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THE UNIVERSITY OF MICHIGAN

COLLEGE OF ENGINEERING
DEPARTMENT OF ELECTRICAL ENGINEERING
Radiation Laboratory

STUDY OF ANGLE OF ARRIVAL ERRORS DUE TO MULTIPATH PROPAGATION EFFECTS (U)

Chiao-Min Chu and John J. LaRue

May 1967

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ABSTRACT

(U) A unified approach using the generalized concept of angular spectra for the study of the radiation received at any point in space from a transmitter due to multipath propagation effects is formulated. The various mechanisms contributing to the multipath effects, such as scattering by discrete objects, and by extended objects such as ground, are formulated in general. Although the formulation is based on CW transmitter and stationary receiver, the results may also be applied to other transmitter signals, scanning and moving receivers (and transmitters) with slight modification.

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(Unclassified)

I

INTRODUCTION AND SUMMARY

(U) The objective of this research is to carry out a general study of the multipath propagation effects on the radiation received at any point in space from a transmitting source. Anticipating the fact that the polarization and distributional characteristics of received signal are important in utilizing the received signal to estimate the range or position of the source, a unified approach using the generalized concept of angular spectra of fields is suggested here.

(U) In Chapter II the characterization of radiation by angular spectra is introduced. The electric field strength at any point $\underline{r}$ associated with radiation coming from a small solid angle $d\Omega$ in the direction $\hat{\Omega}$ may be expressed as

$$d\underline{E}(\underline{r}) = \mathcal{E}_1(\underline{r}, \hat{\Omega})\hat{e}_1 + \mathcal{E}_2(\underline{r}, \hat{\Omega})\hat{e}_2 \qquad (1.1)$$

where $\hat{e}_1$ and $\hat{e}_2$ are directions of horizontal and vertical polarization and $\mathcal{E}_1$ and $\mathcal{E}_2$ are the two components of angular spectra of radiation. From the far zone approximation, the directed radiation from a transmitter located at $\underline{r}_t$ may be expressed as

$$\left.\begin{matrix} \mathcal{E}_1 \\ \\ \mathcal{E}_2 \end{matrix}\right]_{\text{direct}} = \left.\begin{matrix} F_1(\hat{\Omega}) \\ \\ F_2(\hat{\Omega}) \end{matrix}\right] \frac{e^{ik|\underline{r}-\underline{r}_t|}}{|\underline{r}-\underline{r}_t|} \, \delta(\hat{\Omega}-\hat{\Omega}_{dr}) \qquad (1.2)$$

where

$$\hat{\Omega}_{dr} = \frac{(\underline{r}-\underline{r}_t)}{|\underline{r}-\underline{r}_t|}$$

F_1 and F_2 are related to the antenna pattern and gain.

(U) In Chapter III the scattering of the directed signal by discrete objects is discussed. In terms of a scattering matrix, the contribution of the angular spectra due to a scatterer at $\underline{r}_i$ may be characterized by a scattering matrix

$$\left[s(\hat{\Omega}, \hat{\Omega}_{di})\right],$$

where

$$\hat{\Omega}_{di} = \frac{\underline{r}_i - \underline{r}_t}{|\underline{r}_i - \underline{r}_t|} \tag{1.3}$$

is the direction of the incident radiation seen by the scatterer. The scattered field at any point in space can then be represented by the angular spectra

$$\left.\begin{matrix} \mathcal{E}_1 \\ \mathcal{E}_2 \end{matrix}\right]_{\text{scattered}} = \left[s(\hat{\Omega}_i, \hat{\Omega}_{ik})\right] \left.\begin{matrix} F_1(\hat{\Omega}_{di}) \\ F_2(\hat{\Omega}_{di}) \end{matrix}\right] \frac{e^{i|\underline{r}_i - \underline{r}_t|}}{|\underline{r}_i - \underline{r}_t|} \frac{e^{i|\underline{r} - \underline{r}_i|}}{|\underline{r} - \underline{r}_i|} \delta(\hat{\Omega}, \hat{\Omega}_i) \tag{1.4}$$

where

$$\hat{\Omega}_i = \frac{\underline{r} - \underline{r}_i}{|\underline{r} - \underline{r}_i|} \tag{1.5}$$

is the apparent direction of the scattered radiation seen at any point. Theoretical models for the approximate calculations of the scattering matrix are also presented in Chapter III

(U) The reflection due to a rough ground is investigated in Chapter IV. Using the geometric optics approach, the contribution of the angular spectra due to ground reflection may be expressed in terms of an integral such as given by Eq. (4.50). If the transmitter is far from the ground, then this angular spectra may be expressed in terms of ground reflection matrix $\left[R\right]$ such that

$$\left.\begin{matrix} \mathcal{E}_1 \\ \mathcal{E}_2 \end{matrix}\right]_{\text{reflected}} = \left[R\right] \left.\begin{matrix} \mathcal{E}_1 \\ \mathcal{E}_2 \end{matrix}\right]_{\text{direct}} e^{ik\hat{\Omega} \cdot \underline{r}} = \left[R\right] \begin{matrix} F_1(\hat{\Omega}_o) \\ F_2(\hat{\Omega}_o) \end{matrix} \frac{e^{ik|\underline{r}_o - \underline{r}_t|}}{r_t} e^{ik\hat{\Omega} \cdot \underline{r}} \tag{1.6}$$

where $\underline{r}_o$ is some chosen center of the illuminated region of the ground, and

$$\hat{\Omega}_o = \frac{\underline{r}_o - \underline{r}_t}{|\underline{r}_o - \underline{r}_t|} \tag{1.7}$$

is the apparent direction of direct radiation relative to the center.

(U) A study of the ground reflection matrix for a slightly rough ground, using the geometric optics approximation is carried out in Chapter IV. For a slightly rough, random ground, the formulas for the statistical average, and correlations between the elements of the reflection matrix, including the effect of finite index of refraction of the ground are investigated. In Chapter V, a preliminary study on the effect of shadowing on the ground reflection matrix is carried out.

(U) In principle, the radiation at any point in space can then be obtained by adding the direct signal, scattered signal, and reflected signal given by

$$\begin{bmatrix} \mathcal{E}_1(\underline{r}, \hat{\Omega}) \\ \mathcal{E}_2(\underline{r}, \hat{\Omega}) \end{bmatrix} = \begin{bmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \end{bmatrix}_{direct} + \begin{bmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \end{bmatrix}_{scattered} + \begin{bmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \end{bmatrix}_{reflected} \tag{1.8}$$

For a receiver at any point with any receiving pattern, the received signal can then be obtained by integrating over the angular space.

(U) Due to the uncertainties involved in the problem, especially the ground reflection for which various statistical models of ground can be chosen, no specific calculations were made on the theoretical model proposed in this work. It is felt that before meaningful numerical analysis can be made, some experimental results characterizing the statistics of ground reflection are necessary.

(U) It is to be noted that although the present formulation is presented on the basis of CW transmitter and stationary observer, the extension of the formulation to FM and moving observer is relatively uncomplicated. By assuming the transmitted signal to be of the form

$$f(t)\, e^{-i\omega_o t}$$

where ω_o is the carrier frequency, the angular spectra may be expressed in the form

$$\left.\begin{matrix} \mathcal{E}_1(\hat{\Omega}, \underline{r}, t) \\ \mathcal{E}_2(\hat{\Omega}, \underline{r}, t) \end{matrix}\right] e^{-i\omega_0 t}$$

so that the time variation of the power spectra of the radiation may also be included in the formulation.

(U) For a moving detector, the point of observation changes with time, so that

$$\underline{r} = \underline{r}(t) \, ,$$

describing the trajectory of the detector may be used in the angular spectra. The angular spectra then takes the form

$$\left.\begin{matrix} \mathcal{E}_1(\hat{\Omega}, \underline{r}(t) t) \\ \mathcal{E}_2(\hat{\Omega}, \underline{r}(t) t) \end{matrix}\right] e^{-i\omega_0 t} \, .$$

Thus the effect of a moving receiver may also be investigated using this formulation.

(U) Other corrections, including moving and scanning of the transmitter, the tropospheric or meteorological effects may also be incorporated in this formulation.

(U) In summary, a unified approach suitable in the investigation of the radiation at any point from a transmitter due to multipath propagation effects is formulated. This approach, incorporated with experimental results that may yield a reasonable statistical model or models of the ground may be employed to obtain a detailed characterization of the received radiation by a stationary, as well as moving, detector.

II

CHARACTERIZATION OF THE RADIATION FIELD

2.1 Introduction

(U) The radiation observed at any point from a transmitting source generally consists of several components due to the multipath propagation effect. This multipath effect is illustrated in Fig. 2-1. If the transmitter is a low angle radar such that the ionospheric reflection may be neglected, the received radiation may be roughly classified into three categories: i) the direct signal whenever the point of observation is in the illuminated region of the transmitter (main beam or any side lobe); ii) the scattered signal, due to the presence of obstacles in the illuminated region, and iii) the ground reflected signal.

(U) In order to infer from the received signal at any point, the possible configuration of the transmitting system and its surroundings, it is necessary to have a unified, detailed characterization of the radiation at any point.

(U) The basic characteristics of the radiation received at any point that may be subject to detection and analysis are; a) the temporal variation of the signal, b) the angular distribution of the signal, and c) the polarization of the signal.

(U) For CW transmission, the temporal variation may be accounted for by the phase variation of the signal. To incorporate both the information of angular distribution and the polarization of the radiation, which is necessary when ground reflection is important, it seems most natural to employ and generalize the notion of angular spectrum to characterize the radiation. The idea of angular spectrum has been successfully used in the study of ionospheric reflections (Booker, Ratcliffe and Shinn, 1950) and are therefore adapted and generalized in this report for the characterization of the radiation.

2.2 The Angular Spectrum

(U) For a simple introduction of angular spectrum and the notations involved in this work, we shall choose a fixed coordinate system, with the average level of ground taken as the z-plane, as illustrated in Fig. 2-2. Any point in space is then

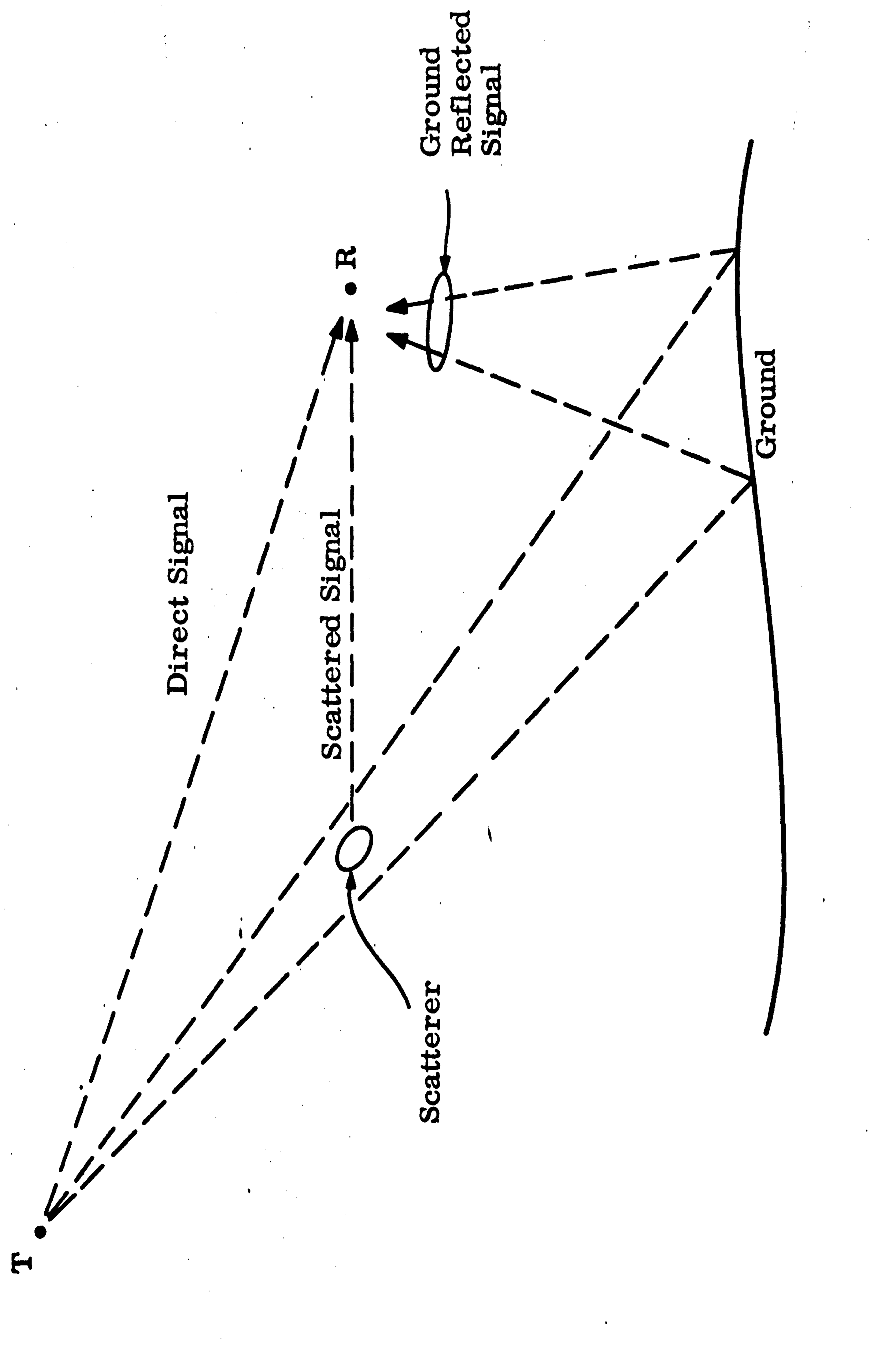

FIG. 2-1: MULTIPATH PROPAGATION EFFECT

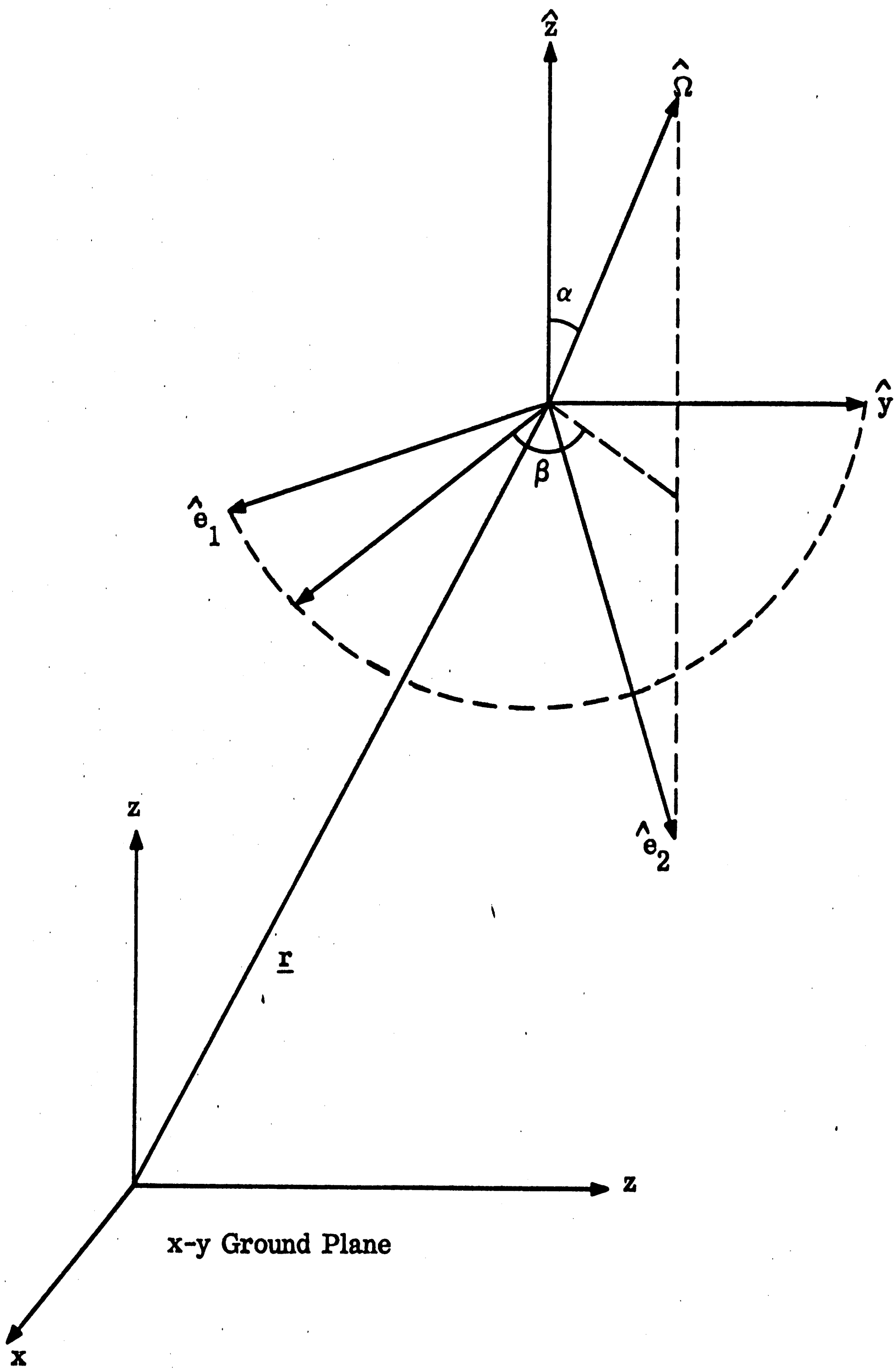

FIG. 2-2: COORDINATE SYSTEM FOR ANGULAR SPECTRUM

represented by a vector

$$\underline{r} = \hat{x}x + \hat{y}y + \hat{z}z \tag{2.1}$$

and any direction may be denoted by a unit vector $\hat{\Omega}$. In terms of the latitude angle α and azimuth angle β the unit vector Ω is

$$\hat{\Omega} = \hat{x}\sin\alpha\cos\beta + \hat{y}\sin\alpha\sin\beta + \hat{z}\cos\alpha \quad . \tag{2.2}$$

(U) Since the radiation reaching $\underline{r}$ may be distributed in all directions, one may define the electric field of the radiation reaching $\underline{r}$ from a small solid angle $d\Omega$ in the direction $\hat{\Omega}$ by

$$d\underline{E} = \underline{\mathcal{E}}(\underline{r}, \hat{\Omega})d\Omega \tag{2.3}$$

where $\underline{\mathcal{E}}(\underline{r}, \hat{\Omega})$ may be called the angular spectrum of the electric field. For CW transmission, the angular spectrum may be expressed in terms of a complex amplitude and phase, so that

$$\underline{\mathcal{E}}(\underline{r}, \hat{\Omega}) \triangleq \underline{A}(\underline{r}, \hat{\Omega})e^{ikd(\underline{r})} \tag{2.4}$$

where $d(\underline{r})$ is the total path travelled by the wave from the transmitter to the point of observation, and $A(\underline{r}, \hat{\Omega})$ is the complex vector amplitude.

(U) For the radiation (far zone) field, the electric vector must be normal to the direction of propagation so that $\underline{A}$ must be a two-dimensional vector normal to $\hat{\Omega}$. Following Green and Wolf (1953), one may define two mutually perpendicular unit vectors $\hat{e}_1$ and $\hat{e}_2$ both normal to $\hat{\Omega}$ by

$$\hat{e}_1 = \frac{\hat{\Omega} \times \hat{z}}{|\hat{z} \times \hat{\Omega}|} = \hat{x}\sin\beta - \hat{y}\cos\beta \tag{2.5}$$

and

$$\hat{e}_2 = \hat{\Omega} \times \hat{e}_1 = \hat{x}\cos\alpha\cos\beta + \hat{y}\cos\alpha\sin\beta - \hat{z}\sin\alpha \quad . \tag{2.6}$$

These vectors are illustrated in Fig. 2-2 . The complex vector amplitude $\underline{A}$ can then be decomposed in the directions $\hat{e}_1$ and $\hat{e}_2$, such as

$$\underline{A}(\underline{r},\hat{\Omega}) \triangleq A_1(\underline{r},\hat{\Omega})\hat{e}_1 + A_2(\underline{r},\hat{\Omega})\hat{e}_2 \quad . \tag{2.7}$$

In other words, the radiation reaching any point may be expressed in terms of two scalar amplitudes (A_1 and A_2) and a phase delay ($e^{ikd(\underline{r})}$) by the angular spectrum

$$\mathcal{E}(\underline{r},\hat{\Omega}) = A_1(\underline{r},\hat{\Omega})\hat{e}_1 e^{ikd(\underline{r})} + A_2(\underline{r},\hat{\Omega})\hat{e}_2 e^{ikd(\underline{r})} . \tag{2.8}$$

Evidently, A_1 is the electric field of the horizontally polarized component while A_2 is the electric field of the vertically polarized component of the radiation.

(U) In most problems, the phase factor due to time delay is relatively easy to determine. Thus, apart from the phase factor, one may describe the radiation field by two scalars, or a two-dimensional vector such as

$$\mathcal{E}(\underline{r},\hat{\Omega}) \underset{\text{(represented by)}}{\sim} \begin{bmatrix} A_1(\hat{\Omega},\underline{r}) \\ A_2(\hat{\Omega},\underline{r}) \end{bmatrix} . \tag{2.9}$$

Representation of the incident field from the transmitter in the components A_1 and A_2, and the study of change of A_1 and A_2 due to various scattering processes shall be of prime importance in the present investigation.

2.3 Direct Signal

(U) The direct signal seen at any point in the far zone approximation is generally represented locally by plane waves. For a plane wave travelling in a direction $\hat{\Omega}_o$, the electric field is given by

$$\underline{E} = \left[E_1\hat{e}_1 + E_2\hat{e}_2\right] e^{ik\hat{\Omega}_o \cdot \underline{r}} \quad . \tag{2.10}$$

Since the radiation for a plane wave appears to come from one direction only, we may represent its spectrum by

$$\mathcal{E} = \left[E_1\hat{e}_1 + E_2\hat{e}_2\right] e^{ik\hat{\Omega}_o \cdot \underline{r}} \quad \delta(\hat{\Omega}-\hat{\Omega}_o) \tag{2.11}$$

where $\delta(\hat{\Omega}-\hat{\Omega}_o)$ is the Kronecker delta function in the angular space.

