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SECOND ORDER DIFFRACTION BY A RING DISCONTINUITY

By

Eugene F. Knott and Thomas B. A. Senior

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1400 Wilson Boulevard
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Ann Arbor, Michigan
48105

ABSTRACT

SECOND ORDER DIFFRACTION BY A RING SINGULARITY

by Eugene F. Knott and Thomas B. A. Senior

For a ring discontinuity in slope as at the base of a right circular cone, the second order (re-) diffracted field is examined in the general case of bistatic scattering. It is shown that the ray paths are specified by a quartic equation whose solution is discussed. Selected results are presented, and an expression for the field contribution of any one such path is derived. An alternative formulation of the problem using equivalent currents leads to a compact expression of the complete second order field as a double line integral which, when evaluated by the stationary phase method, gives precisely the wide angle contributions previously obtained. However, the integral expression is also finite in the direction of the axial caustic and can be used to find the caustic matching functions in second order GTD. These take the form of complementary Fresnel integrals whose practical effectiveness is verified by a comparison of the results of a numerical evaluation of the integral with the caustically-matched expression for the field in the particular case of backscattering.

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INTRODUCTION

The geometrical theory of diffraction is a valuable ray technique for estimating high frequency scattering. It differs from geometrical optics in its inclusion of diffracted rays created whenever an incident ray strikes a surface singularity. These rays may in turn strike another singularity, or another element of the same singularity, producing second order (or re-) diffracted rays, and so on. A particular surface singularity of interest is the discontinuity in slope at the base of a right circular cone or the edge of a disc. Second order diffraction is then produced by rays diffracted across the case of the cone or across either face of the disc, and this report is devoted to a detailed examination of such diffraction in the general case of bistatic scattering.

The first task in any GTD analysis is the determination of the ray paths. This is discussed in Chapter II where it is shown that the second order paths are specified by a quartic equation whose solution can only be obtained numerically. In general there are either two or four such paths whose positions are functions of the angles of incidence and scattering. The problem is greatly simplified in the particular case of bistatic scattering in the plane of incidence. An analytical solution is now possible, and whereas two of the paths remain fixed in the plane of incidence, the other two (when they exist) rotate in opposite directions with increasing angles of incidence or scattering. The four paths always exist in backscattering and the latter pair are then the 'migrating' paths first discovered by Knott et al (1971).

The field contribution attributable to any path can be found using standard GTD techniques and this is determined in Chapter III for an arbitrary second order path. Not surprisingly, the expression is rather complicated, with the main complexity coming from the form of the divergence factors. These factors are an essential feature of all ray-optical techniques and are one source of the numerical inconvenience common to them all. In addition, a caustic occurs whenever a divergence factor becomes infinite, and since the axial backscattering direction is such a caustic, the field expression obtained in Chapter III is basically a wide-angle one.

An alternative approach to the determination of the second order contribution to the field is to use the concept of (GTD) equivalent currents. A method for specifying these currents in first order diffraction was developed by Knott and Senior (1973), and by repeated application of the same procedure, a compact expression of the complete second order field is obtained as a double line integral. This is discussed in Chapter IV and, as required, a stationary phase evaluation of the integral yields the precise wide angle contributions previously found. Moreover, the integral expression is a continuous function of the incidence and scattering angles, and since it is finite even in the direction of the axial caustic, it provides a means for determining the caustic matching functions in second order diffraction. These take the form of complementary Fresnel integrals and, in the particular case of backscattering, are similar in form (but not in detail) to the matching functions postulated by Senior and Uslenghi (1971) based only on physical reasoning. The practical effectiveness of the new functions is verified in Chapter V by a comparison of the results of a numerical evaluation of the integral with the caustically-matched expression for the field.

II

SECOND ORDER RAY PATHS ACROSS A DISK OR CONE

The prescription for finding the diffracted fields from edges via GTD is straightforward enough that the method can be applied repeatedly in order to account for multiple diffraction due to interactions between edges. In principle this can be done for as many interactions as are desired, but the geometrical complexities of doing so, coupled with the asymptotic nature of the diffraction coefficients, have inhibited the exploration of all higher order interactions beyond the second. In this chapter we examine the nature of second order ray paths across flat circular surfaces, such as the base of a right circular cone or the face of a disk. Evaluation of the second order far diffracted fields is reserved for Chapters III and IV.

GTD is a specular theory, in that incident and diffracted rays must subtend equal angles with respect to the local edge tangent. There are only two such points on a ring discontinuity where this special condition is satisfied for singly diffracted rays, and they are called "flash points". The situation is much more complex for double diffraction, however, for there may be as many as eight flash points on the rim, the consequence of four distinct, admissible ray paths crossing the face of the disk or cone. In studying second order backscattering contributions from a metallic disk, Knott et al. (1971) discovered the existence of two pairs of flash points, of which one pair migrated with changing aspect angle while the other pair remained stationary. This happens to be a specialization of the general case of bistatic scattering and we now know that the four flash points actually include a total of four coincident pairs. Moreover, all eight flash points, when they exist, move with changing bistatic angles.

In certain bistatic regions only two pairs of flash points exist and, as will be shown in a moment, this is the consequence of the mutual annihilation of two pairs. Of the four pairs that can exist, three pairs move in a forward direction with changing angle while one pair moves backward; when the retreating pair meets an advancing pair, all four suddenly disappear and only two pairs survive.

The particular orientations of the incident and scattering directions at which this occurs have not yet been determined analytically, but the abrupt disappearance will cause a jump in the far scattered fields, unless, of course, the diffraction coefficients become zero there. This is one of the shortcomings of wide-angle GTD.

Figure 2-1 illustrates the geometry of a single ray crossing the face of a disk or the base of a right circular cone. Although the incident and scattered rays

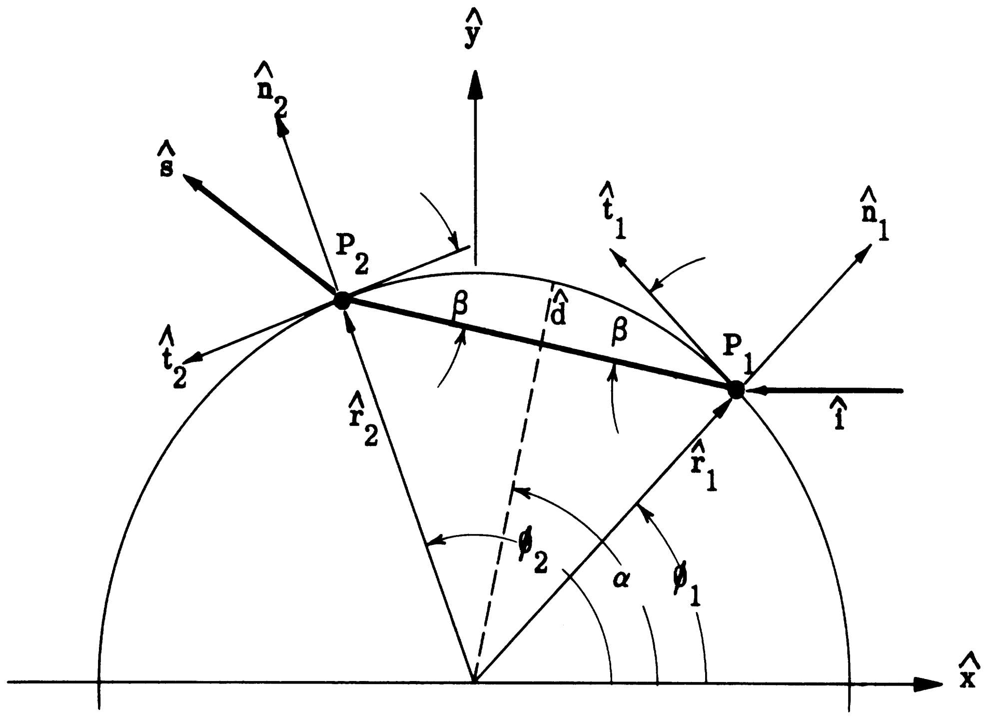

FIG. 2-1: Geometry of a ray crossing the face of a ring discontinuity.

$\hat{i}$ and $\hat{s}$ are shown as lying in the plane of the diagram for purposes of illustration, both $\hat{i}$ and $\hat{s}$ are inclined to this plane in the general case. An incident ray $\hat{i}$ strikes the rim at some angle β measured from the local tangent vector $\hat{t}_1$ and, according to Keller's theory, spawns an infinity of rays all lying on generators of a forward cone whose half angle is precisely β and whose apex lies at the flash point. Only one of the diffracted rays is of interest, namely the one that traverses the face of the disk. It does so along a vector we call $\bar{d}$ and upon reaching another point of the rim excites another cone of rays whose apex is fixed at the second flash point. We are concerned only with the single ray on this second cone that moves off in the direction $\hat{s}$, the direction toward a remote observation point at which we seek to find the total scattered field. The two cones of diffraction must have the same half angle β.

The locations of the flash points may be found by demanding that incident and diffracted rays subtend the same angles at the flash points, which is tantamount to

$$\cos\beta = \hat{i}\cdot\hat{t}_1 = \hat{d}\cdot\hat{t}_1 = \hat{d}\cdot\hat{t}_2 = \hat{s}\cdot\hat{t}_2 \ , \tag{2.1}$$

where the subscripts refer to flash point P_1 or P_2. If we denote by ϕ_1 and ϕ_2 their angular positions, then $\beta = \frac{1}{2}(\phi_2-\phi_1)$, and it is our task to determine all ϕ_1 and ϕ_2 that satisfy (2.1).

The incident and scattered directions may be expressed as

$$\hat{i} = -\hat{x}\sin\gamma_i - \hat{z}\cos\gamma_i \ , \tag{2.2}$$

$$\hat{s} = \hat{x}\sin\gamma_s\cos\phi_s + \hat{y}\sin\gamma_s\sin\phi_s + \hat{z}\cos\gamma_s \ , \tag{2.3}$$

where γ_i, γ_s are polar angles measured from the z-axis, ϕ_s is the azimuthal angle of the scattered direction and, without loss of generality, incidence has been constrained to lie in the xz-plane. At the flash points unit normal and unit tangent vectors may be constructed,

$$\hat{n}_1 = \hat{x}\cos\phi_1 + \hat{y}\sin\phi_1 \ , \qquad \hat{n}_2 = \hat{x}\cos\phi_2 + \hat{y}\sin\phi_2 \ ,$$

$$\hat{t}_1 = -\hat{x}\sin\phi_1 + \hat{y}\cos\phi_1 \ , \qquad \hat{t}_2 = -\hat{x}\sin\phi_2 + \hat{y}\cos\phi_2 \ .$$

Thus, from (2.1) we have

$$\cos\beta = \sin\gamma_i\sin\phi_1 = \sin\gamma_s\sin(\phi_s-\phi_2) \ , \tag{2.4}$$

or, since we may express ϕ_2 in terms of ϕ_1 and β,

$$\cos\beta = \sin\gamma_i \sin\phi_1 = \sin\gamma_s \sin(\phi_s - \phi_1 - 2\beta) \ . \tag{2.5}$$

In this case, the two unknowns to be found are ϕ_1 and β, from which we may also determine ϕ_2.

More symmetry can be obtained than is revealed in (2.4) and (2.5) by choosing another pair of angles as the unknowns. Specifically, let

$$\phi_1 = \alpha - \beta \ , \tag{2.6}$$

so that

$$\cos\beta = \sin\gamma_i \sin(\alpha-\beta) = \sin\gamma_s \sin(\phi_s - \alpha - \beta) \ . \tag{2.7}$$

The first of (2.7) may be expanded as

$$\cos\beta = \sin\gamma_i (\sin\alpha\cos\beta - \cos\alpha\sin\beta) \ , \tag{2.8}$$

while the second is

$$\cos\beta = \sin\gamma_s \left\{\sin(\phi_s - \alpha)\cos\beta - \cos(\phi_s - \alpha)\sin\beta\right\} \tag{2.9}$$

Equating (2.8) and (2.9) produces

$$\tan\beta = \frac{\sin\gamma_i \sin\alpha - \sin\gamma_s \sin(\phi_s - \alpha)}{\sin\gamma_i \cos\alpha - \sin\gamma_s \cos(\phi_s - \alpha)} \ . \tag{2.10}$$

From (2.10) we may express $\sin\beta$ and $\cos\beta$ in terms of α, and substitution of the results into (2.8) or (2.9) now yields

$$\sin\gamma_i \cos\alpha - \sin\gamma_s \cos(\phi_s - \alpha) + \sin\gamma_i \sin\gamma_s \sin(2\alpha - \phi_s) = 0 \tag{2.11}$$

A solution of this transcendental equation for the roots α is now the problem.

