \qquad \text{(F.22)}$$

the necessary computation for the resolvent can be expressed in the form of a power series in λ

$$P(\lambda) = \sum_{n=0}^{\infty} P_n \lambda^n\,, \qquad \text{(F.23)}$$

$$c(u,s;\lambda) = \sum_{n=0}^{\infty} c_n(u,s)\,\lambda^n , \tag{F.24}$$

and the coefficients, P_n and $c_n(u, s)$, can be obtained from a set of recursion equations. Although this formal solution is feasible in principle, it has not been tested numerically as to the complexity of numerical operations necessary. It is felt, however, that for some problems where the c_n's are decreasing rapidly as n increases, the evaluation of the first few terms of the series will be sufficient to obtain an approximate solution of the problem. If the kernel is reformulated in terms of a parameter λ which is directly proportional to frequency or wavelength an approximate solution, appropriate to low frequency or high frequency, respectively, may be obtained by evaluating the first few terms of the series.

[4] One-Dimensional Problem

For simple configuration of the inhomogeneity, such as one-dimensional excitation problem, the general integral equation can be reduced further. As an illustration, consider a plasma medium having a density variation with respect to the z-direction, and neglecting the static magnetic field.

If the Fourier transforms of the following functions are given as

$$\oint \frac{1}{(2\pi)^3}\, e^{-i r\cdot s}\, N_o(z) = \widetilde{N}_o(s_3)\,\delta(s_1)\,\delta(s_2) , \tag{F.25}$$

$$\oint \frac{1}{(2\pi)^3}\, e^{-i r\cdot s} \left[\frac{\omega^2 \epsilon_o m}{e^2} - N_o(z)\right]^{-1} = g(s_3)\,\delta(s_1)\,\delta(s_2) , \tag{F.26}$$

$$\oint \frac{1}{(2\pi)^3} e^{-ir\cdot s} N_o(z) \left[\frac{\omega^2 \epsilon_o m}{e^2} - N_o(z) \right]^{-1}$$

$$= \delta(s_1)\delta(s_2) \int_{-\infty}^{\infty} \widetilde{N}_o(-\ell) g(s_3+\ell)\, d\ell = p(s_3)\delta(s_1)\delta(s_2) \qquad \text{(F.27)}$$

$$\oint \frac{1}{(2\pi)^3} e^{-ir\cdot s} N_o(z)^{-1} = \widetilde{N}_o^{-1}(s_3)\delta(s_1)\delta(s_2)\,, \qquad \text{(F.28)}$$

$$\oint \frac{1}{(2\pi)^3} e^{-ir\cdot s} N_o(z)^{-1} \left[\frac{\omega^2 \epsilon_o m}{e^2} - N_o(z) \right]^{-1}$$

$$= \delta(s_1)\delta(s_2) \int_{-\infty}^{\infty} \widetilde{N}_o^{-1}(-\ell) g(s_3+\ell)\, d\ell = q(s_3)\delta(s_1)\delta(s_2) \qquad \text{(F.29)}$$

Equations (2.52) through (2.57) reduce to

$$-Y_t(u,s) = -\frac{\bar{\bar{s}}}{\omega\mu_o} 1(u,s) \qquad \text{(F.30)}$$

$$-Z_t(u,s) = \frac{\bar{\bar{s}}}{\epsilon_o\omega} \delta(u_1-s_1)\delta(u_2-s_2)p(u_3-s_3) + \frac{\bar{\bar{s}}}{\epsilon_o\omega} \delta(u_1-s_1)\delta(u_2-s_2)\delta(u_3-s_3)$$

$$\text{(F.31)}$$

$$-T_{et}(u,s) = -\frac{i}{e}\bar{\bar{s}}\,\delta(u_1-s_1)\delta(u_2-s_2)g(u_3-s_3) \qquad \text{(F.32)}$$

$$-Z_e(u,s) = -\frac{s'}{\omega} \widetilde{N}_o(u_3-s_3)\delta(u_1-s_1)\delta(u_2-s_2)$$

$$-\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \delta(u_1-s_1)\delta(u_2-s_2) \frac{(u_3-s_3)}{\omega} \widetilde{N}_o(u_3-s_3) \qquad \text{(F.33)}$$

$$-Y_e(u, s) = \frac{-\epsilon_o m\omega U^2 s}{e^2} \delta(u_1 - s_1)\delta(u_2 - s_2)q(u_3 - s_3) \tag{F.34}$$

$$-T_{te}(u, s) = \frac{-imU^2 s}{e} \delta(u_1 - s_1)\delta(u_2 - s_2)g(u_3 - s_3) \,. \tag{F.35}$$