(U) For a transmitting antenna, the field seen at any point may also be represented by the form of (2.11), but the amplitudes and phase factor, of course, vary with direction and distance. To express the direct signal in terms of the radiation pattern, gain, and the transmitted power of the transmitting antenna, let us consider a horizontally polarized antenna with gain G and the power pattern $P(\hat{\Omega})$. If the total power radiated is W_t, then the Poynting vector in any direction $\hat{\Omega}$ is given by

$$p = \frac{W_t G\, P(\hat{\Omega})}{4\pi r^2} \tag{2.12}$$

where r is the distance measured from the antenna. The magnitude of the electric field is then

$$E = \sqrt{2p\sqrt{\frac{\mu_o}{\epsilon_o}}} = \sqrt{240\pi\, p} = \sqrt{60}\,\frac{1}{r}\,\sqrt{W_t}\,\sqrt{G}\,\sqrt{P(\hat{\Omega})}. \tag{2.13}$$

Now let the transmitting antenna be located at position $\underline{r}_t$, then the radiation received at any point appears to be from a direction

$$\hat{\Omega}_t = \frac{\underline{r}-\underline{r}_t}{|\underline{r}-\underline{r}_t|} . \tag{2.14}$$

The magnitude of the electric field is given by

$$E = \sqrt{60}\,\frac{1}{|\underline{r}-\underline{r}_t|}\sqrt{W_t}\,\sqrt{G}\,\sqrt{P(\hat{\Omega}_t)} . \tag{2.15}$$

For a horizontally polarized antenna and if the phase of signal at the antenna is assumed to be zero, we have

$$\underline{E} = \hat{e}_1\,\sqrt{60}\,\sqrt{G}\,\sqrt{P(\hat{\Omega}_t)}\,\frac{\sqrt{W_t}}{|\underline{r}-\underline{r}_t|}\,e^{ik|\underline{r}-\underline{r}_t|} . \tag{2.16}$$

Thus, for a horizontally polarized antenna, we may represent the angular spectrum of the radiation by

$$\mathcal{E}(\underline{r}, \hat{\Omega}) = \hat{e}_1 \frac{1}{|\underline{r}-\underline{r}_t|} F_1(\hat{\Omega}) e^{ik|\underline{r}-\underline{r}_t|} \delta(\hat{\Omega}, \hat{\Omega}_t) \tag{2.17}$$

where

$$F_1(\hat{\Omega}) = \sqrt{60GP(\hat{\Omega})\, W_t} \;. \tag{2.18}$$

(U) In general, the direct signal may be expressed as

$$\mathcal{E}(\underline{r}, \hat{\Omega}) = \left[\hat{e}_1 F_1(\hat{\Omega}) + \hat{e}_2 F_2(\hat{\Omega})\right] \frac{e^{ik|\underline{r}-\underline{r}_t|}}{|\underline{r}-\underline{r}_t|} \delta(\hat{\Omega}, \hat{\Omega}_t) \tag{2.19}$$

in order to specify the angular and spatial variation of the radiation.

(U) It is obvious that; i) for a horizontally polarized antenna, $F_2=0$, ii) for a vertically polarized antenna, $F_1 = 0$, and iii) for a circularly polarized antenna, $F_1 = \pm i\, F_2$.

(U) Elliptically polarized antennas can be expressed in different combinations of the two complex factors, F_1 and F_2 . For mathematical simplicity, we may represent the direct signal by the two-dimensional vector;

$$\mathcal{E}_d(\underline{r}, \hat{\Omega}) \underset{\text{(represented by)}}{\sim} \begin{bmatrix} F_1(\hat{\Omega}) \\ F_2(\hat{\Omega}) \end{bmatrix} \frac{e^{ik|\underline{r}-\underline{r}_t|}}{|\underline{r}-\underline{r}_t|} \delta(\hat{\Omega}-\hat{\Omega}_t) \tag{2.20}$$

2.4 Scattering Matrix

Any obstacle in the beam of the transmitter scatters the incident radiation into different directions. As illustrated in Fig. 2-3, the incident radiation may be assumed coming from the direction $\hat{\Omega}_o$ with the electric field $\underline{E}_o$, while the scattered radiation is distributed over all the directions $\hat{\Omega}_s$ and with different electric fields $\underline{E}_s$. The relation between $\underline{E}_s$ and $\underline{E}_o$, due to the linearity of the Maxwell's equation, may be expressed in terms of a scattering matrix for plane wave scattering (Saxon, 1955). Mathematically, one may deduce, from Maxwell's equation, that the scattered field in the far zone approximation may be expressed by

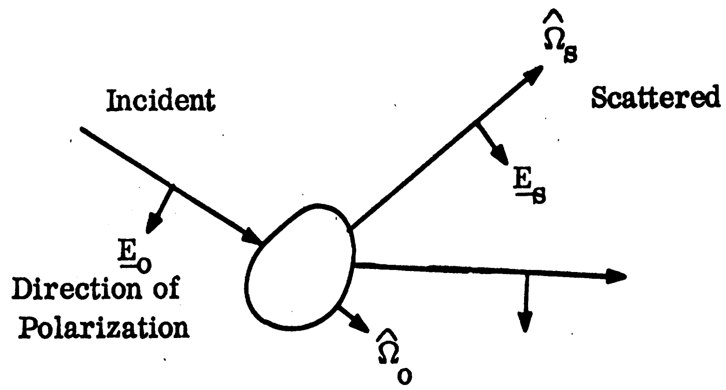

FIG. 2-3: GEOMETRY FOR THE SCATTERED SIGNAL

$$\underline{E}_s(\hat{\Omega}_s) = \underline{\underline{s}}(\hat{\Omega}_s, \hat{\Omega}_o) \cdot \underline{E}_o(\hat{\Omega}_o) \frac{e^{ikr}}{r} \tag{2.21}$$

where

$\underline{\underline{s}}(\hat{\Omega}, \hat{\Omega}_0)$ is known as the scattering matrix.

(U) Explicitly, if one represents the incident field by

$$\underline{E}_o(\hat{\Omega}_o) = \left[A_1^{(o)} \hat{e}_1 + A_2^{(o)} \hat{e}_2\right] e^{i\phi_o} \tag{2.22}$$

and the scattered field in any direction by

$$\underline{E}_s(\hat{\Omega}_s) = \left[A_1^{(s)} \hat{e}_1 + A_2^{(s)} \hat{e}_2\right] \frac{e^{ikr}}{r} e^{i\phi_o} \tag{2.23}$$

then the scattering matrix may be represented by

$$\underline{\underline{s}}(\hat{\Omega}_s, \hat{\Omega}_o) = \begin{bmatrix} s_{11}(\hat{\Omega}_s, \hat{\Omega}_o) & s_{12}(\hat{\Omega}_s, \hat{\Omega}_o) \\ s_{21}(\hat{\Omega}_s, \hat{\Omega}_o) & s_{22}(\hat{\Omega}_s, \hat{\Omega}_o) \end{bmatrix} \tag{2.24}$$

such that

$$\begin{bmatrix} A_1^s \\ A_2^s \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} A_1^o \\ A_2^o \end{bmatrix} . \tag{2.25}$$

(U) If the incident field appears to come from direction $\hat{\Omega}_o$ with phase angle ϕ_o, and if the scatterer is located at $\underline{r}_i$, then the scattered radiation seen at any point $\underline{r}$ appears to come from a direction

$$\hat{\Omega}_s = \hat{\Omega}_i \triangleq \frac{\underline{r}-\underline{r}_i}{|\underline{r}-\underline{r}_i|} , \tag{2.26}$$

In terms of the scattered amplitudes, one may easily represent the scattered signal by the delta function distribution

$$\mathcal{E}_s(\underline{r}, \hat{\Omega}) = \left[A_1^s \hat{e}_1 + A_2^s \hat{e}_2\right] \frac{e^{ik|\underline{r}-\underline{r}_i|}}{|\underline{r}-\underline{r}_i|} e^{i\phi_o} \delta(\hat{\Omega}, \hat{\Omega}_i) . \tag{2.27}$$

(U) In principle, if the scattering properties of various objects, including ground are known, the composite signal that is seen at any point originating from a transmitter can be obtained by summing over all the components. The discussion on the scattering matrices of discrete objects and the ground are given in Chapter III of this report.

III

SCATTERING FORMULATIONS

3.1 Introduction

(U) In Chapter II, it was shown that if the radiation field is represented by two scalars or a two-dimensional vector associated with the angular spectrum, the scattered field may be expressed in terms of a scattering matrix. The description of the scattering properties by a scattering matrix is the natural extension of the ordinary concept of scattering cross section. For example, from Eq. (2.25), for a scatterer located at the origin, the scattered field is given by

$$\begin{bmatrix} E_1^s \\ E_2^s \end{bmatrix} = \frac{e^{ikr}}{r} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} E_1^o \\ E_2^o \end{bmatrix} \tag{3.1}$$

Hence, the conventional bistatic scattering cross section for a horizontally polarized incident wave is given by

$$\sigma = \lim_{r\to\infty} 4\pi r^2 \left[\frac{\left|E_1^s\right|^2 + \left|E_2^s\right|^2}{\left|E_1^o\right|^2} \right] = 4\pi \left\{ \left|S_{11}\right|^2 + \left|S_{21}\right|^2 \right\}. \tag{3.2}$$

Thus, the knowledge of the scattering matrix contains all the information about the conventional cross section. The inverse, however, is not true. Therefore, it is necessary in the present work to discuss the means of evaluating the scattering matrix.

(U) Just as in the case of calculating the scattering cross section, the exact solutions of the problem are only possible in a very few cases. In most cases, approximate methods developed in the evaluation of scattering cross section such as physical optics, Rayleigh approximation, etc., must be used. In this chapter only a general formulation of these approaches are given. For a detailed application of these approximate methods in the calculation of radar cross sections the work of Crispin, Goodrich and Siegel (1959) may be referred to. Extension of their results in the form of scattering matrices seems to be straightforward but in most cases, very detailed.

3.2 The Sphere

(U) Perhaps the only possible shape of finite scatterer whose scattering matrix may be formally written down in relatively simple exact form is the sphere. To illustrate the derivation of scattering matrix from the exact solution of Maxwell's equation, the scattering matrix for a sphere is derived here.

(U) The standard problem for the scattering of a plane wave by a sphere is summarized in detail by Stratton (1941). Refer to Fig. 3-1, a plane wave whose electric field is polarized in the x-direction is impinging on a sphere of radius a and dielectric constant $\sqrt{N}$ (the permeability is assumed to be μ_0). Since

$$\underline{E}_o = E_x \hat{x} e^{ikz} \tag{3.3}$$

the scattered field may be expressed in spherical wave functions as

$$\underline{E}_s = E_o \sum_{n=1}^{\infty} i^n \frac{(2n+1)}{n(n+1)} \left[a_n \underline{M}^{(3)}_{01n} - ib_n \underline{N}^{(3)}_{e1n} \right] \tag{3.4}$$

where the spherical wave functions are expressed in terms of spherical Hankel functions, associated Legendre functions, etc., by

$$\underline{M}^{(3)}_{01n} = \hat{\theta} \left[\frac{P^{(1)}_n(\cos\theta)}{\sin\theta} \cos\phi \, h^{(1)}_n(kr) \right] - \hat{\phi} \left[\frac{dP^{(1)}_n(\cos\theta)}{d\theta} \sin\phi \, h^{(1)}_n(kr) \right] \tag{3.5}$$

$$\underline{N}^{(3)}_{e1n} = \hat{r} \left[P^{(1)}_n(\cos\theta)\cos\phi \frac{n(n+1)}{kr} h^{(1)}_n(kr) \right] + \hat{\theta} \left\{ \frac{dP^{(1)}_n(\cos\theta)}{d\theta} \cos\phi \frac{1}{kr} \left[kr \, h^{(1)}_n(kr) \right]' \right\}$$
$$- \hat{\phi} \left\{ \frac{P^{(1)}_n(\cos\theta)}{\sin\theta} \sin\phi \frac{1}{kr} \left[kr \, h^{(1)}_n(kr) \right]' \right\} . \tag{3.6}$$

The scattering coefficients in (3.4) are expressed in terms of the spherical Bessel functions and the normalized radius of the sphere

$$\alpha = ka \tag{3.7}$$

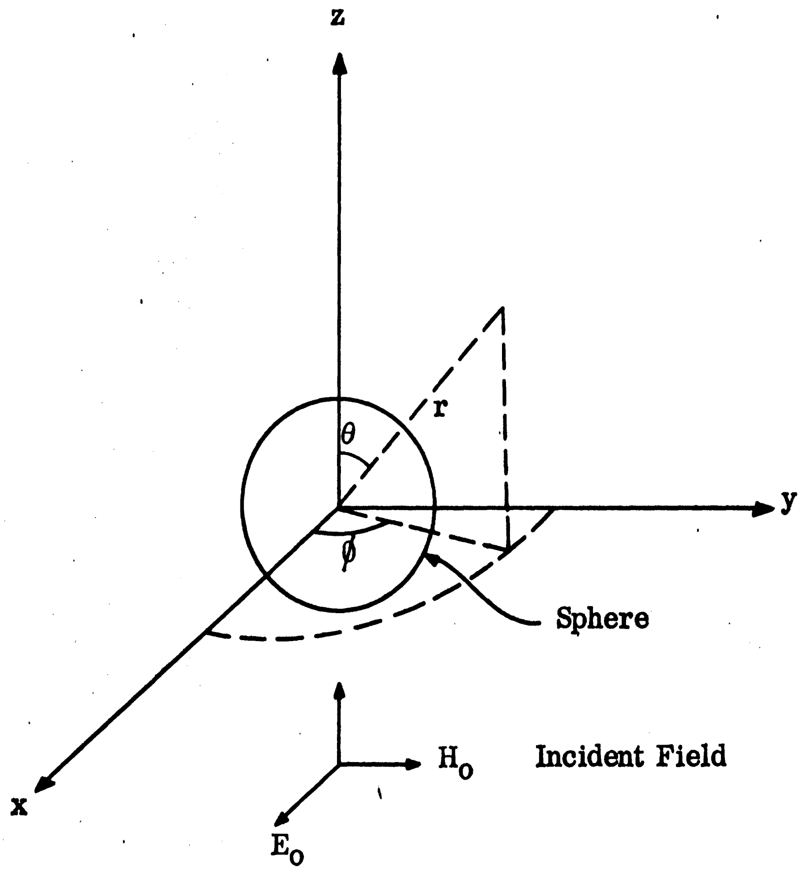

FIG. 3-1: GEOMETRY FOR SCATTERING BY A SPHERE

by

$$a_n = \frac{j_n(N\alpha)\left[\alpha j_n(\alpha)\right]' - j_n(\alpha)\left[N\alpha j_n(N\alpha)\right]'}{j_n(N\alpha)\left[\alpha h_n^{(1)}(\alpha)\right]' - h_n^{(1)}(\alpha)\left[N\alpha h_n^{(1)}(N\alpha)\right]'} \tag{3.8}$$

and

$$b_n = -\frac{j_n(\alpha)\left[N\alpha j_n(N\alpha)\right]' - N^2 j_n(N\alpha)\left[\alpha j_n(\alpha)\right]'}{h_n^{(1)}(\alpha)\left[N\alpha j_n(N\alpha)\right]' - N^2 j_n(N\alpha)\left[\alpha h_n^{(1)}(\alpha)\right]'} \tag{3.9}$$

In the literature, the scattering coefficients have been calculated for various sphere sizes and indices of refraction.

(U) For far zone fields, the asymptotic form of the Hankel function

$$h_n^{(1)}(kr) \sim (-i)^{n+1}\,\frac{e^{ikr}}{kr} \tag{3.10}$$

may be used. This yields

$$\underline{M}_{01n}^{(3)} \sim (-i)^{n+1}\,\frac{e^{ikr}}{kr}\left\{\hat{\theta}\left[\frac{P_n^{(1)}(\cos\theta)}{\sin\theta}\right]\cos\phi - \hat{\phi}\left[\frac{dP_n^{(1)}(\cos\theta)}{d\theta}\right]\sin\phi\right\} \tag{3.11}$$

and

$$\underline{N}_{e1n}^{(3)} \sim (-i)^{n}\,\frac{e^{ikr}}{kr}\left\{\hat{\theta}\left[\frac{dP_n^{(1)}(\cos\theta)}{d\theta}\right]\cos\phi - \hat{\phi}\left[\frac{P_n^{(1)}(\cos\theta)}{\sin\theta}\right]\sin\phi\right\} \tag{3.12}$$

Using these relations, we may express the scattered field due to a plane wave polarized in the x-direction by

$$\underline{E}_s \cong E_x\,\frac{e^{ikr}}{kr}\left\{\hat{\theta}\cos\phi\, S_1(\theta) + \hat{\phi}\sin\phi\, S_2(\theta)\right\} \tag{3.13}$$

where

$$S_1(\theta) = -i\sum_{n=1}^{\infty}\frac{(2n+1)}{n(n+1)}\left\{a_n\,\frac{P_n^{(1)}(\cos\theta)}{\sin\theta} + b_n\,\frac{dP_n^{(1)}(\cos\theta)}{d\theta}\right\} \tag{3.14}$$

and

$$S_2(\theta) = i \sum_{n=1}^{\infty} \frac{(2n+1)}{n(n+1)} \left\{ a_n \frac{dP_n^{(1)}(\cos\theta)}{d\theta} + b_n \frac{P_n^{(1)}(\cos\theta)}{\sin\theta} \right\} . \tag{3.15}$$

Both $S_1(\theta)$ and $S_2(\theta)$ can be calculated for any sphere from the corresponding scattering coefficients.

(U) From Eq. (3.13), it is easy to infer by symmetry that the field scattered by a sphere from an incident wave polarized in the y-direction is given by

$$\underline{E}_s = E_y \frac{e^{ikr}}{kr} \left\{ \hat{\theta} \sin\phi \, S_1(\theta) - \hat{\phi} \cos\phi \, S_2(\theta) \right\} . \tag{3.16}$$

Thus, if the incident field is expressed in terms of the amplitudes

$$\underline{E}_i \sim \begin{bmatrix} E_x \\ E_y \end{bmatrix} , \tag{3.17}$$

while the scattered field is expressed in terms of the amplitudes

$$\underline{E}_s \sim \begin{bmatrix} E_\theta^s \\ E_\phi^s \end{bmatrix} \tag{3.18}$$

one may have .

$$\begin{bmatrix} E_\theta^s \\ E_\phi^s \end{bmatrix} = \frac{e^{ikr}}{kr} \begin{bmatrix} \cos\phi S_1(\theta) & \sin\phi S_1(\theta) \\ \sin\phi S_2(\theta) & -\cos\phi S_2(\theta) \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} . \tag{3.19}$$

(U) In order to adapt (3.19) to the present convention for the direction of polarization, one shall assume that the incident field comes from a direction $\hat{\Omega}_o(\alpha_o, \beta_o)$ instead of the z-direction. The direction of polarization of the incident radiation can then be assumed to be, respectively

$$\hat{e}_{10} = \frac{\hat{\Omega}_o \times \hat{z}}{|\hat{\Omega}_o \times \hat{z}|} = \hat{x} \sin\beta_o - \hat{y} \cos\beta_o \tag{3.20}$$

and

$$\hat{e}_{20}=\hat{\Omega}_o x \hat{e}_{10}=\hat{x}\cos\alpha_o\cos\beta_o+\hat{y}\cos\alpha_o\sin\beta_o-\hat{z}\sin\alpha_o \quad . \tag{3.21}$$

Moreover, the scattered signal in any direction $\hat{\Omega}_s(\alpha_s, \beta_s)$ should be expressed in terms of the components in the two directions

$$\hat{e}_{1s}=\frac{\hat{\Omega}_s x \hat{z}}{\sin\alpha_s}=\hat{x}\sin\beta_s-\hat{y}\cos\beta_s \tag{3.22}$$

and

$$\hat{e}_{2s}=\hat{\Omega}_s x \hat{e}_{1s}=\hat{x}\cos\alpha_s\cos\beta_s+\hat{y}\cos\alpha_s\sin\alpha_s-\hat{z}\sin\beta_s \tag{3.23}$$

instead of the $\hat{\theta}$ and $\hat{\phi}$ components.

(U) Mathematically, this means that one must re-express

$$\underline{E}_s=\hat{\theta}E^s_\theta+\hat{\phi}E^s_\phi \tag{3.24}$$

in the form

$$\underline{E}_s=\hat{e}_{1s}E^s_1+\hat{e}_{2s}E_{2s} \tag{3.25}$$

by a rotation of reference coordinates. It is easy to verify from (3.24) and (3.25) that the vector components are transformed according to

$$\begin{bmatrix} E^s_1 \\ E^s_2 \end{bmatrix}=\begin{bmatrix} (\hat{e}_{1s},\hat{\theta}) & (\hat{e}_{1s},\hat{\phi}) \\ (\hat{e}_{2s},\hat{\theta}) & (\hat{e}_{2s},\hat{\phi}) \end{bmatrix}\begin{bmatrix} E^s_\theta \\ E^s_\phi \end{bmatrix} \tag{3.26}$$

Now, if $\hat{e}_{10}, \hat{e}_{20}, \hat{\Omega}_o$ are identified, respectively, with the x, y, z directions used in the derivation of (3.19), one can rewrite (3.19) in the form

$$\begin{bmatrix} E^s_1 \\ E^s_2 \end{bmatrix}=\frac{e^{ikr}}{kr}\begin{bmatrix} (\hat{e}_{1s}\cdot\hat{\theta}) & (\hat{e}_{1s}\cdot\hat{\phi}) \\ (\hat{e}_{2s}\cdot\hat{\theta}) & (\hat{e}_{2s}\cdot\hat{\phi}) \end{bmatrix}\begin{bmatrix} \cos\phi\, s_1(\theta) & \sin\phi\, s_1(\theta) \\ \sin\phi\, s_2(\theta) & -\cos\phi\, s_1(\theta) \end{bmatrix}\begin{bmatrix} E^o_1 \\ E^o_2 \end{bmatrix} \tag{3.27}$$

(U) To express the matrix relation between the scattered and incident fields in terms of $\hat{\Omega}_o$, $\hat{\Omega}_s$, and the derived directions $\hat{e}_{10}$, $\hat{e}_{20}$, $\hat{e}_{1s}$ and $\hat{e}_{2s}$, one notes that

$$\hat{\phi} = \frac{\hat{\Omega}_o \times \hat{\Omega}_s}{|\hat{\Omega}_o \times \hat{\Omega}_s|} \tag{3.28}$$

$$\hat{\theta} = \hat{\phi} \times \hat{\Omega}_s = \frac{\hat{\Omega}_s(\hat{\Omega}_o \cdot \hat{\Omega}_s) - \hat{\Omega}_o}{|\hat{\Omega}_o \times \hat{\Omega}_s|} \tag{3.29}$$

and

$$\hat{\Omega}_s = \hat{e}_{10} \sin\theta \cos\phi + \hat{e}_{20} \sin\theta \cos\phi + \hat{\Omega}_o \cos\theta \quad . \tag{3.30}$$

It follows, therefore,

$$\hat{\Omega}_s \cdot \hat{\Omega}_o = \cos\theta \tag{3.31}$$

$$|\hat{\Omega}_o \times \hat{\Omega}_s| = \sin\theta \tag{3.32}$$

$$\sin\theta \cos\phi = (\hat{e}_{10} \cdot \hat{\Omega}_s) \tag{3.33}$$

$$\sin\theta \sin\phi = (\hat{e}_{20} \cdot \hat{\Omega}_s) \tag{3.34}$$

$$\hat{e}_{1s} \cdot \hat{\theta} = -\frac{\hat{\Omega}_o \cdot \hat{e}_{1s}}{\sin\theta} = \hat{e}_{2s} \cdot \hat{\phi} \tag{3.35}$$

and

$$\hat{e}_{2s} \cdot \hat{\theta} = -\frac{\hat{\Omega}_o \cdot \hat{e}_{2s}}{\sin\theta} = -\hat{e}_{1s} \cdot \hat{\phi} \tag{3.36}$$