Although the solution of (2.11) is the basis of the results to be given in a moment, an attempt can be made to produce a more symmetrical expression by setting $\alpha = \theta + \delta$, where $\delta = \phi_s/2$. Figure 2-2 shows that the new variable θ is the difference between the bisector of the azimuthal scattering angle and the bisector of the angle between the flash points. Making this substitution we emerge with

$$(\sin\gamma_s + \sin\gamma_i)\sin\delta\sin\theta + (\sin\gamma_s - \sin\gamma_i)\cos\delta\cos\theta - 2\sin\gamma_i\sin\gamma_s\sin\theta\cos\theta = 0 \tag{2.12}$$

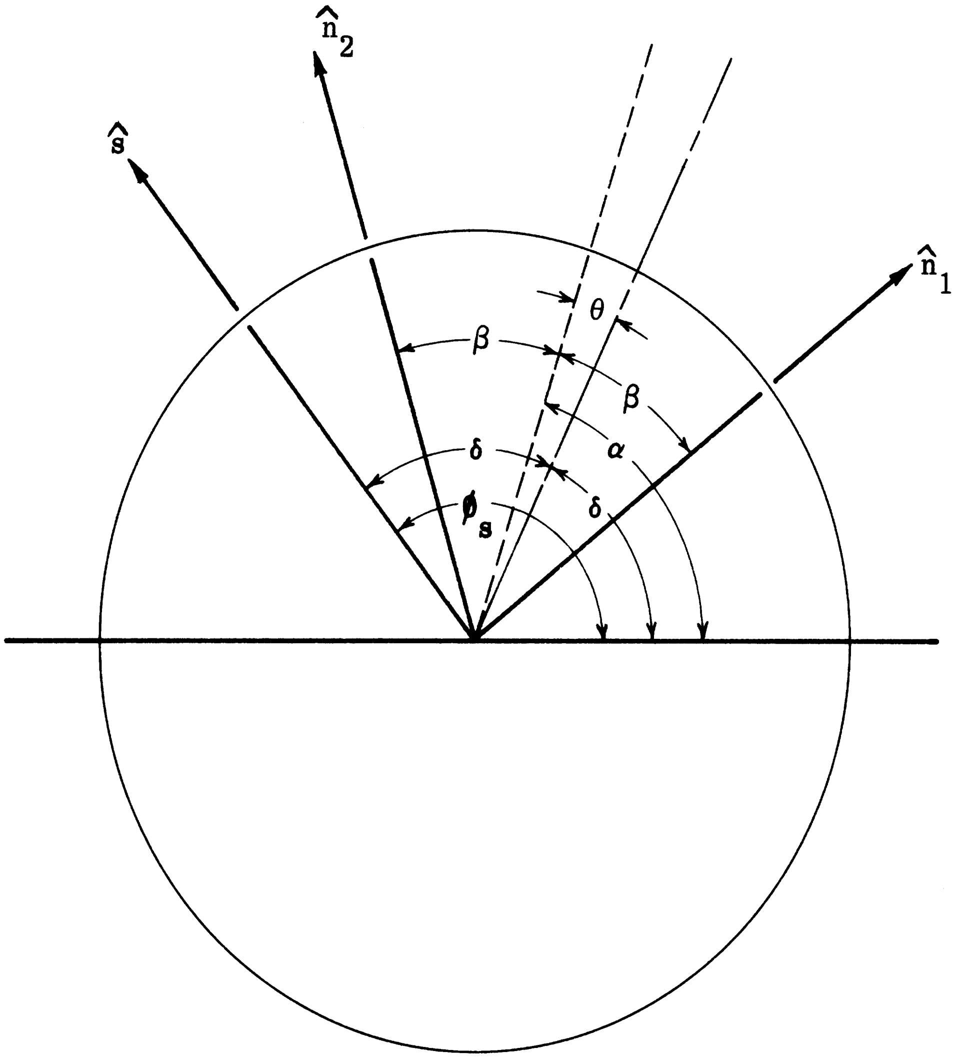

FIG. 2-2: Illustration of the relations between α, β and θ.

If we let

$$p = (\sin\gamma_s + \sin\gamma_i)\sin\delta \quad ,$$

$$q = (\sin\gamma_s - \sin\gamma_i)\cos\delta \quad ,$$

$$r = 2\sin\gamma_i\sin\gamma_s \quad ,$$

then the equation is

$$p\sin\theta + q\cos\theta - r\sin\theta\cos\theta = 0 \quad . \tag{2.13}$$

From (2.13) we may obtain four variants of a quartic equation,

$$\left.\begin{aligned}
&p^2z^4+2pqz^3+(p^2+q^2-r^2)z^2+2pqz+q^2 = 0 \ , \\
&r^2x^4-2prx^3+(p^2+q^2-r^2)x^2+2prx-p^2 = 0 \ , \\
&r^2y^4-2qry^3+(p^2+q^2-r^2)y^2+2qry-q^2 = 0 \ , \\
&rw^4-2(p+iq)w^3+2(p-iq)w-r = 0 \ ,
\end{aligned}\right\} \qquad (2.14)$$

where

$$x = \cos\theta \ ,$$
$$y = \sin\theta \ ,$$
$$z = \tan\theta \ ,$$
$$w = e^{i\theta} \ .$$

Note in the first three equations that the coefficients of the second degree terms are all the same, while the second degree term in the fourth equation is missing. There are always at least two real roots θ and there may be an additional pair under certain conditions. To understand why, it is convenient to return to equation (2.13).

Letting $P = p/r$ and $Q = q/r$, (2.13) is

$$Py+Qx = xy \ ,$$

which is the equation of an equilateral hyperbola whose center lies at $x = P$, $y = Q$. Its focal axis is parallel to the line $x = y$ for $Q > 0$, and parallel to the line $x = -y$ for $Q < 0$, as illustrated in Fig. 2-3. The origin may be shifted to the center of the hyperbola by choosing a coordinate system (u, v)

$$u = x-P \ , \qquad v = y-Q \ ,$$

whereupon the equation of the hyperbola is

$$uv = PQ \ .$$

However, since the variables x and y are rigidly related, not all x and y, and therefore not all u and v, are permitted. Specifically, it is the intersection of the unit circle

$$x^2+y^2 = 1$$

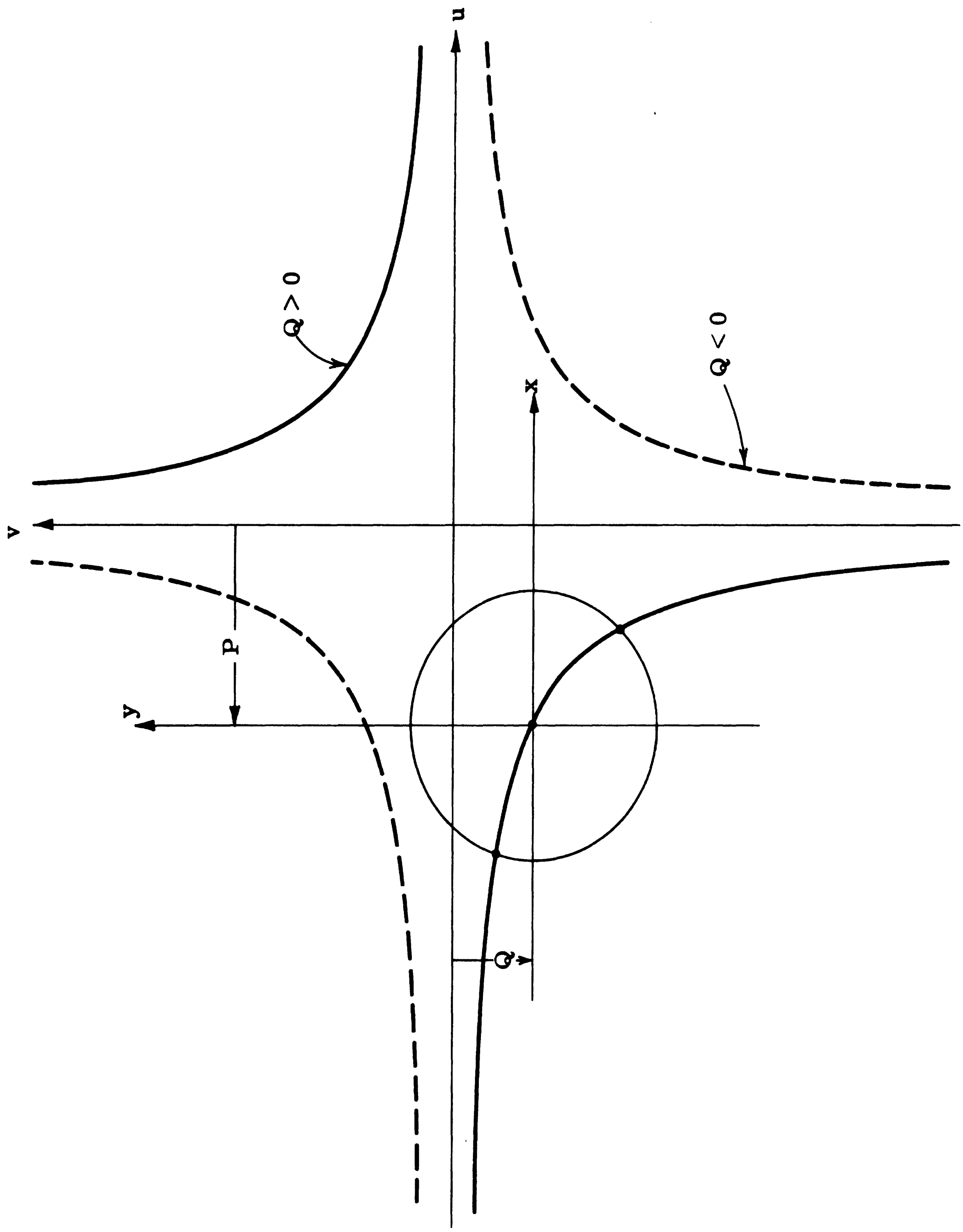

FIG. 2-3: Equation (2.14) represents one of these equilateral hyperbolas, depending on whether Q is positive or negative. The real roots of the quartic equations are given by the intersection of the unit circle and the hyperbola.

with the hyperbola that produces the quartic equations listed above. In the u, v system, the unit circle becomes

$$(u+P)^2+(v+Q)^2 = 1 \ ,$$

and its center (u = -P, v = -Q) lies on the left branch of the hyperbola. The fact that the center lies on the hyperbola itself guarantees that there are always at least two real roots, each corresponding to an intersection of the circle with the left branch, as shown in Fig. 2-3.

Additional roots occur when P and Q are small enough, and the center of the circle close enough to the origin that intersections with the right branch also occur. For this reason it is of interest to introduce scaled coordinates s, t

$$s = u/\sqrt{PQ} \ , \qquad t = v/\sqrt{PQ}$$

so that the hyperbola becomes the unit equilateral hyperbola

$$st = 1 \ .$$

The unit circle becomes

$$\left(s+\sqrt{P/Q}\right)^2+\left(t+\sqrt{Q/P}\right)^2 = 1/PQ \ .$$

Thus the radius of the circle may grow large enough for small enough P and Q that intersections with the upper right branch of the unit hyperbola may now occur. The distinction between the two cases is illustrated in Fig. 2-4.