The matrix components of the kernel are obtained from Eqs. (2.81) through (2.84) and Eqs. (F.30) through (F.35) as

$$\oint Y_t(u, v)Z_t(v, s) = \frac{-c^2 \bar{\bar{u}}s}{\omega^2} \delta(u_1 - s_1)\delta(u_2 - s_2) \left[p(u_3 - s_3) + \delta(u_3 - s_3)\right] \tag{F.36}$$

$$\oint Y_t(u, v)T_{te}(v, s) = \frac{iU^2 m}{e\mu_o \omega} \bar{\bar{u}}s\delta(u_1 - s_1)\delta(u_2 - s_2)g(u_3 - s_3) \tag{F.37}$$

$$\oint Z_e(u, v)T_{et}(v, s)$$

$$= \frac{i}{e\omega} \int_{-\infty}^{\infty} \left[(s_3 - u_3)s_2 \;\; (u_3 - s_3)s_1 \;\; 0\right] \widetilde{N}_o(u_3 - v_3)g(v_3 - s_3)\delta(u_1 - s_1)\delta(u_2 - s_2)dv_3 \tag{F.38}$$

$$\oint Z_e(u, v)Y_e(v, s)$$

$$= \frac{m\epsilon_o U^2}{e^2} \int_{-\infty}^{\infty} (s_1^2 + s_2^2 + s_3 u_3)\widetilde{N}_o(u_3 - v_3)q(v_3 - s_3)\delta(u_1 - s_1)\delta(u_2 - s_2)dv_3 \,. \tag{F.39}$$

Thus the kernel K(u, s) has the partially ideal form as given by the following equation, Eq. (F.40), and the multiple integrations in Eq. (2.70) can be reduced to a single integration with respect to s_3.

$$K(u,s)=\begin{bmatrix} \begin{bmatrix} -(u_3s_3+s_2u_2) & s_1u_2 & s_1u_3 \\ u_1s_2 & -(u_3s_3+u_1s_1) & s_2u_3 \\ u_1s_3 & u_2s_3 & -(u_2s_2+u_1s_1) \end{bmatrix} & \begin{bmatrix} -s_2u_3+u_2s_3 \\ s_1u_3-u_1s_3 \\ -s_1u_2+u_1s_2 \end{bmatrix} \\ \times \frac{c^2}{\omega^2}\left[-\delta(u_3-s_3)-p(u_3-s_3)\right] & \times \frac{iU^2 m}{e\mu_o\omega} g(u_3-s_3) \\ \left[s_2(s_3-u_3) \quad s_1(u_3-s_3) \quad 0\right] & (s_1^2+s_2^2+s_3u_3) \\ \times \frac{i}{e\omega}\int_{-\infty}^{\infty} \widetilde{N_o}(u_3-v_3)g(v_3-s_3)dv_3 & \times \frac{m\epsilon_o U^2}{e^2}\int_{-\infty}^{\infty} \widetilde{N_o}(u_3-v_3)q(v_3-s_3)dv_3 \end{bmatrix} \delta(u_1-s_1)\delta(u_2-s_2) \qquad (F.40)$$

Upon substituting this kernel, given by Eq. (F.40), into the integral Eq. (2.70), four coupled integral equations are obtained which are given by Eqs. (F.41) through (F.44). It is to be noted that in these equations the integrals correspond to the contribution due to inhomogeneity of the medium, and the last terms of the integrands in Eqs. (F.41) and (F.42) and the first term of the integrand in Eq. (F.44) are the coupling terms between plasma wave and electromagnetic wave due to inhomogeneity. The source functions $F_1(u)$, $F_2(u)$, $F_3(u)$ and $F_4(u)$ are given by Eqs. (4.11) through (4.14) by setting $\omega_c=0$.

