10748-3-T = RL-2202

(Radiation Laboratory Report No. 010748-3-T)

May 1972

ANALYTICAL INVESTIGATION OF WAVEFORMS RADIATED BY A RESISTIVELY LOADED LINEAR ANTENNA EXCITED BY A GAUSSIAN PULSE

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ABSTRACT

The far field waveforms radiated by linear antennas loaded non-uniformly and continuously with resistance are investigated analytically. The antennas considered are symmetrical and excited at the centers by slice generators having Gaussian pulse type of time dependence. Expressions have been developed for the far field waveforms produced by such antennas in any direction. The analytic results have been compared with those obtained by direct numerical means. The general agreement between the two results has been found to be satisfactory. It is thus concluded that the various expressions given may be used to study the behavior of waveforms radiated by a resistively loaded linear antenna excited by Gaussian pulse type signals.

1. INTRODUCTION

The present report discusses analytically the far field waveforms produced by symmetrical linear antennas loaded non-uniformly and continuously with resistance. The antennas considered are thin cylinders and excited at the centers by slice generators having Gaussian pulse type of time dependence. Emphasis is given here to a specific type of loading in which the amount of loading increases continuously towards the antenna end-points.

Direct numerical investigation of such a problem has been discussed in a separate report.[1] Here we derive theoretical expressions for the waveforms produced by the loaded linear antenna excited by Gaussian pulse type time dependent signals. Fourier transform technique is utilized in obtaining the final time dependent results. Numerical results are obtained by computing the various expressions derived. These results are then compared with those obtained by direct numerical means.

2. BASIC RELATIONS

Let us assume that the linear antenna of length 2 L be aligned along the z-axis of a rectangular coordinate system with the origin located at the center of the antenna and that it is excited by a slice generator located at the origin. The far electric field produced by the antenna when excited by a harmonically time dependent slice generator of unit strength consists of only a θ-component and is given by the following expression:

$$F = \tilde{F}_\theta(r, \theta, \omega)\, e^{j\omega t} \quad , \tag{1}$$

where,

$$\tilde{F}(r, \theta, \omega) = \frac{j\omega\eta_o \sin\theta}{4\pi c} \, \frac{e^{-j\frac{\omega}{c}r}}{r} \int_{-L}^{L} \tilde{I}(z', \omega)\, e^{j\frac{\omega}{c}z'\cos\theta}\, dz' \quad , \tag{2}$$

$\eta_o = \sqrt{\frac{\mu_o}{\epsilon_o}}$ is the intrinsic impedance of free space,

$c = \frac{1}{\sqrt{\mu_o \epsilon_o}}$ is the velocity of light in free space,

r, θ, ϕ are the spherical polar coordinates of the far field point with origin located at the center of the antenna,

$\tilde{I}(z', \omega)$ is the current distribution on the antenna due to the harmonically time dependent slice generator.

$\tilde{F}_\theta(r, \theta, \omega)$ is the time independent far electric field produced by the antenna and may be looked upon as the transfer function of the antenna. For convenience we define the following modified form for the antenna transfer function:

$$\tilde{f}_\theta(\theta, \omega) = r\tilde{F}_\theta(r, \theta, \omega)\, e^{i\frac{\omega}{c}r}$$

$$= \frac{j\omega\eta_o \sin\theta}{4\pi c} \int_{-L}^{L} \tilde{I}(z', \omega)\, e^{j\frac{\omega}{c} z' \cos\theta}\, dz' \quad , \tag{3}$$

Notice that in the modified transfer function given by Eq.(3), the dependence on the factor r as well as the phase shift suffered by the signal in traveling from the antenna to the field point are both suppressed.

Let the time waveform of the input voltage signal be given by:

$$V(t) = e^{-t^2/2\sigma^2} \quad , \tag{4}$$

where σ is proportional to the width of the input Gaussian pulse. The Fourier transform $\tilde{V}(\omega)$ of $V(t)$ is obtained as follows:

$$\tilde{V}(\omega) = \int_{-\infty}^{\infty} V(t)\, e^{-j\omega t}\, dt = \sqrt{2\pi}\, \sigma\, e^{-\omega^2\sigma^2/2} \quad . \tag{5}$$

By making use of the linearity of the system along with superposition theorem and the concepts of Fourier transform technique, it can be shown that the modified time dependent far field produced by the antenna excited by a Gaussian voltage pulse is given by the following:

$$e_\theta(\theta, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{e}_\theta(\theta, \omega)\, e^{j\omega t}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}_\theta(\theta, \omega)\, \tilde{V}(\omega)\, e^{j\omega t} \quad . \tag{6}$$

In Eq. (6) the quantity $\tilde{e}_\theta(\theta, \omega) = \tilde{f}_\theta(\theta, \omega)\tilde{V}(\omega)$ may be looked upon as the spectral density of the far field wave. Notice that by definition $e_\theta(\theta, t)$ and $\tilde{e}_\theta(\theta, \omega)$ are related to each other by the following transform relationship

$$\tilde{e}_\theta(\theta, \omega) = \int_{-\infty}^{\infty} e_\theta(\theta, t)\, e^{-j\omega t}\, dt$$

$$= \tilde{f}_\theta(\theta, \omega)\, \tilde{V}(\omega) \quad . \tag{7}$$

3. TRANSFER FUNCTION OF THE ANTENNA $\tilde{f}_\theta(\theta, \omega)$.

In this section we discuss the evaluation of the modified transfer function of the antenna as given by Eq. (3). To obtain $\tilde{f}_\theta(\theta, \omega)$ it is necessary to know the current distribution $\tilde{I}(z, \omega)$ on the antenna excited by the harmonically time dependent unit slice generator.

Let the antenna be resistively loaded in the following fashion:

$$r_s(z) = \frac{C}{L - |z|} \text{ ohms/meter} \quad , \tag{8}$$

where C is a constant expressed in ohms. For some special value of C, the loaded linear antenna sustains a pure outward traveling wave of current for the time harmonic case[2,3,4]. In particular, for the non-reflecting case the necessary loading is,

$$r_s(z) = \frac{\eta_o \psi}{2\pi} \frac{1}{L-|z|} \tag{9}$$

which means that the constant $C = \frac{\eta_o \psi}{2\pi}$, where ψ is obtained from the expansion parameter as defined by King and Wu.[2]. As discussed in the references cited, the current distribution on the antenna loaded according to Eq. (8) and excited by a harmonically time dependent unit slice generator is given by:

$$\tilde{I}(z, \omega) = \frac{2\pi}{\eta_o \psi} \left[\frac{j\omega L/c}{1+j\omega L/c} \right] \left[1 - \frac{|z|}{L} \right] e^{-j\frac{\omega}{c}|z'|} . \tag{10}$$

We shall not go into more discussion of the current distribution on the antenna. The parameter ψ in Eq. (10) is very slowly varying function[2] of ω and will be assumed from now on that ψ is a constant for an antenna of given length L and $\Omega = 2 \ln \frac{2L}{a}$, where a is the radius of the antenna element. For the non-reflecting case[5] $\psi \simeq 5.3$. Thus Eq. (10) gives the complete description of the current distribution on the non-reflectively loaded linear antenna excited by a harmonically time dependent slice generator of unit strength.

After introducing Eq. (10) into Eq. (3) the following expression is obtained for the antenna transfer function:

$$\tilde{f}_\theta(\theta, \omega) = \frac{\sin\theta}{2\psi} \frac{(j\omega\tau)(j\omega\tau)}{1+j\omega\tau} \, x$$

$$x \int_{-1}^{1} (1-|\nu|) \, e^{-j\omega\tau|\nu|} \, e^{j\omega\tau\nu\cos\theta} d\nu \,, \tag{11}$$

where,

$\tau = \frac{L}{c}$ = is the transit time of the signal on the antenna.

The integral involved in Eq. (11) can be carried out in closed form and is given by the following:

$$\int_{-1}^{1} (1-|\nu|)\, e^{-j\omega\tau|\nu|}\, e^{j\omega\tau\nu\cos\theta}\, d\nu$$

$$= \frac{1}{j\omega\tau(1-\cos\theta)} \left[\frac{e^{-j\omega\tau(1-\cos\theta)}-1}{j\omega\tau\,(1-\cos\theta)} + 1 \right]$$

$$+ \frac{1}{j\omega\tau\,(1+\cos\theta)} \left[\frac{e^{-j\omega\tau\,(1+\cos\theta)}-1}{j\omega\tau\,(1+\cos\theta)} + 1 \right] \quad . \qquad (12)$$

After introducing Eq. (12) into Eq. (11) we obtain the complete expression for the antenna transfer function $\tilde{f}_\theta(\theta,\omega)$.