The positions of the flash points depend, of course, on γ_i, γ_s and ϕ_s, so that their motion is difficult to display unless two parameters are fixed and the third is regarded as an independent variable. Fixing γ_i and γ_s at 30 and 35 degrees, respectively, produces the plot of Fig. 2-5, with ϕ_s ranging from zero to 180 degrees. In the plane of incidence (i.e., for $\phi_s = 0$) there are a total of eight flash points, implying the existence of four distinct ray paths, and we have assigned them the names A, B, C and D in order to distinguish one from another. Note that, as ϕ_s increases, six flash points move forward and that two move backward. Near ϕ_s = 42 degrees flash point ϕ_{2c} meets ϕ_{2d}, and ϕ_{1c} meets ϕ_{1d}, whereupon all

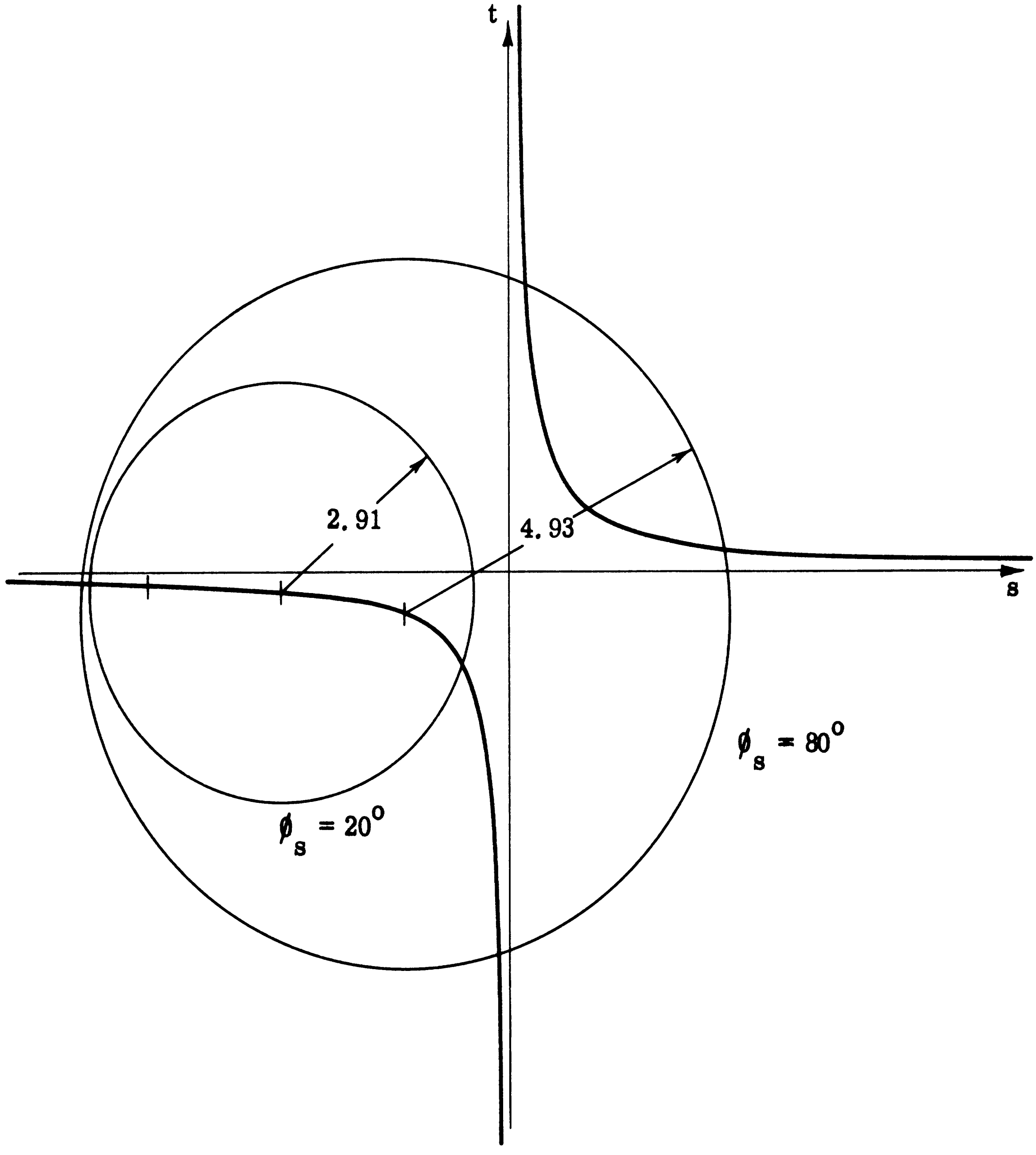

FIG. 2-4: **Depending on the specific values of P and Q, the scaled circle may intersect the unit hyperbola at either two or four points. The circles shown here are for** $\gamma_i = 30^{\circ}$ **and** $\gamma_s = 35^{\circ}$.

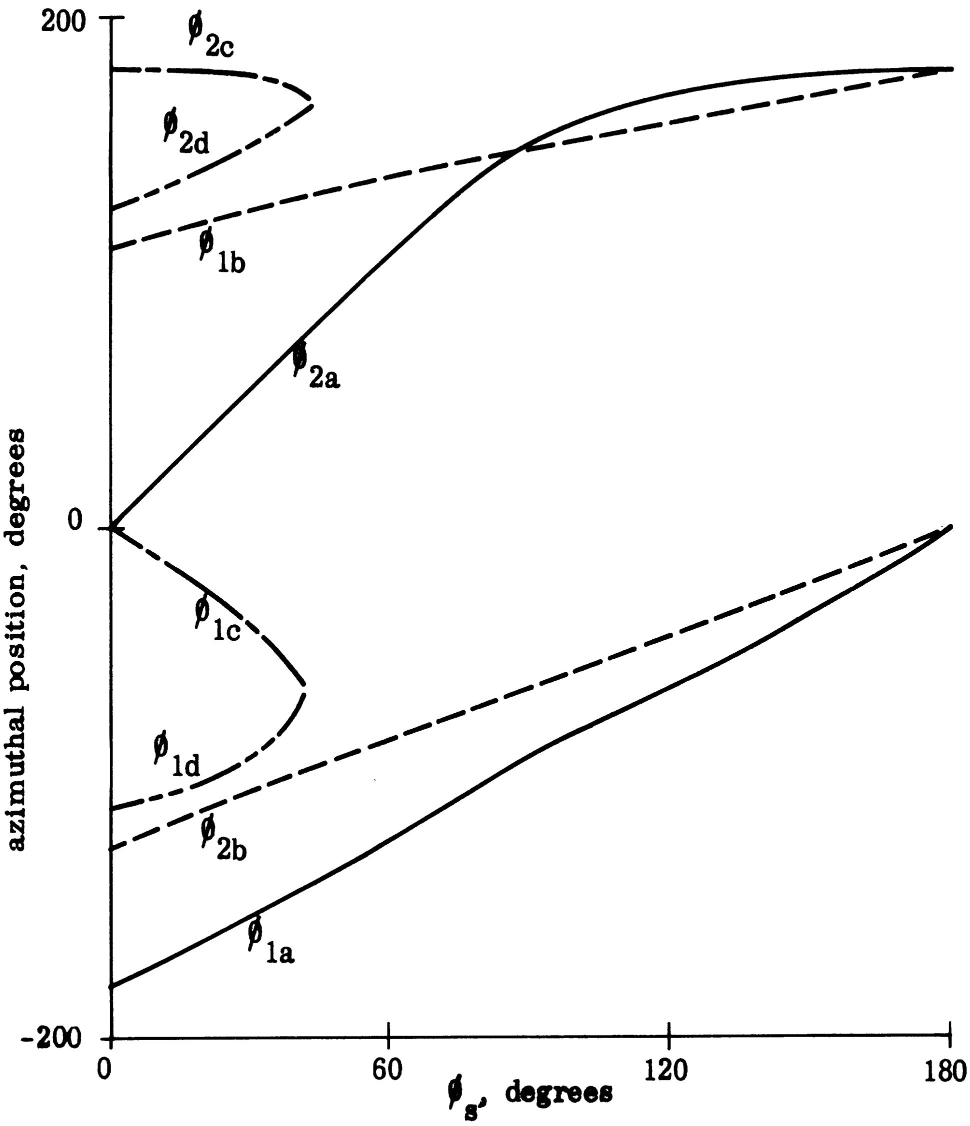

FIG. 2-5: **Flash point positions as a function of ϕ_s for $\gamma_i = 30^o$, $\gamma_s = 35^o$.**

four disappear. Among the advancing flash points, ϕ_{2a} moves faster than the others, at least for ϕ_s less than 90 degrees or so.

A more graphical display of the ray path motion is shown in Fig. 2-6. When $\phi_s = 0$, ray paths A and C coincide and the remaining pair are symmetrically disposed; all four have a common point of intersection. As ϕ_s increases, ray path A swings quickly in a positive direction (counterclockwise) while B moves more slowly. Note that C and D move toward each other and are fairly close at $\phi_s = 38$ degrees, and that the ray paths, in general, do not have a common point of intersection. Beyond $\phi_s = 42$ degrees, C and D no longer exist, but A and B continue to move. As they do, their point of intersection moves toward the rim of the disk and eventually A and B cease to intersect. The motion continues until, at last, A and B both lie on a diameter (in the plane of incidence), and propagation across the ring is in opposite directions. Note that A moves through a full half circle from $\phi = 0$ to $\phi = 180$, but that B moves only slightly more than half this angular range.

Fixing γ_i at 30 degrees and ϕ_s at 35 degrees, with γ_s being the variable, produces similar changes in the ray paths, as shown in Fig. 2-7. Initially (for $\gamma_s = 0$) there are but two paths and they move clockwise with increasing γ_s. Near $\gamma_s = 27$ degrees, rays A and D are "born" and they commence moving apart rapidly, A in a clockwise direction and D counterclockwise. D soon meets C, and they both disappear, leaving only ray paths A and B, which move very little after the disappearance of C and D.

Although the above plots were obtained directly from equation (2.11) by indexing α from zero to 2π in a search for the roots, it is possible to solve any of the four quartic equations by following established procedures. The resulting solutions are not expressible in any convenient analytic form and, by way of demonstration, we examine the quartic

$$r^2x^4 - 2prx^3 + (p^2 + q^2 - r^2)x^2 + 2prx - p^2 = 0 \ , \qquad (2.15)$$

where $x = \cos\theta$. If $r = 0$, the expression reduces to a quadratic whose solution is trivial and need not concern us here. Assuming $r^2 \neq 0$, we may divide by r^2

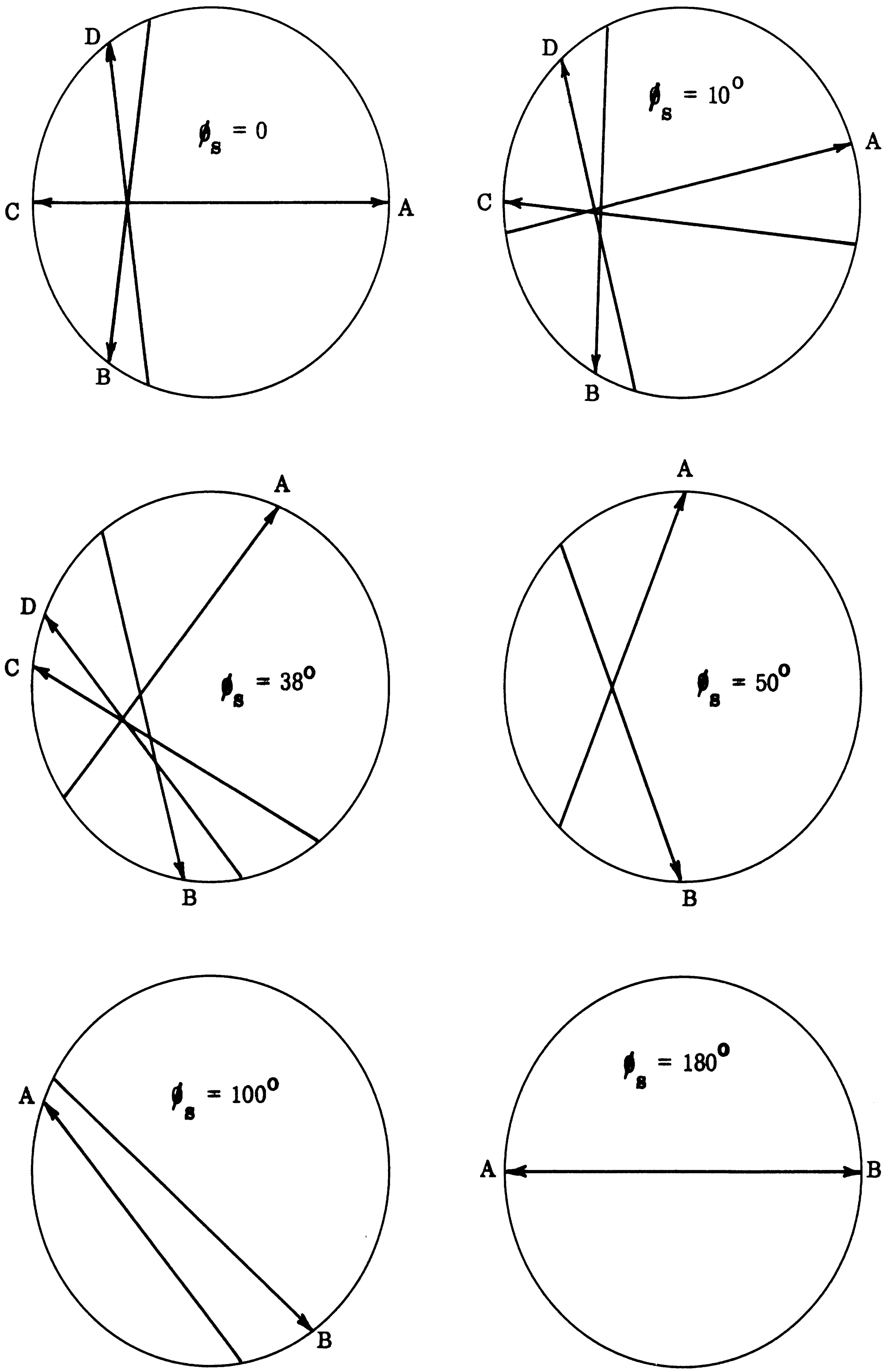

FIG. 2-6: "Snapshots" of the ray paths for $\gamma_i = 30^o$ and $\gamma_s = 35^o$, with ϕ_s being a variable.