$$I_{tx}(u) = F_1(u) - \frac{c^2}{\omega^2}\left[-(u_3^2+u_2^2)\,I_{tx}(u)+u_1u_2\,I_{ty}(u)+u_1u_3\,I_{tz}(u)\right]$$

$$+\int_{-\infty}^{\infty}\left\{\frac{c^2}{\omega^2}\left[(u_3s_3+u_2^2)\,I_{tx}(s_3)-u_1u_2\,I_{ty}(s_3)-u_1u_3\,I_{tz}(s_3)\right]p(u_3-s_3)\right.$$

$$\left.+\frac{iU^2m}{e\mu_o\omega}\,u_2(s_3-u_3)\,g(u_3-s_3)\,V_e(s_3)\right\}ds_3 \qquad \text{(F.41)}$$

$$I_{ty}(u) = F_2(u) - \frac{c^2}{\omega^2}\left[u_1u_2\,I_{tx}(u)-(u_3^2+u_1^2)\,I_{ty}(u)+u_2u_3\,I_{tz}(u)\right]$$

$$+\int_{-\infty}^{\infty}\left\{\frac{c^2}{\omega^2}\left[-u_1u_2\,I_{tx}(s_3)+(s_3u_3+u_1^2)\,I_{ty}(s_3)-u_2u_3\,I_{tz}(s_3)\right]p(u_3-s_3)\right.$$

$$\left.+\frac{iU^2m}{e\mu_o\omega}\,u_1(u_3-s_3)\,g(u_3-s_3)\,V_e(s_3)\right\}ds_3 \qquad \text{(F.42)}$$

$$I_{tz}(u) = F_3(u) - \frac{c^2}{\omega^2}\left[u_1 u_3 I_{tx}(u) + u_2 u_3 I_{ty}(u) - (u_1^2 + u_2^2) I_{tz}(u)\right]$$

$$+ \int_{-\infty}^{\infty} \frac{c^2}{\omega^2}\left[-u_1 s_3 I_{tx}(s_3) - u_2 s_3 I_{ty}(s_3) + (u_1^2 + u_2^2) I_{tz}(s_3)\right] p(u_3 - s_3)\, ds_3$$

(F.43)

$$V_e(u) = F_4(u) + \int_{-\infty}^{\infty}\left\{\frac{i}{e\omega}\left[u_2(s_3 - u_3) I_{tx}(s_3) + u_1(u_3 - s_3) I_{ty}(s_3)\right]\int_{-\infty}^{\infty} \widetilde{N}_o(u_3 - v_3) g(v_3 - s_3)\, dv_3\right.$$

$$\left. + \frac{m\epsilon_o U^2}{e^2}(u_1^2 + u_2^2 + s_3 u_3) V_e(s_3) \int_{-\infty}^{\infty} \widetilde{N}_o(u_3 - v_3) q(v_3 - s_3)\, dv_3\right\} ds_3$$

(F.44)

NOTATIONS

$N_o = N_e = N_i$:	Undisturbed electron and ion number densities.
m_e and m :	The electron mass.
U_e and U :	The acoustic velocity for electron gas.
$\bar{B}_o$:	Externally applied constant magnetic field.
ω :	Radian frequency.
$\bar{h}$:	Varying component of the magnetic field.
$\bar{E}$:	Varying component of the electric field.
$\bar{V}_e$ and $\bar{V}$:	Fluid velocity of the electron gas.
n_e and n :	Varying component of the electron number density.
ϵ_o :	Dielectric constant of free space.
μ_o :	Permeability of free space.
$\bar{J}$:	Electric current source
$\bar{K}$:	Magnetic current source
e :	Absolute value of the charge of an electron.
$\bar{F}_e$ and $\bar{F}$:	Mechanical body source for the electron gas.
Q_e and Q :	Fluid flux source for the electron gas.
$\psi(r)$:	Field vector.
$\phi(r)$:	Source vector.
$\mathcal{W}$:	Matrix differential operator.
$\not{S}$:	Generic summation symbol.