(U) Using the above relations in (3.27) the components of the scattering matrix may be expressed explicitly by

$$s_{11}(\hat{\Omega}_s, \hat{\Omega}_o) = \frac{-1}{\left[1-(\hat{\Omega}_s \cdot \hat{\Omega}_o)^2\right] k} \left\{ +(\hat{\Omega}_o \cdot \hat{e}_{1s})(\hat{\Omega}_s \cdot \hat{e}_{10}) s_1(\hat{\Omega}_s \cdot \hat{\Omega}_o) - (\hat{\Omega}_o \cdot \hat{2}s)(\hat{\Omega}_s \cdot \hat{e}_{10}) s_2(\hat{\Omega}_s \cdot \hat{\Omega}_o) \right\} \tag{3.37}$$

$$s_{12}(\hat{\Omega}_s, \hat{\Omega}_o) = \frac{-1}{\left[1-(\hat{\Omega}_s\cdot\hat{\Omega}_o)^2\right]k} \quad (\hat{\Omega}_o\cdot\hat{e}_{1s})(\hat{\Omega}_s\cdot\hat{e}_{20})s_1(\hat{\Omega}_s\cdot\hat{\Omega}_o)$$

$$+(\hat{\Omega}_o\cdot\hat{e}_{2s})(\hat{\Omega}_s\cdot\hat{e}_{10})s_2(\hat{\Omega}_s\cdot\hat{\Omega}_o) \tag{3.38}$$

$$s_{21}(\hat{\Omega}_s, \hat{\Omega}_o) = \frac{-1}{\left[1-(\hat{\Omega}_s\cdot\hat{\Omega}_o)^2\right]\left[k\right]} \left\{(\hat{\Omega}_o\cdot\hat{e}_{2s})(\hat{\Omega}_s\cdot\hat{e}_{10})s_1(\hat{\Omega}_s\cdot\hat{\Omega}_o)\right.$$

$$\left.+(\hat{\Omega}_o\cdot\hat{e}_{1s})(\hat{\Omega}_s\cdot\hat{e}_{20})s_2(\hat{\Omega}_s\cdot\hat{\Omega}_o)\right\} \tag{3.39}$$

and

$$s_{22}(\hat{\Omega}_s, \hat{\Omega}_o) = -\frac{1}{\left[1-(\hat{\Omega}_s\cdot\hat{\Omega}_o)^2\right]\left[k\right]} \left\{(\hat{\Omega}_o\cdot\hat{e}_{2s})(\hat{\Omega}_s\cdot\hat{e}_{20})s_1(\hat{\Omega}_s\cdot\hat{\Omega}_o)\right.$$

$$\left.-(\hat{\Omega}_o\cdot\hat{e}_{1s})(\hat{\Omega}_s\cdot\hat{e}_{20})s_2(\hat{\Omega}_s\cdot\hat{\Omega}_o)\right\}. \tag{3.40}$$

3.3 Dipole Scattering

(U) For obstacles with dimensions much less than the wavelength the Rayleigh approximation may be used in calculating the scattering matrix. Roughly, when a static electric field is applied to an isotropic body, the body is polarized. In the low frequency approximation (wavelength large in respect to the dimension of the body), the dominant terms of the scattered field may be approximated by the field radiated from the induced oscillating dipole. From the solutions of electrostatic problems involving spheres, spheroids, etc., one generally recognizes that a small body has three mutually perpendicular principal axes of polarization. In referring to a fixed coordinate system, these directions are denoted by $\hat{n}_1$, $\hat{n}_2$ and $\hat{n}_3$, then the induced polarization caused by any incident field are given by

$$\underline{p} = \sum_{i=1}^{3} (\underline{E}_o\cdot\hat{n}_i)\alpha_i\,\hat{n}_i \tag{3.41}$$

where α_i are known as polarizability of the body (van de Hulst, 1957).

(U) If

$$E_o = E_{10}\,\hat{e}_1 + E_{20},\,\hat{e}_2 \tag{3.42}$$

then

$$\underline{p} = \sum_{i=1}^{3} \alpha_i \hat{n}_i (\hat{n}_i \cdot \hat{e}_{10}) E_{10} + \sum_{i=1}^{3} \alpha_i \hat{n}_i (\hat{n}_i \cdot \hat{e}_{20}) E_{20} \quad . \tag{3.43}$$

(U) Since the far zone field due to an oscillating dipole located at the origin is given by (Stratton 1941),

$$\underline{E} = -\frac{k^2}{4\pi\epsilon_o} \frac{1}{r} e^{ikr} \left[\hat{\Omega}_s \times (\hat{\Omega}_s \times \underline{p})\right] \tag{3.44}$$

The components of the scattered field are, therefore, given by

$$E_{1s} = \underline{E} \cdot \hat{e}_{1s} = \frac{k^2}{4\pi\epsilon_o} \frac{e^{ikr}}{r} \underline{p} \cdot \hat{e}_{1s} \tag{3.45}$$

and

$$E_{2s} = \underline{E} \cdot \hat{e}_{2s} = \frac{k^2}{4\pi\epsilon_o} \frac{e^{ikr}}{r} \underline{p} \cdot \hat{e}_{2s} \quad . \tag{3.46}$$

Substitution of (3.43) in the above, yields

$$E_{1s} = \frac{k^2}{4\pi\epsilon_o} \frac{1}{r} e^{ikr} \left[\sum_{i=1}^{3} \alpha_i (\hat{e}_{10} \cdot \hat{n}_i)(\hat{n}_i \cdot \hat{e}_{1s}) E_{10} + \sum_{i=1}^{3} \alpha_i (\hat{e}_{20} \cdot \hat{n}_i)(\hat{n}_i \cdot \hat{e}_{1s}) E_{20}\right] \tag{3.47}$$

and

$$E_{2s} = \frac{k^2}{4\pi\epsilon_o} \frac{1}{r} e^{ikr} \left[\sum_{i=1}^{3} \alpha_i (\hat{e}_{10} \cdot \hat{n}_i)(\hat{n}_i \cdot \hat{e}_{2s}) E_{10} + \sum_{i=1}^{3} \alpha_i (\hat{e}_{20} \cdot \hat{n}_i)(\hat{n}_i \cdot \hat{e}_{2s}) E_{20}\right] \tag{3.48}$$

The scattering matrix for small bodies are therefore given by

$$s_{ij}(\Omega_s, \hat{\Omega}_o) = \frac{k^2}{4\pi\epsilon_o} \frac{1}{r} \sum_{k=1}^{3} \alpha_k (\hat{e}_{is} \cdot \hat{n}_{is})(\hat{n}_i \cdot \hat{e}_{jo}) \quad . \tag{3.49}$$

(U) For the special case of a sphere, $\alpha_1=\alpha_2=\alpha_3=\alpha$, one has

$$s(\hat{\Omega}_s, \hat{\Omega}_o)= \frac{e^{ikr}}{4\pi\epsilon_o} \frac{k^2}{r} \alpha \begin{bmatrix} (\hat{e}_{10}\cdot\hat{e}_{1s}) & (\hat{e}_{20}\cdot\hat{e}_{1s}) \\ (\hat{e}_{10}\cdot\hat{e}_{2s}) & (\hat{e}_{20}\cdot\hat{e}_{2s}) \end{bmatrix} . \tag{3.50}$$

(U) For most regular bodies such as ellipsoids, spheroids, cylinders, etc., values of polarizability are known, so that (3.50) may be used to calculate the scattering matrix. For bodies of other irregular shapes, approximate values of α may be obtained by using the Born approximation which yields

$$\underline{\underline{\alpha}} \cong V\mathcal{E}_o (\underline{\underline{\mathcal{E}}}-1) \tag{3.51}$$

where V is the volume of the body and $\underline{\underline{\mathcal{E}}}$ is the dielectric tensor. The limitations of such an approximation have been discussed by van de Hulst (1957).

3.4 Physical Optics Approximation

(U) When an obstacle of infinite conductivity is illuminated by an incident wave, currents are induced on the surface of the body. The scattered field can then be interpreted as the fields radiated from the surface currents. In general, the surface currents are not known unless one can solve the scattering problem involving the obstacle exactly. In the physical optics formulation, the following two physically plausible approximations are made regarding the surface currents (see Fig. 3-2).

(U) a) For a body of finite size, a part of the surface is in the shadow region, where the field is small. Therefore, for a first order approximation one assumes that the surface current is zero in the shadow region.

(U) b) To calculate the current on the illuminated surface of the obstacle, one assumes the local radius of curvature of the obstacle to be much larger than the wavelength. Under this approximation, the surface currents may be approximated everywhere by the currents that would be induced on a plane tangent to the surface. This approximate current may be expressed in terms of the incident magnetic field strength by

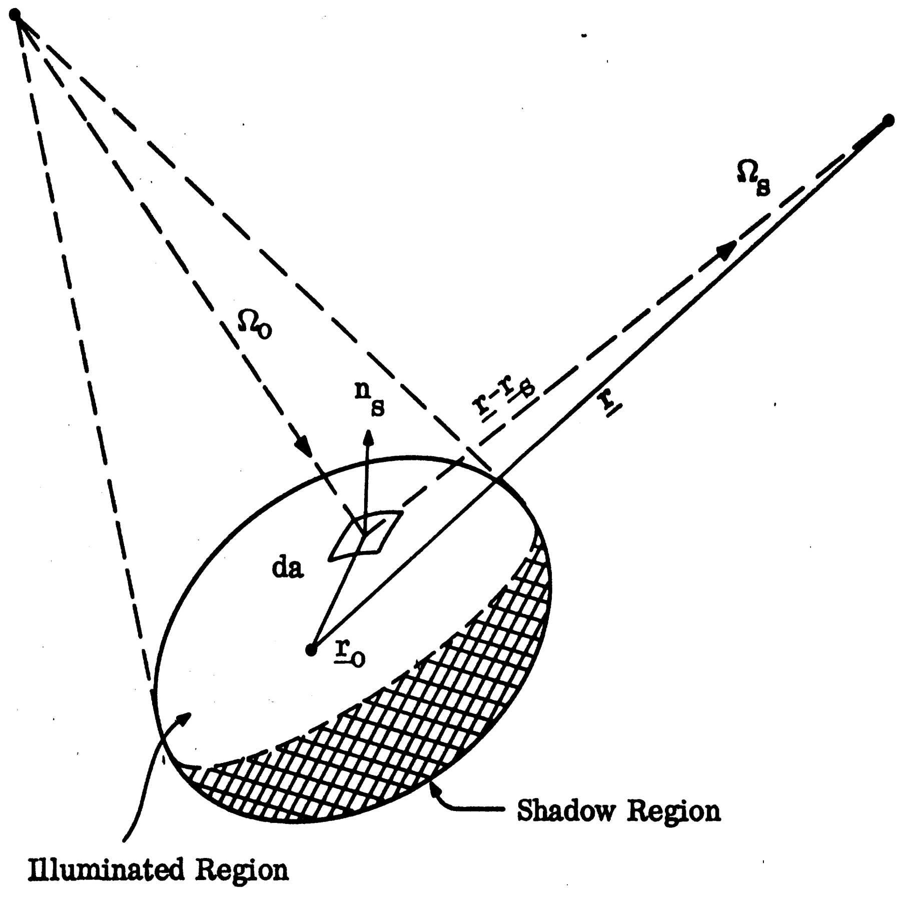

FIG. 3-2: GEOMETRY FOR PHYSICAL OPTICS APPROXIMATION

$$\underline{K}_s = +2\hat{n}_s \times \underline{H}_o \qquad \text{amp/m}^2 \tag{3.52}$$

where $\hat{n}_s$ is the normal to the surface.

(U) Using this approximate value of surface currents, the scattered magnetic field due to any surface element is given by

$$d\underline{H}_s(\underline{r}) = +\frac{1}{2\pi}\left[\hat{n}_s \times \underline{H}_i(\underline{r}_o)\right] \times \nabla_s \frac{e^{ik|\underline{r}-\underline{r}_s|}}{|\underline{r}-\underline{r}_s|}\, da \quad . \tag{3.53}$$

If $|\underline{r}-\underline{r}_s|$ is large,

$$\nabla_s \frac{e^{ik|\underline{r}-\underline{r}_s|}}{|\underline{r}-\underline{r}_s|} \cong -ik \frac{e^{ik|\underline{r}-\underline{r}_s|}}{|\underline{r}-\underline{r}_s|}\, \hat{\Omega}_s \tag{3.54}$$

where $\hat{\Omega}_s$ is the direction of the scattered field. Thus,

$$d\underline{H}_s(\underline{r}) = \frac{-ik}{2\pi}\, \frac{e^{ik|\underline{r}-\underline{r}_s|}}{|\underline{r}-\underline{r}_s|}\left[\hat{n}_s \times \underline{H}_o(\underline{r}_s)\right] \times \hat{\Omega}_s\, da \quad . \tag{3.55}$$

(U) If the incident field intercepted by the area comes from the direction $\hat{\Omega}_o$, then (3.55) may be expressed in terms of the electric field by the relations,

$$d\underline{E}_s(\underline{r}) = d\underline{H}_s(\underline{r}) \times \hat{\Omega}_s \sqrt{\frac{\mu_o}{\epsilon_o}} \tag{3.56}$$

and

$$\underline{H}_o(\underline{r}_s) = \hat{\Omega}_i \times \underline{E}_o(\underline{r}_s)\sqrt{\frac{\epsilon_o}{\mu_o}} \quad . \tag{3.57}$$

Using these relations in (3.55) one obtains

$$d\underline{E}_s(\underline{r}) = \frac{-ik}{2\pi}\, \frac{e^{ik|\underline{r}-\underline{r}_s|}}{|\underline{r}-\underline{r}_s|}\left\{(\hat{n}_s \cdot \underline{E}_o)\left[\hat{\Omega}_s(\hat{\Omega}_o \cdot \hat{\Omega}_s) - \hat{\Omega}_o\right] - (\hat{n}_s \cdot \hat{\Omega}_o)\left[\hat{\Omega}_s(\hat{\Omega}_o \cdot \underline{E}_o) - \underline{E}_o\right]\right\} da \quad . \tag{3.58}$$

In terms of the horizontally and vertically polarized component,

$$d\underline{E}_s(\underline{r}) = \hat{e}_{1s}\, dE_{1s} + \hat{e}_{2s}\, dE_{2s} \quad , \tag{3.59}$$

and

$$E_o(\underline{r}_s) = \hat{e}_{10} E_{10}(\underline{r}_s) + \hat{e}_{20} E_{20}(\underline{r}_s) \quad , \tag{3.60}$$

one finds that

$$\begin{bmatrix} dE_{1s} \\ dE_{2s} \end{bmatrix} = \frac{-ik}{2\pi} \frac{e^{ik|\underline{r}-\underline{r}_s|}}{|\underline{r}-\underline{r}_s|} da \begin{bmatrix} \hat{n}_s \cdot (\hat{e}_{1s} x \hat{e}_{20}) & -\hat{n}_s \cdot (\hat{e}_{1s} x \hat{e}_{10}) \\ \hat{n}_s \cdot (\hat{e}_{2s} x \hat{e}_{20}) & -\hat{n}_s \cdot (\hat{e}_{2s} x \hat{e}_{10}) \end{bmatrix} \begin{bmatrix} E_{10} \\ E_{20} \end{bmatrix} \tag{3.61}$$

(U) For any arbitrary incident field and an extended surface, $\hat{e}_{10}$, $\hat{e}_{20}$, $\hat{e}_{1s}$, $\hat{e}_{2s}$ are not constant vectors, so that the integration of (3.61) over the illuminated region of the obstacle is somewhat cumbersome. In most calculations, one assumes that the incident field is a plane wave, thus the vectors $\hat{\Omega}_o$, $\hat{e}_{10}$, $\hat{e}_{20}$ are constant and

$$\begin{bmatrix} E_{10}(\underline{r}_s) \\ E_{20}(\underline{r}_s) \end{bmatrix} = \begin{bmatrix} E_{10} \\ E_{20} \end{bmatrix} e^{ik\hat{\Omega}_o \cdot \underline{r}_s} \quad . \tag{3.62}$$

Moreover, if the far zone approximation is introduced into the scattered field, then

$$\hat{\Omega}_s \cong \underline{r} \, , \tag{3.63}$$

$$\frac{e^{ik|\underline{r}-\underline{r}_s|}}{|\underline{r}-\underline{r}_s|} \cong \frac{e^{ikr}}{r} e^{-ik\hat{\Omega}_s \cdot \underline{r}_s} \tag{3.64}$$

and the vectors $\hat{e}_{1s}$ and $\hat{e}_{2s}$ are constant. Thus,

$$\begin{bmatrix} E_{1s}(\underline{r}) \\ E_{2s}(\underline{r}) \end{bmatrix} = \frac{-ik}{2\pi} \frac{e^{ikr}}{r} \left\{ \int_{\substack{\text{lit} \\ \text{region}}} da \begin{bmatrix} \hat{n}_s \cdot (\hat{e}_{1s} x \hat{e}_{20}) & -\hat{n}_s \cdot (\hat{e}_{1s} x \hat{e}_{10}) \\ \hat{n}_s \cdot (\hat{e}_{2s} x \hat{e}_{20}) & -\hat{n}_s \cdot (\hat{e}_{2s} x \hat{e}_{10}) \end{bmatrix} x \right.$$

$$\left. x \, e^{ik\underline{r}_s \cdot (\hat{\Omega}_o - \hat{\Omega}_s)} \right\} \begin{bmatrix} E_{10} \\ E_{20} \end{bmatrix} \quad . \tag{3.65}$$

Therefore the scattering matrix of the obstacle is given by

$$S(\hat{\Omega}_s, \hat{\Omega}_o) = \frac{-ik}{2\pi} \int_{\substack{\text{lit} \\ \text{region}}} da\, e^{ik\underline{r}_s \cdot (\hat{\Omega}_o \cdot \hat{\Omega}_s)} \begin{bmatrix} \hat{n}_s \cdot (\hat{e}_{1s} x \hat{e}_{20} & -\hat{n}_s \cdot (\hat{e}_{1s} x \hat{e}_{10}) \\ \hat{n}_s \cdot (\hat{e}_{2s} x \hat{e}_{20} & -\hat{n}_s \cdot (\hat{e}_{2s} x \hat{e}_{10}) \end{bmatrix} \qquad (3.66)$$

(U) As an example, consider the scattering matrix of a plane defined by

$$-\frac{a}{2} \le x \le \frac{a}{2} \quad , \quad -\frac{b}{2} \le y \le \frac{b}{2}$$

and oriented in the z-direction. The scattering matrix is easily calculated to be

$$S(\alpha_s, \beta_s, \alpha_o, \beta_o) = -\frac{ik}{2\pi} A \,\mathrm{sinc}\left[\frac{a}{2}(\sin\alpha_o \cos\beta_o - \sin\alpha_s \cos\beta_s)\right]$$

$$\mathrm{sinc}\left[\frac{b}{2}(\sin\alpha_o \sin\beta_o - \sin\alpha_o \sin\beta_o)\right] \quad \mathrm{X}$$

$$\begin{bmatrix} \cos\alpha_o \cos(\beta_s - \beta_o) & -\sin(\beta_o - \beta_s) \\ \cos\alpha_o \cos\alpha_s \sin(\beta_o - \beta_s) & +\cos\alpha_s \cos(\beta_o - \beta_s) \end{bmatrix} , \qquad (3.67)$$

where A is the area of the plate and

$$\mathrm{sinc}\, y \triangleq \frac{\sin y}{y} \;.$$

IV

GROUND REFLECTION

4.1 Introduction

(U) The reflection of waves by a rough surface such as the ground has been a subject of investigation by many authors. The approaches used by various investigators and their results have been summarized by Beckmann and Spizzichino (1963). In general, most formulations deal with the scattering of scalar waves using the Kirchhoff approximation. In the case of reflection of electromagnetic waves, the scalar formulation has been applied individually to the vertically and horizontally polarized components of the electromagnetic wave. A practical, explicit formulation which considers the inter-polarization coupling (depolarization effect due to the non-planar nature of the reflection) has yet to be developed.

(U) Only recently (Fung, 1966) the vector reflection problem has been formulated in terms of the vector form of the Kirchhoff-Huygen principle. In this work, however, a different formulation of the problem using the concept of angular spectra is given in order to study the polarization as well as the angular distribution characteristics of the ground reflected wave. Approximate boundary conditions using geometric optics are then used to deduce the reflection matrix of the ground.

4.2 Angular Spectra of the Reflected Radiation

(U) The ground reflected wave may be considered as the radiation due to 'induced sources' on the ground as a result of interaction of incident fields with the ground. Thus, the reflected field satisfies the source-free Maxwell equations everywhere above the ground.

