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THE UNIVERSITY OF MICHIGAN

COLLEGE OF ENGINEERING

DEPARTMENT OF ELECTRICAL ENGINEERING

Radiation Laboratory

ACOUSTIC AND ELECTROMAGNETIC SCATTERING PROBLEMS FOR LOW FREQUENCIES

FINAL REPORT ON NSF GRANT GP-4581

by

Ergun Ar

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I. Introduction

The classic three-dimensional scalar scattering problem consists of determining a function ϕ^s exterior to a smooth finite boundary B, which function is a solution of a scalar Helmholtz equation, satisfies Dirichlet or Neumann boundary conditions on B, and obeys a radiation condition at infinity, i.e.,

$$(\nabla^2 + k^2)\, \phi^s = 0 \tag{1}$$

$$\phi^s = -\phi^i \quad \text{or} \quad \frac{\partial \phi^s}{\partial n} = -\frac{\partial \phi^i}{\partial n} \quad \text{on B,} \tag{2}$$

$$\lim_{r \to \infty} r\left(\frac{\partial}{\partial r}\phi^s - ik\phi^s\right) = 0\ , \tag{3}$$

where ϕ^i is the incident field which is known everywhere including the boundary G.

The study of the relation between this problem and potential problems (boundary value problems for the Laplace's equation, $\nabla^2 \phi = 0$) goes back to 1897. The general problem is one of generating solutions of the Helmholtz equation (vector or scalar), which satisfy prescribed conditions on a given boundary in terms of solutions of Laplace's equation. Physically, this amounts to an attempt to infer the manner in which an obstacle perturbs the field due to a source of wave motion from a knowledge of how the same object perturbs a stationary (non-oscillatory) field, e.g., determining an electromagnetic field from an electrostatic field. The advantage of such a procedure derives from the fact that stationary fields are physically simpler than wave phenomena and the associated mathematical problems, thought often still formidable, are always more easily handled.

Interest in this problem has gained new momentum in recent years. The major drawback in most of the methods heretofore proposed is in their intrinsic dependence on a particular geometry. That is, the techniques result from the exploitation of the geometric properties of the surface on which the boundary conditions are specified. For those shapes where the Helmholtz equation is separable, of course, the low frequency expansion may always be obtained from the series solution provided sufficient knowledge of the special functions involved is available.

Most low frequency techniques, however, have as their starting point the formulation of scattering problems as integral equations using the Helmholtz representation of the solution in terms of its properties on the boundary and the free space Green's function; i.e.,

$$\phi^s(p) = \frac{1}{4\pi}\int_B \left\{ \phi^s(p_B)\frac{\partial}{\partial n}u(p,p_B) - u(p,p_B)\frac{\partial}{\partial n}\phi^s(p_B)\right\} dB \tag{4}$$

where

$$u = -\frac{e^{ikR(p,p_B)}}{R(p,p_B)} \quad ,$$

the integration is carried out over the entire scattering surface B, the normal here is taken out of B, p is the general field point, and p_B a point on B whose coordinates are the integration variables, and R is the distance between them. This formulation is also vital to the proof of existence of solutions for a general boundary. Noble[7] shows how the integral formulation

(4) may be used to obtain a representation of the solution of a scattering problem for a general bounday as a perturbation of the solution of the corresponding potential problem. Each term in the low frequency expansion is the solution of an integral equation which differs only in its inhomogeneous part from term to term. However, this formulation does not yield an explicit representation for successive terms in general except as the formal inverse.

Long sought has been the development of a systematic procedure which will generate the solution of the Helmholtz equation, satisfying a particular boundary condition, from the solution of Laplace's equation which satisfies the same boundary condition.