$d(s, r)$	:	Transformation kernel.
$c(r, s)$	:	Inverse transformation kernel.
$\mathcal{W}(u, s)$	:	Transform of $\mathcal{W}$.
$\Psi(s)$	:	Transform of $\psi(r)$.
$\Phi(s)$	:	Transform of $\phi(r)$.
$K(u, s)$	:	Kernel of the Fredholm integral equation of the second kind.
$\hat{b}$	:	Unit vector in the direction of the externally applied constant magnetic field.
ω_{ce} and ω_c	:	Electron cyclotron frequency.
ω_{pe}^2 and ω_p^2	:	Electron plasma frequency.
$\bar{\bar{1}}(u, s)$	:	A Dirac delta function or a Kronecker delta and a unit dyadic.
$I_t(s)$	:	Transform of $\bar{h}$.
$V_e(s)$	:	Transform of n.
$V_t(s)$	:	Transform of $\bar{E}$.
$I_e(s)$	:	Transform of $\bar{V}$.
$J_t(s)$	:	Corresponding to transform of $\bar{K}$.
$W_e(s)$	:	Corresponding to transform of Q.
$W_t(s)$	:	Transform of the vector source function $\bar{S}_1$.
$J_e(s)$	:	Transform of the vector source function $\bar{S}_2$.
$Y_t(u, s)$	:	Three-dyadic admittance function.

$Z_t(u, s)$	:	Three-dyadic impedance function.
$T_{et}(u, s)$	:	Three-dyadic transfer function.
$Z_e(u, s)$	:	Three-row-vector impedance function.
$Y_e(u, s)$	:	Three-column-vector admittance function.
$T_{te}(u, s)$	:	Three-column-vector transfer function.
s_1, s_2, s_3	:	Rectangular coordinates of the propagation constant s.
c	:	Velocity of light in free space.
θ	:	The angle between the direction of a propagation constant and the direction of the static magnetic field.
$\beta_o = \omega/c$		
$\beta_e = \omega/U$		
$\Omega = \omega_c/\omega$		
$\omega_o = \omega_p/\omega$		
j(s)	:	Transform of the electric current source $\bar{J}$.
f(s)	:	Transform of the mechanical body source $\bar{F}$.
$H_o^{(1)}(sr)$	:	Hankel function of the first kind and order zero.
$\mathcal{J}_1$	:	Inversion integral given by Eq. (4.28).
$\mathcal{J}_2$	:	Inversion integral given by Eq. (4.29).
K	:	Gaussian curvature.
ϕ	:	Angle between the direction of a ray and the direction of the static magnetic field.

$s_\rho^2 = s_1^2 + s_3^2$

E_ρ, E_ϕ, E_y : Three components of the electric field in a cylindrical coordinate system.

BIBLIOGRAPHY

(1) L. Spitzer, Physics of Fully Ionized Gases, Second Edition, Interscience John Wiley and Sons, New York; 1962.

(2) J. A. Ratcliffe, The Magneto-ionic Theory and Its Applications to the Ionosphere, Cambridge University Press; 1959.

(3) L. Oster, "Linearized Theory of Plasma Oscillations," Reviews of Modern Physics, Vol. 32, No. 1, pp. 141-168; 1960.

(4) V. L. Ginzburg, Propagation of Electromagnetic Waves in Plasma. Translated from the Russian by Royer and Roger, Gordon and Breach, New York; 1961.

(5) K. G. Budden, Radio Waves in the Ionosphere, Cambridge University Press; 1961.

(6) S. I. Pai, "Wave Motions of Small Amplitude in a Fully Ionized Plasma Under Applied Magnetic Field," The Physics of Fluids, Vol. 5, pp. 234-240; 1962.

(7) B. S. Tanenbaum and D. Mintzer, "Wave Propagation in a Partly Ionized Gas," The Physics of Fluids, Vol. 5, No. 10, pp. 1226-1237; 1962.

(8) J. F. Denisse and J. L. Delcroix, Plasma Waves, Translated from the French by Interscience Publishers; 1963.

(9) V. L. Ginzburg, "On the Electrodynamics of Anisotropic Media," J. Exp. Theor. Phys., U. S S. R., Vol. 10, p. 601; 1940.

(10) A. A. Kolomenskii, "Radiation from a Plasma Electron in Uniform Motion in a Magnetic Field," Doklady Akad. Nauk. S. S. S. R., Vol. 106, p. 982; 1956.

(11) F. V. Bunkin, "On Radiation in Anisotropic Media," J. Exp. Theor. Phys. U. S. S. R., Vol. 32, pp. 338-346; 1957.