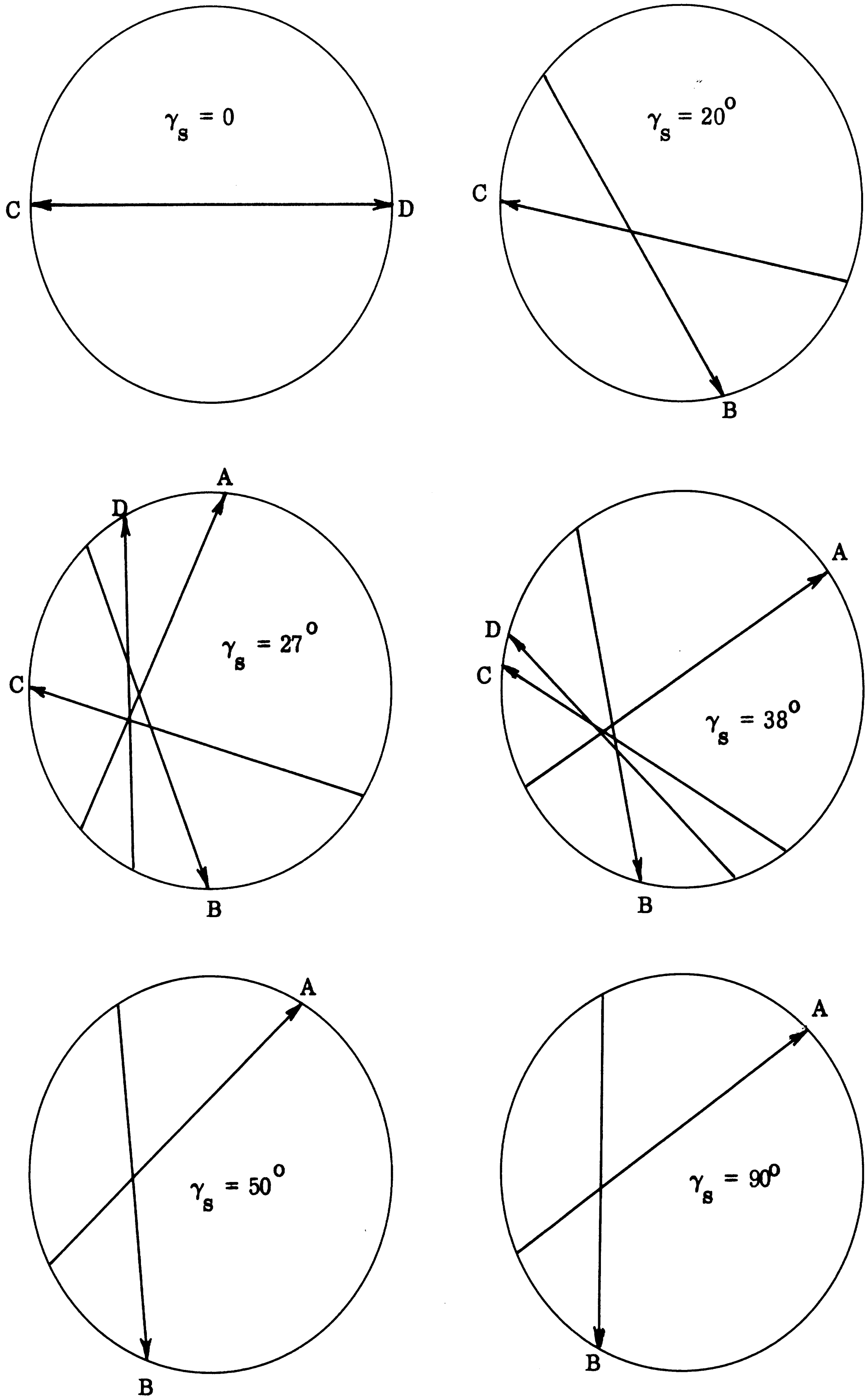

FIG. 2-7: More "snapshots", this time for $\gamma_i = 30^o$ and $\phi_s = 30^o$, with γ_s being variable.

and move all save the first two terms to the right side to obtain

$$x^4 - 2Px^3 = (1 - P^2 - Q^2)x^2 - 2Px + P^2 . \qquad (2.16)$$

The method of solution is to now add a function to both sides of (2.16) such that the left side becomes a perfect square of a second degree expression in x while the right side becomes a perfect square of a first degree expression. One such function is

$$(P^2 + u)x^2 - Pux + u^2/4 ,$$

where u is yet to be determined, whereupon (2.16) becomes

$$(x^2 - Px + u/2)^2 = (1 - Q^2 + u)x^2 - P(2 + u)x + (P^2 + u^2/4) . \qquad (2.17)$$

The left side is, of course, a perfect square and the right side can also be if u is adjusted such that the discriminant vanishes. This implies that u must satisfy

$$u^3 + (1 - P^2 - Q^2)u^2 - 4P^2Q^2 = 0 . \qquad (2.18)$$

Equation (2.18) is known as the reducing cubic for the quartic and any one of its three roots can be used to obtain the roots of (2.15).

For convenience in notation we set $b = (1 - P^2 - Q^2)/3$ and $d = PQ$ so that (2.15) can be written

$$u^3 + 3bu^2 - 4d^2 = 0 . \qquad (2.19)$$

A standard procedure for solving any cubic is to first remove its second degree term. This is easily done by defining a new variable $v = u + b$, whence

$$v^3 - 3b^2 v + (2b^3 - 4d^2) = 0 . \qquad (2.20)$$

We now form the coefficients

$$A = (2d^2 - b^3) + 2d(d^2 - b^3)^{1/2} ,$$

$$B = (2d^2 - b^3) - 2d(d^2 - b^3)^{1/2} ,$$

and compute the three roots of (2.20) as

$$v_1 = (A)^{1/3} + (B)^{1/3}$$

$$v_2 = \omega(A)^{1/3} + \omega^2(B)^{1/3},$$

$$v_3 = \omega^2(A)^{1/3} + \omega(B)^{1/3} ,$$

where $\omega = e^{i2\pi/3}$. Thus the roots of (2.19) are

$$u_1 = (A)^{1/3} + (B)^{1/3} - b ,$$

$$u_2 = \omega(A)^{1/3} + \omega^2(B)^{1/3} - b ,$$

$$u_3 = \omega^2(A)^{1/3} + \omega(B)^{1/3} - b .$$

The coefficients A and B can be complex but since B is the complex conjugate of A, the root u_1 will be purely real. If $d^2 < b^3$, implying that $b > 0$, then

$$A = b^3 e^{i\eta} , \qquad B = b^3 e^{-i\eta} ,$$

where

$$\tan\eta = \frac{2d(b^3 - d^2)^{1/2}}{2d^2 - b^3} .$$

Hence

$$u_1 = b(2\cos\frac{\eta}{3} - 1) ,$$

a real number. If $d^2 > b^3$, then both A and B are real and again u_1 is real. Thus we shall use only u_1, and ignore u_2 and u_3, to find the roots of (2.15).

Since u_1 satisfies (2.18), the right side of (2.17) will be a perfect square for $u = u_1$. This implies that the right side can be written as

$$(-ex+f)^2 \quad \text{or} \quad (ex-f)^2 , \qquad P(2+u_1) > 0 ,$$

$$(-ex-f)^2 \quad \text{or} \quad (ex+f)^2 , \qquad P(2+u_1) < 0 ,$$

where

$$e = (1+u_1-Q^2)^{1/2} , \qquad f = (u_1^2+4P^2)^{1/2}/2$$

Thus the roots of (2.15) are

$$\begin{aligned} x_{1,2} &= \frac{1}{2}\left\{(P+e) \pm \sqrt{(P+e)^2-2(u_1+2f)}\right\} \\ x_{3,4} &= \frac{1}{2}\left\{(P-e) \pm \sqrt{(P-e)^2-2(u_1-2f)}\right\}, \end{aligned} \qquad (2.21)$$

for $P(2+u) > 0$, or

$$\begin{aligned} x_{1,2} &= \frac{1}{2}\left\{(P+e) \pm \sqrt{(P+e)^2-2(u_1-2f)}\right\} , \\ x_{3,4} &= \frac{1}{2}\left\{(P-e) \pm \sqrt{(P-e)^2-2(u_1+2f)}\right\}, \end{aligned} \qquad (2.22)$$

for $P(2+u) < 0$.

Observe that

$$1+u_1-Q^2 = \frac{P^2(2+u_1)^2}{u_1^2+4P^2} ,$$

which, being the ratio of two positive quantities, is always positive so that e and f, and therefore all parameters, in (2.21) and (2.22) are real. The radicals, however, can produce complex terms. The results (2.21) and (2.22) are computable of course, but are not convenient expressions to use in gaining any insight into the nature of the roots or the ray paths.

On the other hand, the quartic equations can be solved quite easily when the directions of incidence and scattering are co-planar with the ring axis, for then either p or q may vanish. For example, if one seeks the diffracted field for directions in the forward half plane of incidence $\phi_s = \pi$, the second of eq. (2.14) reduces to

$$(rx-p)^2 \ (x^2-1) = 0,$$

whose roots are simply

$$x = \cos\theta = \pm 1, \ \ p/r, \ p/r$$

The double root exists only if $|p/r| \leq 1$, and since

$$\frac{p}{r} = \frac{1}{2}\left(\frac{1}{\sin\gamma_i} + \frac{1}{\sin\gamma_s}\right),$$

this can occur only for $\gamma_i = \gamma_s = \pi/2$, corresponding to incidence in the plane of the ring. Since this set of directions corresponds to a caustic, we immediately discount the existence of <u>any</u> additional pairs of flash points. Thus, for diffraction in the forward half-plane, there are but two ray paths and their flash points are fixed at

$$\phi_1^{(1)} = 0, \qquad \phi_2^{(1)} = \pi,$$

$$\phi_1^{(2)} = \pi, \qquad \phi_2^{(2)} = 0.$$

The situation is more interesting for scattering in the backward half plane, $\phi_s = 0$. In this case $p = 0$ and the third of equations (2.14) becomes

$$(ry-q)^2 (y^2-1) = 0,$$

having roots

$$y = \sin\theta = \pm 1, \quad q/r, \quad q/r.$$

Since

$$\frac{q}{r} = \frac{1}{2}\left(\frac{1}{\sin\gamma_i} - \frac{1}{\sin\gamma_s}\right),$$

the repeated roots exist only for

$$\left|\frac{1}{\sin\gamma_i} - \frac{1}{\sin\gamma_s}\right| \leq 2, \tag{2.23}$$

which is a relatively small range of angles for modest values of γ_i and γ_s.

Provided the condition (2.23) is satisfied, the roots of eq. (2.11) are

$$\alpha^{(1)} = \pi/2, \quad \alpha^{(2)} = -\pi/2, \quad \alpha^{(3)} = \alpha', \quad \alpha^{(4)} = \pi - \alpha',$$

where

$$\alpha' = \sin^{-1}\left\{\frac{1}{2}\left(\frac{1}{\sin\gamma_i} - \frac{1}{\sin\gamma_s}\right)\right\}.$$

From eq. (2.10) we find that

$$\beta^{(1)} = \beta^{(2)} = \pi/2, \quad \beta^{(3)} = \pi - \beta', \quad \beta^{(4)} = \beta',$$

where

$$\beta' = \tan^{-1}\left\{\frac{\frac{1}{2}\left(\frac{1}{\sin\gamma_i} + \frac{1}{\sin\gamma_s}\right)}{\left[1 - \frac{1}{4}\left(\frac{1}{\sin\gamma_i} - \frac{1}{\sin\gamma_s}\right)^2\right]^{1/2}}\right\}.$$

Thus the flash point locations are

$$\phi_1^{(1)} = 0, \qquad \phi_2^{(1)} = \pi,$$

$$\phi_1^{(2)} = \pi, \qquad \phi_2^{(2)} = 0,$$

$$\phi_1^{(3)} = \alpha' - \pi + \beta', \qquad \phi_2^{(3)} = \alpha' + \pi - \beta',$$

$$\phi_1^{(4)} = \pi - \alpha' - \beta', \qquad \phi_2^{(4)} = \pi - \alpha' + \beta'.$$

The disposition of the ray paths is typically as shown in the upper left diagram of Figure 2-6. In this particular instance, $\gamma_i < \gamma_s$ and if the values γ_i and γ_s were to be interchanged, we would find that path B would become the opposite of path D, and vice versa; paths A and C would remain unchanged.

The special case of backscattering can now be obtained trivially by setting $\gamma_i = \gamma_s = \gamma$, whereupon $\alpha' = 0$ and $\beta' = \tan^{-1}(1/\sin\gamma)$; note that β' can never be less than 45 degrees for this case. The flash point locations are then

$$\phi_1^{(1)} = 0, \qquad \phi_2^{(1)} = \pi,$$

$$\phi_1^{(2)} = \pi, \qquad \phi_2^{(2)} = 0,$$

$$\phi_1^{(3)} = \pi + \tan^{-1}\left(\frac{1}{\sin\gamma}\right), \qquad \phi_2^{(3)} = \pi - \tan^{-1}\left(\frac{1}{\sin\gamma}\right),$$

$$\phi_1^{(4)} = \pi - \tan^{-1}\left(\frac{1}{\sin\gamma}\right), \qquad \phi_2^{(4)} = \pi + \tan^{-1}\left(\frac{1}{\sin\gamma}\right).$$

These positions, which will come into play in the remaining chapters in the calculation of doubly diffracted fields, are illustrated in Figure 2-8.