(U) It is well known that at a single frequency $\omega = 2\pi f$ (or each Fourier component of a time varying field), the solutions of the homogeneous Maxwell equation may be represented by two scalar functions. Those functions, $\phi(\underline{r})$ and $\psi(\underline{r})$ satisfy

$$\nabla^2 \begin{Bmatrix} \phi(\underline{r}) \\ \psi(\underline{r}) \end{Bmatrix} + k^2 \begin{Bmatrix} \phi(\underline{r}) \\ \psi(\underline{r}) \end{Bmatrix} = 0 \quad , \tag{4.1}$$

where

$$k \triangleq \frac{\omega}{c} = \frac{2\pi}{\lambda} \quad . \tag{4.2}$$

(U) To conform with the geometry appropriate to the problem of ground reflection, it is convenient to choose some reference plane z = 0 near the ground, and describe the ground profile by a function

$$z = z_s(x, y) \quad . \tag{4.3}$$

The reflected fields, therefore, exist in the region of space defined by

$$z_s \leq z < \infty \quad . \tag{4.4}$$

In terms of the two scalar functions, one may express the field in the following forms

$$\underline{E} = \nabla \times (\hat{z}\phi) + \frac{1}{k} \nabla \times \nabla \times (\hat{z}\psi) \tag{4.5}$$

and

$$\underline{H} = \frac{1}{i\omega\mu_o} \nabla \times \underline{E} = \frac{1}{i\omega\mu_o} \left[\nabla \times \nabla \times (\hat{z}\phi) + k \nabla \times (\hat{z}\psi) \right] . \tag{4.6}$$

(U) By taking the spatial Fourier transform of (4.1) with respect to x and y coordinates, one finds that $\phi(\underline{r})$ and $\psi(\underline{r})$ may be represented by

$$\phi(\underline{r}) = \int dk_x \int dk_y \, \Phi(k_x, k_y) e^{ik_x x} e^{ik_y y} e^{i\sqrt{k^2 - k_x^2 - k_y^2}\, z} \tag{4.7}$$

and

$$\Psi(\underline{r}) = \int dk_x \int dk_y \, \Psi(k_x, k_y) e^{ik_x x} e^{ik_y y} e^{i\sqrt{k^2 - k_x^2 - k_y^2}\, z} \tag{4.8}$$

respectively. In the above, we choose

$$\mathrm{Im} \sqrt{k^2 - k_x^2 - k_y^2} \geq 0 \tag{4.9}$$

in order to satisfy the radiation condition as $z \to \infty$. Although, mathematically, the integrations in (4.7) and (4.8) extend from $-\infty$ to $+\infty$ for both k_x and k_y, physically meaningful solutions for the field far (several wavelengths) from the ground plane may be obtained by carrying out the integration over the range of k_x and k_y such that $\sqrt{k^2-k_x^2-k_y^2}$ is real. Thus, we may denote

$$k_x = k\sin\alpha\cos\beta \tag{4.10}$$

$$k_y = k\sin\alpha\sin\beta \tag{4.11}$$

and

$$\sqrt{k^2-k_x^2-k_y^2} = k\cos\alpha \tag{4.12}$$

where α and β are real angles. Physically, if we denote by $\hat{\Omega}$ the unit vector in the direction defined by the latitude angle α and the azimuth angle β, then (4.7) and (4.8) may be reduced to more meaningful terms

$$\phi(\underline{r}) = k^2 \int_0^{\pi/2} \sin\alpha\, d\alpha \int_0^{2\pi} d\beta \cos\alpha\, \Phi(\alpha,\beta) e^{ik\hat{\Omega}\cdot\underline{r}} \tag{4.13}$$

$$\psi(\underline{r}) = k^2 \int_0^{\pi/2} \sin\alpha\, d\alpha \int_0^{2\pi} d\beta \cos\alpha\, \Psi(\alpha,\beta) e^{ik\hat{\Omega}\cdot\underline{r}} \tag{4.14}$$

(U) Substituting these equations in (4.5) and (4.6) yields

$$\underline{E}(\underline{r}) = ik^3 \int_0^{\pi/2} \sin^2\alpha \cos\alpha\, d\alpha \int_0^{2\pi} d\beta \left[\hat{e}_1 \Phi(\alpha,\beta) + i\hat{e}_2 \Psi(\alpha,\beta)\right] e^{ik\hat{\Omega}\cdot\underline{r}} \tag{4.15}$$

and

$$\underline{H}(\underline{r}) = \eta_0 ik^3 \int_0^{\pi/2} \sin^2\alpha \cos\alpha\, d\alpha \int_0^{2\pi} d\beta \left[\hat{e}_2 \Phi(\alpha,\beta) - i\hat{e}_1 \Psi(\alpha,\beta)\right] e^{ik\hat{\Omega}\cdot\underline{r}} \tag{4.16}$$

where

$$\eta_0 = \sqrt{\frac{\epsilon_0}{\mu_0}} = \frac{1}{120\,\pi} \tag{4.17}$$

These expressions for $\underline{E}(\underline{r})$ and $\underline{H}(\underline{r})$ may be interpreted as the angular spectra representation of the radiation. It is easy to see that

$$\mathcal{E}_1(\alpha,\beta) = ik^3 \Phi(\alpha,\beta)\sin\alpha\cos\alpha \tag{4.18}$$

and

$$\mathcal{E}_2(\alpha,\beta) = -k^3 \Psi(\alpha,\beta)\sin\alpha\cos\alpha \tag{4.19}$$

are, respectively, the components of vertically and horizontally polarized electric fields of the radiation coming to a point $\underline{r}$ from a small solid angle $d\Omega$ in the direction $\hat{\Omega}$. Mathematically, one may rewrite (4.15) and (4.16) as

$$d\underline{E}(\underline{r})=\left[\mathcal{E}_1(\alpha,\beta)\hat{e}_1+\mathcal{E}_2(\alpha,\beta)\hat{e}_2\right]e^{ik\hat{\Omega}\cdot\underline{r}}\,d\Omega \tag{4.20}$$

and

$$d\underline{H}(\underline{r})=\eta_o\left[\mathcal{E}_1(\alpha,\beta)\hat{e}_2-\mathcal{E}_2(\alpha,\beta)\hat{e}_1\right]e^{ik\hat{\Omega}\cdot\underline{r}}\,d\Omega \quad . \tag{4.21}$$

From (4.20) and (4.21), it is easy to see the advantage of using angular spectrum in the characterization of the radiation. In the calculation of ground reflected radiation, the angular spectrum expressed in the form of (4.20) and (4.21) are functions of direction only and are independent space coordinates.

(U) To evaluate the angular spectrum in terms of the boundary condition, let the ground be defined by

$$\underline{r}_s=\hat{x}x_s+\hat{y}y_s+\hat{z}z_s \tag{4.22}$$

and the $\hat{z}$-component of the reflected field oh the ground is given by $E_z(\underline{r}_s)$ and $H_z(\underline{r}_s)$. Then, integrating the $\hat{z}$-component of (4.21) and (4.22) yields

$$\begin{aligned} E_z(\underline{r}_s)&=-\int\sin\alpha\, d\alpha\int d\beta\sin\alpha\; e^{ik\hat{\Omega}\cdot\underline{r}_s}\mathcal{E}_2(\alpha,\beta)\\ &=-\frac{1}{k^2}\int dk_x\int dk_y\,\sin\alpha\frac{\mathcal{E}_2(\alpha,\beta)}{\cos\alpha}\,e^{ik_xx_s}\,e^{ik_yy_s}\,e^{i\sqrt{k^2-k_x^2-k_y^2}\,z_s} \end{aligned} \tag{4.23}$$

and

$$H_z(\underline{r}_s)=\eta_o\,\frac{-1}{k^2}\int dk_x\int dk_y\,\frac{\mathcal{E}_1(\alpha,\beta)\sin\alpha}{\cos\alpha}\,e^{ik_xx_s}\,e^{ik_yy_s}\,e^{i\sqrt{k^2-k_x^2-k_y^2}\,z_s} \tag{4.24}$$

(U) Equations (4.23) and (4.24) may be considered a two-dimensional Fourier transform involving the functions $\mathcal{E}_1$ and $\mathcal{E}_2$. For slightly rough ground, where z_s does not differ greatly from zero, one may argue that the inverse transforms are approximately given by

$$\mathcal{E}_1(\alpha,\beta) = \frac{-k^2}{(2\pi)^2}\,\frac{\cos\alpha}{\sin\alpha}\int dx_s \int dy_s\, \frac{H_z(\underline{r}_s)}{\eta_o}\, e^{-ik\hat{\Omega}\cdot\underline{r}_s} \tag{4.25}$$

and

$$\mathcal{E}_2(\alpha,\beta) = \frac{-k^2}{(2\pi)^2}\,\frac{\cos\alpha}{\sin\alpha}\int dx_s \int dy_s\, E_z(\underline{r}_s)\, e^{-ik\hat{\Omega}\cdot\underline{r}_s} \quad . \tag{4.26}$$

Thus, approximately, the knowledge of the z-components of the electric and magnetic field strength of the radiation determines the angular spectra completely.

4.3 Ground Reflection Matrix

(U) From the results of the last section, it is seen that a study of the ground reflected wave may begin with the knowledge of the z-component of the reflected electric and magnetic field at the ground, or at some reference plane. In general, these two components are not known, so that approximate evaluations, or even direct postulations concerning these components (in the case of randomly rough surfaces) must be used. Three possible approaches to the estimation of $E_z(\underline{r}_s)$ and $H_z(\underline{r}_s)$ are given below.

a) The Layer Approach

(U) Borrowing the idea from the random screen approach for wave propagation through the ionosphere, we may postulate directly the phase, amplitude and polarization variation due to a plane wave reflected from the surface layer of the ground. Such a model has been used successfully in ionospheric diffraction, but a great deal of measurement is necessary to determine the parameters involved in such a model.

b) The Multiple Scattering Approach

(U) Assuming that the surface of the ground is composed of a random distribution of scatters of appropriate properties, the reflected waves due to some plane wave incident on the ground may be calculated by the method of multiple scattering. Such a model has been used successfully for the transmission of solar radiation through the atmosphere. However, the correct model of the scatterers, and meaningful solutions for engineering use are both difficult to obtain.

c) The Reflection Approach

(U) Assuming that the ground is locally smooth so that the law of plane wave reflection is applicable locally, then for any ground surface defined by $z_s(x_s, y_s)$, the reflected waves may be obtained. A postulated statistics of the surface (and the electric properties of the ground) would then suffice to specify the problem.

(U) In the present work we shall follow approach c) due to its successful application in scalar wave reflection.

(U) Referring to Fig. 4-1, the radiation from a source located at $\underline{r}_o(x_o, y_o, z_o)$ is reflected by the ground surface defined by

$$z_s = z_s(x_s, y_s) \quad . \tag{4.27}$$

At any point of reflection as indicated in the figure, the radiation appears to come from a direction $\hat{\Omega}_o$. This $\hat{\Omega}_o$ is given by

$$\hat{\Omega}_o = \frac{1}{|\underline{r}-\underline{r}_s|}\left[\hat{x}(x_s-x_o)+\hat{y}(y_s-y_o)+\hat{z}(z_s-z_o)\right] \triangleq \hat{x}\sin\alpha_o\cos\beta_o+\hat{y}\sin\alpha_o\sin\beta_o+\hat{z}\cos\alpha_o \tag{4.28}$$

where

$$\cos\alpha_o = \frac{z_s(x_s, y_s)-z_o}{|\underline{r}-\underline{r}_o|} \tag{4.29}$$

and

$$\cos\beta_o = \frac{x_s-x_o}{|\underline{r}-\underline{r}_o|\sin\alpha_o} \quad . \tag{4.30}$$

The incident radiation, except for a phase factor, may be represented by

$$\underline{E}_o(\underline{r}_s) = E_{10}\hat{e}_{10} + E_{20}\hat{e}_{20} \tag{4.31}$$

where

$$\hat{e}_{10} = \frac{\hat{\Omega}_o \times \hat{z}}{\sin\alpha_o} = \hat{x}\sin\beta_o - \hat{y}\cos\beta_o \tag{4.32}$$

$$\hat{e}_{20} = \hat{\Omega}_o \times \hat{e}_{10} = \hat{x}\cos\alpha_o\cos\beta_o + \hat{y}\cos\alpha_o\sin\beta_o - \hat{z}\sin\alpha_o \tag{4.33}$$

and E_{10} and E_{20} are the complex amplitudes of the vertically and horizontally polarized components, respectively, of the incident radiation.

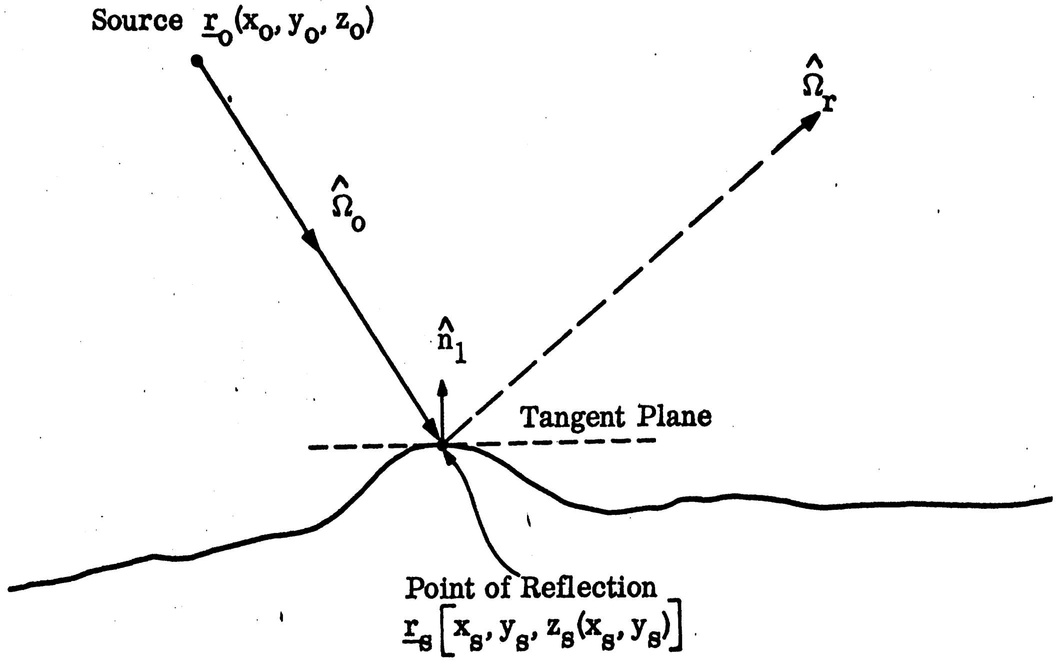

FIG. 4-1: GEOMETRY FOR REFLECTION APPROACH

(U) At the point of reflection, $\underline{r}_s$, the normal to the surface is given by

$$\hat{n}_1 = \left[1+\left(\frac{\partial z_s}{\partial x_s}\right)^2+\left(\frac{\partial z_s}{\partial y_s}\right)^2\right]^{-1/2}\left[-\hat{x}\frac{\partial z_s}{\partial x_s}-\hat{y}\frac{\partial z_s}{\partial y_s}+\hat{z}\right]$$

$$\triangleq \hat{x}\sin\alpha_1\cos\beta_1+\hat{y}\sin\alpha_1\sin\beta_1+\hat{z}\cos\alpha_1 \qquad (4.34)$$

therefore, if

$$\hat{\Omega}_o \cdot \hat{n}_1 < 0,$$

the incident field is reflected. The reflected field is now assumed to be that reflected by a local tangent plane at $\underline{r}_s$. The direction of the reflected wave is therefore

$$\hat{\Omega}_r = \hat{\Omega}_o - 2\hat{n}_1(\hat{n}_1\Omega_o) \quad . \qquad (4.35)$$

(U) In order to express the reflected radiation in terms of the local reflection coefficient for plane waves, one may resolve the incident field in the direction of perpendicular and parallel components. Represent these directions associated with the incident radiation by

$$e_{o\perp} = \frac{\hat{\Omega}_o x \hat{n}_1}{|\hat{\Omega}_o x \hat{n}_1|} = \frac{\hat{\Omega}_o x \hat{n}_1}{\sin\gamma} \qquad (4.36)$$

$$\hat{e}_{o//} = \hat{\Omega}_o \; x \; e_{o\perp} \qquad (4.37)$$

where

$$\cos\gamma = -\hat{\Omega}_o \cdot \hat{n}_1 = -\left[\cos\alpha_o\cos\alpha_1+\sin\alpha_o\sin\alpha_1\cos(\beta_o-\beta_1)\right] \qquad (4.38)$$

By simple rotation of coordinates, one may have

$$\underline{E}_o = \left[(\hat{e}_{o\perp}\cdot\hat{e}_{10})E_{10}+(\hat{e}_{o\perp}\cdot\hat{e}_{20})E_{20}\right]\hat{e}_{o\perp} + \left[(\hat{e}_{o//}\cdot\hat{e}_{10})E_{10}+(\hat{e}_{o//}\cdot\hat{e}_{20})E_{20}\right]\hat{e}_{o//} \qquad (4.39)$$

Similarly, the reflected field at the point of reflection may be resolved in the parallel and perpendicular directions of polarization defined below.

$$\hat{e}_{r\perp} = \frac{\hat{\Omega}_r x \; n_1}{|\hat{\Omega}_r x n_1|} = \hat{e}_{o\perp} \qquad (4.40)$$

$$\hat{e}_{r//} = \hat{\Omega}_r x e_{r\perp} = \hat{e}_{o//} + 2\hat{n}_1 x \hat{e}_{o\perp} \cos\gamma \quad . \tag{4.41}$$

The fields of the reflected radiation at the point of reflection are therefore given by

$$\underline{E}(\underline{r}_s) = R_\perp \hat{e}_{o\perp}\left[(\hat{e}_{o\perp}\cdot\hat{e}_{10})E_{10} + (\hat{e}_{o\perp}\cdot\hat{e}_{20})E_{20}\right] - R_{//}\left[\hat{e}_{o//} + 2\hat{n}_1 x \hat{e}_{o\perp}\cos\gamma\right]$$
$$\left[(\hat{e}_{o//}\cdot\hat{e}_{10})E_{10} + (\hat{e}_{o//}\cdot\hat{e}_{20})E_{20}\right] \tag{4.42}$$

and

$$\underline{H}(\underline{r}_s) = \eta_o R_\perp\left[(\hat{e}_{o\perp}\cdot\hat{e}_{10})E_{10} + (\hat{e}_{o\perp}\cdot e_{20})E_{20}\right]\left[\hat{e}_{o//} + 2\hat{n}_1 x \hat{e}_{o\perp}\cos\gamma\right]$$
$$+\eta_o R_{//}\hat{e}_{o\perp}\left[(\hat{e}_{o//}\cdot\hat{e}_{10})E_{10} + (\hat{e}_{o//}\cdot\hat{e}_{20})E_{20}\right] \tag{4.43}$$

where $R_\perp$ and $R_{//}$ are the plane wave reflection coefficients, given by

$$R_\perp = \frac{\cos\gamma - \sqrt{N^2 - \sin^2\gamma}}{\cos\gamma + \sqrt{N^2 - \sin^2\gamma}} \tag{4.44}$$

and

$$R_{//} = \frac{N^2\cos\gamma - \sqrt{N^2 - \sin^2\gamma}}{N^2\cos\gamma - \sqrt{N^2 - \sin^2\gamma}} \tag{4.45}$$

where N is the index of refraction of the ground.

(U) From Eqs. (4.42) and (4.43) it is easily seen that

$$\begin{bmatrix} \dfrac{-H_z(\underline{r}_s)}{\eta_o} \\ \\ -E_z(\underline{r}_s) \end{bmatrix} = \frac{1}{\sin\alpha_o \sin^2\gamma}\begin{bmatrix} R_\perp ac + R_{//}b^2 & R_\perp bc - R_{//}ab \\ \\ R_{//}ab - R_{//}bc & R_\perp b^2 + R_{//}ac \end{bmatrix}\begin{bmatrix} E_{10}(\underline{r}_s) \\ \\ E_{20}(\underline{r}_s) \end{bmatrix} \tag{4.46}$$

where

$$a = \cos\alpha_1 + \cos\gamma\cos\alpha_o \tag{4.47}$$

$$b = \sin\alpha_o \sin\alpha_1 \sin(\beta_1 - \beta_o) \tag{4.48}$$

and

$$c = -\cos\gamma\cos\alpha_0' + \cos\alpha_1(1-2\cos^2\gamma) \quad . \tag{4.49}$$

Introducing (4.46) into (4.48) and (4.49), one finds the angular spectra of the ground reflected radiation to be related to the incident radiation (approximately) by the following,

$$\begin{bmatrix} \mathcal{E}_1(\alpha,\beta) \\ \mathcal{E}_2(\alpha,\beta) \end{bmatrix} = \frac{k^2}{(2\pi)^2} \frac{\cos\alpha}{\sin\alpha_0 \sin\alpha} \int dx_s \int dy_s \frac{e^{-ik\hat{\Omega}\cdot\underline{r}_s}}{\sin^2\gamma}$$

$$\begin{bmatrix} R_\perp ac+R_{//}b^2 & R_\perp bc-R_{//}ab \\ R_\perp ab-R_{//}bc & R_\perp b^2-R_{//}ac \end{bmatrix} \begin{bmatrix} E_{10}(\underline{r}_s) \\ E_{20}(\underline{r}_s) \end{bmatrix} \quad . \tag{4.50}$$

(U) If the incident field is a plane wave, then

$$\begin{bmatrix} E_{01}(\underline{r}_s) \\ E_{02}(\underline{r}_s) \end{bmatrix} = \begin{bmatrix} E_{01} \\ E_{02} \end{bmatrix} e^{ik\underline{r}_s\cdot\hat{\Omega}_0} \tag{4.51}$$

where E_{01} and E_{02} are respectively the complex amplitudes of the electric fields of the horizontally and vertically polarized components of the incident radiation. For plane wave incidence, one may then characterize the reflection properties of the ground by a ground reflection matrix such that

$$\begin{bmatrix} \mathcal{E}_1(\hat{\Omega}) \\ \mathcal{E}_2(\hat{\Omega}) \end{bmatrix} = \begin{bmatrix} R \end{bmatrix} \begin{bmatrix} E_{01} \\ E_{02} \end{bmatrix} \quad . \tag{4.52}$$

(U) This reflection matrix is then given by

$$\left[R\right] = \frac{k^2}{(2\pi)^2} \frac{\cos\alpha}{\sin\alpha_o \sin\alpha} \int dx_s \int dy_s \frac{e^{ik\underline{r}_s \cdot (\hat{\Omega}_o - \hat{\Omega})}}{\sin^2\gamma}$$

$$\begin{bmatrix} R_\perp ac + R_{/\!/} b^2 & R_\perp bc - R_{/\!/} ab \\ R_\perp ab - R_{/\!/} bc & R_\perp b^2 + R_{/\!/} ac \end{bmatrix} . \qquad (4.53)$$

(U) For any given definitive ground profile, this reflection matrix, in principle, may be evaluated by integration. In general, however, due to the uncertainties in the exact ground profile configuration, one has to use statistical approaches, and obtain the statistical description of $[R]$. The statistical study of the reflection matrix $[R]$ shall be treated in Chapter V.

4.4 Reflection Matrix for Plane Ground

(U) The approximate form of the angular spectra for the ground reflected wave and the reflection matrix discussed in the previous section is basically deduced from the geometric optics approximation. In order to investigate the accuracy of such approximations, the limiting cases of reflection of a plane wave by a plane ground are carried out using this approximate formulation for comparison with the known results.

(U) For the case of a plane ground,

$$\cos \alpha_1 = 1 \tag{4.54}$$

$$\cos \gamma = -\cos \alpha_o \tag{4.55}$$

hence

$$a = c = \sin^2 \alpha_o \tag{4.56}$$

$$b = 0 \quad . \tag{4.57}$$

Equation (4.53) then becomes

$$[R] = \frac{k^2}{(2\pi)^2} \frac{\cos \alpha \sin \alpha_o}{\sin \alpha} \int dx_s \int dy_s \, e^{ik\underline{r}_s \cdot (\hat{\Omega}_o - \hat{\Omega})} \begin{bmatrix} R_\perp & \\ & R_{//} \end{bmatrix} \tag{4.58}$$

For an infinite plane, one notes that

$$\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dx_s \int_{-\infty}^{\infty} dy_s e^{ik\underline{r}_s \cdot (\hat{\Omega}_o - \hat{\Omega})}$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dx_s \int_{-\infty}^{\infty} dy_s \, e^{ikx_s(\sin\alpha_o \cos\beta_o - \sin\alpha\cos\beta)}$$

$$e^{iky_s(\sin\alpha_o \sin\beta_o - \sin\alpha\sin\beta)} = \delta\left[k\sin\alpha_o\cos\beta_o - k_x\right] \delta\left[k\sin\alpha_o\sin\beta_o - k_y\right]$$

$$= \frac{1}{k^2 \sin\alpha\cos\alpha} \delta(\alpha - \pi + \alpha_o)\delta(\beta - \beta_o) \quad . \tag{4.59}$$

Thus

$$[R] = \frac{\sin\alpha_0}{\sin^2\alpha}\,\delta(\alpha-\pi+\alpha_0)\delta(\beta-\beta_0)\ . \tag{4.60}$$

This indicates that the reflected wave is indeed a plane wave given by

$$\underline{E}_{\text{refl.}}(\underline{r}) = R_\perp \hat{e}_{10} E_{10} e^{ik\hat{\Omega}_r\cdot\underline{r}} + R_{//}\hat{e}_{20}E_{20}e^{ik\hat{\Omega}_r\cdot\underline{r}} \tag{4.61}$$

where $\hat{\Omega}_r$ is given by the angles $(\pi-\alpha_0), \beta_0$ (which of course is the specular direction of reflection). Therefore it is seen that the present formulation reduces to the known form of speculdr reflection in the case of an infinite plane ground.