In this connection Kleinman[5] has made the following key observation. If

1(a) V is the volume exterior to a smooth, closed, and bounded surface B,

1(b)
$$G_o(p,p_o) = -\frac{1}{4\pi R(p,p_o)} + u_o(p,p_o)$$

is the potential Green's function of the first kind ($G_o(p_B, p_o) = 0$), and

1(c)
$$G_k(p,p_o) = -\frac{e^{ikR(p,p_o)}}{4\pi R(p,p_o)} + u_k(p,p_o)$$

is the Green's function for the Helmholtz equation, also satisfying a Dirichlet condition on B, then the scattered field $u_k(p,p_o)$ satisfies the integral equation

$$u_k(p,p_o) = -2ik\,e^{ikr} \int_V \frac{G_o(p,p_1)}{r_1} \frac{\partial}{\partial n} \left[r_1 e^{-ikr_1} u_k(p_1,p_o) \right] dv_1$$

$$+ \frac{e^{ikr}}{4\pi} \int_B \frac{e^{-ikr_B + ikR(p_B,p_o)}}{R(p_B\,,\,p_o)} \frac{\partial}{\partial n} G_o(p,p_B)\; d\sigma \;. \tag{5}$$

Here dv_1 is a volume element in coordinates $p_1 = (\theta_1, \phi_1, r_1)$ and $d\sigma$ is a surface element and $\partial/\partial n$ the normal derivative directed out of V expressed in coordinates p_B. $R(p,p_o)$ is the distance between the points p and p_o. The origin of the spherical coordinates $p = (r, \theta, \phi)$ is situated inside the body.

II. Background

On the basis of Kleinman's(5) work the investigation of the following problems was proposed:

2(a) Rigorous solution of the integral equation (5), thereby providing a low frequency technique for the scattering problems for acoustically soft objects.

2(b) Derivation of a similar integral equation for the Neumann problem and its rigorous solution.

2(c) Solution of the non-separable problems, e.g., those problems which are unsolved due to the non-separability of the Helmholtz equation.

2(d) Extension to vector (electromagnetic) problems.

2(e) The abstract mathematical results which include functional analytic aspects of the problem and the new existence proofs.

2(f) Extension to two dimensional low frequency scattering problems.

2(g) Studies in connection with the radius of convergence of the low frequency expansion.

III. The Progress

3(a) The integral equation for the Neumann problem has been found by Ar and Kleinman[2]. With the geometry and the notation indicated in the Introduction, if G_o is the static Green's function of the second kind and G_k the Neumann Green's function for the Helmholtz equation then the integral equation in question is given by

$$u_k(p) = -2ike^{ikr} \int_V \frac{G_o(p,p_1)}{r_1} \frac{\partial}{\partial r_1} \left[r_1 e^{-ikr_1} u_k(p_1) \right] dv_1$$

$$+ ike^{ikr} \int_B G_o(p,p_B)\, \hat{n} \cdot \hat{r}_B\, e^{-ikr_B} u_k(p_B)\, d\sigma \tag{6}$$

$$- e^{ikr} \int_B G_o(p,p_B)\, e^{-ikr_B} \frac{\partial u_k(p_B)}{\partial n}\, d\sigma \quad .$$

3(b) The rigorous solution of this equation (and of that for the Dirichlet case) has been found by Ar[4]. This is done by defining a certain function space with a proper norm in which the Neumann series arising from the equations (5) and (6) is convergent to the solution sought.

3(c) A surface for which the Helmholtz equation is non-separable is an ogive. However, the Laplace's equation $\nabla^2 \phi = 0$ is partially separable (and [redacted] solvable) in the exterior region of this body. Applying the above mentioned techniques the Helmholtz equation has been solved in the "closed" form (for sufficiently small wave numbers) for this case by Ar[1].

3(d) The solution of electromagnetic scattering problems involving a smooth finite three-dimensional scatterer was presented by Stevenson[8] in terms of solutions of standard potential problems involving the same boundary. Kleinman[6] has shown that Stevenson's general procedure leads to erronous field expressions, and in the case when the scatterer is perfectly conducting he has provided a modification which corrects this heretofore-unnoticed deficiency.

IV. Continuation and the Anticipated New Areas of Investigation

4(a) The function spaces for the Dirichlet as well as the Neumann problems mentioned above, though sufficient to solve these problems, are not complete. However, it has been recently discovered by Ar[3] that a Banach space can be found which provides new existence (and uniqueness) proofs and rigorous low frequency techniques as well. The preliminary work on this has been completed.

4(b) Another body for which the Helmholtz equation is unsolvable due to its non-separability is the torus. The solution of the torus problem by means similar to that used in solving the ogive problem mentioned above is being found.

These problems are being given the immediate attention at the present, while the other areas mentioned in the original proposal still remain under consideration.

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