(12) H. Kogelnik, "On Electromagnetic Radiation in Magneto-ionic Media," J. Research, NBS, Vol. 64D, No. 5, pp. 515-523; 1960.

(13) E. Arbel, "Radiation From a Point Source in an Anisotropic Medium," Polytechnic Institute of Brooklyn, Microwave Research Institute, Report No. PIB MRI-861-60; 1960.

(14) H. Kuehl, "Radiation from an Electric Dipole in an Anisotropic Cold Plasma," California Institute of Technology, Antenna Laboratory Report 24; 1960.
"Electromagnetic Radiation from a Dipole in an Anisotropic Plasma," University of Southern California Electrical Engineering Department Report No. USCED 79-203; 1961.

(15) R. Mittra, "Solution of Maxwell's Equations in a Magneto-ionic Medium with Sources," University of Illinois Antenna Laboratory Report No. 4; 1962.

(16) R. Mittra and G. A. Deschamps, "Field Solution for a Dipole in an Anisotropic Medium," Electromagnetic Theory and Antennas, Proceedings of the URSI Symposium, June 1962. Pergamon Press, New York, pp. 495-512; 1963.

(17) P. C. Clemmow, "On the Theory of Radiation from a Source in a Magneto-ionic Medium," Electromagnetic Theory and Antennas, Proceedings of the URSI Symposium, Copenhagen, June 1962. Pergamon Press, New York, pp. 461-475; 1963.

(18) C. P. Wu, "A Study on the Radiation from Elementary Dipoles in a Magneto-ionic Medium," The Ohio State University Department of Electrical Engineering Report 1021-20; 1962.

(19) H. Motz and H. Kogelnik, "Electromagnetic Radiation from Sources Embedded in an Infinite Anisotropic Medium and the Significance of the Poynting Vector," Electromagnetic Theory and Antennas, Proceedings of the URSI Symposium, Copenhagen, June 1962. Pergamon Press, New York, pp. 477-493; 1963.

(20) E. Arbel and L. B. Felsen, "Theory of Radiation from Sources in Anisotropic Media," Electromagnetic Theory and Antennas, Proceedings of the URSI Symposium, Copenhagen, June 1962. Pergamon Press, New York, pp. 391-459; 1963.

(21) Y. Chow, "A Note on Radiation in a Gyro-electric-magnetic Medium - An Extension of Bunkin's Calculation," IRE Trans. PGAP, Vol. AP-10, pp. 464-469; 1962.

(22) H. A. Whale, "The Excitation of Electroacoustic Waves by Antennas in the Ionosphere," J. Geophysical Research, Vol. 68, No. 2, pp. 415-422; 1963.

(23) A. Hessel and J. Shmoys, "Excitation of Plasma Waves in a Homogeneous Isotropic Plasma by a Dipole," Proceedings of the Symposium on Electromagnetic and Fluid Dynamics of Gaseous Plasma, Polytechnic Press, New York, pp. 173-184; 1962.

(24) S. R. Seshadri, "Excitation of Plasma Waves in an Unbounded Homogeneous Plasma by a Line Source," IEEE Trans. G-MTT, Vol. MTT-11, pp. 39-50; 1963.

(25) A. D. Bresler and N. Marcuvitz, "Operator Methods in Electromagnetic Field Theory," Polytechnic Institute of Brooklyn, Microwave Research Institute Report R-495-56; 1956.

(26) A. D. Bresler and N Marcuvitz, "Operator Methods in Electromagnetic Field Theory, " Polytechnic Institute of Brooklyn, Microwave Research Institute Report R-565-57; 1957.

(27) H. E Moses, "A Spinor Representation of Maxwell's Equations, " New York University, Institute of Mathematical Sciences Report IMM-NYU 238; 1957.

(28) P. Diament, "A Formal Solution to Maxwell's Equations for General Linear Media, " Columbia University Scientific Report 78; 1963

(29) T. Watanabe, "Waves in a Rarefied Ionized Gas Propagated Transverse to an External Magnetic Field, " Canadian J. of Physics, Vol. 39, No. 7, pp. 1044-1057;1961.

(30) M. H Cohen, "Radiation in a Plasma-I: Cerenkov Effect, " The Physical Review, Vol. 123, No. 3, pp. 711-721; 1961 .