(U) In general, if a transmitter illuminates a part of ground defined by an area A_T, then the contribution of the transmitted radiation to the angular spectra of the reflected radiation within the geometric optics approximation for a plane ground can be expressed in terms of the reflection matrix

$$[R] = \frac{k^2}{(2\pi)^2}\,\frac{\cos\alpha\sin\alpha_0}{\sin\alpha}\begin{bmatrix} R_\perp & 0 \\ 0 & R_{//}\end{bmatrix}\int_{A_T} dx_s \int dy_s\, e^{ik\underline{r}_s\cdot(\hat{\Omega}_0-\hat{\Omega})}$$

$$\triangleq A_T\,\frac{k^2\cos\alpha\sin\alpha_0}{\sin\alpha}\,F(\hat{\Omega},\hat{\Omega}_0)\begin{bmatrix} R_\perp & 0 \\ 0 & R_{//}\end{bmatrix}. \tag{4.62}$$

For the special case that A_T is defined by the region

$$-\frac{a}{2} \le x_s \le \frac{a}{2}\ , \qquad -\frac{b}{2} \le y_s \le \frac{b}{2}\ ,$$

one finds that

$$F(\hat{\Omega},\hat{\Omega}_0) = \text{sinc}\left[\frac{ka}{2}(\sin\alpha_0\cos\beta_0 - \sin\alpha\cos\beta)\right]\text{sinc}\left[\frac{kb}{2}(\sin\alpha_0\cos\beta_0 - \sin\alpha\cos\beta)\right] \tag{4.63}$$

(U) In terms of this reflection matrix, the ground reflected wave is given by

$$\underline{E}(\underline{r})_{refl.} = \frac{k^2}{(2\pi)^2} A_T \sin\alpha_o \int_0^{\pi/2} d\alpha \cos\alpha \int_0^{2\pi} d\beta\, F(\hat{\Omega}, \hat{\Omega}_o) e^{ik\underline{r}\cdot\hat{\Omega}} \begin{bmatrix} \begin{bmatrix} R_\perp & 0 \\ 0 & R_{//} \end{bmatrix} & \begin{matrix} E_{10} \\ E_{20} \end{matrix} \end{bmatrix} \tag{4.64}$$

In the far zone approximation, if a receiver is far from the illuminated region, and whose coordinate is given by $\Omega_s r$, then one may use the asymptotic expression

$$e^{ik\underline{r}\hat{\Omega}} \cong \frac{2\pi i}{k} \delta(\hat{\Omega}+\hat{\Omega}_s) \frac{e^{-ikr}}{r} - \frac{2\pi i}{k} \delta(\hat{\Omega}-\hat{\Omega}_s) \frac{e^{ikr}}{r} \quad . \tag{4.65}$$

Thus, the scattered field observed at a point far from the reflection area is given by

$$\underline{E}(r)_{refl.} = -\frac{ik}{2\pi}, \frac{e^{ikr}}{r} A_T \frac{\sin\alpha_o \cos\alpha_s}{\sin\alpha_s} F(\hat{\Omega}_s, \hat{\Omega}_o)\left[\hat{e}_{1s} R_\perp E_{10} + \hat{e}_{2s} R_{//} E_{20}\right] \tag{4.66}$$

(U) From (4.66) it is easily seen that the scattering matrix for a plane ground, within geometric optics, is given by

$$\left[S\right] = \frac{-ik}{2\pi} A \frac{\sin\alpha_o \cos\alpha_o}{\sin\alpha_s} F(\hat{\Omega}_s, \hat{\Omega}_o) \begin{bmatrix} R_\perp & 0 \\ 0 & R_{//} \end{bmatrix} \tag{4.67}$$

In the case of a planar rectangle, this becomes

$$\left[S\right] = \frac{-ik}{2\pi} A \,\mathrm{sinc}\left[\frac{ka}{2}(\sin\alpha_o \cos\beta_o - \sin\alpha_s \cos\beta_s)\right] \mathrm{sinc}\left[\frac{kb}{2}(\sin\alpha_o \sin\beta_o - \sin\alpha_s \sin\beta_s)\right]$$

$$\cdot \begin{bmatrix} \frac{\cos\alpha_s \sin\alpha_o}{\sin\alpha_s} R_\perp & 0 \\ 0 & \frac{\cos\alpha_o \sin\alpha_o}{\sin\alpha_o} R_{//} \end{bmatrix} \tag{4.68}$$

(U) This equation does not agree exactly with the result of physical optics as given by Eq. (3. 67). The primary reason for the discrepancy is due to the fact that the surface current assumed in the physical optics approach is inconsistent with the result predicted by the geometric optics approach, using local tangent plane reflection even in the case of infinitely conducting flat planes, as pointed out by Fung (1966). Insofar that both approaches are approximate, it is difficult to argue precisely which is the more correct one. In the present work, therefore the geometric optics approach and the resulting ground reflection matrix given in the previous section shall be used throughout because of its relative simplicity.*

*It is to be noted that the scattering matrix given by (4. 68) contains singularities in the normal direction $\alpha_s=0$. For distributed radiation, where one has to carry the integration of angular spectra over all directions, this singularity does not cause any trouble in the integration. For the calculation of scattering matrix, this singularity, caused by the simple formulation of using z-components of the fields, may be avoided by an alternate formulation using the tangential components of the electric field. This alternate formulation is given in Appendix B.

V

REFLECTION FROM RANDOM ROUGH SURFACES

5.1 Statistical Averages

(U) In general, most surfaces such as terrain, sea surfaces, etc., are either too complicated to be characterized by a simple contour, or are time varying such that characterizing by a single contour is inadequate. For such surfaces, it is common practice to consider them as random rough surfaces specified by their stochastic properties. For a random rough surface, the contour of the surface may be represented by a random function in two dimensions, such as

$$z_s = \xi\,(x_s, y_s) \quad . \tag{5.1}$$

Since each element of the reflecting matrix is a function of ξ and its derivatives, it is therefore necessary to consider the reflecting matrix as a random quantity. In most practical cases, based on physical grounds, one may assume the function ξ to be second order stationary. In such case, a statistical investigation of the reflection matrix may be carried out by evaluating some of the statistical average quantities .

(U) To carry out formally some of the statistical averages, consider each element of the reflection matrix given by (4.53). These may be written as

$$R_{ij} = \frac{k^2}{(2\pi)^2}\,\frac{\cos\alpha}{\sin\alpha_o \sin\alpha}\int dx_s\, e^{ikx_s(\sin\alpha_o\cos\beta_o - \sin\alpha_1\cos\beta_1)} \int dy_s\, e^{iky_s(\sin\alpha_o\sin\beta_o - \sin\alpha_1\sin\beta_1)}\; C_{ij} \tag{5.2}$$

where

$$\left\{ i, j \right\} = 1, 2$$

and the coefficients

$$c_{11} = \frac{1}{\sin^2\gamma}\, e^{ik(\cos\alpha_o - \sin\alpha)\xi}\left[R_{\perp} ac + R_{/\!/} b^2\right] \tag{5.3}$$

$$c_{12} = \frac{1}{\sin^2\gamma} e^{ik(\cos\alpha_o - \sin\alpha)\xi} \left[R_\perp bc - R_{/\!/} ab\right] \tag{5.4}$$

$$c_{21} = \frac{1}{\sin^2\gamma} e^{ik(\cos\alpha_o - \sin\alpha)\xi} \left[R_\perp ab - R_{/\!/} bc\right] \tag{5.5}$$

$$c_{22} = \frac{1}{\sin^2\gamma} e^{ik(\cos\alpha_o - \sin\alpha)\xi} \left[R_\perp ab + R_{/\!/} b^2\right] \tag{5.6}$$

are functions of the random variable ξ . Explicitly, if one denotes the partial derivatives of ξ with respect to x_s by ξ_x and the partial derivative of ξ with respect to y_s by ξ_y, then the quantities

$$\hat{n}_1 = \left[1+\xi_x^2 + \xi_y^2\right]^{-1/2}\left[-\hat{x}\xi_x - \hat{y}\xi_y + \hat{z}\right] \tag{5.7}$$

$$\cos\alpha_1 = \left[1+\xi_x^2+\xi_y^2\right]^{-1/2} \tag{5.8}$$

$$\cos\alpha_1 \cos\beta_1 = -\left[\xi_x^2+\xi_y^2\right]^{-1/2} \xi_x \tag{5.9}$$

$$\sin\alpha_1 \sin\beta_1 = -\left[\xi_x^2+\xi_y^2\right]^{-1/2} \xi_y \tag{5.10}$$

contained in C_{ij} are all random variables. Functionally, therefore,

$$C_{ij} = C_{ij}\left[\xi(x_s, y_s), \xi_x(x_s, y_s)\xi_y(x_s, y_s)\right] . \tag{5.11}$$

(U) The mean reflected field and the correlation of the field components are therefore dependent on the expected values $\mathcal{E}\left[R_{ij}\right]$ and $\mathcal{E}\left[R_{ij}'\right]$. Here, we use the symbol $\mathcal{E}[x]$ to denote the expected value of the random variable x instead of the conventional E to avoid confusion with the electric field. Formally, one has

$$\mathcal{E}\left[R_{ij}\right] = \frac{k^2}{(2\pi)^2}\frac{\cos\alpha}{\sin\alpha_o \sin\alpha}\int e^{ik(\sin\alpha_o\cos\beta_o - \sin\alpha\cos\beta)x_s} dx_s \int e^{ik(\sin\alpha_o\cos\beta_o - \sin\alpha\sin\beta)y_s} dy_s\, \mathcal{E}\left[C_{ij}\right] \tag{5.12}$$

and

$$\mathcal{E}\left[R_{ij}, R_{ij}'\right] = \frac{k^4}{(2\pi)^4} \frac{\cos^2\alpha}{\sin^2\alpha_o \sin^2\alpha} \int dx_s \int dx_s' \, e^{ik(\sin\alpha_o\cos\beta_o - \sin\alpha(\cos\beta)(x_s - x_s')}$$

$$\int dy_s \int dy_s' \, e^{ik(\sin\alpha_o\sin\beta_o - \sin\alpha\sin\beta)(y_s - y_s')}$$

$$\mathcal{E}\left[C_{ij}(\xi, \xi_x, \xi_y) C_{ij}'(\xi', \xi_x', \xi_y')\right] \qquad (5.13)$$

where, for simplicity, the prime is used to indicate that the function ξ, etc., are evaluated at x_s', y_s'. For example,

$$\xi' \triangleq \xi\,(x_s', y_s') \quad .$$

(U) Equations (5.12) and (5.13) can be somewhat simplifed from the assumptions that ξ is a second order stationary random variable. In (5.12), since $\mathcal{E}\left[C_{ij}\right]$ is not a function of x_s and y_s (due to stationary) one may carry out the integrals involving x_s and y_s separated. Just as in the case of plain, smooth ground, one may carry out the integral over the area of illumination, and denote

$$\frac{k^2}{(2\pi)^2} \frac{\cos\alpha}{\sin\alpha_o \sin\alpha} \int\limits_{\substack{\text{area of} \\ \text{illumination}}} dx_s \int dy_s \left[e^{ik\left[\sin\alpha_o\cos\beta_o - \sin\alpha\cos\beta\right]}\right] \left[e^{ik\left[\sin\alpha_o\sin\beta_o - \sin\alpha\sin\beta\right]}\right]$$

$$\triangleq A\, F\,(\hat{\Omega}, \hat{\Omega}_o) \qquad . \qquad (5.14)$$

where A is the area of integration. Thus, one may express

$$\mathcal{E}\left[R_{ij}\right] = A\, F\,(\hat{\Omega}, \hat{\Omega}_o) \mathcal{E}\left[C_{ij}\right] \quad . \qquad (5.15)$$

(U) In (5.13) if one denotes that

$$x_s - x_s' = \tau_x \quad , \quad y_s - y_s' = \tau_y \qquad (5.15)$$

due to stationary assumption, functionally, one may denote

$$\mathcal{E}\left[C_{ij}(\xi,\xi_x,\xi_y)C_{ij}{}'(\xi',\xi_x',\xi_y')\right]=g_{ij,ij'}(\tau_x\tau_y) \quad . \tag{5.17}$$

Thus (5.13) may be reduced to

$$\mathcal{E}\left[R_{ij},R_{ij}{}'\right]=\frac{k^4}{(2\pi)^4}\frac{\cos^2\alpha}{\sin^2\alpha_o\sin^2\alpha}\int dx_s\int dy_s \int d\tau_x\, e^{ik(\sin\alpha_o\cos\beta_o-\sin\alpha\cos\beta)\tau_x} \int d\tau_y\, e^{ik(\sin\alpha_o\sin\beta_o-\sin\alpha\sin\beta)\tau_y}\, g_{ij,ij'}(\tau_x,\tau_y) \tag{5.18}$$

(U) The evaluation of the integrals involved in the average values of reflection matrices for any arbitrary stationary random surface is extremely complicated. Reduction of these equations for a relatively simple case of a slightly rough surface with gaussian statistics is described in the following section.

5.2 Slightly Rough Surfaces

(U) The expressions for the average values of the reflection matrix is not useful in practice unless some approximations are introduced so that one may have a scheme to evaluate these integrals numerically. Some approximations introduced in reducing these equations are given in the steps below.

(U) A. The surface is slightly rough, so that in all the expressions, it is only necessary to retain terms up to an including the second power of ξ_x and ξ_y. Thus, from

$$\hat{n}_1=\left[1+\xi_x^2+\xi_y^2\right]^{-1/2}\left[\hat{x}(-\xi_x)+\hat{y}(-\xi_y)+\hat{z}\right] \tag{5.19}$$

one may have approximately

$$\hat{n}_1\sim\hat{z}+\left[-\hat{x}\xi_x-\hat{y}\xi_y-\hat{z}\,\frac{1}{2}(\xi_x^2+\xi_y^2)\right] \tag{5.20}$$

Within the same order of approximation one has

$$\cos\gamma \cong -\cos\alpha_o - \sin\alpha_o \sin\beta_o \xi_x - \sin\alpha_o \sin\beta_o \xi_y + \frac{1}{2}\cos\alpha_o (\xi_x^2 + \xi_y^2) \qquad (5.21)$$

$$\sin^2\gamma \cong \sin^2\alpha_o - 2\cos\alpha_o \sin\alpha_o \cos\beta_o \xi_x - 2\cos\alpha_o \sin\alpha_o \sin\beta_o \xi_y$$
$$+ \left[\cos^2\alpha_o - \sin^2\alpha_o \cos^2\beta_o\right]\xi_x^2 + \left[\cos^2\alpha_o - \sin^2\alpha_o^2 \sin^2\beta_o\right]\xi_y^2$$
$$-2\sin^2\alpha_o \cos\beta_o \sin\beta_o \xi_x \xi_y \; . \qquad (5.22)$$

$$a \cong \sin^2\alpha_o - \cos\alpha_o \sin\alpha_o \cos\beta_o \xi_x - \cos\alpha_o \sin\alpha_o \sin\beta_o \xi_y - \frac{1}{2}\sin^2\alpha_o \left[\xi_x^2 + \xi_y^2\right] \qquad (5.23)$$

$$b \cong -\sin\alpha_o \sin\beta_o \xi_x + \sin\alpha_o \cos\beta_o \xi_y \qquad (5.24)$$

and

$$c \cong \sin^2\alpha_o - 3\cos\alpha_o \sin\alpha_o \xi_x - 3\cos\alpha_o \sin\alpha_o \sin\beta_o \xi_y$$
$$+ \left[\frac{5}{2}\cos^2\alpha_o - \frac{1}{2} - 2\sin^2\alpha_o \cos^2\beta_o\right]\xi_x^2 + \left[\frac{5}{2}\cos^2\alpha_o - \frac{1}{2} - 2\sin^2\alpha_o \sin^2\beta_o\right]\xi_y^2$$
$$-4\sin^2\alpha_o \sin\beta_o \cos\beta_o \xi_x \xi_y \; . \qquad (5.25)$$

(U) B. The index of refraction of the surface is high, i.e. $|N| \gg 1$. Then one may have

$$R_\perp \cong R\,(\alpha_o) + \frac{2}{N}\left[\cos\alpha_o \cos\beta_o \xi_x + \sin\alpha_o \sin\beta_o \xi_y - \frac{\cos\alpha_o}{2}(\xi_x^2 + \xi_y^2)\right] \qquad (5.26)$$

$$R_{//} \cong R\,(\alpha_o) + \frac{2}{N\cos\alpha_o}\left[\sin\alpha_o \cos\beta_o \xi_x + \sin\alpha_o \sin\beta_o \xi_y - \frac{\cos\alpha_o}{2}(\xi_x^2 + \xi_y^2)\right] \qquad (5.27)$$

where $R_\perp(\alpha_o)$ and $R_{//}(\alpha_o)$ are respectively the plane wave reflection coefficient for the incident wave if the reflecting surface is smooth, i.e. $\xi = 0$.

(U) With these approximations, one has,

$$
\begin{aligned}
c_{11} \cong e^{ik\left[\cos\alpha_o - \cos\alpha\right]\xi} \Bigg\{ & R_\perp(\alpha_o)\sin^2\alpha_o + \left[2\sin\alpha_o\cos\alpha_o - \frac{2}{N}\sin\alpha_o\left(\sin^2\alpha_o - 2\cos^2\alpha_o\right)\right]\cos\beta_o\,\xi_x \\
& + \left[2\sin\alpha_o\cos\alpha_o + \frac{2\sin\alpha_o}{N}\left(\sin^2\alpha_o - 2\cos^2\alpha_o\right)\right]\sin\beta_o\,\xi_y + \frac{4\sin\beta_o\cos\beta_o}{N\cos\alpha_o}\left[\cos^2\alpha_o\left(2\sin^2\alpha_o + 1\right) - 1\right]\xi_x\xi_y \\
& + \left[2\sin^2\alpha_o + \frac{2\cos\alpha_o}{N}\left[\frac{\sin^2\alpha_o}{2} - 2\cos^2\alpha_o + 1 + \left(2\sin^2\alpha_o + 1\right)\cos^2\beta_o + \frac{\sin^2\beta_o}{\cos^2\alpha_o}\right]\right]\xi_x^2 \\
& + \left[2\sin^2\alpha_o + \frac{2\cos\alpha_o}{N}\left[\frac{\sin^2\alpha_o}{2} - 2\cos^2\alpha_o + 1 \pm \left(2\sin^2\alpha_o + 1\right)\sin^2\beta_o + \frac{\cos^2\beta_o}{\cos^2\alpha_o}\right]\right]\xi_y^2 \Bigg\}
\end{aligned}
\tag{5.28}
$$

$$
c_{21} \cong e^{+ik\left[\cos\alpha_o - \cos\alpha\right]\xi} \Bigg\{ \left[2\sin\alpha_o + \frac{2}{N}\tan\alpha_o\left(1+\cos^2\alpha_o\right)\right]\sin\beta_o\xi_x - \left[+2\sin\alpha_o + \frac{2}{N}\tan\alpha_o\left(1+\cos^2\alpha_o\right)\right]\cos\beta_o\xi_y
$$

$$
+\frac{2}{N}\tan^2\alpha_o\left(\cos^2\beta_o - \sin^2\beta_o\right)\xi_x\xi_y - \frac{2}{N}\tan^2\alpha_o\sin\beta_o\cos\beta_o\,\xi_x^2
$$

$$
+\frac{2}{N}\tan^2\alpha_o\sin\beta_o\cos\beta_o\,\xi_y^2 \Bigg\} \tag{5.29}
$$

$$
\begin{aligned}
c_{12} \cong e^{ik\left[\cos\alpha_o - \cos\alpha\right]\xi} \Bigg\{ & \left[2\sin\alpha_o + \frac{2\sin\alpha_o\cos\alpha_o}{N\cos^2\alpha_o}(\cos^2\alpha_o + 1)\right] \sin\beta_o \xi_x \\
& - \left[2\sin\alpha_o + \frac{2\sin\alpha_o\cos\alpha_o(\cos^2\alpha_o + 1)}{N\cos^2\alpha_o}\right]\cos\beta_o\xi_y + \left[\frac{2\sin^2\alpha_o}{N\cos^2\alpha_o}(\sin^2\alpha_o - \cos^2\alpha_o)(\cos^2\beta_o - \sin^2\beta_o)\right]\xi_x\xi_y \\
& - \frac{2}{N}\tan^2\alpha_o(\sin^2\alpha_o - \cos^2\alpha_o)\sin\beta_o\cos\beta_o\,\xi_x^2 + \frac{2}{N}\tan^2\alpha_o(\sin^2\alpha_o - \cos^2\alpha_o)\sin\beta_o\cos\beta_o\,\xi_y^2 \Bigg\}
\end{aligned}
\tag{5.30}
$$

$$
\begin{aligned}
c_{22} \cong e^{ik\left[\cos\alpha_o - \cos\alpha\right]\xi} \Bigg\{ & R_{/\!/}(\alpha_o)\sin^2\alpha_o - \left[2\sin\alpha_o\cos\alpha_o + \frac{2\sin\alpha_o}{N\cos^2\alpha_o}\left(2\cos^2\alpha_o + \sin^2\alpha_o\right)\right]\cos\beta_o\,\xi_x \\
& - \left[2\sin\alpha_o\cos\alpha_o + \frac{2\sin\alpha_o}{N\cos^2\alpha_o}\left(2\cos^2\alpha_o + \sin^2\alpha_o\right)\right]\sin\beta_o\,\xi_y - \frac{4\sin\beta_o\cos\beta_o\sin^2\alpha_o}{N\cos\alpha_o}\,\xi_x\xi_y \\
& - \left[2\sin^2\alpha_o - \frac{2}{N\cos^2\alpha_o}\left[\cos\alpha_o\left(\cos^2\alpha_o - \frac{\sin^2\alpha_o}{2}\right) + \cos\alpha_o\left(2\sin^2\alpha_o - 1\right)\cos^2\beta_o - \cos^3\alpha_o\sin^2\beta_o\right]\right]\xi_x^2 \\
& - \left[2\sin^2\alpha_o + \frac{2}{N\cos^2\alpha_o}\left[\cos\alpha_o\left(\cos^2\alpha_o - \frac{\sin^2\alpha_o}{2}\right) + \cos\alpha_o\left(2\sin^2\alpha_o - 1\right)\sin^2\beta_o - \cos^3\alpha_o\cos^2\beta_o\right]\right]\xi_y^2 \Bigg\}
\end{aligned}
\tag{5.31}
$$

To simplify notation, one may write

$$c_{11} \cong \exp\left[ik(\cos\alpha_o - \cos\alpha)\right]\left[R_{\perp}(\alpha_o)\sin^2\alpha_o + d_1\cos\beta_o\xi_x + d_1\sin\beta_o\xi_y + d_2\xi_x\xi_y + d_3\xi_x^2 + d_4\xi_y^2\right] \tag{5.32}$$

$$c_{12} \cong \exp\left[ik(\cos\alpha_o - \cos\alpha)\right]\left[c_1\sin\beta_o\xi_x - c_1\cos\beta_o\xi_y + c_2\xi_x\xi_y + c_3\xi_x^2 + c_4\xi_y^2\right] \tag{5.33}$$

$$c_{21} \cong \exp\left[ik(\cos\alpha_o \cos\alpha)\right]\left[b_1\sin\beta_o\xi_x - b_1\cos\beta_o\xi_y + b_2\xi_x\xi_y + b_3\xi_x^2 + b_4\xi_y^2\right] \tag{5.34}$$

$$c_{22} \cong \exp\left[ik(\cos\alpha_o + \cos\alpha)\right]\left[R_{//}(\alpha_o)\sin^2\alpha_o + a_1\cos\beta_o\xi_x + a_1\sin\beta_o\xi_y + a_2\xi_x\xi_y + a_3\xi_x^2 + a_4\xi_y^2\right] \tag{5.35}$$

to show the explicit dependence of C_{ij} on the random variable ξ. The coefficients a_i, b_j, etc., of course can easily be identified from (5.9) through (5.12).