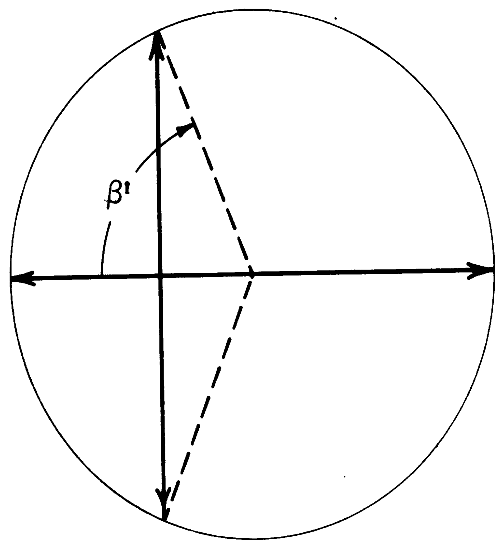

FIG. 2-8: There are 4 ray paths for backscattering overlaid by pairs. The vertical pair shown here shift to the right (toward the center of the circle) as incidence approaches the ring axis.

III

WIDE ANGLE CONTRIBUTION

The direct derivation of the second order contribution to the far zone bistatic field at wide angles, i.e., away from any caustic in the field, can be carried out using standard GTD techniques. The first analysis of this type was performed by Knott et al. (1971) in the particular case of backscattering from a circular disk and was followed shortly afterwards by a similar analysis by Senior and Uslenghi (1971, 1973) for the base of a right circular cone. These investigations first revealed the existence of migrating ray paths in addition to the fixed path along the diameter of the ring lying in the plane formed by its normal and the direction of incidence and diffraction.

Although the ray paths in bistatic scattering are much more complicated and, in general, cannot be analytically described, it is possible to develop an analytical expression for the contribution to the field of any two points on the ring defining a permissible second order path. Such a path is illustrated in Fig. 2-1, and by inserting the azimuthal coordinates ϕ_1 and ϕ_2 of the specific points P_1 and P_2 on the ring at which the diffraction occurs, the explicit contribution of this path can be found. For given directions of incidence and diffraction, there will be either two or four such paths, and the complete second order expression for the bistatic field then follows by adding the contributions of the individual paths.

As in Chapter II, the ring singularity of radius a is chosen to lie in the xy plane of a Cartesian coordinate system x, y, z. The origin is at the center of the ring and the ray path considered is across the face z = -0. The plane wave is assumed incident in the xz plane and is written as

$$\underline{E}^i = \hat{e}_i e^{ik\hat{i}\cdot\underline{r}}, \qquad \underline{H}^i = Y_0 \hat{h}_i e^{ik\hat{i}\cdot\underline{r}} \tag{3.1}$$

where

$$\hat{e}_i = -\hat{x}\cos\gamma_i \sin p - \hat{y}\cos p + \hat{z}\sin\gamma_i \sin p \ ,$$

$$\hat{h}_i = -\hat{x}\cos\gamma_i \cos p + \hat{y}\sin p + \hat{z}\sin\gamma_i \cos p \ , \qquad (3.2)$$

in which p and q are polarization angles as defined in Fig. A-1 of the Appendix. The propagation vector $\hat{i}$ is defined in eq. (2.2) and, of course,

$$\hat{i} = \hat{e}_i \wedge \hat{h}_i \ .$$

The incident electric field at the point P_1 having position vector

$$\underline{r}_1 = a(\hat{x}\cos\phi_1 + \hat{y}\sin\phi_1) \qquad (3.3)$$

is therefore

$$\hat{e}_{i1}\, e^{ik\hat{i}\cdot\underline{r}_1} \ .$$

The diffracted rays which are produced all lie on the surface of a cone of half angle β_1 with vertex at P_1 and at a point P_2 a distance d ($\gg\lambda$) along one of these rays, the diffracted electric field is

$$-\frac{e^{i\pi/4}}{\sin\beta_1\sqrt{2\pi kd}}\,\Gamma_{i12}\,\bar{\bar{\Delta}}\cdot\hat{e}_{i1}\,e^{ik(\hat{i}\cdot\underline{r}_1+d)} \qquad (3.4)$$

(Senior and Uslenghi, 1971), where Γ_{i12} is the divergence factor and $\bar{\bar{\Delta}}$ is the diffraction tensor which, for a wedge-like singularity, is given in the Appendix to the above reference. Although (3.4) represents the total diffracted field at P_2 and would normally constitute the incident field for purposes of re-diffraction, the fact that P_2 is itself on the same planar surface as P_1 requires us to suppress the reflected field. The incident electric field $\underline{e}_{i2}$ at P_2 is therefore one-half that in (3.4), and by a further use of the tensor $\bar{\bar{\Delta}}$, the electric field at a distance s from P_2 along a (re-)diffracted ray is found to be

$$\left(-\frac{e^{i\pi/4}}{\sin\beta_2\sqrt{2\pi ks}}\right)\Gamma_{12s}\,\bar{\bar{\Delta}}\cdot\underline{e}_{i2}\,e^{iks} \tag{3.5}$$

where Γ_{12s} is the divergence factor and β_2 is the half angle of the Keller cone at P_2.

For sufficiently large s,

$$s \simeq r_0 - \hat{s}\cdot\underline{r}_2$$

where r_0 is the distance of the observation point from the center of the ring, $\hat{s}$ is the unit vector in the direction of observation defined in eq. (2.3), and

$$\underline{r}_2 = a(\hat{x}\cos\phi_2 + \hat{y}\sin\phi_2) \tag{3.6}$$

is the position vector of P_2. If we also introduce the concept of a linearly polarized receiver whose electric and magnetic polarizations are described by the unit vectors $\hat{e}_r$ and $\hat{h}_r$ aligned such that $\hat{s} = \hat{e}_r \wedge \hat{h}_r$, the received signal associated with the rediffracted field will be

$$\underline{E}\cdot\hat{e}_r = S e^{ikr_0}/kr_0 \tag{3.7}$$

where S is a scalar scattering function. From eqs. (3.4) through (3.7) it now follows that

$$S = \frac{i}{4\pi}\left(\sin\beta_1\sin\beta_2\right)^{-1}(s/d)^{1/2}\,\Gamma_{i12}\Gamma_{12s}\left\{\bar{\bar{\Delta}}\cdot(\bar{\bar{\Delta}}\cdot e_{i1})\right\}\cdot\hat{e}_r\,e^{ik(\hat{i}\cdot\underline{r}_1 - \hat{s}\cdot\underline{r}_2 + d)} . \tag{3.8}$$

Each factor in this result can be expressed in terms of the angles defining the directions of incidence and diffraction and the azimuthal angles ϕ_1 and ϕ_2 of P_1 and P_2, and for some of the factors the task is rather trivial. From eqs. (2.2), (2.3), (3.3) and (3.6),

$$\hat{i}\cdot\underline{r}_1 = \hat{s}\cdot\underline{r}_2 = -a\left\{\sin\gamma_i\cos\phi_1 + \sin\gamma_s\cos(\phi_s - \phi_2)\right\} ,$$

and since d is, by definition, the (positive) distance $\left[P_1P_2\right]$, the phase factor in Eq. (3.8) is

$$\exp\left[ika\left\{2\left|\sin\frac{\phi_2-\phi_1}{2}\right| - \sin\gamma_i\cos\phi_1 - \sin\gamma_s\cos(\phi_s-\phi_2)\right\}\right] \quad . \tag{3.9}$$

The angles β_1 and β_2 of the Keller cones can be found by using the conditions for the existence of the diffraction path P_1P_2. If $\hat{t}$ is a unit vector tangent to the ring in the direction of increasing ϕ, so that at P_1

$$\hat{t} = \hat{t}_1 = -\hat{x}\sin\phi_1 + \hat{y}\cos\phi_1 \tag{3.10}$$

the condition that must be satisfied for energy to be diffracted towards P_2 is

$$\hat{i}\cdot\hat{t}_1 = \hat{d}\cdot\hat{t}_1 = \cos\beta_1$$

where

$$\hat{d} = \Omega\left(-\hat{x}\sin\frac{\phi_1+\phi_2}{2} + \hat{y}\cos\frac{\phi_1+\phi_2}{2}\right) \tag{3.11}$$

with

$$\Omega = \frac{\sin\frac{\phi_2-\phi_1}{2}}{\left|\sin\frac{\phi_2-\phi_1}{2}\right|} \quad , \tag{3.12}$$

implying

$$\sin\gamma_i\sin\phi_1 = \Omega\cos\frac{\phi_2-\phi_1}{2} = \cos\beta_1 \quad . \tag{3.13}$$

Similarly, at P_2,

$$\hat{t} = \hat{t}_2 = -\hat{x}\sin\phi_2 + \hat{y}\cos\phi_2 \tag{3.14}$$

and the condition for diffraction in the direction $\hat{s}$ is

$$\hat{d}\cdot\hat{t}_2 = \hat{s}\cdot\hat{t}_2 = \cos\beta_2 \quad ,$$

which can be written as

$$\sin\gamma_s\sin(\phi_s-\phi_2) = \Omega\cos\frac{\phi_2-\phi_1}{2} = \cos\beta_2 \quad . \tag{3.15}$$

Hence,

$$\beta_1 = \beta_2 \tag{3.16}$$

and

$$\sin\beta_1 \sin\beta_2 = \sin^2 \frac{\phi_2 - \phi_1}{2} \quad . \tag{3.17}$$

Turning now to the divergence factors, we have

$$\Gamma_{i12} = \left(\frac{\rho_{i12}}{\rho_{i12} + d} \right)^{1/2} \tag{3.18}$$

where ρ_{i12} is the radius of curvature of the diffracted wavefront at P_1 in the plane of the ring. Since the incident field is a plane wave

$$\rho_{i12} = \frac{a \sin^2 \beta_1}{(\hat{i} - \hat{d}) \cdot \hat{n}_1} \tag{3.19}$$

where $\hat{n}_1 = -\hat{\tau}_1$ is the inward (principal) unit vector normal to the singularity at P_1. Hence, from eqs. (2.2), (3.3), (3.11) and (3.17),

$$\rho_{i12} = \frac{a \sin^2 \frac{\phi_2 - \phi_1}{2}}{\sin\gamma_i \cos\phi_1 - \left| \sin \frac{\phi_2 - \phi_1}{2} \right|} \tag{3.20}$$

and by using eq. (3.13) it can be verified that $\rho_{i12} \leq 0$. On the other hand,

$$\rho_{i12} + d = a \left| \sin \frac{\phi_2 - \phi_1}{2} \right| \frac{2 \sin\gamma_i \cos\phi_1 - \left| \sin \frac{\phi_2 - \phi_1}{2} \right|}{\sin\gamma_i \cos\phi_1 - \left| \sin \frac{\phi_2 - \phi_1}{2} \right|} \tag{3.21}$$

can be either positive or negative, and this fact must be borne in mind in interpreting the final expression for Γ_{i12}, viz .

$$\Gamma_{i12} = \left\{ \frac{\left| \sin \frac{\phi_2 - \phi_1}{2} \right|}{2 \sin \gamma_i \cos \phi_1 - \left| \sin \frac{\phi_2 - \phi_1}{2} \right|} \right\}^{1/2} \tag{3.22}$$

The divergence factor for the rays which are rediffracted at P_2 is

$$\Gamma_{12s} = \left(\frac{\rho_{12s}}{\rho_{12s} + s} \right)^{1/2} \tag{3.23}$$

where ρ_{12s} is the initial radius of curvature of the rediffracted wavefront in the plane of the ring. At large distances from the singularity, ρ_{12s} can be neglected in comparison with s, in which case

$$(s/d)^{1/2} \Gamma_{12s} = \left(\frac{\rho_{12s}}{d} \right)^{1/2} \quad . \tag{3.24}$$

Since the field incident at P_2 is not a plane wave, the expression for ρ_{12s} is

$$\rho_{12s} = \frac{a \sin^2 \beta_2}{(\hat{d} - \hat{s}) \cdot \hat{n}_2 + a\hat{t}_2 \cdot \frac{\partial \hat{d}}{\partial t_2}} \tag{3.25}$$

where $\hat{n}_2 = -\hat{r}_2$ is the inward unit vector normal at P_2 and t_2 is the circumferential distance at P_2 taken positively in the direction $\hat{t}_2$. Knowing the focal point of the rays diffracted at P_1, which is therefore the point in the plane of the ring where the rays incident at P_2 appear to originate, and by considering two adjacent rays one of which arrives at P_2 whereas the other arrives at a point P_2' displaced a small distance t_2 around the ring, it can be shown that

$$a\hat{t}_2 \cdot \frac{\partial \hat{d}}{\partial t_2} = a \, \frac{\sin^2 \frac{\phi_2 - \phi_1}{2}}{\rho_{i12} + d} \quad . \tag{3.26}$$