(31) M. H. Cohen, "Radiation in a Plasma-II: Equivalent Sources, " The Physical Review, Vol. 126, No. 2, pp. 389-397 ; 1962.

(32) M. H. Cohen, "Radiation in a Plasma-III: Metal Boundaries, " The Physical Review, Vol. 126, No. 2, pp. 398-404; 1962.

(33) S. I. Pai, "Wave Motions of Small Amplitude in a Fully-ionized Plasma Without External Magnetic Field, " Reviews of Modern Physics, Vol. 32, pp. 882-887;1960.

(34) L. I. Schiff, Quantum Mechanics, McGraw-Hill Book Co., New York, p. 128;1955.

(35) S. A. Schelkunoff, "Conversion of Maxwell's Equations into Generalized Telegraphist's Equations, " Bell System Technical Journal, Vol. 34, pp. 995-1043; 1955.

(36) N. Marcuvitz, Waveguide Handbook, Radiation Laboratory Series, No. 10, McGraw-Hill Book Co., New York, Sect. 3. 5c; 1951.

(37) A. Hessel, N. Marcuvitz and J. Shmoys, "Scattering and Guided Waves at an Interface Between Air and a Compressible Plasma, " IRE Trans. PGAP, Vol. AP-10, pp. 48-54; 1962.

(38) M. J. Lighthill, "Studies on Magneto-hydrodynamic Waves and Other Anisotropic Wave Motions, " Philos. Trans. Roy. Soc. London, Vol. 252, p. 397; 1960.

(39) L. B. Felsen, "On the Use of Refractive Index Diagrams for Source-excited Anisotropic Regions, " J. Research, NBS., Vol. 69D, No. 2, pp. 155-170; 1965 .

(40) W. P. Allis, S. J. Buchsbaum and A. Bers, Waves in Anisotropic Plasmas, MIT Press, Cambridge; 1963.

(41) T. H. Stix, The Theory of Plasma Waves, McGraw-Hill Book Co.; 1962.

(42) L. Tonks and I. Langmuir, "Oscillations in Ionized Gases," The Physical Review, Vol. 33, p. 195; 1929.

(43) D. Bohm and E. P. Gross, "Theory of Plasma Oscillations," The Physical Review, Vol. 75, No. 12, pp. 1851-1876; 1949.

(44) W. P. Allis, "Waves in a Plasma," Sherwood Conference on Controlled Fusion, Gatlinburg, April 27-28, 1959, TID-7582, p. 32; 1959. Also in MIT Research Laboratory for Electronics Quarterly Progress Report 54:5; 1959.

(45) K-M Chen, "Interaction of a Radiating Source with a Plasma," Proc. IEE London, Vol. 111, No. 10, pp. 1668-1678; 1964. Also appeared as University of Michigan Institute of Science and Technology Report 4563-39-T; 1963.

(46) J. R. Wait, "Theory of a Slotted-sphere Antenna Immersed in a Compressible Plasma: Parts I and II," Radio Science J. of Research, NBS, Vol. 68D, No. 10, pp. 1127-1143 ; 1964.

(47) M. L. Stone, L. E. Bird and M. Balser, "A Faraday Rotation Measurement on the Ionospheric Perturbation Produced by a Burning Rocket," J. Geophys. Res., Vol. 69, No. 5, pp. 971-978 ; 1964.

(48) S. J. Bauer and L. J. Blumle, "Mean Diurnal Variation of the Topside Ionosphere at Mid-latitudes," J. Geophys. Res., Vol. 69, No. 17, pp. 3613-3618; 1964.

(49) N. W. Spencer, L. H. Brace and G. R. Carignan, "Electron Temperature Evidence for Nonthermal Equilibrium in the Ionosphere," J. Geophys. Res., Vol. 67, pp. 157-175; 1962.

(50) A. F. Nagy, L. H. Brace, G. R. Carignan and M. Kanal, "Direct Measurements Bearing on the Extent of Thermal Nonequilibrium in the Ionosphere," J. Geophys. Res., Vol. 68, No. 24, pp. 6401-6412; 1963.

(51) A. Sommerfeld, Partial Differential Equations in Physics, Academic Press, New York ; 1949.

(52) F. Smithies, Integral Equations, Cambridge University Press, 1958.