(U) C. In order to carry out the expected values of C_{ij} and their products it is necessary to use the joint probability distribution densities of ξ, ξ', ξ_x, etc. In order that all these high order joint probability distribution functions can be written down with relative ease, one shall assume that the surface is isotropic and obeys gaussian statistics. For such surfaces, the statistical description can be expressed in terms of one correlation function defined by

$$B(\tau) = \mathcal{E}\left[\xi(s+\tau)\xi(s)\right] \tag{5.36}$$

where s, $s+\tau$ are distances measured along any straight line. For such surfaces the statistical average for the variables involving ξ, ξ_x, ξ_y and their products can, in principle, be carried out explicitly. In Appendix A explicit formulas necessary for evaluating the expected values of C_{ij} and their products are deduced.

(U) Using these approximations and the formulas in Appendix A, the statistical averages of the reflection matrix can be written down explicitly.

(U) For the expectation value of the reflection matrix, one has

$$\mathcal{E}\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = AF(\hat{\Omega}, \hat{\Omega}_o) \cdot \exp\left[-\frac{k^2(\cos\alpha_o - \cos\alpha)^2 B(0)}{2}\right]$$

$$\begin{bmatrix} R_\perp(\alpha_o) - \frac{B''(0)}{\sin^2\alpha_o}(d_3+d_4) & -\frac{B''(0)}{\sin^2\alpha_o}(c_3+c_4) \\ -\frac{B''(0)}{\sin^2\alpha_o}(b_3+b_4) & R_{/\!/}(\alpha_o) - \frac{B''(0)}{\sin^2\alpha_o}(a_3+a_4) \end{bmatrix} \tag{5.37}$$

(U) For the products involving $C_{ij}\, C_{ij}'$, if one represents

$$y_s - y_s' = \tau \sin\theta\tau \ , \tag{5.38}$$

then their expectation values may be expressed in terms of τ and $\theta\tau$. Some explicit results are given in Eqs. (5.39) to (5.42) inclusive. Due to the systematic representations of C_{ij} given by (5.32) to (5.35) it is easy to write down the expected value for the products between any other two C_{ij} 's. For example, $\mathcal{E}\left[c_{11}c_{11}^*\right]$ can be obtained from $\mathcal{E}\left[c_{22}c_{22}^*\right]$ by replacing $R_{/\!/}(\alpha_o)$ and a_i with $R_\perp(\alpha_o)$ and d_i.

$$
\begin{aligned}
\mathcal{E}\left[c_{22}c'^{\,*}_{221}\right] = \exp\left\{-k^2(\cos\alpha_o + \cos\alpha)^2\left[B(0) - B(\tau)\right]\right\} \; X \\
\left\{\sin^2\alpha_o\left\{R_{||}(\alpha_o)\,R^*_{||}(\alpha_o)\sin^2\alpha_o - B''(0)\left[R_{||}(\alpha_o)(a_3^* + a_4^*) + R^*_{||}(\alpha_o)(a_3 + a_4)\right]\right\}\right. \\
+ ik(\cos\alpha_o - \cos\alpha)B'(\tau)\sin^2\alpha_o\cos\beta_o\cos\theta_\tau\left[R_{||}(\alpha_o)\,a_1^* + R^*_{||}(\alpha_o)\,a_1\right] \\
+ ik(\cos\alpha_o + \cos\alpha)B'(\tau)\sin^2\alpha_o\sin\beta_o\sin\theta_\tau\left[R_{||}(\alpha_o)\,a_1^* + R^*_{||}(\alpha_o)\,a_1\right] \\
+ \sin\theta_\tau\cos\theta_\tau\left\{k^2(\cos\alpha_o - \cos\alpha)^2\,\overline{B'(\tau)}^2\left[R_{||}(\alpha_o)\sin^2\alpha_o\,a_2^* + R^*_{||}(\alpha_o)\sin^2\alpha_o\,a_2 + 2a_1a_1^*\sin\beta_o\cos\beta_o\right]\right. \\
\left. + 2a_1a_1^*\sin\beta_o\cos\beta_o\left[\frac{B'(\tau)}{\tau} - B''(\tau)\right]\right\} \\
+ \cos^2\theta_\tau\left\{k^2(\cos\alpha_o - \cos\alpha)^2\,\overline{B'(\tau)}^2\left[R_{||}(\alpha_o)\sin^2\alpha_o\,a_3^* + R^*_{||}(\alpha_o)\sin^2\alpha_o\,a_3 + a_1a_1^*\cos^2\beta_o\right]\right. \\
\left. - a_1a_1^*\cos^2\beta_o\,B''(\tau) - a_1a_1^*\sin^2\beta_o\,\frac{B''(\tau)}{\tau}\right\} \\
+ \sin^2\theta_\tau\left\{k^2(\cos\alpha_o + \cos\alpha)^2\,\overline{B'(\tau)}^2\left[R_{||}(\alpha_o)\sin^2\alpha_o\,a_4^* + R^*_{||}(\alpha_o)\sin^2\alpha_o\,a_4 + a_1a_1^*\sin^2\beta_o\right]\right. \\
\left.\left. - a_1a_1^*\sin^2\beta_o\,B''(\tau) - a_1a_1^*\cos^2\beta_o\,\frac{B'(\tau)}{\tau}\right\}\right\} \qquad (5.39)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E}\left[c_{22}c_{21}'^{*}\right] = \exp\Big\{-k^2(\cos\alpha_o - \cos\alpha)^2\left[B(0) - B(\tau)\right]\Big\} \quad X \\
\Big\{ - B''(0)\, R_{||}(\alpha_o)\sin^2\alpha_o\,(b_3^* + b_4^*) \\
+ ik(\cos\alpha_o - \cos\alpha)\, B'(\tau)\sin^2\alpha_o\cos\beta_o\cos\theta_\tau\, R_{||}(\alpha_o)\, b_1^* \\
+ ik(\cos\alpha_o - \cos\alpha)\, B'(\tau)\sin^2\alpha_o\sin\beta_o\sin\theta_\tau\, R_{||}(\alpha_o)\, b_1^* \\
+ \sin\theta_\tau\cos\theta_\tau\Big\{k^2(\cos\alpha_o - \cos\alpha)^2\,\overline{B'(\tau)}\left[R_{||}(\alpha_o)\sin^2\alpha_o b_2^* + 2a_1 b_1^*\cos\beta_o\sin\beta_o\right] \\
+ 2a_1 b_1\sin\beta_o\cos\beta_o\left[\frac{B'(\tau)}{\tau} - B''(\tau)\right] \\
+ \cos^2\theta_\tau\Big\{k^2(\cos\alpha_o - \cos\alpha)^2\,\overline{B'(\tau)}^{\,2}\left[R_{||}(\alpha_o)\sin^2\alpha_o b_3^* + a_1 b_1^*\cos^2\beta_o\right] \\
- a_1 b_1^*\cos^2\beta_o B''(\tau) - b_1 b_1^*\sin^2\beta_o\frac{B'(\tau)}{\tau}\Big\} \\
+ \sin^2\theta_\tau\Big\{k^2(\cos\alpha_o - \cos\alpha)^2\,\overline{B'(\tau)}^{\,2}\left[R_{||}(\alpha_o)\sin^2\alpha_o b_4^* + a_1 b_1^*\sin^2\beta_o\right] \\
- a_1 b_1^*\sin^2\beta_o B''(\tau) - a_1 b_1^*\cos^2\beta_o\frac{B'(\tau)}{\tau}\Big\}\Big\}
\end{aligned}
\qquad (5.40)
$$

$$
\begin{aligned}
\mathcal{E}\left[c_{22}c_{11}'^{*}\right] = \exp\left\{-k^2(\cos\alpha_o - \cos\alpha)^2\left[B(0)-B(\tau)\right]\right\} \times \\
\Bigg\{\sin^2\alpha_o\Bigg\{R_{||}(\alpha_o)R_{\perp}^{*}(\alpha_o)\sin^2\alpha_o - B''(0)\left[R_{||}(\alpha_o)(d_3^{*}+d_4^{*}) + R_{\perp}^{*}(\alpha_o)(a_3+a_4)\right]\Bigg\} \\
+ ik(\cos\alpha_o - \cos\alpha)B'(\tau)\sin^2\alpha_o\cos\beta_o\cos\theta_\tau\left[R_{||}(\alpha_o)d_1^{*} + R_{\perp}^{*}(\alpha_o)a_1\right]\Bigg\} \\
+ ik(\cos\alpha_o - \cos\alpha)B'(\tau)\sin^2\alpha_o\sin\beta_o\sin\theta_\tau\left[R_{||}(\alpha_o)d_1^{*} + R_{\perp}^{*}(\alpha_o)a_1\right]\Bigg\} \\
+ \sin\theta_\tau\cos\theta_\tau\Bigg\{k^2(\cos\alpha_o - \cos\alpha)^2\,\overline{B'(\tau)}^2\,R_{||}(\alpha_o)\sin^2\alpha_o d_2^{*} + R_{\perp}^{*}(\alpha_o)\sin^2\alpha_o a_2 \\
+ 2a_1d_1^{*}\sin\beta_o\cos\beta_o + 2a_1d_1^{*}\sin\beta_o\cos\beta_o\left[\frac{B'(\tau)}{\tau} - B''(\tau)\right]\Bigg\} \\
+ \cos^2\theta_\tau\Bigg\{k^2(\cos\alpha_o - \cos\alpha)^2\,\overline{B'(\tau)}^2\Big[R_{||}(\alpha_o)\sin^2\alpha_o d_3^{*} + R_{\perp}^{*}(\alpha_o)\sin^2\alpha_o a_3 + a_1d_1^{*}\cos^2\beta_o \\
- a_1d_1^{*}\cos^2\beta_o B''(\tau) - a_1d_1^{*}\sin^2\beta_o\frac{B'(\tau)}{\tau}\Bigg\} \\
+ \sin^2\theta_\tau\Bigg\{k^2(\cos\alpha_o - \cos\alpha)^2\,\overline{B'(\tau)}^2\Big[R_{||}(\alpha_o)\sin^2\alpha_o d_4^{*} + R_{\perp}^{*}(\alpha_o)\sin^2\alpha_o a_4 + a_1d_1^{*}\sin^2\beta_o \\
- a_1d_1^{*}\sin^2\beta_o B''(\tau) - a_1d_1^{*}\cos^2\beta_o\frac{B'(\tau)}{\tau}\Bigg\}\Bigg\}
\end{aligned}
\qquad (5.41)
$$

$$
\begin{aligned}
\mathcal{E}\left[c_{21}c'^{*}_{12}\right] \cong \exp\left\{-k^2(\cos\alpha_o - \cos\alpha)^2\left[B(0) - B(\tau)\right]\right\} X \\
\left\{2b_1c_1^{*}\sin\beta_o\cos\beta_o\left[\frac{B'(\tau)}{\tau} - B''(\tau)\right]\sin\theta_\tau\cos\theta_\tau\right. \\
+ b_1c_1^{*}\cos^2\theta_\tau\left\{k^2(\cos\alpha_o - \cos\alpha)^2\,\overline{B'(\tau)}^2\cos^2\beta_o - \cos^2\beta_o B''(\tau) - \sin^2\beta_o\frac{B'(\tau)}{\tau}\right\} \\
\left. + b_1c_1^{*}\sin^2\theta_\tau\left\{k^2(\cos\alpha_o - \cos\alpha)^2\,\overline{B(\tau)}^2\sin^2\beta_o - \sin^2\beta_o B''(\tau) - \cos^2\beta_o\frac{B'(\tau)}{\tau}\right\}\right\}
\end{aligned}
\tag{5.42}
$$

(U) In general, to carry out the integrals involved in (5.13) one may represent the function

$$g_{ij,ij'} \stackrel{\Delta}{=} \mathcal{E}\left[C_{ij}(\xi,\xi_x,\xi_y)C_{ij'}(\xi',\xi_x',\xi_y')\right]$$

by the form

$$g_{ij,ij'}(\tau_x,\tau_y) = \exp\left\{-k^2(\cos\alpha_0-\cos\alpha)^2\left[B(0)-B(\tau)\right]\right\} \times$$
$$\left[A_0(\tau)+A_1(\tau)\cos\theta_\tau+A_2(\tau)\sin\theta_\tau+A_3(\tau)\sin\theta_\tau\cos\theta_\tau\right.$$
$$\left.+A_4(\tau)\cos^2\theta_\tau+A_5(\tau)\sin^2\theta_\tau\right] \qquad (5.43)$$

where for each set of ij, ij', the functions $A_i(\tau)$ can be obtained from (5.39) - (5.42). Using this representation the correlation between the elements of the reflection matrix may be represented in the integral form

$$\mathcal{E}\left[R_{jk}R_{jk}'\right] = \frac{k^4}{(2\pi)^3}\frac{\cos^2\alpha}{\sin^2\alpha_0\sin^2\alpha} A\int_0^\infty \tau d\tau$$
$$\left\{J_0(\tilde{k}\tau)\left[2A_0(\tau)+A_4(\tau)+A_5(\tau)\right] -2iJ_1(\tilde{k}\tau)A_1(\tau)\cos\theta_0\right.$$
$$-2iJ_1(\tilde{k}\tau)A_2(\tau)\sin\theta_0-J_2(\tilde{k}\tau)A_3(\tau)\sin^2\theta_0$$
$$\left.-J_2(\tilde{k}\tau)\left[A_4(\tau)+A_5(\tau)\right]\right\}\exp\left\{-k^2(\cos\alpha_0-\cos\alpha)^2\left[B(0)-B(\tau)\right]\right\}$$
$$(5.44)$$

where

$$\tilde{k} = k\sqrt{\sin^2\alpha_0+\sin^2\alpha-2\mathrm{dina}\alpha\sin\alpha_0\cos(\beta-\beta_0)}$$

$$\cos\theta_0 = \frac{\sin\alpha\cos\beta-\sin\alpha_0\cos\beta_0}{\sqrt{\sin^2\alpha_0+\sin^2\alpha-2\sin\alpha\sin\alpha_0\cos(\beta-\beta_0)}}$$

and J_n's are Bessel functions.

(U) Thus, within the approximation of slightly rough, high index of refraction, and guassian statistical description of the surface, the statistics of the field reflected from the surface may be obtained by carrying out a set of integrations. If a suitable model for $B(\tau)$ is chosen, such integrations can be carried out at least by numerical methods.

5.3 The Shadow Effect

(U) The statistical description of the reflection matrix given in the previous section neglects the effect of shadowing, i.e. a part of the ground that is being shadowed by other parts. These parts therefore have negligible contribution to the reflected radiation. To the first order approximate solution of the shadow effect, Beckmann (1965) introduced the notion of shadow fraction, i.e. the average fraction of the nominally illuminated part of the ground that is actually being illuminated by the incident wave. In terms of this shadow fraction, the reflection matrix such as derived in the previous section can then be corrected appropriately. For a more sophisticated treatment, the statistical averages given by Section 5.2 should be carried only over the illuminated part of the ground. Hence a study of the statistics of the illuminated part of the ground becomes necessary.

(U) Due to the complicated nature of the statistical problem involved in the shadow problem, even the estimation of the shadow fraction becomes controversal. Beckmann (1965) derived an expression for the estimation of the shadow fraction. Due to some oversight in his statistical analysis, his result does not seem to be correct. Brockelman and Hagfors (1966) computed the shadow fraction by computer simulation and found that their results do not check with Beckmann's formulation. In this section, a general study of the problem of shadowing is carried out, but due to lack of time only some preliminary results are reported.

(U) In the last stages of this contract, the report by Wagner (1966) was made available to us. In that report some preliminary results on the theoretical investigation of the shadow effect are given. Our approach seems to be different from that work, but it is interesting to note that both approaches yield the same first order correction to the shadow factor.

(U) A study of the shadow effect can be carried out by generalizing the classical approach of investigating the zero crossing of random variables (Rice, 1954). Referring to Fig. 5-1, consider the cross section of a random surface given by a ξ-τ

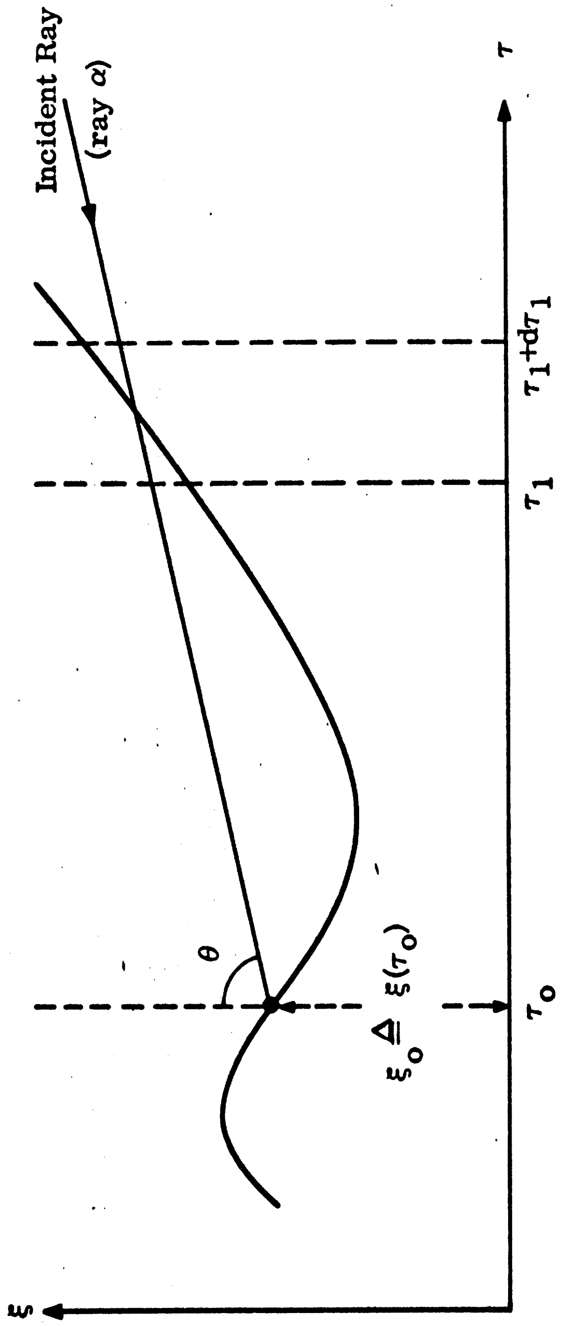

FIG. 5-1: GEOMETRY FOR THE SHADOW EFFECT

curve. If a ray inclined at an angle θ to the vertical (ξ-axis) reaches the point (τ_0, ξ_0) i.e. the point (τ_0, ξ_0) is not shadowed, this ray must not intersect the ξ-τ curve at finite τ. For simplicity, this ray shall be hereafter referred to as the ray α. Mathematically, for a ray not to be shadowed in the range $(\tau_1, \tau_1+d\tau_1)$

$$\xi(\tau_1) \triangleq \xi_1 < \xi_0+(\tau_1-\tau_0)\cot\theta \tag{5.45}$$

This condition may be written as

$$\xi_1-\tau_1\alpha < \xi_0-\tau_0\alpha \tag{5.46}$$

where for simplicity, we denote $\alpha \triangleq \cot\alpha$. If one introduces a new variable

$$y \triangleq \xi(\tau)-\tau\alpha \tag{5.47}$$

then (5.46) may be reduced to

$$y_1 < y_0 \ . \tag{5.48}$$

In terms of the random variable y, the fundamental problem of shadowing therefore may be reduced to: given $y=y_0$ at $\tau=\tau_0=0$, find the probability that y in the range of τ given by $(\tau_1, \tau_1+d\tau_1)$ crosses the level y_0. For the crossing to occur, one must have for small $d\tau_1$,

$$y_1=y_0 \tag{5.49}$$

$$y_1' = \left[\frac{dy}{d\tau}\bigg|_{\tau=\tau_1}\right] > 0 \tag{5.50}$$

and

$$dy_1=y_1'\,d\tau_1 \ . \tag{5.51}$$

If one denotes the conditional probability that given $y=y_0$ at $\tau=\tau_0=0$,

$$y_1 \ll y(\tau_1) < y_1+dy_1$$

and

$$y_1' < \frac{dy}{d\tau}\bigg|_{\tau=\tau_1} < y_1' + dy_1$$

by

$$\bar{f}(y_1, y_1')dy_1dy_1'$$

then the conditional probability that given $y=y_o$ (or $\xi=\xi_o$) at $\tau=\tau_o=0$, the ray α would cross the level ξ_o in the range $\tau_1<\tau<\tau_1+d\tau_1$ is given by

$$g_1(\tau_1, \alpha/\xi_o\xi_o')d\tau_1 = \int_0^{\infty} \bar{f}(\xi_o, y_1')y_1'dy_1'd\tau_1 \tag{5.52}$$

which of course is the familiar result of the crossing problem.