When this is substituted into eq. (3.25) and eqs. (2.3), (3.6), (3.11), (3.16) and (3.17) employed, we have

$$(s/d)^{1/2}\,\Gamma_{12s} =$$

$$\left\{ \frac{\left|\sin\frac{\phi_2-\phi_1}{2}\right| \left\{2\sin\gamma_i\cos\phi_1 - \left|\sin\frac{\phi_2-\phi_1}{2}\right|\right\}}{\sin\gamma_s\cos(\phi_s-\phi_2)\left\{2\sin\gamma_i\cos\phi_1 - \left|\sin\frac{\phi_2-\phi_1}{2}\right|\right\} - \sin\gamma_i\cos\phi_1\left|\sin\frac{\phi_2-\phi_1}{2}\right|} \right\}^{1/2} \tag{3.27}$$

To evaluate the factor $\bar{\bar{\Delta}}\cdot(\bar{\bar{\Delta}}\cdot\hat{e}_{i1})$ in eq. (3.8) using the diffraction matrix for a wedge-like singularity, it is necessary to choose edge-based coordinate systems at each of the diffraction points P_1 and P_2. Thus, at P_1, we choose a set of mutually orthogonal base vectors $\hat{T}_1$, $\hat{N}_1$, $\hat{B}_1$ with

$$\hat{T}_1 = \hat{t}_1, \qquad \hat{N}_1 = \hat{r}_1, \qquad \hat{B}_1 = -\hat{z}$$

in terms of which

$$\hat{i} = \hat{T}_1\sin\gamma_i\sin\phi_1 - \hat{N}_1\sin\gamma_i\cos\phi_1 + \hat{B}_1\cos\gamma_i$$

$$\hat{d} = \left(\hat{T}_1\cos\frac{\phi_2-\phi_1}{2} - \hat{N}_1\sin\frac{\phi_2-\phi_1}{2}\right)\Omega$$

$$\hat{e}_{i1} = \hat{T}_1(\cos\gamma_i\sin p\sin\phi_1 - \cos p\cos\phi_1) + \hat{N}_1(-\cos\gamma_i\sin p\cos\phi_1 - \cos p\sin\phi_1) + \hat{B}_1(-\sin\gamma_i\sin p)\ .$$

We can now invoke the results in Appendix A of Senior and Uslenghi (1971) with $\delta = \pi(1-n/2)$. Comparison of the incident ray directions leads to the identification

$$\cos\beta = \sin\gamma_i\sin\phi_1$$
$$\sin\beta\sin\alpha = \sin\gamma_i\cos\phi_1$$
$$\sin\beta\cos\alpha = \cos\gamma_i\quad ,$$

and a similar comparison for the diffracted ray directions gives

$$\theta = 3\pi/2$$

$$\cos\gamma = \Omega\cos\frac{\phi_2-\phi_1}{2}$$

$$\sin\beta = \left|\sin\frac{\phi_2-\phi_1}{2}\right| \quad .$$

Hence

$$\beta = \beta_1$$

as expected, and

$$\sin\alpha = \sin\gamma_i\cos\phi_1\left|\mathrm{cosec}\,\frac{\phi_2-\phi_1}{2}\right|$$

$$\cos\alpha = \cos\gamma_i\left|\mathrm{cosec}\,\frac{\phi_2-\phi_1}{2}\right| \quad ,$$

implying

$$\tan\alpha = \tan\gamma_i\cos\phi_1 \quad . \tag{3.28}$$

With the above values of α and θ, the scalar diffraction coefficients $X \pm Y$ are such that

$$X = Y = \frac{1}{n}\sin\frac{\pi}{n}\left\{\cos\frac{\pi}{n} - \cos\frac{1}{n}\left(\alpha - \frac{3\pi}{2}\right)\right\}^{-1} = X_1 \ , \ \text{say} \tag{3.29}$$

and

$$\Delta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2X_1\cos\alpha & -2X_1\sin\alpha \end{pmatrix}$$

from which we obtain

$$\bar{\bar{\Delta}}\cdot\hat{e}_{i1} = 2X_1\left\{(\cos\gamma_i\sin p\cos\phi_1 + \cos p\sin\phi_1)\cos\alpha + \sin\gamma_i\sin p\sin\alpha\right\}\hat{B}_1 \quad . \tag{3.30}$$

The diffracted ray in the direction $\hat{d}$ proceeds to the point P_2 where it constitutes the incident ray and is again diffracted. At P_2 the appropriate base vectors are $\hat{T}_2$, $\hat{N}_2$, $\hat{B}_2$ where

$$\hat{T}_2 = -\hat{t}_2, \qquad \hat{N}_2 = \hat{r}_2, \qquad \hat{B}_2 = \hat{z} = -\hat{B}_1 \quad .$$

The choice of $\hat{N}_2$ is obvious and the reversal of $\hat{T}_2$ follows from the reversal of $\hat{B}_2$. In terms of these vectors

$$\hat{d} = \left(-\hat{T}_2 \cos\frac{\phi_2-\phi_1}{2} + \hat{N}_2 \sin\frac{\phi_2-\phi_1}{2}\right)\Omega$$

$$\hat{s} = -\hat{T}_2 \sin\gamma_s \sin(\phi_s-\phi_2) + \hat{N}_2 \sin\gamma_s \cos(\phi_s-\phi_2) + \hat{B}_2 \cos\gamma_s$$

and

$$\underline{e}_{i2} = e_{i2}\hat{B}_2 = \frac{1}{2}\bar{\bar{\Delta}}\cdot\hat{e}_{i1} \tag{3.31}$$

with

$$e_{i2} = -X_1\left\{(\cos\gamma_i \sin p \cos\phi_1 + \cos p \sin\phi_1)\cos\alpha + \sin\gamma_i \sin p \sin\alpha\right\} \quad . \tag{3.32}$$

When the incident and diffracted ray directions, $\hat{d}$ and $\hat{s}$ respectively, are compared with those implied by the diffraction tensor, we have

$$\beta = \pi - \beta_2 \, , \qquad \alpha = -\pi/2$$

and

$$\sin\theta = \sin\gamma_s \cos(\phi_s - \phi_2)\left|\operatorname{cosec}\frac{\phi_2-\phi_1}{2}\right|$$

$$\cos\theta = -\cos\gamma_s \left|\operatorname{cosec}\frac{\phi_2-\phi_1}{2}\right| \quad ,$$

so that

$$\tan\theta = -\tan\gamma_s \cos(\phi_s - \phi_2) \quad . \tag{3.33}$$

For the above values of α and θ with $\delta = -\pi(1 - n/2)$,

$$X = Y = \frac{1}{n}\sin\frac{\pi}{n}\left\{\cos\frac{\pi}{n} - \cos\frac{1}{n}\left(\theta + \frac{\pi}{2}\right)\right\}^{-1} = X_2, \text{ say} \tag{3.34}$$

and

$$\Delta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2X_2\cos\theta \\ 0 & 0 & -2X_2\sin\theta \end{pmatrix}$$

from which we obtain

$$\begin{aligned} \bar{\bar{\Delta}}\cdot\underline{e}_{i2} &= -2X_2 e_{12}(\hat{N}_2\cos\theta + \hat{B}_2\sin\theta) \\ &= -2X_2 e_{12}(\hat{x}\cos\theta\cos\phi_2 + \hat{y}\cos\theta\sin\phi_2 + \hat{z}\sin\theta) \ . \end{aligned} \tag{3.35}$$

From eqs. (3.30) - (3.33) and (3.35) it now follows that

$$\begin{aligned} \bar{\bar{\Delta}}\cdot(\bar{\bar{\Delta}}\cdot\hat{e}_{i1}) = -4X_1X_2\,\text{cosec}^2\frac{\phi_2-\phi_1}{2}\,(\sin p\cos\phi_1 + \cos\gamma_i\cos p\sin\phi_1) \\ \times\left\{\hat{x}\cos\gamma_s\cos\phi_2 + \hat{y}\cos\gamma_s\sin\phi_2 - \hat{z}\sin\gamma_s\cos(\phi_s - \phi_2)\right\} \end{aligned} \tag{3.36}$$

where $X_1 = X_1(\alpha)$ and $X_2 = X_2(\theta)$ are given by eqs. (3.29) and (3.34). It is of interest to note that (3.36) is simply

$$\bar{\bar{\Delta}}\cdot(\bar{\bar{\Delta}}\cdot\hat{e}_{i1}) = 4X_1X_2\,\text{cosec}^2\beta_1(\hat{h}_i\cdot\hat{t}_1)(\hat{s}\wedge\hat{t}_2)$$

implying

$$\left\{\bar{\bar{\Delta}}\cdot(\bar{\bar{\Delta}}\cdot\hat{e}_{i1})\right\}\cdot\hat{e}_r = -4X_1X_2\,\text{cosec}^2\beta_1(\hat{h}_i\cdot\hat{t}_1)(\hat{h}_r\cdot\hat{t}_2) \tag{3.37}$$

and this more compact form is an immediate consequence of the equivalent current approach discussed in the next chapter.

Having now determined the individual factors in the expression (3.8) for a general second order contribution, the only remaining task is to assemble the final result. Although the expressions for Γ_{i12} and Γ_{12s} have a factor in common, some care is necessary in cancelling this out since, with the present time convention, $\sqrt{-1} = -i$. However, by considering the various possible cases we have, from eqs. (3.8), (3.9), (3.22), (3.27) and (3.37),

$$S = \mp \frac{i}{\pi\sqrt{2}} X_1 X_2 \left|\cos\frac{\phi_2-\phi_1}{2}\right|^3 \left\{ \sin\gamma_s \cos(\phi_s-\phi_2)\left[2\sin\gamma_i\cos\phi_1 - \left|\sin\frac{\phi_2-\phi_1}{2}\right|\right]\right.$$
$$\left. -\sin\gamma_i\cos\phi_1\left|\sin\frac{\phi_2-\phi_1}{2}\right|\right\}^{-1/2} (\hat{h}_i\cdot\hat{t}_1)(\hat{h}_r\cdot\hat{t}_2)\exp\left[ika\left\{2\left|\sin\frac{\phi_2-\phi_1}{2}\right|\right.\right.$$
$$\left.\left. -\sin\gamma_i\cos\phi_1 - \sin\gamma_s(\phi_s-\phi_2)\right\}\right] \tag{3.38}$$

where the upper sign must be chosen unless $\rho_{i12}+d > 0$ and $\rho_{12s} < 0$.

In the particular case of backscattering ($\gamma_s = \gamma_i = \gamma$, say; $\phi_s = 0$), the above expression becomes

$$S = \mp \frac{i}{\pi\sqrt{2}} X_1 X_2 \left|\operatorname{cosec}\frac{\phi_2-\phi_1}{2}\right|^3 (\sin\gamma)^{-1/2}\left\{2\sin\gamma\cos\phi_1\cos\phi_2\right.$$
$$\left. -(\cos\phi_1+\cos\phi_2)\left|\sin\frac{\phi_2-\phi_1}{2}\right|\right\}^{-1/2}$$
$$\times (\hat{h}_i\cdot\hat{t}_1)(\hat{h}_r\cdot\hat{t}_2)\exp\left[ika\left\{2\left|\sin\frac{\phi_2-\phi_1}{2}\right| - (\cos\phi_1+\cos\phi_2)\sin\gamma\right\}\right] \tag{3.39}$$

and this can be checked with the results previously obtained by Senior and Uslenghi (1971, 1973). For the fixed ray path across the diameter of the ring in the plane of incidence, $\phi_1 = 0$ and $\phi_2 = \pi$ implying $\alpha = \gamma$ and $\theta = \pi+\gamma$, or $\phi_1 = \pi$ and $\phi_2 = 0$ implying $\alpha = -\gamma$ and $\theta = \pi-\gamma$. In either case

$$X_1X_2 = \left(\frac{1}{n}\sin\frac{\pi}{n}\right)^2\left\{\cos\frac{\pi}{n} - \cos\frac{1}{n}\left(\frac{3\pi}{2}+\gamma\right)\right\}^{-1}\left\{\cos\frac{\pi}{n} - \cos\frac{1}{n}\left(\frac{3\pi}{2}-\gamma\right)\right\}^{-1} = G(\gamma), \tag{3.40}$$

where G is the function defined in eq. (31) of Senior and Uslenghi (1971). Since $\rho_{i12}+d$ and ρ_{12s} both have the same sign (positive if $\sin\gamma < 1/2$, negative if $\sin\gamma > 1/2$), the upper sign in eq. (3.39) is required, and

$$S = \frac{G(\gamma)}{2\pi\sin\gamma}\sin p(\hat{x}\cos\gamma - \hat{z}\sin\gamma)\cdot\hat{e}_r\, e^{2ika} \quad . \tag{3.41}$$

This is zero when $\sin p = 0$ corresponding to an E-polarized incident field.