4. SPECTRAL DENSITY OF THE FAR FIELD WAVEFORM: $\tilde{e}_\theta(\theta, \omega)$.

In this section we obtain the spectral density function $\tilde{e}_\theta(\theta, \omega)$ for the antenna when excited by a Gaussian pulse. This is done by making use of Eqs. (5), (7), (11) and (12) and the final result is given by the following expression:

$$\tilde{e}_\theta(\theta, \omega) = \frac{\sigma}{\psi}\sqrt{\frac{\pi}{2}}\,\sin\theta\,\frac{(j\omega\tau)\,(j\omega\tau)}{1+j\omega\tau}\, e^{-\sigma^2\omega^2/2}$$

$$\times \left[\frac{1}{j\omega\tau}\,\frac{1}{1-\cos\theta} \left\{ \frac{e^{-j\omega\tau\,(1-\cos\theta)}-1}{(j\omega\tau)\,(1-\cos\theta)} + 1 \right\} \right. \qquad (13)$$

$$\left. + \frac{1}{j\omega\tau}\,\frac{1}{1+\cos\theta} \left\{ \frac{e^{-j\omega\tau\,(1+\cos\theta)}-1}{(j\omega\tau)\,(1+\cos\theta)} + 1 \right\} \right] \quad .$$

In particular, in the broadside direction of the antenna $\theta = \pi/2$ and we obtain the following simplified expression for the spectral density of the far field waveform:

$$\tilde{e}_\theta(\frac{\pi}{2}, \omega) = \sqrt{2\pi}\ \frac{\sigma}{\psi}\ e^{-\sigma^2\omega^2/2}\ \frac{(j\omega\tau)\,(j\omega\tau)}{1 + j\omega\tau}\ \times$$

$$\times \left[\frac{1}{j\omega\tau} - \frac{1}{j\omega\tau}\ \frac{\sin\omega\tau/2}{\omega\tau/2}\ e^{-j\omega\tau/2}\right] \quad . \tag{14}$$

All the important frequency domain results pertinent to the antenna may be obtained from Eq. (13). It is useful to investigate the high and low frequency behavior of $\tilde{e}_\theta(\theta, \omega)$ analytically. After studying the behavior of $\tilde{e}_\theta(\theta, \omega)$ as given by Eq. (13) in the low and high frequency limits it is found that

$$\tilde{e}_\theta(\theta, \omega) \sim \frac{\sigma}{\psi}\sqrt{\frac{\pi}{2}}\ \ (j\omega\tau)\ \ \sin\theta\ e^{-\sigma^2\omega^2/2} \quad , \qquad \text{for } \omega\tau \ll 1 \tag{15}$$

$$\tilde{e}_\theta(\theta, \omega) \sim \frac{\sigma}{\psi}\sqrt{2\pi}\ \ \frac{1}{\sin\theta}\ e^{-\sigma^2\omega^2/2} \quad , \qquad \text{for } \omega\tau \gg 1 \quad . \tag{16}$$

The above two relations indicate that for all $\theta \neq 0$, $\tilde{e}_\theta(\theta, \omega) \rightarrow 0$ as $\omega \rightarrow 0$ and ∞. After study Eq. (13) for $\theta \rightarrow 0$ it is found that $\tilde{e}_\theta(\theta, \omega) = 0$. Thus it is concluded that for all values θ, the spectral density function tends to zero in both the low and high frequency limits.

5. EVALUATION OF THE FAR FIELD WAVEFORM $e_\theta(\theta, t)$.

After introducing Eq. (13) into Eq. (6) and some algebraic manipulations the far field waveform can be written in the following form:

$$e_\theta(\theta, t) = \frac{\sigma}{\psi}\sqrt{\frac{\pi}{2}}\ \sin\theta\left[I(\nu, t) + I(\nu', t)\right] \quad , \tag{17}$$

where

$$I(\nu, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\sigma^2\omega^2/2} \frac{(j\omega\tau)(j\omega\tau)}{1+j\omega\tau} \left[\frac{1}{j\omega\tau\nu} + \frac{e^{-j\omega\tau\nu}-1}{(j\omega\tau)^2\nu^2} \right] e^{j\omega\tau} d\omega \quad , \quad (18)$$

$$\left.\begin{aligned} \nu &= 1 - \cos\theta \\ \nu' &= 1 + \cos\theta \end{aligned}\right\} . \quad (19)$$

The integral in Eq. (18) may be carried out by applying the convolution theorem. For this purpose it has been found convenient to write Eq. (18) in the following form:

$$I(\nu, t) = I_1(\nu, t) + I_2(\nu, t) \quad . \quad (20)$$

where,

$$I_1(\nu, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\sigma^2\omega^2/2} \frac{(j\omega\tau)(j\omega\tau)}{1+j\omega\tau} \frac{1}{j\omega\tau\nu} e^{j\omega\tau} d\omega \quad , \quad (21)$$

$$I_2(\nu, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\sigma^2\omega^2/2} \frac{(j\omega\tau)(j\omega\tau)}{1+j\omega\tau} \frac{e^{-j\omega\tau\nu}-1}{(j\omega\tau)^2\nu^2} e^{j\omega\tau} d\omega \quad . \quad (22)$$

Let us consider the integral given by Eq. (21) at first. It can be shown that $I_1(\nu, t)$ can be written as follows:

$$I_1(\nu, t) = I_1'(\nu, t) + I_1''(\nu, t) \quad , \quad (23)$$

where,

$$I_1'(\nu, t) = \frac{1}{\nu\tau} \frac{1}{2\pi} \int_{-\infty}^{\infty} (j\omega) \frac{1}{\frac{1}{\tau^2} + \omega^2} e^{-\sigma^2\omega^2/2} e^{j\omega\tau} d\omega \quad , \quad (24)$$

$$I_1''(\nu, t) = -\tau \frac{\partial I_1'(\nu, t)}{\partial t} \quad . \quad (25)$$

We now make use of the following two Fourier transform relations:[6]

$$e^{-t^2/2\sigma^2} \longleftrightarrow 2\pi\,\sigma\, e^{-\omega^2\sigma^2/2} \quad , \tag{26}$$

$$\pi\,\tau e^{-t/\tau} \longleftrightarrow \frac{1}{\frac{1}{\tau^2}+\omega^2} \quad . \tag{27}$$

After using the relations (26) and (27) and making use of the convolution theorem Eq. (24) can be transformed into the following form:

$$I_1'\,(\nu,t) = \sqrt{\frac{\pi}{2}}\;\frac{1}{\sigma\nu}\;\frac{\partial}{\partial t}\int_{-\infty}^{\infty} e^{-(t-\alpha)^2/2\sigma^2}\, e^{-|\alpha|/\tau}\, d\alpha \quad . \tag{28}$$

From Eq. (25) we obtain $I_1''(\nu,t)$ as follows:

$$I_1''(\nu,t) = -\sqrt{\frac{\pi}{2}}\;\frac{\tau}{\sigma\nu}\;\frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty} e^{-(t-\alpha)^2/2\sigma^2}\, e^{-|\alpha|/\tau}\, d\alpha \quad . \tag{29}$$

Before evaluating the integrals in Eqs. (28) and (29) we discuss the integral $I_2(\nu,t)$ given by Eq. (22). It has been found convenient to write $I_2(\nu,t)$ in the following form:

$$I_2(\nu,t) = I_{21}(\nu,t) - I_{21}(\nu,t-\tau\nu) \tag{30}$$

where,

$$I_{21}(\nu,t) = -\frac{1}{2\pi\,\nu^2}\int_{-\infty}^{\infty} \frac{e^{-\sigma^2\omega^2/2}}{1+j\omega\tau}\, e^{j\omega\tau}\, d\omega \tag{31}$$

It can be shown that $I_{21}(\nu,t)$ can be written as follows:

$$I_{21}(\nu,t) = I_{21}'(\nu,t) + I_{21}''(\nu,t) \quad , \tag{32}$$

where

$$I'_{21}(\nu, t) = -\frac{1}{\nu^2}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-\sigma^2\omega^2/2}}{1+\omega^2\tau^2}\,e^{j\omega\tau}\,d\omega \quad , \tag{33}$$

$$I''_{21}(\nu, t) = -\tau\frac{\partial}{\partial t}I'_{21}(\nu, t) \quad . \tag{34}$$

Following the similar procedure as before, Eq. (33) can be transformed into the following form:

$$I'_{21}(\nu, t) = -\sqrt{\frac{\pi}{2}}\,\frac{1}{\nu^2\tau}\,\frac{1}{\sigma}\int_{-\infty}^{\infty} e^{-(t-\alpha)^2/2\sigma^2}\,e^{-|\alpha|/\tau}\,d\alpha \quad . \tag{35}$$

$I''_{21}(\nu, t)$ can be evaluated by making use of Eqs. (33) and (34).