(U) If ξ is a second order normally distributed random function, then joint probability density distributions such as $f_o(\xi_o, \xi_o')$, $f_1(\xi_o, \xi_o', \xi_1, \xi_1'\ \tau)$ .... $f_n(\xi_o, \xi_o', \xi_1, \xi_1', \ldots\ldots \xi_n, \xi_n', \tau_1, \tau_2 \ldots \tau_n)$, etc for

$$\xi_n < \xi(\tau_n) < \xi_n + d\xi_n, \quad \xi_n' < \xi'(\tau_n) < \xi_n' + d\xi_n',$$

$$\tau_1 > \tau_2 > \tau_3 \ldots > \tau_n$$

can be formally written down. Then the conditional probability density that given ξ_o, ξ_o' at $\tau=\tau_o$,

$$\xi_1 + d\xi_1 > \xi(\tau_1) > \xi_1, \qquad \xi_1' + d\xi_1' > \xi'(\tau_1) > \xi_1'$$

is

$$\bar{f}_1(\xi_o, \xi_o', \xi_1, \xi_1', \tau_1) = f_1/f_o \tag{5.53}$$

since by definition, $y_1 = \xi_1 - \tau_1\alpha$, $y_1' = \xi_1' - \alpha$, it is easily seen from (5.52) that given ξ_o, ξ_o' the conditional probability that the ray α is shadowed in the range $\tau_1 < \tau < \tau_1 + d\tau_1$ is

$$g_1(\tau_1, \alpha, /\xi_o, \xi_o')d\tau_1$$

$$\triangleq d\tau_1 \int_0^{\infty} \bar{f}_1\left[\xi_o, \xi_o'(\xi_o + \tau_1\alpha), (y' + \alpha), \tau_1\right] y_1'dy_1' \tag{5.54}$$

(U) Similarly, given ξ_o, ξ_o' the conditional probability density that $\xi(\tau_1)$ is in the range $(\xi_1, \xi_1 + d\tau_1)$, $\xi(\tau_2)$ is in the range $(\xi_2, \xi_2 + d\tau_2)$...., etc., is

$$\bar{f}_n(\xi_o, \xi_o', \xi_1, \xi_1', \ldots \xi_n, \xi_n', \tau_1, \tau_2, \ldots \tau_n) = f_n/f_o \tag{5.55}$$

Following arguments similar to those used in deriving (5.55), it is easily seen that given ξ_o, ξ_o', the event that the ray α would be multiply shadowed at $(\tau_1, \tau_1+d\tau_1), (\tau_2+d\tau_2)\ldots.$ $(\tau_n+d\tau_n)$ has a probability

$$g_n(\tau_1, \tau_2 \ldots \tau_n, \alpha/\xi_o, \xi_o')d\tau_1 d\tau_2 \ldots d\tau_n$$

where

$$g_n(\tau_1, \tau_2, \ldots \tau_n, \alpha/\xi_o, \xi_o') = \int_0^\infty y_1' dy_1' \int_0^\infty y_2' dy_2' \ldots \int_0^\infty y_n' dy_n'$$

$$\bar{f}_n\left[\xi_o, \xi_o'(\xi_o+\alpha\tau_1), (y_1'+\alpha) \ldots. (\xi_o+d\tau_n), (y_n'+\alpha_n), \tau_1, \tau_2, \ldots \tau_n\right] \quad . \tag{5.56}$$

(U) Using (5.55) and (5.56) and following Lonquet-Higgens (1962), it is then obvious that given ξ_o, ξ_o' the event that a ray from ξ_o is being shadowed <u>the first time</u> in the range $(\tau_1, \tau_1+d\tau_1)$ has a probability given by the following infinite series:

$$
\begin{aligned}
g(\tau_1, \alpha/\xi_o, \xi_o')d\tau_1 &= g_1(\tau_1, \alpha/\xi_o, \xi_o')d\tau_1 \\
&- \int_0^{\tau_1} d\tau_2 g_2(\tau_1, \tau_2, \alpha/\xi_o, \xi_o')d\tau_1 \\
&+ \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 g_3(\tau_1, \tau_2, \tau_3 \alpha/\xi_o, \xi_o')d\tau_1 \\
&+ \ldots\ldots\ldots
\end{aligned} \tag{5.57}
$$

(U) In terms of g, one may deduce that given ξ_o, ξ_o' the conditional probability that the ray α is being shadowed at all is

$$P_s(\text{shadow}/\xi_o, \xi_o') = \int_0^\infty g(\tau_1, \alpha/\xi_o, \xi_o')d\tau_1 \quad . \tag{5.58}$$

It follows therefore, that given ξ_o, ξ_o' the ray α illuminates the point ξ_o is

$$P_i(\text{illuminated}/\xi_o, \xi_o') = 1 - P_s(\text{shadow}/\xi_o, \xi_o') \quad . \tag{5.59}$$

The probability that any point on the surface is being illuminated by the ray α, therefore, is given by

$$s(\alpha) = \int_{-\infty}^{\infty} d\xi_o \int_{-\alpha}^{\infty} d\xi_o' \, P_i(\text{illuminated}/\xi_o, \xi_o') f(\xi_o, \xi_o') \tag{5.60}$$

which should be the correct form of the 'shadow fraction'.

(U) Equation (5.59) also yields some statistics of the illuminated region. Since $P_i(\text{illuminated}/\xi_o, \xi_o') f_o(\xi_o, \xi_o')$ is the joint distribution density of height (ξ_o), slope (ξ_o') and the event being illuminated by ray α the height and slope distribution for the illuminated region is evidently

$$f_{o\,(\text{illuminated})}(\xi_o, \xi_o'/\text{ill.}) = \frac{P_i(\text{illu.}/\xi_o \xi_o') f_o(\xi_o, \xi_o')}{s(\alpha)} \tag{5.61}$$

(U) Owing to the complicated multiple integrals and series involved in the expression (5.57), the rigorous, formal solution of the shadowing problem, is difficult to apply. In order to obtain some acceptable numerical results, approximations simplifying the numerical process seem to be necessary. The simplest types of approximation involve the use of approximate joint distribution functions f_n or $\bar{f}_n$. For a first approximation,

$$f_n \cong f_o(\xi_o \xi_o') f_o(\xi_1, \xi_1') \ldots f_n(\xi_n, \xi_n') \tag{5.62}$$

which neglects the correlation between height and slope distribution at different values of τ_o. For a second approximation, one may take into account the correlation between two adjacent values of τ, so that

$$\bar{f}_n \cong \frac{f_1(\xi_o, \xi_o', \xi_1, \xi_1') f_1(\xi_1, \xi_1', \xi_2, \xi_2') \ldots f_n(\xi_{n-1}, \xi_{n-1}', \xi_n, \xi_n')}{f_o(\xi_o, \xi_o') f_o(\xi_1, \xi_1') f_o(\xi_2, \xi_2') \ldots f_o(\xi_{n-1}, \xi_{n-1}')} \tag{5.63}$$

Such processes may be continued indefinitely. The advantage of such an approximation lies in the fact that the distribution densities are now approximated by a product involving factors with less variables, and thereby simplifying the integrations involved.

(U) As an example for the first approximation, one notes that

$$\bar{f}_n \cong \prod_{k=1}^{n} f_o(\xi_k, \xi_k')$$

$$= \prod_{k=1}^{n} \frac{1}{\sqrt{2\pi\, B(0) |B''(0)|}} \exp\left[-\frac{\xi_k^2}{2B(0)} - \frac{\xi_k^{2'}}{2|B''(0)|} \right] \tag{5.64}$$

$$g_n = \frac{1}{(2\pi)^n \left[B(0) |B''(0)| \right]^{n/2}}$$

$$\prod_{k=1}^{n} e^{-(\xi_o + \tau_k)^2 / 2|B(0)|} \int_0^\infty dy_k' y_k' \, e^{-(y_k' + \alpha)^2 / 2 |B''(0)|} \tag{5.65}$$

It is easy to verify that

$$\int_0^\infty dy_k' \, y_k' \exp\left[- \frac{(y_k' + \alpha)^2}{2 |B''(0)|} \right]$$

$$= |B''(0)| \exp\left[- \frac{\alpha^2}{2\, B''(0)} \right] - \sqrt{|B''(0)|} \sqrt{\frac{\pi}{2}} \, \alpha \, \mathrm{erfc} \left[\frac{\alpha}{\sqrt{2 |B''(0)|}} \right] \tag{5.66}$$

To simplify notations, let

$$V \triangleq \frac{\alpha}{\sqrt{2 |B''(0)|}}$$

so that g_n may be expressed as

$$g_n = \frac{1}{(2\pi)^n} \left[\frac{|B''(0)|}{B(0)} \right]^{n/2} \left[e^{-V^2} - \sqrt{\pi} \; V \, \mathrm{erfc}\, V \right]^n \prod_{k=1}^{n} e^{-(\xi_o + \tau_k \alpha)^2 / 2B(0)} \tag{5.67}$$

Therefore, from (5.57), (5.58) and (5.59) one finds that

$$P_i(\text{illu.}/\xi_o, \xi_o') = 1 + \sum_{n=1}^{\infty} (-1)^n \int_0^{\infty} d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{n-1}} d\tau_n \, g_n \tag{5.68}$$

(U) To find P_i, one has to evaluate the integral

$$I_n = \int_0^{\infty} d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{n-1}} d\tau_n \prod_{k=1}^{n} e^{-(\xi_o+\tau_k)^2/2B(0)} \tag{5.69}$$

If one denotes

$$F(\tau) = \int_0^{\tau} e^{-(\xi_o+\tau)^2/2B(0)} d\tau \tag{5.70}$$

so that

$$F'(\tau) = e^{-(\xi_o+\tau)^2/2B(0)} \tag{5.71}$$

Equation (5.69) may be evaluated easily. The result is

$$I_n = \int_0^{\infty} F'(\tau_1) d\tau_1 \int_0^{\tau_1} F'(\tau_2) d\tau_2 \ldots \int_0^{\tau_{n-1}} F'(\tau_n) d\tau_n = \frac{1}{\underline{/n}} \left[F(\infty)\right]^n \tag{5.72}$$

Now

$$F(\infty) = \int_0^{\infty} e^{-(\xi_o+\tau\alpha)^2/2B(0)} d\tau = \frac{\sqrt{\pi B(0)}}{\sqrt{2}\,\alpha} \operatorname{erfc}\left[\frac{\xi_o}{\sqrt{2B(0)}}\right] \tag{5.73}$$

Hence, one has

$$P_i(\text{illu.}/\xi_o, \xi_o') = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1}{\underline{/n}}$$

$$\left\{ \frac{\sqrt{|B''(0)|}}{2\sqrt{2\pi}\,\alpha} \left[e^{-V^2} - \sqrt{\pi}\ \text{erfc}\ V \right] \text{erf}\ \frac{\xi_o}{\sqrt{2B(0)}} \right\}^n$$

$$\doteq \exp\left[- \frac{\sqrt{|B''(0)|}}{2\sqrt{2\pi}\ \alpha} \left(e^{-V^2} - \sqrt{\pi}\ V\ \text{erfc}\ V \right) \text{erf}\ \frac{\xi_o}{\sqrt{2B(0)}} \right]$$

$$= \exp\left[-A\ \text{erfc}\ \frac{\xi_o}{\sqrt{2B(0)}} \right] \tag{5.74}$$

where

$$A \triangleq \frac{e^{-V^2} - \sqrt{\pi}\ V\ \text{erfc}\ V}{4\sqrt{\pi}\ V} \tag{5.75}$$

Using (5.74) and (5.60) one may express the shadow fraction as

$$s(\alpha) = \int_{-\alpha}^{\infty} d\xi_o' \int_{-\infty}^{\infty} d\xi_o \frac{\exp\left[- \frac{\xi_o^2}{2B(0)} - \frac{\xi_o^{2\prime}}{2|B''(0)|} - A\ \text{erfc}\ \frac{\xi_o}{\sqrt{2B(0)}} \right]}{2\pi\ \sqrt{B(0)}\ \sqrt{|B''(0)|}} \tag{5.76}$$

Equation (5.76) can easily be integrated by using these two relations

$$\int_{-\infty}^{\infty} d\xi_o' \frac{1}{\sqrt{2\pi\,|B''(0)|}} \exp\left[- \frac{\xi_o'^{\,2}}{2|B''(0)|} \right]$$

$$= \frac{1}{\sqrt{\pi}} \int_{-V}^{\infty} e^{-x^2}\, dx = \frac{1}{2} \left[1 + \text{erf}\ V \right]$$

and $$\int_{-\infty}^{\infty} d\xi_o \frac{1}{\sqrt{2\pi B(0)}} \exp\left[-\frac{\xi_o}{2B(0)} - A \operatorname{erf} \frac{\xi_o}{\sqrt{2B(0)}}\right] = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left[-x^2 - A \operatorname{erf} x \; dx\right]$$

$$= -\frac{1}{2} \int_{-\infty}^{\infty} \exp\left[-A \operatorname{erf} x\right] \frac{d}{dx}\left[\operatorname{erf} x\right] dx = \frac{1}{2A}\left[1 - e^{-2A}\right].$$

Using these relations, one finds that

$$s(\alpha) = \frac{1}{4A}\left[1 + \operatorname{erf} V\right]\left[1 - e^{-2A}\right]. \tag{5.77}$$

(U) This expression has been derived by Wagner (1966) and was shown to check closely with the simulated computer results of Brockelman and Hagfors (1966). In this report, this result is obtained as a first order approximation of the rigorous result, expressed in the form (5.57 - 5.60). Higher order approximations to the shadow fraction within the present formulation, in principle, seem to be possible. Due to lack of time, these are not completely carried out.

APPENDIX A

NOTES ON STATISTICAL AVERAGES

(This Appendix is Unclassified)

The reflecting surface defined by the random function $\xi(x, y)$ will be assumed to have the following properties.

a. The mean value is zero

$$\mathcal{E}(\xi) = 0 \tag{A.1}$$

b. The surface is isotropic, i. e. the statistics of ξ are invariant to a rotation of coordinates about the z-axis. Thus, the statistics of ξ may be specified by $\xi(s)$ where s is a straight line in the xy-plane.

c. The function is stationary and the correlation function given by

$$B(\tau) = \mathcal{E}\left[\xi(s+\tau)\xi(s)\right] \tag{A.2}$$

is known. In most physical problems we may assume that $B(\tau)$ is an analytic function of τ. It is obvious that $B(\tau)$ is an even function of τ to that

$$B'(0) = 0 \tag{A.3}$$

$$B''(0) < 0 \tag{A.4}$$

and

$$\left.\frac{B'(\tau)}{\tau}\right|_{\tau=0} = B''(0) . \tag{A.5}$$

d. ξ is a Guassian random variable and the probability density function is

$$f(\xi) = \frac{1}{\sqrt{2\pi B(0)}} \exp\left[-\frac{\xi^2}{2B(0)}\right] . \tag{A.6}$$

The statistical averages evaluated in Chapter V, Section 5.3, are

$$\mathcal{E}(C_{ij}) \tag{A.7}$$

and

$$\mathcal{E}(C_{ij}, C_{ij}^*) . \tag{A.8}$$

C_{ij} may be written as

$$C_{ij} = \exp(-ia\xi) \sum_{n=o}^{\infty} \sum_{m=o}^{\infty} C_{ij}^{mn} \xi_x^m \xi_y^n \tag{A.9}$$

where

$$a = k(\cos\alpha + \cos\alpha_o) \tag{A.10}$$

$$\xi_x = \frac{\partial\xi}{\partial x} \tag{A.11}$$

$$\xi_y = \frac{\partial\xi}{\partial y}, \tag{A.12}$$

and the C_{iy}^{mn} are given in Eqs. (5.32) tnrough (5.35) inclusive.

For the conditions discussed in Chapter V of this report, only the terms up to the second order in ξ_x and ξ_y need be considered in (A.7) and (A.8).

In order to evaluate $\mathcal{E}(C_{ij})$ the joint probability density function $f(\xi, \xi_x, \xi_y)$ is required. It is easy to see that

$$\mathcal{E}(\xi) = \mathcal{E}(\xi_x) = \mathcal{E}(\xi_y) = 0 \tag{A.13}$$

$$\mathcal{E}(\xi^2) = B(0) \tag{A.14}$$

$$\mathcal{E}(\xi\xi_x) = \mathcal{E}(\xi\xi_y) = \mathcal{E}(\xi_x\xi_y) = 0 \tag{A.15}$$

and

$$\mathcal{E}(\xi_x^2) = \mathcal{E}(\xi_y^2) = -B''(0). \tag{A.16}$$

Therefore, the joint probability density is

$$f(\xi, \xi_x, \xi_y) = \frac{1}{(2\pi)^{3/2}\sqrt{B(0)B''(0)}} \exp\left\{-\frac{\xi^2}{2B(0)} + \frac{\xi_x^2}{2B''(0)} + \frac{\xi_y^2}{2B''(0)}\right\} \tag{A.17}$$

From (A.17) it is clear that

$$\left[\exp(-ia\xi)\right] = \exp\left[-\frac{a^2B(0)}{2}\right] \tag{A.18}$$

The statistical averages necessary to evaluate (A. 7) can be obtained from (A. 17) and (A. 18) and are

$$\mathcal{E}\left[\exp(-ia\xi)\xi_x\right] = 0 \qquad \text{(A. 19)}$$

$$\mathcal{E}\left[\exp(-ia\xi)\xi_y\right] = 0 \qquad \text{(A. 20)}$$

$$\mathcal{E}\left[\exp(-ia\xi)\xi_x\xi_y\right] = 0 \qquad \text{(A. 21)}$$

and

$$\mathcal{E}\left[\exp(-ia\xi)\xi_x^2\right] = \mathcal{E}\left[\exp(-ia\xi)\xi_y^2\right] = -B''(0)\exp\left(-\frac{a^2B(0)}{2}\right) \qquad \text{(A. 22)}$$

In order to evaluate $\mathcal{E}(C_{ij}\, C^*_{ij'})$ we consider a general term of the form

$$\exp\left[-(\xi-\xi')\right](\xi_x)^m(\xi_y)^n(\xi'_x)^{m'}(\xi'_y)^{n'} \qquad \text{(A. 23)}$$

where $\xi' = \xi(x-\tau_x, y-\tau_y)$ and ξ'_x and ξ'_y are the partial derivatives of ξ evaluated at $x-\tau_x$ and $y-\tau_y$ respectively. To simplify the evaluation we introduce the notation

$$\tau_x = \tau\cos\theta_\tau \qquad \text{(A. 24)}$$

$$\tau_y = \tau\sin\theta_\tau\,, \qquad \text{(A. 25)}$$

The joint probablity density $f(\xi-\xi', \xi_x, \xi_y, \xi'_x, \xi'_y)$ can be obtained from the correlation matrix. The elements of the correlation matrix are determined by the following averages which are obtained by straightforward differentiation:

$$\mathcal{E}(\xi, \xi'_x) = -\mathcal{E}(\xi_x, \xi') = -B'(\tau)\cos\theta_\tau$$

$$\mathcal{E}(\xi, \xi'_y) = -\mathcal{E}(\xi', \xi_y) = -B'(\tau)\sin\theta_\tau$$

$$\mathcal{E}(\xi_x, \xi'_x) = -B''(\tau)\cos^2\theta_\tau - \frac{B'(\tau)}{\tau}\sin^2\theta_\tau$$

$$\mathcal{E}(\xi_y\xi'_y) = -B''(\tau)\cos^2\theta_\tau - \frac{B'(\tau)}{\tau}\cos^2\theta_\tau$$

$$\mathcal{E}(\xi_y, \xi'_x) = \mathcal{E}(\xi_x, \xi'_y) = -\frac{\partial^2}{\partial\tau_x\partial\tau_y}B(\tau) = \left[\frac{B'(\tau)}{\tau} - B''(\tau)\right]\sin\theta_\tau\cos\theta_\tau\,.$$

The correlation matrix M is of order five with elements ρ_{ij} given by

$$\rho_{11} = 2\, B(0) - B(\tau)$$

$$\rho_{22} = \rho_{33} = \rho_{44} = \rho_{55} = B''(0)$$

$$\rho_{12} = \rho_{14} = \rho_{21} = \rho_{41} = -B'(\tau)\cos\theta_\tau$$

$$\rho_{13} = \rho_{15} = \rho_{31} = \rho_{51} = B'(\tau)\sin\theta_\tau$$

$$\rho_{23} = \rho_{32} = \rho_{45} = \rho_{54} = 0$$

$$\rho_{24} = \rho_{42} = B''(\tau)\cos^2\theta_\tau - \frac{B'(\tau)}{\tau}\sin^2\theta_\tau$$

$$\rho_{25} = \rho_{52} = \rho_{34} = \rho_{43} = \frac{B'(\tau)}{\tau} - B''(\tau)\ \sin\theta_\tau\cos\theta_\tau$$

$$\rho_{35} = \rho_{53} = -B''(\tau)\sin^2\theta_\tau - \frac{B'(\tau)}{\tau}\cos^2\theta_\tau$$

Although in principle, we may always evaluate the expected values from the joint probability density of these five variables, the average values for the first few terms of the expansion when any one of the m, m', n, n' is zero can be evaluated in a less complicated manner. In Chapter V, when we take the powers of expansion in ξ_x etc to the second power, some special cases for the average can be deduced as follows:

(a) If $m = n = m' = n' = 0$, we need only the distribution density for $u_1 \triangleq h$,

$$f(u_1) = \frac{1}{\sqrt{2\pi}\sqrt{2[\beta(0) - \beta(\tau)]}} \cdot \exp\left[-\frac{u_1^2}{2[\beta(0) - \beta(\tau)]}\right].$$

$$I_a \triangleq \int_{-\infty}^{\infty} f(u_1)\, e^{-iau_1}\, du_1 = \exp\left[-a^2\left[\beta(0) - \beta(\tau)\right]\right] \qquad \text{(A. 26)}$$

(b) if any of the powers m, m', n, n' is non-zero, we may consider only the correlation matrix M between u_1 and u_2, where u_2 is the variable (ξ_x, ξ'_x, ζ_y, ζ'_y). Denote the determinants of this matrix by $M_1^{(2)}$ and its cofactors by $M_{iju}^{(2)}$ then

$$f(u_1, u_2) = \frac{1}{2\pi\sqrt{M^{(2)}}} \exp\left[-\frac{1}{2M^{(2)}}\left(M_{11}^{(2)}\, u_1^2 + 2M_{12}^{(2)}\, u_1 u_2 + M_{22}^{(2)}\, u_2^2\right)\right].$$

(A. 27)

The integral

$$I_b = \int_{-\infty}^{\infty} du_1 \int_{-\infty}^{\infty} u_2^{(\hat{a})}\, f(u_1, u_2)\, e^{-iau_1}$$

can be evaluated as follows: If we write the exponent in (A. 27) as

$$-\frac{1}{2M^{(2)}}\left[M_{11}^{(2)} u_1^{\;2} + 2M_{12}^{(2)} u_1 u_2 + M_{22}^{(2)} u_2^{2}\right] - i a u_1$$

$$= -\frac{1}{2M^{(2)}}\left[\sqrt{M_{11}^{(2)}}\, u_1 + \frac{M_{12}^{(2)} u_2 + i a M^{(2)}}{\sqrt{M_{11}^{(2)}}_{11}}\right]^2$$

$$-\frac{1}{2M_{11}^{(2)}}\left(u_2 - i\alpha M_{12}^{(2)}\right)^2 - a^2\left[B(0) - B(\tau)\right] .$$