The migrating ray paths provide two differing contributions. If $\phi_1 = \tilde{\phi}$, $\phi_2 = 2\pi - \tilde{\phi}$ where

$$\sin\tilde{\phi} = (1+\sin^2\gamma)^{-1/2}, \qquad \cos\tilde{\phi} = -\sin\gamma(1+\sin^2\gamma)^{-1/2}, \tag{3.42}$$

implying $\alpha = \alpha_0$, $\theta = \pi+\alpha_0$ with $\alpha_0 = \sin^{-1}(\sin^2\gamma)$, then

$$X_1X_2 = \left(\frac{1}{n}\sin\frac{\pi}{n}\right)^2\left\{\cos\frac{\pi}{n} - \cos\frac{3\pi+2\alpha_0}{2n}\right\}^{-2} = E(\gamma) \tag{3.43}$$

where E is the function defined in eq. (38) of Senior and Uslenghi (1971). Since $\rho_{i12}+d > 0$ but $\rho_{12s} < 0$ for all γ, the lower sign in eq. (3.39) is required and

$$S = i\,\frac{E(\gamma)}{2\pi\sin\gamma}(1+\sin^2\gamma)^{1/2}\cos(\gamma+p)(-\hat{x}\sin\gamma\cos\gamma - \hat{y}\cos\gamma + \hat{z}\sin^2\gamma)\cdot\hat{e}_r$$
$$\times\ \exp\left\{2ika(1+\sin^2\gamma)^{1/2}\right\} \quad . \tag{3.44}$$

Similarly, if ϕ_2 and ϕ_1 are interchanged, the lower sign in eq. (3.39) is again required, giving

$$S = -i\,\frac{E(\gamma)}{2\pi\sin\gamma}(1+\sin^2\gamma)^{1/2}\cos(\gamma-p)(-\hat{x}\sin\gamma\cos\gamma + \hat{y}\cos\gamma + \hat{z}\sin^2\gamma)\cdot\hat{e}_r$$
$$\times\ \exp\left\{2ika(1+\sin^2\gamma)^{1/2}\right\} \quad . \tag{3.45}$$

The results in eqs. (3.41), (3.44) and (3.45) are in agreement with those of Senior and Uslenghi (1971) as corrected (Senior and Uslenghi, 1973).

IV

THE EQUIVALENT CURRENT METHOD

Many of the analytical difficulties associated with the second order contribution can be avoided by using the concept of equivalent currents. The currents for first order diffraction by a ring discontinuity in slope were derived by Knott and Senior (1973) and lead to an expression for the field as a line integral around the singularity. Although the currents were deduced by reference to the wide-angle GTD result, the integral expression is finite even in the vicinity of the axial caustic, and can therefore be used to determine the caustic matching functions that must be applied to the GTD result to produce a smooth transition into the axial caustic. The detailed derivation is given in the Appendix.

These same currents also provide a simple and revealing way of obtaining the second order contribution as a double line integral around the singularity. As in the case of the first order result, its determination does not require a knowledge of the ray paths, and since the expression is finite everywhere, it can be used to obtain the hitherto unknown caustic matching functions applicable in second order theory.

As shown by Knott and Senior (1973), the diffracted field at a point $\underline{r}'$ resulting from first order diffraction at the singularity is

$$\begin{aligned} \underline{E}^{(1)}(\underline{r}') &= \nabla' \wedge \nabla' \wedge \underline{\pi}_e^{(1)} + ikZ_0 \nabla' \wedge \underline{\pi}_m^{(1)} \\ \underline{H}^{(1)}(\underline{r}') &= \nabla' \wedge \nabla' \wedge \underline{\pi}_m^{(1)} - ikY_0 \nabla' \wedge \underline{\pi}_e^{(1)} \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} \underline{\pi}_e^{(1)}(\underline{r}') &= \frac{1}{2\pi k^2} \int_C \hat{t}(\underline{E}^i \cdot \hat{t}) \frac{X-Y}{\sin^2\beta} \frac{e^{ik|\underline{r}'-\underline{r}|}}{|\underline{r}'-\underline{r}|} \, d\ell \\ \underline{\pi}_m^{(1)}(\underline{r}') &= \frac{1}{2\pi k^2} \int_C \hat{t}(\underline{H}^i \cdot \hat{t}) \frac{X+Y}{\sin^2\beta} \frac{e^{ik|\underline{r}'-\underline{r}|}}{|\underline{r}'-\underline{r}|} \, d\ell \ . \end{aligned} \tag{4.2}$$

The integration is around the ring C, $\underline{r}$ being a variable point on it; $X \pm Y$ are the scalar diffraction coefficients and are known functions of $\hat{r}$ and $\hat{r}'$. First order stationary phase evaluations of these integrals yield the precise wide angle GTD expressions, whereas numerical evaluations with β treated as constant gives results which are indistinguishable from those of the caustically-corrected expressions. The constant value assigned to β is that associated with the ray paths:

$$\sin^2\beta = 1 - (\hat{i}\cdot\hat{t})^2 = 1 - (\hat{r}'\cdot\hat{t})^2 \ ,$$

but for the purposes of the present discussion it is desirable to write $\beta = \beta_1$ in eq. (4.2) with β_1 still unspecified.

If the point $\underline{r}'$ is on the plane surface bounded externally by the ring, $Y = X$ as can be seen from the form of the diffraction coefficients. This corresponds to diffraction across the metallic surface and implies that $\underline{\pi}_e^{(1)}(\underline{r}') = 0$. In particular, if $\underline{r}'$ is a point on the ring

$$\underline{\pi}_e^{(1)}(\underline{r}') = 0$$

$$\underline{\pi}_m^{(1)}(\underline{r}') = \frac{1}{\pi k^2}\int_C \hat{t}(\underline{H}^i\cdot\hat{t})\,\frac{X}{\sin^2\beta_1}\,\frac{e^{ikd}}{d}\,d\ell$$

where $d = |\underline{r}' - \underline{r}|$, and on the assumption that $kd \gg 1$,

$$\nabla'\wedge F \sim ik\hat{d}\wedge\underline{F},$$

giving

$$\underline{E}^{(1)}(\underline{r}') \sim -\frac{Z_0}{\pi}\int_C \hat{d}\wedge\hat{t}(\underline{H}^i\cdot\hat{t})\,\frac{X}{\sin^2\beta_1}\,\frac{e^{ikd}}{d}\,d\ell$$

$$\underline{H}^{(1)}(\underline{r}') \sim -\frac{1}{\pi}\int_C \hat{d}\wedge(\hat{d}\wedge\hat{t})(\underline{H}^i\cdot\hat{t})\,\frac{X}{\sin^2\beta_1}\,\frac{e^{ikd}}{d}\,d\ell \ . \qquad (4.3)$$

When divided by a factor 2 (because of grazing incidence), this represents the field incident on the singularity at the point $\underline{r}'$ after diffraction at $\underline{r}$, and a further application of the single diffraction formulae (4.2) with (say) $\beta = \beta_2$ now leads to the expression for the second order contribution.

If $\hat{t}'$ is the unit tangent vector at $\underline{r}'$,

$$(\hat{d} \wedge \hat{t}) \cdot \hat{t}' = 0$$

since $\hat{t}$ and $\hat{t}'$ are coplanar, and thus

$$\underline{E}^i \cdot \hat{t}' = 0$$

showing that the electric Hertz vector $\underline{\pi}_e^{(2)}(\underline{r}_0)$ for the second order contribution is identically zero. Also

$$\frac{1}{d}\left\{\hat{d} \wedge (\hat{d} \wedge \hat{t})\right\} \cdot \hat{t}' = \frac{1}{d}\left\{(\hat{d} \cdot \hat{t})(\hat{d} \cdot \hat{t}') - (\hat{t} \cdot \hat{t}')\right\} = \frac{1}{2a} \sin\beta$$

since $d = 2a \sin\beta$, implying

$$\underline{H}^i \cdot \hat{t}' = -\frac{1}{4\pi a} \int_C (\underline{H}^i \cdot \hat{t}) X \frac{\sin\beta}{\sin^2\beta_1} e^{2ika\sin\beta} \, d\ell$$

Hence, for the doubly diffracted contribution at the point $\underline{r}_0$ we have

$$\underline{\pi}_e^{(2)}(\underline{r}_0) = 0$$

$$\underline{\pi}_m^{(2)}(\underline{r}_0) = -\frac{1}{4\pi^2 k^2 a} \int_C \int_C \hat{t}'(\underline{H}^i \cdot \hat{t}) XX' \frac{\sin\beta}{\sin^2\beta_1 \sin^2\beta_2} e^{2ika\sin\beta} \frac{e^{ik|\underline{r}_0 - \underline{r}'|}}{|\underline{r}_0 - \underline{r}'|} \, d\ell d\ell' \tag{4.4}$$

where we have used the fact that for diffraction at $\underline{r}'$, $Y' = X'$. In particular, in the far field,

$$|\underline{r}_0 - \underline{r}'| \sim r_0 - \hat{s} \cdot \underline{r}'$$

giving

$$\underline{\pi}_m^{(2)}(\underline{r}_0) \sim \frac{e^{ikr_0}}{r_0} \left\{ -\frac{1}{4\pi^2 k^2 a} \int_C \int_C \hat{t}'(\underline{H}^i \cdot \hat{t}) XX' \frac{\sin\beta}{\sin^2\beta_1 \sin^2\beta_2} e^{ik(2a\sin\beta - \hat{s} \cdot \underline{r}')} d\ell d\ell' \right\}$$

from which we obtain

$$S = -\frac{k}{4\pi^2 a}\int_C\int_C (\hat{h}_r\cdot\hat{t})(\hat{h}_i\cdot\hat{t})\, XX'\, \frac{\sin\beta}{\sin^2\beta_1 \sin^2\beta_2}\, e^{ik(2a\sin\beta + \hat{i}\cdot\underline{r} - \hat{s}\cdot\underline{r}')} d\ell d\ell' \,. \tag{4.5}$$

Note that the second order contribution is determined only by the magnetic current.

In spite of the attractive symmetry and simplicity of the above formula, an evaluation of the double integral expression is impossible without a specification of the angles β, β_1 and β_2. From the geometry, however,

$$\sin\beta = \frac{d}{2a} \tag{4.6}$$

and for any valid doubly-diffracted ray,

$$\beta_1 = \beta_2 = \beta \;.$$

This identification is analogous to the one made in the integral expression for first order diffraction and is sufficient for an analytical evaluation of (4.5) by the stationary phase method. Nevertheless, it is inadequate for a numerical evaluation since the integral is then proportional to d^{-3} and becomes infinite when $\underline{r}' = \underline{r}$. We take up this matter again in Chapter V.

To evaluate the double integral in eq. (4.5) either analytically or numerically it is convenient to use the azimuthal angles ϕ and ϕ', $0 \le \phi,\ \phi' < 2\pi$, as variables of integration around the ring. Since we now have $\beta = \frac{1}{2}(\phi - \phi')$, eq. (4.5) takes the form

$$S = -\frac{ka}{4\pi^2}\int_0^{2\pi}\!\!\int f(\phi,\phi')\, e^{ikag(\phi,\phi')}\, d\phi d\phi' \tag{4.7}$$

where

$$f(\phi,\phi') = \left[\hat{h}_r \cdot (\hat{x}\sin\phi' - \hat{y}\cos\phi')\right]\left[\hat{h}_i \cdot (\hat{x}\sin\phi - \hat{y}\cos\phi)\right] \frac{\left|\sin\frac{\phi-\phi'}{2}\right|}{\sin^2\beta_1 \sin^2\beta_2} XX' \tag{4.8}$$

$$g(\phi,\phi') = 2\left|\sin\frac{\phi-\phi'}{2}\right| - \sin\gamma_i\cos\phi - \sin\gamma_s\cos(\phi_s-\phi') \quad . \tag{4.9}$$

Given a double integral of the form (4.7) having an interior stationary phase point $\phi = \phi_1$, $\phi' = \phi_2$ at which* $g_\phi = g_{\phi'} = 0$, the double stationary phase (D. S. P.) method yields the asymptotic expression

$$S \sim -\frac{i}{2\pi}\,(g_{\phi\phi})^{-1/2}\left(g_{\phi'\phi'} - \frac{g_{\phi\phi'}^2}{g_{\phi\phi}}\right)^{-1/2} f(\phi_1,\phi_2)\, e^{ikag(\phi_1,\phi_2)} \tag{4.10}$$

(Papoulis, 1968; p. 241) for large ka, where all derivatives are evaluated at the D. S. P. point. The manner in which (4.10) is written is that which results if the ϕ integration is performed first and for the expression to be valid it is necessary that $g_{\phi\phi}(\phi_1,\phi')$ does not vanish identically for all ϕ', and that

$$\left.\left(g_{\phi'\phi'} - \frac{g_{\phi\phi'}^2}{g_{\phi\phi}}\right)\right|_{\substack{\phi=\phi_1\\ \phi'=\phi_2}} \neq 0 \quad .$$

---

* Suffices ϕ, ϕ' are used to denote differentiation.