It can be shown that $I'_1(\nu, t)$ as given by Eq. (28) may be expressed in the following form:

$$I'_1(\nu, t) = \frac{\pi}{2}\frac{1}{\nu\tau}e^{\sigma^2/2\tau^2} \times \left[-\left\{1+\operatorname{erf}\frac{1}{\sqrt{2}}\left(\frac{t}{\sigma}+\frac{\sigma}{\tau}\right)\right\}e^{t/\tau} + \left\{1-\operatorname{erf}\frac{1}{\sqrt{2}}\left(\frac{t}{\sigma}+\frac{\sigma}{\tau}\right)\right\}e^{t/\tau}\right] , \tag{36}$$

where the error function is defined as follows

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt \quad . \tag{37}$$

After using Eqs. (36) and (29), $I_1''(\nu, t)$ is obtained as in the following

$$I_1''(\nu, t) = \frac{\pi}{2} \frac{1}{\nu\tau} \left[\frac{2}{\sqrt{2}} \frac{\tau}{\sigma} e^{-t^2/2\sigma^2} - e^{-\sigma^2/2\tau^2} \left\{ (1 + \operatorname{erf} \frac{1}{\sqrt{2}} (\frac{t}{\sigma} - \frac{\sigma}{\tau})) e^{-t/\tau} + (1 - \operatorname{erf} \frac{1}{\sqrt{2}} (\frac{t}{\sigma} + \frac{\sigma}{\tau})) e^{t/\tau} \right\} \right] . \qquad (38)$$

Introducing Eqs. (38) and (36) into Eq. (23), we obtain the following expression for $I_1(\nu, t)$:

$$I_1(\nu, t) = \frac{\pi}{\nu\sigma} \left[-\frac{\sigma}{\tau} e^{\sigma^2/2\tau^2} \left\{ 1 + \operatorname{erf} \frac{1}{\sqrt{2}} (\frac{t}{\sigma} - \frac{\sigma}{\tau}) \right\} e^{-t/\tau} + \frac{1}{\sqrt{2}} e^{-t^2/2\sigma^2} \right] . \qquad (39)$$

$I_{21}'(\nu, t)$ as given by Eq. (35) can be evaluated in closed form and is given by:

$$I_{21}'(\nu, t) = -\frac{\pi}{2} \frac{1}{\nu^2\tau} e^{\sigma^2/2\tau^2} \left[\left\{ 1 + \operatorname{erf} \frac{1}{\sqrt{2}} (\frac{t}{\sigma} - \frac{\sigma}{\tau}) \right\} e^{-t/\tau} + \left\{ 1 - \operatorname{erf} \frac{1}{\sqrt{2}} (\frac{t}{\sigma} + \frac{\sigma}{\tau}) \right\} e^{t/\tau} \right] . \qquad (40)$$

After using Eqs. (34) and (40) $I_{21}''(\nu, t)$ is obtained as follows:

$$I_{21}''(\nu, t) = \frac{\pi}{2} \frac{1}{\nu^2\tau} e^{\sigma^2/2\tau^2} \left[-\left\{ 1 + \operatorname{erf} \frac{1}{\sqrt{2}} (\frac{t}{\sigma} - \frac{\sigma}{\tau}) \right\} e^{-t/\tau} + \left\{ 1 - \operatorname{erf} \frac{1}{\sqrt{2}} (\frac{t}{\sigma} + \frac{\sigma}{\tau}) \right\} e^{t/\tau} \right] . \qquad (41)$$

By making use of Eqs. (40), (41) and (32) we obtain:

$$I_{21}(\nu, t) = -\pi \frac{1}{\nu^2 \tau} e^{\sigma^2/2\tau^2} \left\{ 1 + \mathrm{erf} \frac{1}{\sqrt{2}} \left(\frac{t}{\sigma} - \frac{\sigma}{\tau} \right) \right\} e^{-t/\tau} \quad . \tag{42}$$

Introducing Eq. (42) into Eq. (30) we obtain,

$$\begin{aligned} I_2(\nu, t) = \frac{\pi}{\sigma \nu} \cdot \frac{\sigma}{\nu \tau} e^{\sigma^2/2\tau^2} \Bigg[& - \left\{ 1 + \mathrm{erf} \frac{1}{\sqrt{2}} \left(\frac{t}{\sigma} - \frac{\sigma}{\tau} \right) \right\} e^{-t/\tau} + \\ & + \left\{ 1 + \mathrm{erf} \frac{1}{2} \left(\frac{t - \tau\nu}{\sigma} - \frac{\sigma}{\tau} \right) \right\} e^{-\frac{t-\tau\nu}{\tau}} \Bigg] \quad . \end{aligned} \tag{43}$$

After introducing Eqs. (42) and (43) into Eq. (20) we finally obtain the following:

$$\begin{aligned} I(\nu, t) = \frac{\pi}{\nu\sigma} \Bigg[& \frac{1}{\sqrt{2}} e^{-t^2/2\sigma^2} - \frac{\sigma}{\tau} \left(1 + \frac{1}{\nu} \right) \left\{ 1 + \mathrm{erf} \frac{1}{\sqrt{2}} \left(\frac{t}{\sigma} - \frac{\sigma}{\tau} \right) \right\} e^{-t/\tau} e^{\sigma^2/2\tau^2} \\ & + \frac{\sigma}{\nu\tau} e^{\sigma^2/2\tau^2} \left\{ 1 + \mathrm{erf} \frac{1}{\sqrt{2}} \left(\frac{t - \tau\nu}{\sigma} - \frac{\sigma}{\tau} \right) \right\} e^{-\frac{t-\tau\nu}{\tau}} \Bigg] . \end{aligned} \tag{44}$$

Changing ν to ν' in Eq. (44) we obtain:

$$\begin{aligned} I(\nu', t) = \frac{\pi}{\nu'\sigma} \Bigg[& \frac{1}{\sqrt{2}} e^{-t^2/2\sigma^2} - \frac{\sigma}{\tau} \left(1 + \frac{1}{\nu'} \right) \left\{ 1 + \mathrm{erf} \frac{1}{\sqrt{2}} \left(\frac{t}{\sigma} - \frac{\sigma}{\tau} \right) \right\} e^{-t/\tau} e^{\sigma^2/2\tau^2} \\ & + \frac{\sigma}{\nu'\tau} e^{\sigma^2/2\tau^2} \left\{ 1 + \mathrm{erf} \frac{1}{\sqrt{2}} \left(\frac{t - \tau\nu'}{\sigma} - \frac{\sigma}{\tau} \right) \right\} e^{-\frac{t-\tau\nu'}{\tau}} \Bigg] . \end{aligned} \tag{45}$$

Thus we obtain the following expression for $I(\nu, t) + I(\nu', t)$:

$$I(\nu, t) + I(\nu', t) = \frac{\pi}{\sigma} \Bigg[\frac{1}{\sqrt{2}} \left(\frac{1}{\nu} + \frac{1}{\nu'} \right) e^{-t^2/2\sigma^2}$$

$$- \frac{\sigma}{\tau} e^{-t/\tau} e^{\sigma^2/2\tau^2} \left\{1 + \text{erf} \frac{1}{\sqrt{2}} \left(\frac{t}{\sigma} - \frac{\sigma}{\tau}\right)\right\} \left(\frac{\nu+1}{2} + \frac{\nu'+1}{\nu'^2}\right)$$

$$+ \frac{\sigma}{\tau} e^{\sigma^2/2\tau^2} \left\{ \left(1 + \text{erf} \frac{1}{\sqrt{2}} \left(\frac{t-\nu\tau}{\sigma} - \frac{\sigma}{\tau}\right)\right) \frac{e^{-\frac{t-\nu\tau}{\tau}}}{\nu^2} \right.$$

$$\left. \left. + \left(1 + \text{erf} \frac{1}{\sqrt{2}} \left(\frac{t-\nu'\tau}{\sigma} - \frac{\sigma}{\tau}\right)\right) \frac{e^{-\frac{t-\nu'\tau}{\tau}}}{\nu'^2} \right\} \right] \qquad (46)$$

Introducing Eq. (40) into Eq. (17) we obtain the following expression for the far field waveform produced by the antenna:

$$e_\theta(\theta, t) = \frac{\pi}{\psi} \sqrt{\frac{\pi}{2}} \sin\theta \left[\frac{1}{\sqrt{2}} \left(\frac{1}{\nu} + \frac{1}{\nu'}\right) e^{-t^2/2\sigma^2} \right.$$

$$- \frac{\sigma}{\tau} e^{-t/\tau} e^{\sigma^2/2\tau^2} \left\{1 + \text{erf} \frac{1}{\sqrt{2}} \left(\frac{t}{\sigma} - \frac{\sigma}{\tau}\right)\right\} \left(\frac{\nu+1}{\nu^2} + \frac{\nu'+1}{\nu'^2}\right)$$

$$+ \frac{\sigma}{\tau} e^{\sigma^2/2\tau^2} \left\{1 + \text{erf} \frac{1}{\sqrt{2}} \left(\frac{t-\nu\tau}{\sigma} - \frac{\sigma}{\tau}\right)\right\} \frac{e^{-\frac{t-\nu\tau}{\tau}}}{\nu^2}$$

$$\left. + \frac{\sigma}{\tau} e^{\sigma^2/2\tau^2} \left\{1 + \text{erf} \frac{1}{\sqrt{2}} \left(\frac{t-\nu'\tau}{\sigma} - \frac{\sigma}{\tau}\right)\right\} \frac{e^{-\frac{t-\nu'\tau}{\tau}}}{\nu'^2} \right] , \qquad (47)$$

where,

$$\nu = 1 + \cos\theta ,$$

$$\nu' = 1 - \cos\theta ,$$

In the direction $\theta = \pi/2$, $\nu = \nu' = 1$ and we obtain from (47):

$$e_\theta(\frac{\pi}{2}, t) = \frac{\pi}{\psi}\sqrt{2\pi}\left[\frac{e^{-t^2/2\sigma^2}}{\sqrt{2}}\right.$$

$$- \frac{2\sigma}{\tau} e^{\sigma^2/2\tau^2}\left\{1 + \mathrm{erf}\frac{1}{\sqrt{2}}(\frac{t}{\sigma} - \frac{\sigma}{\tau})\right\} e^{-t/\tau}$$

$$\left. + \frac{\sigma}{\tau} e^{\sigma^2/2\tau^2}\left\{1 + \mathrm{erf}\frac{1}{\sqrt{2}}(\frac{t-\tau}{\sigma}) - \frac{\sigma}{\tau})\right\} e^{-\frac{t-\tau}{\tau}}\right] . \quad (48)$$