The integration over u_1 yields

$$I_b = \frac{1}{\sqrt{2\pi}\sqrt{M_{12}^{(2)}}}\int_{-\infty}^{\infty} u_2^n\, du_2\, e^{-\frac{1}{2M_{11}^{(2)}}\left[u_2 - i a M_{12}^{(2)}\right]^2}\, e^{-a^2\left[B(0) - B(\tau)\right]} .$$

For $n = 1$, we have

$$I_b = i a M_{12}^{(2)} \exp\left[-a^2\left[B(0) - B(\tau)\right]\right] \tag{A.28}$$

and for $n = 2$, we have

$$I_b = \left[M_{11}^{(2)} - a^2\, \overline{M_{12}^{(2)}}^{\,2}\right] \exp\left[-a\left[B(0) - B(\tau)\right]\right] \tag{A.29}$$

using these expressions, it is straight forward to obtain the following integrals that are used in the statistical averages,

$$\int_{-\infty}^{\infty} \xi_x\, d\xi_x \int_{-\infty}^{\infty} dh\, f(h, \xi_x)\, e^{-iah}$$

$$= \int_{-\infty}^{\infty} \xi'_x \, d\,\xi_x \int_{-\infty}^{\infty} dh \, f(h, \xi'_x) \, e^{-iah}$$

$$= i a \beta' (\tau) \cos \theta_\tau \; \exp \left\{ -a^2 \left[B(0) - B'(\tau) \right] \right\}$$

$$\int_{-\infty}^{\infty} \xi_y \, d\xi_y \int_{-\infty}^{\infty} dhf(h, \xi_y) e^{-iah}$$

$$= \int_{-\infty}^{\infty} \xi'_y \, d\xi'_y \int_{-\infty}^{\infty} dh \, f(h, \xi'_y) \, e^{-iah}$$

$$= i a \beta' (\tau) \; \sin \theta_\tau \; \exp \left\{ -a^2 \left[B(0) - B(\tau) \right] \right\}$$

$$\int_{-\infty}^{\infty} \xi_x^2 \, d\xi_x \int_{-\infty}^{\infty} dh \, f(h, \xi_x) \, e^{-iah}$$

$$\int_{-\infty}^{\infty} \xi'^2_x \, d\xi'_x \int_{-\infty}^{\infty} d\xi'_x \; dh \, f(h, \xi'_x) \, e^{-iah}$$

$$= \left[-B''(0) - a^2 \, \overline{B'(\tau)}^2 \cos^2 \theta_\tau \right] \; \exp \left\{ -a^2 \left[B(0) - B(\tau) \right] \right\}$$

$$\int_{-\infty}^{\infty} \xi_y^2 \, d\xi_y \int_{-\infty}^{\infty} dh \, f(h, \xi_y) \, e^{-iah}$$

$$= \int_{-\infty}^{\infty} \xi'^2_y \, d\xi'_y \int_{-\infty}^{\infty} dh \, f(h, \xi'_y) \, e^{-iah}$$

$$= \left[-B''(0) - a^2 \overline{B'(\tau)}^2 \sin^2 \theta_\tau\right] \exp\left\{-a^2 \left[B'(0) - B'(\tau)\right]\right\}.$$

(c) If any two of the exponents m, n, m', n' are non-zero, we have to evaluate the expected values by using the correlation matrix of the three variables, $u_1 \triangleq h$, u_1 and u_2 and u_3 respectively for the two values of ξ_x, ξ'_x, ξ_y, ξ'_y whose exponents are non-vanishing. Denote the elements of the matrix by ρ_{ij} and, the determinants by M, and cofactors by $M_{ij}^{(3)}$, then,

$$f(u_1, u_2, u_3) = \frac{1}{(2\pi)^{3/2}\sqrt{M^{(3)}}} \exp\left[-\frac{\sum_{j=1}^{3}\sum_{k=1}^{3} M_{ij}\, u_j\, u_k}{2\, M^{(3)}}\right] . \quad \text{(A.30)}$$

To evaluate the integral

$$I_c = \int_{-\infty}^{\infty} du_1 \int_{-\infty}^{\infty} u_2^n\, du_2 \int_{-\infty}^{\infty} u_3^m\, du_3\, f(u_1, u_2, u_3)\, e^{-iah}$$

we expand the exponents in A.30 in form,

$$\frac{1}{2M^{(3)}} \sum_{j=1}^{3}\sum_{k=1}^{3} M_{jk}^{(3)}\, u_j u_k + iau_1$$

$$= \frac{1}{2M^{(3)}}\left[\sqrt{M_{11}^{(3)}}\, u_1 + \frac{M_{12}^{(3)}\, u_2 + M_{13}^{(2)}\, u_3 + iaM}{\sqrt{M_{11}^{(3)}}}\right]^2 -$$

$$+ \frac{1}{2N^{(2)}}\left[\sqrt{N_{22}^{(2)}}\, u_3 + \frac{N_{12}^{(2)}\, u_2 - iaM_{13}^{(3)}}{\sqrt{N_{22}^{(2)}}}\right]^2$$

$$+ \frac{1}{2\rho_{22}} \left[u_2 - ia\rho_{12}\right]^2 + \frac{a^2}{2} \rho_{11} \tag{A.31}$$

where

$$N^{(2)} = \begin{bmatrix} \rho_{22} & \rho_{23} \\ \rho_{32} & \rho_{33} \end{bmatrix} . \tag{A.32}$$

If we substitute,

$$u_3 = \frac{\sqrt{2N^{(2)}}}{\sqrt{N_{22}^{(2)}}} y_3 + \sqrt{2\rho_{22}} \frac{\rho_{23}}{\rho_{22}} y_2 - ia\rho_{13}$$

$$u_2 = \sqrt{2\rho_{22}} y_2 + ia\rho_{12}$$

then after integrating over u_1 we obtain

$$I_c = \frac{1}{\pi} \exp - \frac{a^2}{2} \rho_{11} \int_{-\infty}^{\infty} \left(\frac{\sqrt{2N^{(2)}}}{\sqrt{N_{22}^{(2)}}} y_3 + \sqrt{2\rho_{22}} \frac{\rho_{23}}{\rho_{22}} y_2 - ia\rho_{13}\right)^n dy_3$$

$$\int_{-\infty}^{\infty} (\sqrt{2\rho_{22}} y_2 - ia\rho_{12})^m dy_2 \, e^{-y_2^2 - y_3^2} .$$

These integrals can be carried out explicitly, the results are:

$$\int_{-\infty}^{\infty} du_1 \int_{-\infty}^{\infty} u_2 du_2 \int_{-\infty}^{\infty} u_3 du_3 \, f(u_1, u_2, u_3) \, e^{-iau_1}$$

$$= \left[\rho_{23} + a^2 \rho_{12}\rho_{13}\right] \exp\left[-\frac{a^2 \rho_{11}}{2}\right] \tag{A.33}$$

$$\int_{-\infty}^{\infty} du_1 \int_{-\infty}^{\infty} u_2^2 \int_{-\infty}^{\infty} u_3 du_3\, f(u_1, u_2, u_3)\, e^{-iau_1}$$

$$= i\left[a^3 \rho_{13}^2 - a\rho_{22}\rho_{13} + 2a\rho_{12}\rho_{23}\right] \exp\left[-\frac{a^2\rho_{11}}{2}\right] \tag{A.34}$$

$$\int_{-\infty}^{\infty} du_1 \int_{-\infty}^{\infty} u_2^2 \int_{-\infty}^{\infty} u_3^2 du_3\, f(u_1, u_2, u_3)\, e^{-iau_1}$$

$$= \left[N^{(2)} + 3\rho_{23}^2 - a^2\left[\rho_{13}^2\rho_{22} + \rho_{12}^2\rho_{33} - 4\rho_{12}\rho_{13}\rho_{23}\right] + a^4\rho_{12}^2\rho_{13}^2\right] \cdot \exp\left[-\frac{a^2\rho_{11}}{2}\right] \tag{A.35}$$

d, if any three of the exponents are non-zero, we use the correlation matrix t between 4 variables u_1, u_2, u_3, and u_4. Denote determinant of this matrix by $M^{(4)}$, we have,

$$f(u_1, u_2, u_3, u_4) = \frac{1}{2\pi\sqrt{M^{(4)}}} \exp\left[-\sum_{j=1}^{4}\sum_{k=1}^{4} M_{jk}^{(4)} u_j u_k\right]. \tag{A.36}$$

To evaluate the integral

$$I_a = \int_{-\infty}^{\infty} du_1\, e^{-iau_1} \int_{-\infty}^{\infty} u_2^n\, du_2 \int_{-\infty}^{\infty} u_3\, du_3 \int_{-\infty}^{\infty} u_4\, du_4\, f(u_1, u_2, u_3, u_4) \quad ,$$

we expand the exponential in (A. 36)

$$\sum_{j=1}^{4} \sum_{k=1}^{4} M_{jk}^{(4)} u_j u_k + iau_1 \quad ,$$

$$= \frac{1}{2M^{(4)}} \left[\sqrt{M_{11}^{(4)}}\, u_1 + \frac{M_{12}^{(4)} u_2 + M_{13}^{(4)} u_3 + M_{14}^{(4)} u_4 + iaM}{\sqrt{M_{11}^{(4)}}} \right]^2$$

$$+ \frac{1}{2N^{(3)}} \left[\sqrt{N_{33}^{(3)}}\, u_4 + \frac{N_{13}^{(3)} u_2 + N_{23}^{(3)} u_3 - iaM_{14}^{(4)}}{\sqrt{N_{33}^{(3)}}} \right]^2$$

$$+ \frac{1}{2N^{(2)}} \left[\sqrt{N_{22}^{(2)}}\, u_3 + \frac{N_{12}^{(2)} u_2 - iaM_{13}^{(3)}}{\sqrt{N_{22}^{(2)}}} \right]^2$$

$$+ \frac{1}{2\rho_{22}} \left[u_2 - ia\rho_{12} \right]^2 + \frac{a^2}{2} \rho_{11} \qquad \text{(A.37)}$$

where $N^{(3)}$ is the determinant of

$$N^{(3)} = \begin{bmatrix} \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{42} & \rho_{43} & \rho_{44} \end{bmatrix} . \tag{A.38}$$

By change of variables,

$$u_2 = \sqrt{2\rho_{22}}\, y_2 + i a \rho_{12}$$

$$u_3 = \sqrt{\frac{2N^{(2)}}{\rho_{22}}}\, y_3 + \sqrt{2\rho_{22}}\left(\frac{\rho_{23}}{\rho_{22}}\right) y_2 + i a \rho_{13}$$

$$u_4 = \sqrt{\frac{2N^{(3)}}{N_{33}^{(3)}}}\, y_4 - \frac{\sqrt{2N^{(2)}}}{\sqrt{\rho_{22}}} \frac{N_{23}^{(3)}}{N_{33}^{(3)}}\, y_3 + \sqrt{2\rho_{22}}\left(\frac{\rho_{24}}{\rho_{22}}\right) y_2 - i a \rho_{14}$$

and after the u_1 integral is carried out,

$$I_d = \frac{1}{\pi^{3/2}} \int_{-\infty}^{\infty} dy_2 \int_{-\infty}^{\infty} dy_2 \int_{-\infty}^{\infty} dy_3\, e^{-y_1^2 - y_2^2 - y_3^2}\, u_1^n u_2 u_3 \quad .$$

For $n = 1$,

$$\int_{-\infty}^{\infty} du_1\, e^{-i a u_1} \int_{-\infty}^{\infty} u_2\, du_2 \int_{-\infty}^{\infty} u_3\, du_3 \int_{-\infty}^{\infty} u_4\, du_4\, f(u_1, u_2, u_3, u_4)$$

$$= \left\{ i a^3 \rho_{12} \rho_{13} \rho_{14} - i a \left[\rho_{12}\rho_{34} + \rho_{13}\rho_{24} + \rho_{14}\rho_{23} \right] \right\} \exp\left[-\frac{a^2 \rho_{11}}{2} \right] \tag{A.39}$$

For $n = 2$ we have,

$$\int_{-\infty}^{\infty} du_1 \, e^{-iau_1} \int_{-\infty}^{\infty} u_2^2 \, du_2 \int_{-\infty}^{\infty} u_3 \, du_3 \int_{-\infty}^{\infty} u_4 \, du_4 \; f(u_1, u_2, u_3, u_4)$$

$$= \left\{ \left[\rho_{22}\rho_{43} + 2\rho_{23}\rho_{24} \right] + a^2 \left[\rho_{12}\rho_{14}\rho_{22} + 2\rho_{12}\rho_{14}\rho_{23} - 2\rho_{12}\rho_{13}\rho_{24} - \rho_{12}^2 \rho_{34} \right] + a^4 \rho_{12}^2 \rho_{13} \rho_{14} \right\} \exp\left[-\frac{a^2 \rho_{11}}{2} \right] \tag{A.40}$$

(e) For the expectated values of the functions involving all the powers, we need the complete correlation matrix between the five variables. Denote the determinant of the matrix by $M^{(5)}$, and its nth order sub-matrix involving the second n rows and columns by $N^{(n)}$, the integral

$$I_e = \int_{-\infty}^{\infty} du_1 \int_{-\infty}^{\infty} u_2 du_2 \int_{-\infty}^{\infty} u_3 du_3 \int_{-\infty}^{\infty} u_4 du_4 \int_{-\infty}^{\infty} u_5 du_5 \, f(u_1, u_2, u_3, u_4, u_5) e^{-iau}$$

can be simplified by substitutions similar to those given previously. If we substitute,

$$u_2 \sqrt{2\rho_{22}}\, y_2 + i a \rho_{12}$$

$$u_3 = \sqrt{\frac{2N^{(2)}}{\rho_{22}}}\, y_3 + \sqrt{2\rho_{22}}\left(\frac{\rho_{23}}{\rho_{22}}\right) y_2 + ia\rho_{13}$$

$$u_4 = \sqrt{\frac{2N^{(3)}}{N_{33}^{(3)}}}\, y_4 - \sqrt{\frac{2N^{(2)}}{\rho_{22}}}\,\frac{N_{23}^{(3)}}{N_{33}^{(3)}}\, y_3 + \sqrt{2\rho_{22}}\left(\frac{\rho_{24}}{\rho_{22}}\right) y_2 + ia\rho_{14}$$

$$u_5 = \sqrt{\frac{2N^{(4)}}{N_{44}^{(4)}}}\, y_5 - \sqrt{\frac{2N^{(3)}}{N_{33}^{(3)}}}\,\frac{N_{34}^{(4)}}{N_{44}^{(4)}}\, y_4 + \sqrt{\frac{2N^{(2)}}{\rho_{22}}}\,\frac{(\rho_{22}\rho_{53} - \rho_{52}\rho_{23})}{N_{33}^{(3)}}\, y_3$$

$$+ \sqrt{2\rho_{22}}\,\frac{\rho_{52}}{\rho_{22}}\, y_2 + ia\rho_{15} \;,$$

the integral, after carrying out the u_1 - integration is reduced to

$$I_e = \exp\left[-\frac{a^2\rho_{11}}{2}\right] \int_{-\infty}^{\infty} dy_2 \int_{-\infty}^{\infty} dy_3 \int_{-\infty}^{\infty} dy_4 \int_{-\infty}^{\infty} dy_5 \exp\left[-y_2^2 - y_3^2 - y_4^2 - y_5^2\right] u_1 u_2 u_3$$

$$= \exp\left[-\frac{a^2\rho_{11}}{2}\right] \Big\{ \rho_{34}\rho_{25} + \rho_{24}\rho_{35} + \rho_{23}\rho_{45}$$

$$-a^2 \Big[\rho_{14}\rho_{15}\rho_{23} + \rho_{13}\rho_{15}\rho_{24} + \rho_{12}\rho_{15}\rho_{43}$$

$$+ \rho_{14}\rho_{13}\rho_{52} + \rho_{14}\rho_{12}\rho_{53} + \rho_{13}\rho_{12}\rho_{54}\Big]$$

$$+ a^4 \rho_{12}\rho_{13}\rho_{14}\rho_{15} \Big\} \qquad \text{(A.41)}$$

APPENDIX B

ALTERNATE FORMULATION OF GROUND REFLECTION

(This Appendix is Unclassified)

In the derivation of the angular spectra of the ground reflected radiation given in Chapter IV, the z-components of the incident field at the ground were used in the interests of simplicity. However, the employment of the z-components of the field causes some difficulties in the limiting case of normal incidence, since in this case the values of E_z and H_z are zero, although the angular spectra is finite by taking limiting values. In order to consider such cases, an alternate derivation of the angular spectra, based on the tangential components of electric and magnetic fields, are given here.

Starting from

$$d\underline{E}(r) = \left[\mathcal{E}_1(\alpha,\beta)\,\hat{e}_1 + \mathcal{E}_2(\alpha,\beta)\,\hat{e}_2\right] e^{ik\,\Omega\cdot\underline{r}} \tag{B.1}$$

$$\hat{e}_1 = \hat{x}\sin\beta - \hat{y}\cos\beta \tag{B.2}$$

and

$$\hat{e}_2 = \hat{x}\cos\alpha\cos\beta + \hat{y}\cos\alpha\sin\beta - \hat{z}\sin\alpha \tag{B.3}$$

one finds that

$$E_x(\underline{r}) = \int \left[\mathcal{E}_1(\alpha,\beta)\sin\beta + \mathcal{E}_2(\alpha,\beta)\cos\alpha\cos\beta\right] e^{ik\hat{\Omega}\cdot\underline{r}}\, d\Omega \tag{B.4}$$

$$E_y(\underline{r}) = \int \left[-\mathcal{E}_1(\alpha,\beta)\cos\beta + \mathcal{E}_2(\alpha,\beta)\cos\alpha\sin\beta\right] e^{ik\hat{\Omega}\cdot\underline{r}}\, d\Omega\ . \tag{B.5}$$

Let $\underline{r} \rightarrow \underline{r}_s$, and consider the above approximately as a two dimensional Fourier transform, then one finds that

$$\left[\mathcal{E}_1(\alpha,\beta)\sin\beta + \mathcal{E}_2(\alpha,\beta)\cos\alpha\cos\beta\right]\frac{1}{\cos\alpha}$$

$$= \frac{k^2}{(2\pi)^2}\int dx_s \int dy_s\ E_x(\underline{r}_s)\, e^{-ik\hat{\Omega}\cdot\underline{r}_s} \tag{B.6}$$

and

$$\left[-\mathcal{E}_1(\alpha,\beta)\cos\beta+\mathcal{E}_2(\alpha,\beta)\cos\alpha\sin\beta\right]\frac{1}{\cos\alpha}$$
$$=\frac{k^2}{(2\pi)^2}\int dx_s\int dy_s\; E_y(\underline{r}_s)e^{-ik\hat{\Omega}\cdot\underline{r}_s} \tag{B.7}$$

From the above it follows

$$\mathcal{E}_1(\alpha,\beta)=\frac{k^2}{(2\pi)^2}\int dx_s\int dy_s\left[E_x(\underline{r}_s)\sin\beta-E_y(\underline{r}_s)\cos\beta\right]\cos\alpha\; e^{-i\hat{\Omega}\cdot\underline{r}_s} \tag{B.8}$$

and

$$\mathcal{E}_2(\alpha,\beta)=\frac{k^2}{(2\pi)^2}\int dx_s\int dy_s\left[E_x(\underline{r}_s)\cos\beta+E_y(\underline{r}_s)\sin\beta\right]e^{-ik\hat{\Omega}\cdot\underline{r}_s} \tag{B.9}$$

Now, for an incident field approaching the ground in a direction $\hat{\Omega}_o$, the incident electric field may be expressed as

$$\underline{E}_o(\underline{r}_s)=\left[E_{10}\hat{e}_{10}+E_{20}\hat{e}_{20}\right]e^{+ik\hat{\Omega}_o\cdot\underline{r}_s} \tag{B.10}$$

where E_{10} and E_{20} are respectively the amplitude of the perpendicular and parallel polarized components of the incident field referring to a plane ground z = 0. The amplitude of the reflected field on the ground is therefore given by

$$E_x(\underline{r}_s)=\left[E_{10}R_{\perp}\sin\beta_o+E_{20}R_{/\!/}\cos\alpha_o\cos\beta_o\right]e^{ik\hat{\Omega}_o\cdot\underline{r}_s} \tag{B.11}$$

and

$$E_y(\underline{r}_s)=\left[-E_{10}R_{\perp}\cos\beta_o+E_{20}R_{/\!/}\cos\alpha_o\sin\beta_o\right]e^{ik\hat{\Omega}_o\cdot\underline{r}_s} \tag{B.12}$$

Thus

$$E_x(\underline{r}_s)\cos\beta+E_y\sin\beta = \left[E_{10}R_\perp\sin(\beta_o-\beta)+E_{20}R_{//}\cos\alpha_o\cos(\beta_o-\beta)\right] e^{ik\hat{\Omega}_o\cdot\underline{r}_s} \tag{B.13}$$

and

$$\left[E_x(\underline{r}_s)\sin\beta-E_y(\underline{r}_s)\cos\beta\right]\cos\alpha$$
$$=\left[E_{10}R_\perp\cos\alpha\cos(\beta_o-\beta)-E_{20}R_{//}\cos\alpha\cos\alpha_o\sin(\beta_o-\beta)\right]e^{i\hat{\Omega}_o\cdot\underline{r}_s} \tag{B.14}$$

Introducing the above to (B.8) and (B.9), one obtains the following matrix form for the reflected angular spectra of the reflected radiation.

$$\begin{bmatrix}\mathcal{E}_1(\alpha,\beta)\\ \mathcal{E}_1(\alpha,\beta)\end{bmatrix} = \frac{k^2}{(2\pi)^2}\int dx_s\int dy_s\; e^{ik(\hat{\Omega}_o-\hat{\Omega})\cdot\underline{r}_s}$$
$$\begin{bmatrix}R_\perp\cos\alpha\cos(\beta_o-\beta) & -R_{//}\cos\alpha\cos\alpha_o\sin(\beta_o-\beta)\\ R_\perp\sin(\beta_o-\beta) & R_{//}\cos\alpha_o\cos(\beta_o-\beta)\end{bmatrix}\begin{bmatrix}E_{10}\\ E_{20}\end{bmatrix} \tag{B.15}$$

This result seems to be closer, but not identical to the physical optics results, but the singularities in the scattering matrix are avoided.

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13. ABSTRACT

(U) A unified approach, using the generalized concept of angular spectra, for the study of the radiation received at any point in space from a transmitter due to multipath propagation effects is formulated. The various mechanisms contributing to the multipath effects, such as scattering by discrete objects, and by extended objects such as ground are formulated in general. Although the formulation is based on CW transmitter and stationary receiver, the results may also be used for other transmitter signals, scanning and moving receivers (and transmitters) with slight modification.

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