At time $t = 0$ the Eq. (47) reduces to the following:

$$e_\theta(\theta, 0) = \frac{\pi}{\psi}\sqrt{\frac{\pi}{2}} \sin\theta\left[\frac{1}{\sqrt{2}}(\frac{1}{\nu} + \frac{1}{\nu'})\right.$$

$$- \frac{\sigma}{\tau} e^{\sigma^2/2\tau^2}\left\{1 - \mathrm{erf}\frac{\sigma}{\sqrt{2}\tau}\right\}\left(\frac{\nu+1}{\nu^2} + \frac{\nu'+1}{\nu'^2}\right)$$

$$+ \frac{\sigma}{\tau} e^{\sigma^2/2\tau^2}\left\{1 - \mathrm{erf}\frac{1}{\sqrt{2}}\left(\frac{\nu\tau}{\sigma} + \frac{\sigma}{\tau}\right)\right\}\frac{e^{\nu}}{\nu^2}$$

$$\left. + \frac{\sigma}{\tau} e^{\sigma^2/2\tau^2}\left\{1 - \mathrm{erf}\frac{1}{\sqrt{2}}\left(\frac{\nu'\tau}{\sigma} + \frac{\sigma}{\tau}\right)\right\}\frac{e^{\nu'}}{\nu'^2}\right] . \quad (49)$$

The broadside field produced by the antenna at time $t = 0$ is obtained by setting $\theta = \pi/2$ and $\nu = \nu' = 1$ in Eq. (48) and is given by:

$$e_\theta(\frac{\pi}{2}, 0) = \frac{\pi}{\psi}\sqrt{2\pi}\left[\frac{1}{\sqrt{2}} - \frac{2\sigma}{\tau} e^{\sigma^2/2\tau^2}\left\{1 - \mathrm{erf}\,(\frac{\sigma}{\sqrt{2}\tau})\right\}\right.$$

$$\left. + \frac{\sigma}{\tau} e^{\frac{\sigma^2}{2\tau^2}+1}\left\{1 - \mathrm{erf}\frac{1}{\sqrt{2}}(\frac{\tau}{\sigma} + \frac{\sigma}{\tau})\right\}\right] . \quad (50)$$

For both large positive and negative time, i.e., $t \gg \tau$, and $\tau > \sigma$, it can be shown that Eq. (48) can be approximated by the following:

$$e_\theta\left(\frac{\pi}{2}, t\right) \underset{|t| \gg \tau}{\sim} \frac{\pi\sqrt{2\pi}}{\psi} \left[\frac{e^{-t^2/2\sigma^2}}{\sqrt{2}} + \underset{|t| \gg \tau,\ \tau > \sigma}{\frac{\sigma}{\tau} e^{\sigma^2/2\tau^2} \left\{ 1 + \operatorname{erf}\frac{t}{2\sigma} \right\} (e-2)} \right] e^{-t/\tau} . \tag{51}$$

It can be seen from Eq. (50) that $e_\theta(\frac{\pi}{2}, t)$ 0 as $t \to \infty$ which is as it should be.

It can be seen from Eqs. (47) and (48) that the radiated waveform is different from Gaussian and also that it is a function of the parameter σ/τ. It is thus anticipated that the radiated pulse shape will strongly depend on the ratio of the input pulse width to the transit time on the antenna.

6. TIME DEPENDENT CURRENT DISTRIBUTION $i(z, t)$

In some cases it may be of interest to know explicitly the time dependent current distribution on the Gaussian pulse excited loaded antenna. The current distribution on the antenna excited by a harmonically time dependent unit slice generator is given by Eq. (10). For a Gaussian input signal of the form given by Eqs. (4) and (5), the time dependent current distribution on the antenna is obtained by using the following relations:

$$i(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{I}(z, \omega) \tilde{V}(\omega) e^{j\omega t} \, d\omega \quad , \tag{52}$$

where $\tilde{I}(z, \omega)$ and $\tilde{V}(\omega)$ are given by Eqs. (10) and (5) respectively. The integral in Eq. (52) can be evaluated in closed form in a manner discussed in section 5. The final expression for the time dependent current distribution is given by the following:

$$i(z,t) = \frac{\pi\sqrt{2\pi}}{\eta_o \psi}\left[1 - \frac{|z|}{L}\right] \times$$

$$\times \left[\frac{e^{-(t-|z|/c)^2/2\sigma^2}}{\sqrt{2}} - \right.$$

$$\left. - \frac{\sigma}{\tau}\, e^{\sigma^2/2\tau^2}\left\{1 + \mathrm{erf}\frac{1}{\sqrt{2}}\left(\frac{t - z/c}{\sigma}\right)\right\} e^{-\frac{t-|z|/c}{\tau}}\right] . \quad (53)$$

The input current $i(0,t)$ on the antenna when excited by an input signal of the form $V(t) = e^{-t^2/2\sigma^2}$ is given by:

$$i(0,t) = \frac{\pi\sqrt{2\pi}}{\eta_o \psi}\left[\frac{e^{-t^2/2\sigma^2}}{\sqrt{2}} - \frac{\sigma}{\tau}\, e^{\sigma^2/2\sigma^2}\left\{1 + \mathrm{erf}\frac{1}{\sqrt{2}}\left(\frac{t}{\sigma} - \frac{\sigma}{\tau}\right)\right\} e^{-t/\tau}\right] . \quad (54)$$

7. NUMERICAL RESULTS

In this section we discuss some of the selected results obtained by numerical computation of the waveform expressions derived in section 5. Figs. 1 and 2 show $e_\theta(\frac{\pi}{2}, t)$ vs t for $\sigma = 1.0$ ns and $\sigma = 0.471$ ns respectively and for two different values of the parameter ψ. In both cases the transit time on the antenna is $\tau = 3.33$ ns. The corresponding values of $e_\theta(\frac{\pi}{2}, t)$ obtained by direct numerical investigation are also shown in the same figures for comparison.

The direct numerical investigation of the problem has been discussed in a separate note[1]. However, it is appropriate here to give a very brief outline of the numerical method. In this method, at first, the current distribution $\tilde{I}(z,\omega)$ on the harmonically excited loaded antenna is obtained by numerically solving an appropriate form of Hallen's integral equation. The transfer function $\tilde{f}_\theta(\theta,\omega)$ of the antenna and the spectral density $\tilde{e}_\theta(\theta, \omega)$ of the radiated waveform are then obtained by numerically evaluating Eq. (6) with the help of fast Fourier inversion technique.

The results shown in Figs. 1 and 2 indicate that the agreement between the analytical and numerical solutions may be considered to be satisfactory. In particular, in the main pulse region the two results agree very well.

Figs. 3 and 4 show the radiated waveforms obtained from Eq. (47) for three different values of θ and for $\sigma = 1$ ns, and $\sigma = 0.471$ ns. The transit time on the antenna is $\tau = 3.33$ ns.

8: CONCLUSION

In the above we have developed analytic expressions for the waveforms radiated by a resistively loaded linear antenna excited by Gaussian pulse. The antenna is assumed to be loaded such that it sustains a pure traveling wave of current. The analytic results have been compared with those obtained by direct numerical means. The general agreement between the two results has been found to be satisfactory. It is thus concluded that the various expressions given above may be used to study the behavior of waveforms radiated by a resistively loaded linear antenna excited by a Gaussian pulse.

9. REFERENCES

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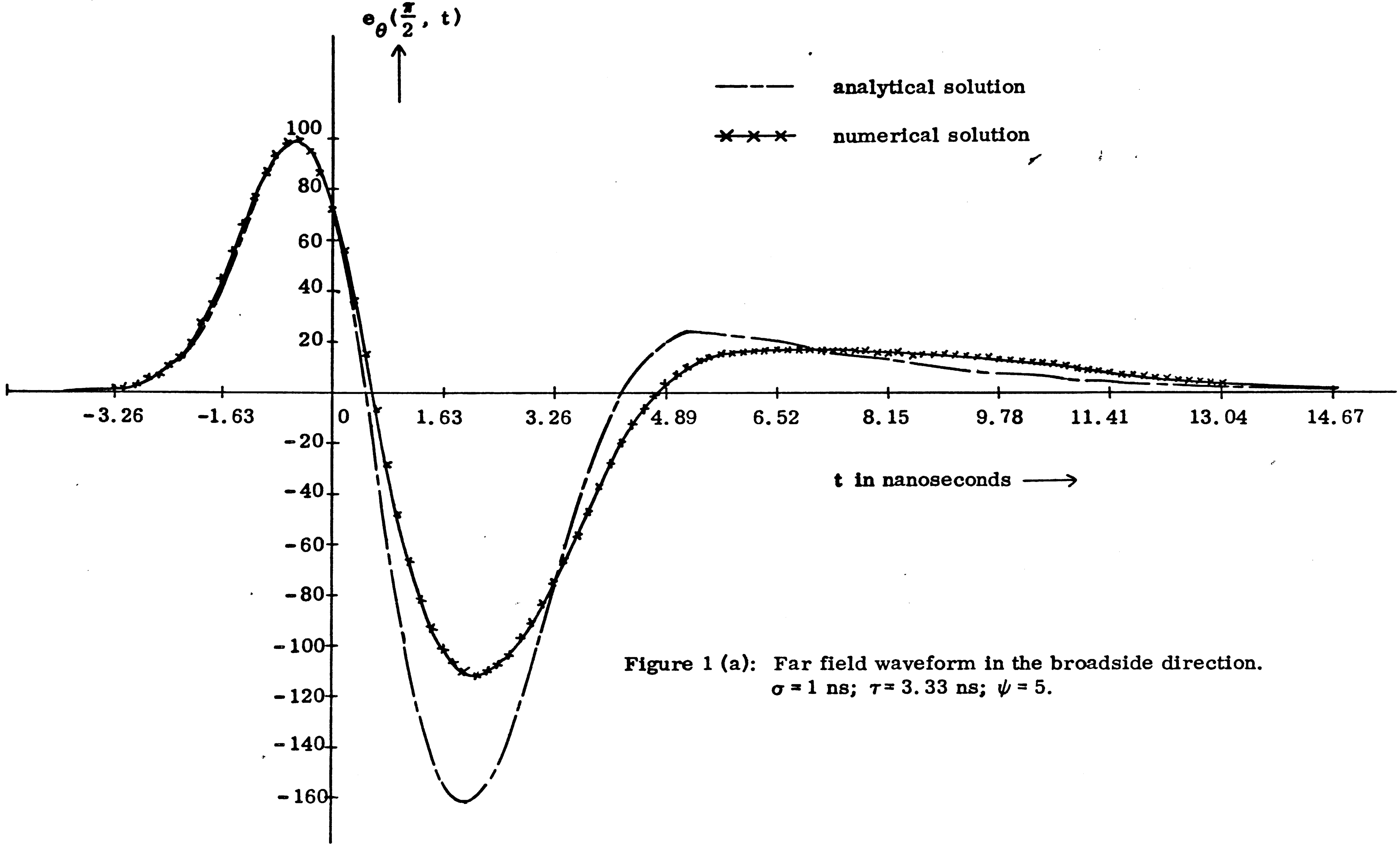

Figure 1 (a): Far field waveform in the broadside direction. $\sigma = 1$ ns; $\tau = 3.33$ ns; $\psi = 5$.

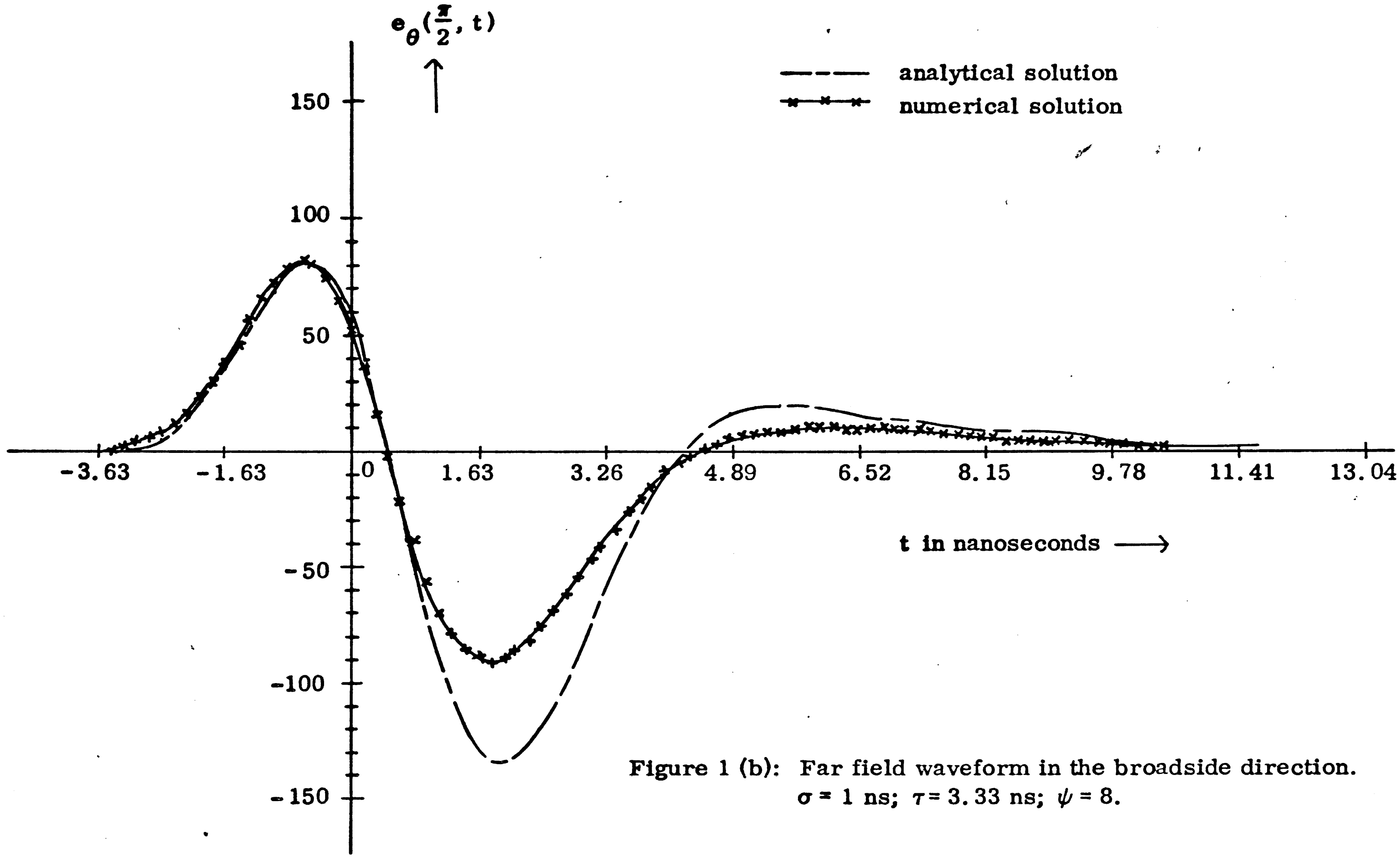

Figure 1 (b): Far field waveform in the broadside direction. $\sigma = 1$ ns; $\tau = 3.33$ ns; $\psi = 8$.

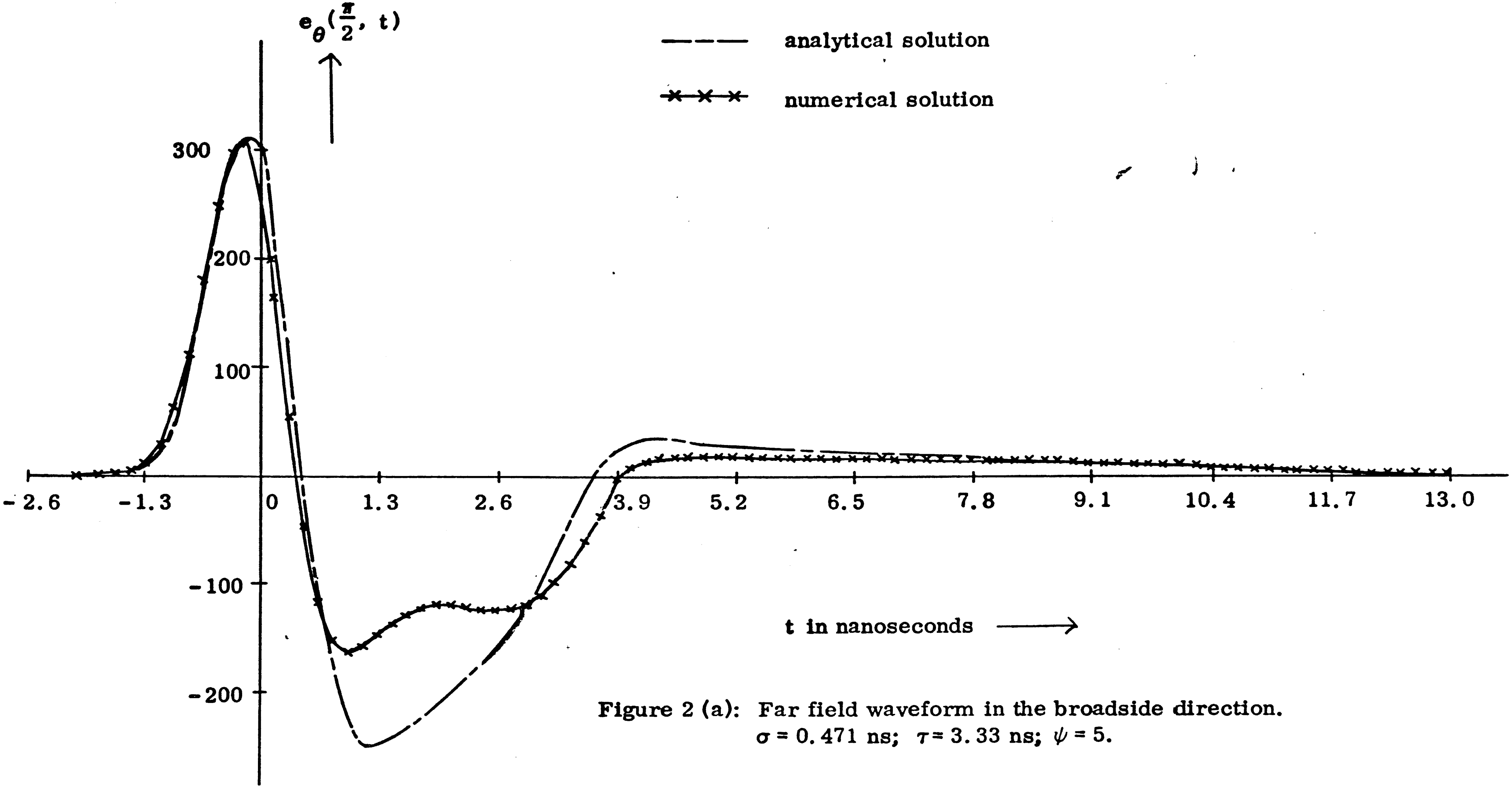

Figure 2 (a): Far field waveform in the broadside direction.
$\sigma = 0.471$ ns; $\tau = 3.33$ ns; $\psi = 5$.

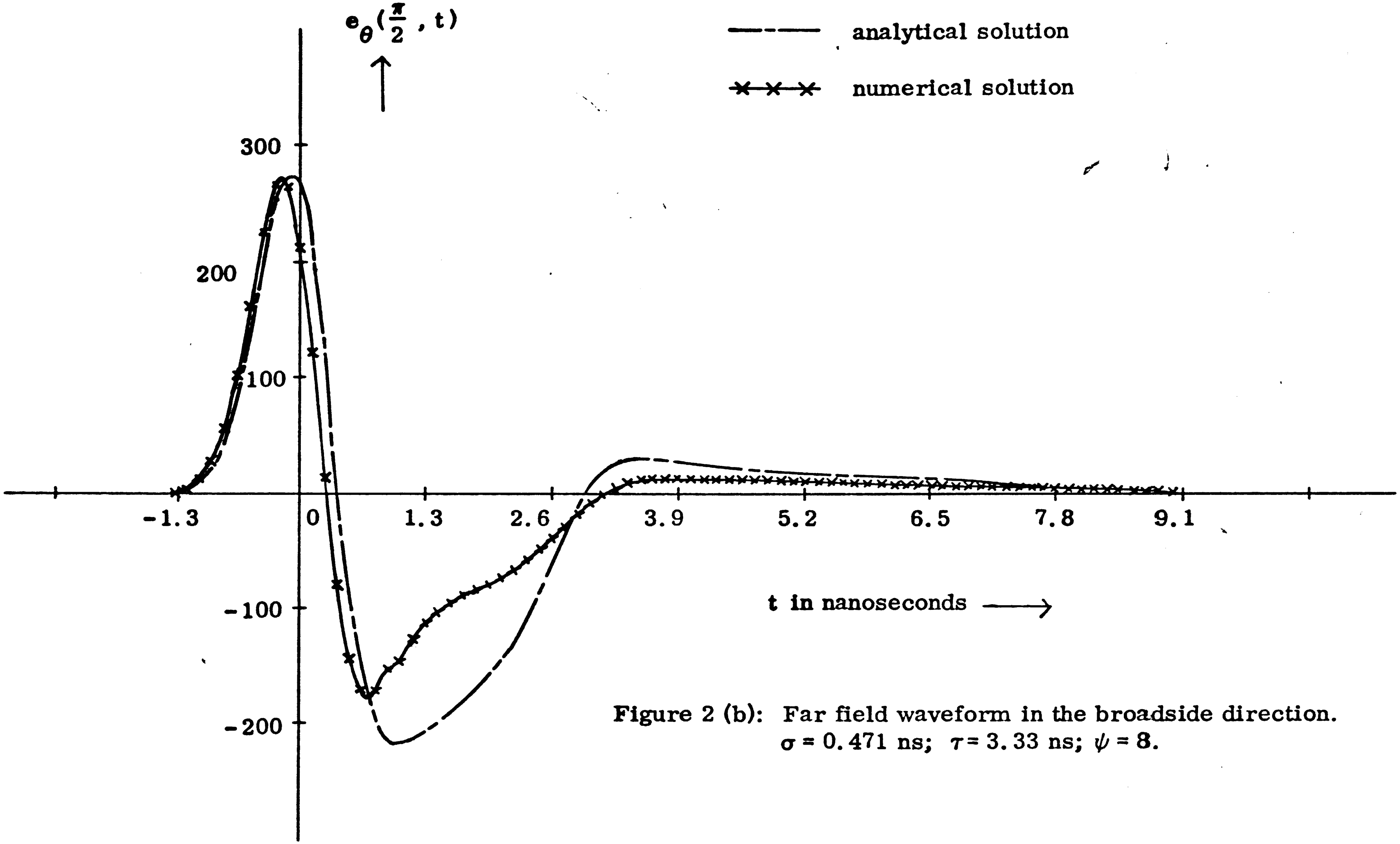

Figure 2 (b): Far field waveform in the broadside direction.
$\sigma = 0.471$ ns; $\tau = 3.33$ ns; $\psi = 8$.

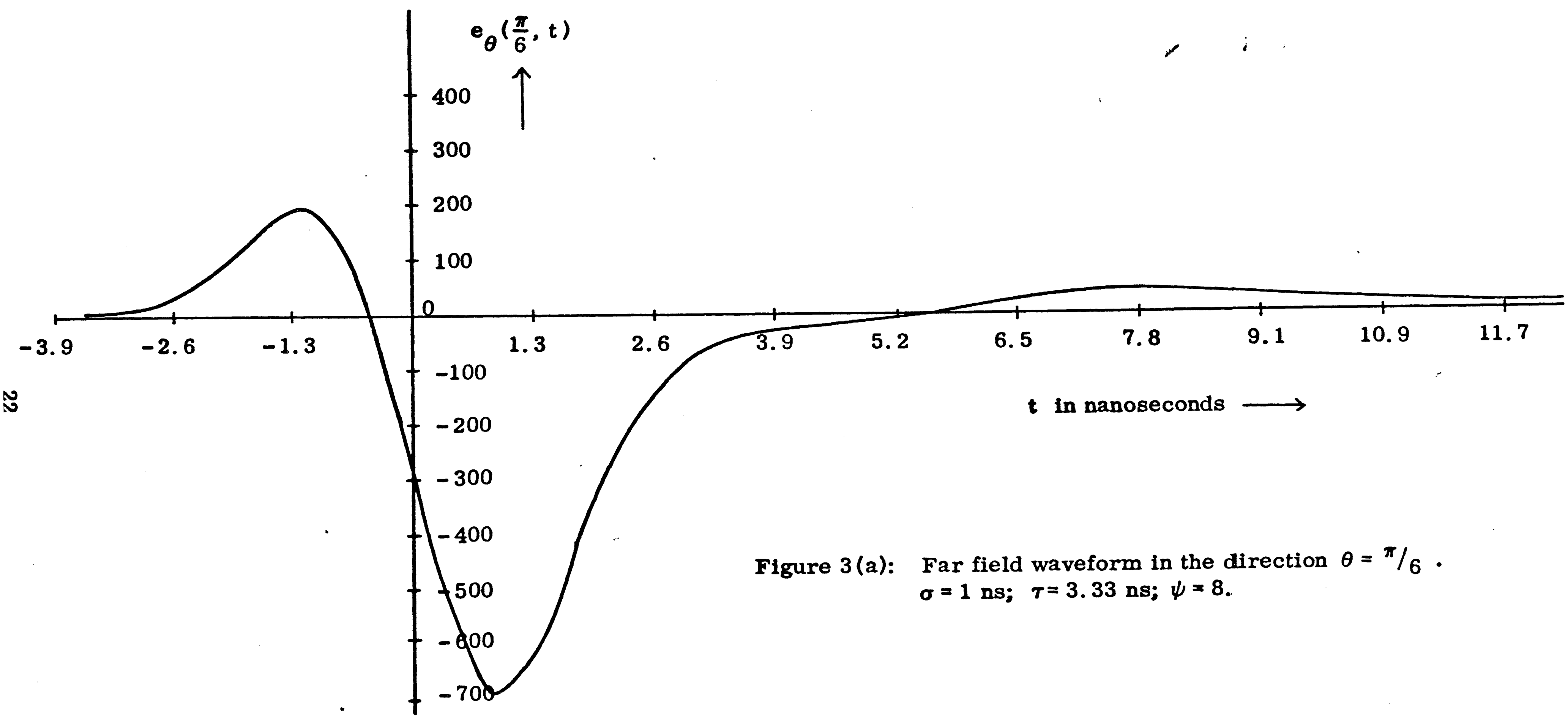

Figure 3(a): Far field waveform in the direction $\theta = \pi/6$.
$\sigma = 1$ ns; $\tau = 3.33$ ns; $\psi = 8$.

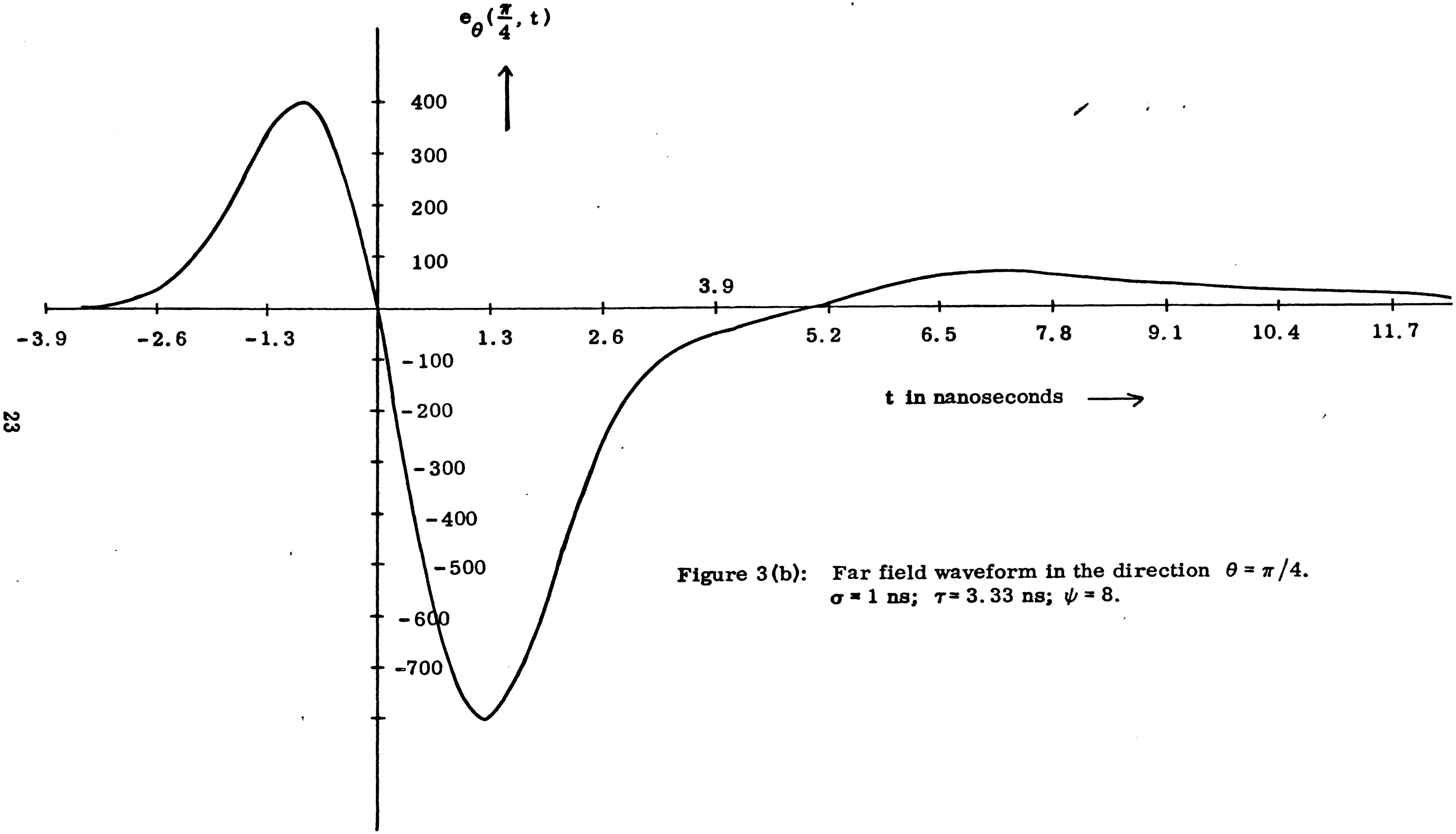

Figure 3(b): Far field waveform in the direction $\theta = \pi/4$.
$\sigma = 1$ ns; $\tau = 3.33$ ns; $\psi = 8$.

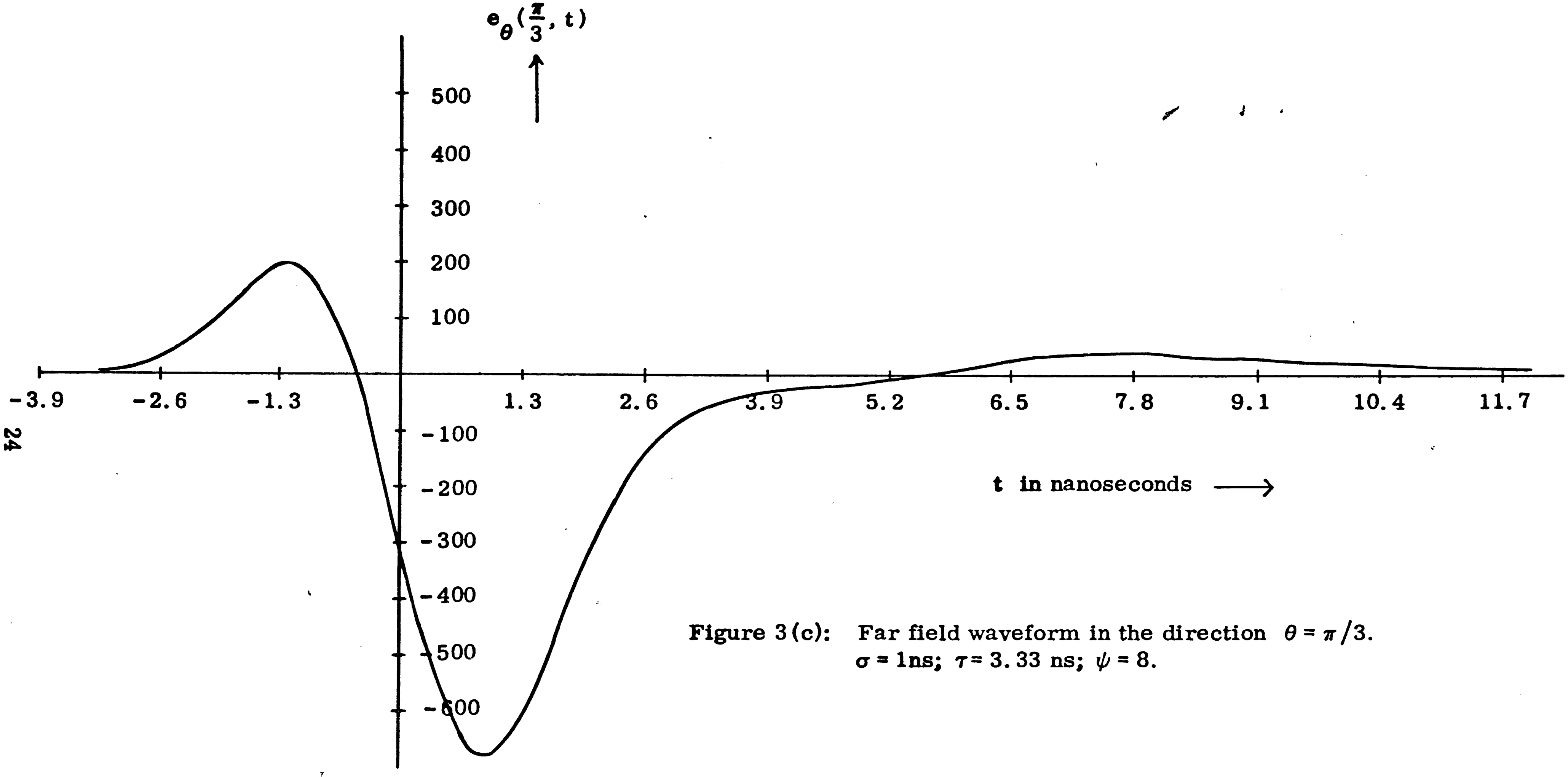

Figure 3(c): Far field waveform in the direction $\theta = \pi/3$. $\sigma = 1\text{ns}$; $\tau = 3.33$ ns; $\psi = 8$.

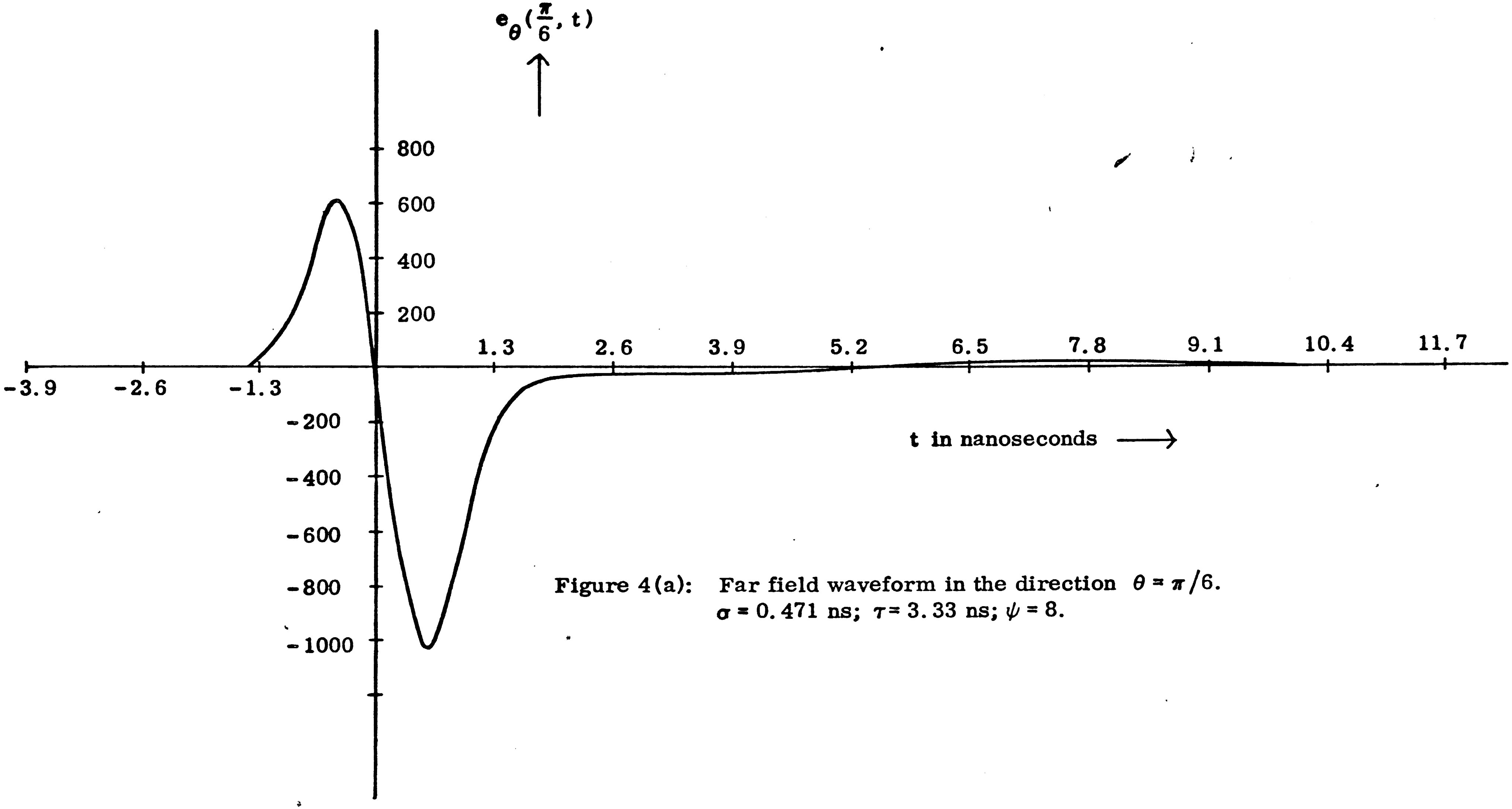

Figure 4(a): Far field waveform in the direction $\theta = \pi/6$.
$\sigma = 0.471$ ns; $\tau = 3.33$ ns; $\psi = 8$.

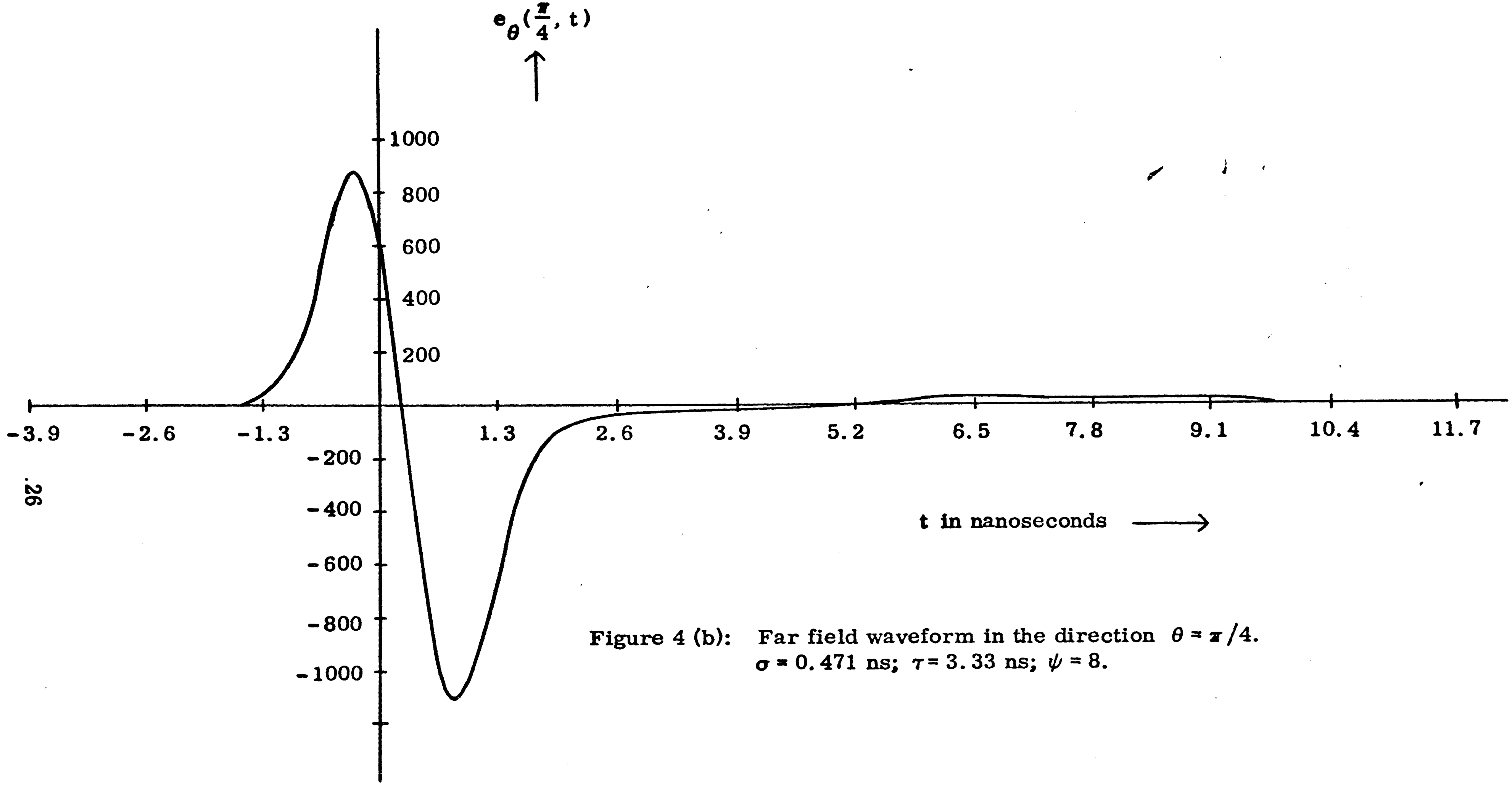

Figure 4 (b): Far field waveform in the direction $\theta = \pi/4$.
$\sigma = 0.471$ ns; $\tau = 3.33$ ns; $\psi = 8$.

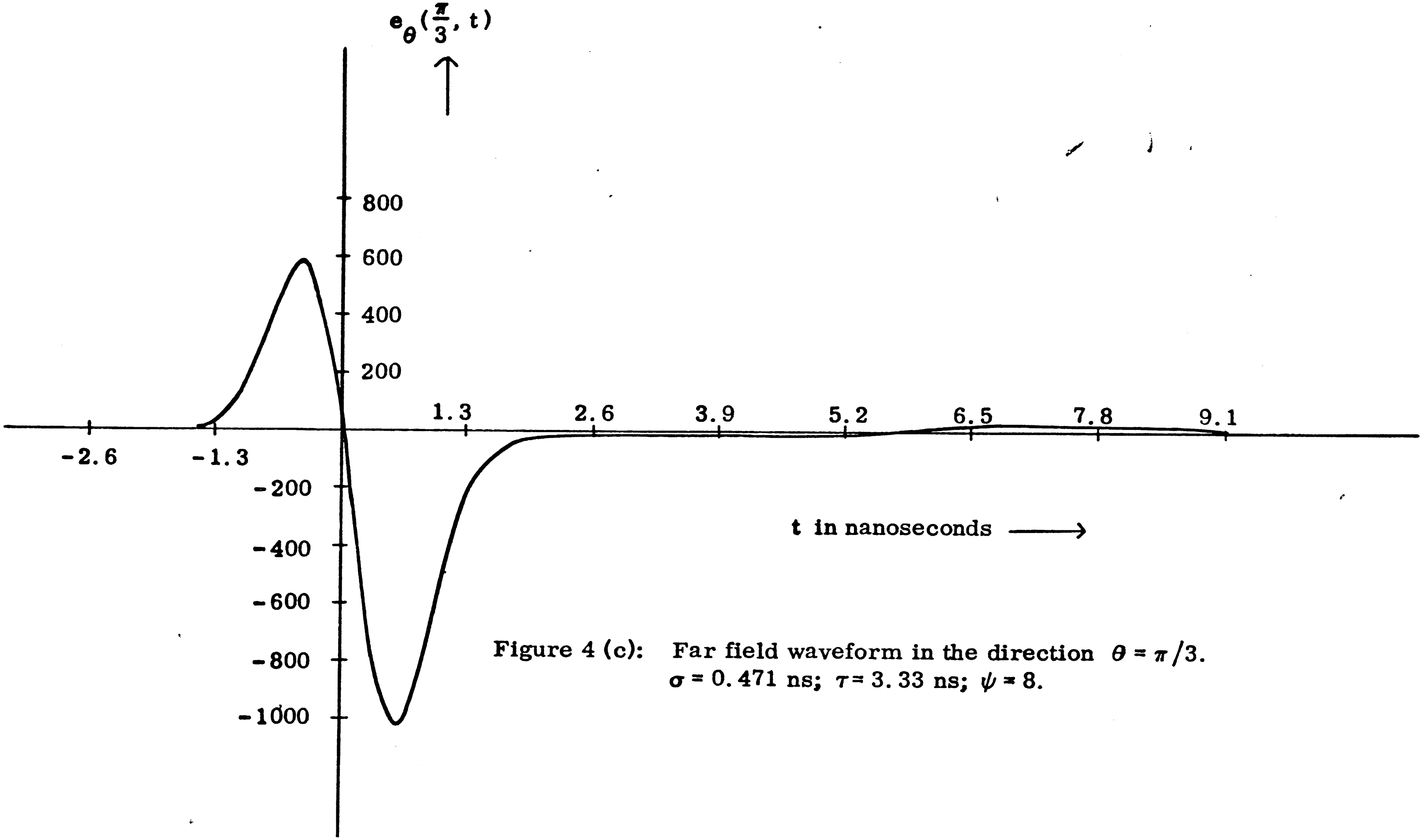

Figure 4 (c): Far field waveform in the direction $\theta = \pi/3$. $\sigma = 0.471$ ns; $\tau = 3.33$ ns; $\psi = 8$.