From eq. (4.12),

$$g_{\phi}(\phi,\phi') = -\Omega\cos\frac{\phi-\phi'}{2} + \sin\gamma_i\sin\phi$$
$$g_{\phi'}(\phi,\phi') = \Omega\cos\frac{\phi-\phi'}{2} - \sin\gamma_s\sin(\phi_s-\phi') \tag{4.11}$$

where Ω is defined in eq. (3.12). Equating the right hand sides of the eqs. (4.11) to zero yields two equations specifying a D.S.P. point $\phi = \phi_1$, $\phi' = \phi_2$. Comparison with eqs. (3.13) and (3.15) shows, as expected, that the D.S.P. point is simply a coupled pair of flash points in second order diffraction. Also.

$$g_{\phi\phi}(\phi,\phi') = -\frac{1}{2}\left|\sin\frac{\phi-\phi'}{2}\right| + \sin\gamma_i\cos\phi$$
$$g_{\phi'\phi'}(\phi,\phi') = -\frac{1}{2}\left|\sin\frac{\phi-\phi'}{2}\right| + \sin\gamma_s\cos(\phi_s-\phi') \tag{4.12}$$
$$g_{\phi\phi'}(\phi,\phi') = \frac{1}{2}\left|\sin\frac{\phi-\phi'}{2}\right|$$

giving

$$g_{\phi\phi}g_{\phi'\phi'} - g_{\phi\phi'}^2 = \frac{1}{2}\left\{\sin\gamma_s\cos(\phi_s-\phi')\left[2\sin\gamma_i\cos\phi - \left|\sin\frac{\phi-\phi'}{2}\right|\right]\right.$$
$$\left. - \sin\gamma_i\cos\phi\left|\sin\frac{\phi-\phi'}{2}\right|\right\}. \tag{4.13}$$

According to the square root rule, $\sqrt{-1}$ must be interpreted as $-i$. If we now invoke this, it follows that

$$(g_{\phi\phi})^{-1/2}\left(g_{\phi'\phi'} - \frac{g_{\phi\phi'}^2}{g_{\phi\phi}}\right)^{-1/2} = \pm\,(g_{\phi\phi}g_{\phi'\phi'} - g_{\phi\phi'}^2)^{-1/2}$$

with the lower sign only if

$$g_{\phi\phi} \text{ and } g_{\phi'\phi'} - \frac{g_{\phi\phi'}^2}{g_{\phi\phi}}$$

are both negative, and this enables us to write eq. (4.10) in the more compact and symmetrical form

$$S \sim \mp \frac{i}{2\pi}\left(g_{\phi\phi}g_{\phi'\phi'} - g^2_{\phi\phi'}\right)^{-1/2} f(\phi_1, \phi_2)\, e^{ikag(\phi_1,\phi_2)} \quad . \tag{4.14}$$

From eqs. (3.20) and (3.21), however,

$$g_{\phi\phi} = \frac{1}{2}\left|\sin\frac{\phi_2-\phi_1}{2}\right| \frac{\rho_{i12}+d}{\rho_{i12}}$$

at the D. S. P. point, and since $\rho_{i12} \leq 0$, we have $g_{\phi\phi} < 0$ if and only if $\rho_{i12}+d > 0$. Likewise

$$g_{\phi'\phi'} - \frac{g^2_{\phi\phi'}}{g_{\phi\phi}} = \frac{1}{2}\left|\sin\frac{\phi_2-\phi_1}{2}\right| \frac{d}{\rho_{12s}}$$

which is negative if and only if ρ_{12s} is. The condition for having the lower sign in eq. (4.14) is therefore $\rho_{i12}+d > 0$ with $\rho_{12s} < 0$. This requirement is the same as that for eq. (3.38) and on using eqs. (4.8), (4.9) and (4.13) it can be verified that eq. (4.17) is, indeed, identical to eq. (3.38).

We have thus shown that a D. S. P. point of the integral expression (4.5) for S coincides with the coupled flash points in second order diffraction and that a stationary phase evaluation of the integral yields precisely the wide angle GTD result derived in Chapter III. These properties are analogous to those demanded of the first order expression in order that the currents be truly "equivalent".

A particular advantage of an integral expression for S is that it remains finite even at the axial caustic. It is therefore automatically a caustically-corrected result and, in the case of first order diffraction, this fact can be used to develop an analytical form for S valid uniformly in angle. The analysis is given in the Appendix and the result displayed in eq. (A.7). It is observed that the caustic matching functions which must be applied to a wide angle GTD expression to reproduce the correct caustic behavior are the Bessel functions J_0, J_1 and J_2. This particular combination of functions was originally obtained by Ufimtsev (1958) in analyzing bistatic scattering from a disk in the plane of incidence.

In second order diffraction, the required caustic matching functions are unknown. Based on physical reasoning, Senior and Uslenghi (1971) postulated Fresnel integrals in their study of backscattering by a finite cone, but we can now use the integral expression (4.10) for S to obtain an analytic form valid uniformly in angle from which the caustic matching functions can be deduced. The procedure is similar to that described in the Appendix and is in one respect simpler: since the diffraction coefficient Y does not appear in the integrand, all amplitude factors are given their values at the D. S. P. point.

To illustrate the method, consider

$$I(\phi') = \int_0^{2\pi} f(\phi,\phi')\, e^{ika\, g(\phi,\phi')}\, d\phi \quad . \tag{4.15}$$

Let $\phi = \phi_1(\phi')$ be a stationary phase point such that

$$g_\phi(\phi_1,\phi') = 0$$

for all ϕ'. When $\phi' = \phi_2$, $\phi_1(\phi')$ is simply the angle ϕ_1 used originally. Expanding $g(\phi,\phi')$ about ϕ_1,

$$g(\phi,\phi') = g(\phi_1,\phi') + \frac{1}{2}(\phi-\phi_1)^2 g_{\phi\phi}(\phi_1,\phi') + \ldots$$

and if ϕ_1 is the only stationary phase point in the range $(0, 2\pi)$,

$$I(\phi') \sim f(\phi_1,\phi') \int_{-\pi+\phi_1}^{\pi+\phi_1} e^{ika\, g(\phi,\phi')}\, d\phi$$

where we have used the fact that $g(\phi,\phi')$ is a periodic function of ϕ of period 2π. Hence,

$$I(\phi') \sim f(\phi_1, \phi') e^{ika\, g(\phi_1, \phi')} \int_{-\pi+\phi_1}^{\pi+\phi_1} e^{\frac{ika}{2}(\phi - \phi_1)^2 g_{\phi\phi}(\phi_1, \phi')} d\phi$$

$$= 2f(\phi_1, \phi') e^{ika\, g(\phi_1, \phi')} \int_0^{\pi} e^{i\theta^2 A} d\theta \qquad (4.16)$$

where

$$A = \frac{1}{2} ka\, g_{\phi\phi}(\phi_1, \phi') \ . \qquad (4.17)$$

If the process is repeated for the ϕ' integration assuming $\phi' = \phi_2$ is the only stationary phase point in $(0, 2\pi)$, we find

$$S \sim -\frac{ka}{\pi^2} f(\phi_1, \phi_2) e^{ika\, g(\phi_1, \phi_2)} \int_0^{\pi}\!\!\int_0 e^{i(\theta^2 A + \theta'^2 B)} d\theta\, d\theta'$$

where

$$B = \frac{1}{2} ka \left(g_{\phi'\phi'} - \frac{g^2_{\phi\phi'}}{g_{\phi\phi}} \right) \ , \qquad (4.18)$$

and both A and B are evaluated at the D.S.P. point $\phi = \phi_1$, $\phi' = \phi_2$. Defining now the complementary Fresnel integral

$$\mathfrak{F}(\tau) = \int_0^{\tau} e^{i\mu^2} d\mu \qquad (4.19)$$

in terms of which the standard Fresnel integral is

$$F(\tau) = \int_{\tau}^{\infty} e^{i\mu^2} d\mu = \frac{1}{2}\sqrt{\pi}\, e^{i\pi/4} - \mathfrak{F}(\tau) \; , \tag{4.20}$$

it is a trivial matter to show that

$$S \sim -\frac{ka}{2\pi} \left| A B \right|^{-1/2} f(\phi_1, \phi_2)\, e^{ikag(\phi_1, \phi_2)}\, h(\phi_1, \phi_2) \tag{4.21}$$

where, for $A, B > 0$,

$$h(\phi_1, \phi_2) = \mathfrak{F}(\pi \left|A\right|^{1/2})\, \mathfrak{F}(\pi \left|B\right|^{1/2}) \; . \tag{4.22}$$

If A and/or B is negative, the corresponding complementary Fresnel integral must be replaced by its complex conjugate (denoted by an asterisk in the following). For sufficiently large $|A|$ and $|B|$, the Fresnel integrals in eq. (4.22) can be replaced by the leading terms in their asymptotic expansions, giving

$$h(\phi_1, \phi_2) \sim \begin{cases} i\,\frac{\pi}{4} & \text{if } A > 0,\; B > 0 \\ -i\,\frac{\pi}{4} & \text{if } A < 0,\; B < 0 \\ \frac{\pi}{4} & \text{if } A < 0 \text{ and } B > 0, \text{ or } A > 0 \text{ and } B < 0, \end{cases}$$

Eq. (4.21) then reduces to (4.17) with the sign alternatives concretely displayed.

The above analysis has been carried out under the assumption that there is just one D.S.P. point in the range $0 \le \phi, \phi' < 2\pi$. In practice, there are either two or four such points since each corresponds to a valid second order path (see Chapter II). If two D.S.P. points exist, the ϕ integral (4.15) must be split into two parts with each range of integration spanning one point only. From each integral $f(\phi, \phi')$ is removed with ϕ given its appropriate value, and though it is impossible to arrive at a result which is precisely equivalent to (4.16) for arbitrary locations of the D.S.P. points $\phi_1^{(1)}$ and $\phi_1^{(2)}$, an approximation which is equally valid for large and small $A^{(1),(2)}$ is

$$I(\phi') \sim 2\sum_{n=1}^{2} f(\phi_1^{(n)},\phi')\, e^{ikag(\phi_1^{(n)},\phi')} \int_0^{\pi/2} e^{i\theta^2 A^{(n)}} d\theta \quad . \tag{4.23}$$

The ϕ' integration can be carried out in a similar manner.

The extension to n D.S.P. points is now obvious and leads to the following approximation for S:

$$S \sim -\frac{ka}{\pi^2} \sum_n \left| A^{(n)} B^{(n)} \right|^{-1/2} f(\phi_1^{(n)},\phi_2^{(n)})\, e^{ikag(\phi_1^{(n)},\phi_2^{(n)})}\, h(\phi_1^{(n)},\phi_2^{(n)}) \quad (4.24$$

c.f. eq. (4.21), where, for example,

$$h(\phi_1^{(n)},\phi_2^{(n)}) = \mathfrak{F}\left(\frac{\pi}{n}\left|A^{(n)}\right|^{1/2}\right) \mathfrak{F}\left(\frac{\pi}{n}\left|B^{(n)}\right|^{1/2}\right) \tag{4.25}$$

if $A^{(n)}$, $B^{(n)} > 0$.

The general result (4.24) can be illustrated by considering the special case of backscattering for which $\gamma_s = \gamma_i$ (= γ, say) and $\phi_s = 0$. There are now four D.S.P. points whose locations are known (see Senior and Uslenghi, 1971). For simplicity, we shall assume $\sin\gamma < 1/2$.

(i) $\phi_1^{(1)} = 0$, $\phi_2^{(1)} = \pi$:

$$A^{(1)} = -\frac{1}{2} ka \left(\frac{1}{2} - \sin\gamma\right) \quad < 0$$

$$B^{(1)} = \frac{1}{2} ka \sin^2\gamma \left(\frac{1}{2} - \sin\gamma\right)^{-1} \quad > 0$$

implying

$$A^{(1)}\,B^{(1)} = -\left(\frac{1}{2}\,ka\,\sin\gamma\right)^2$$

$$f\left(\phi_1^{(1)},\,\phi_2^{(1)}\right) = -\,G(\gamma)\left[\hat{h}_r\cdot\hat{y}\right]\left[\hat{h}_i\cdot\hat{y}\right]$$

where $G(\gamma)$ is defined in eq. (3.40), and

$$g\left(\phi_1^{(1)},\,\phi_2^{(1)}\right) = 2.$$

Since $n = 4$, the resulting contribution to S is

$$S^{(1)} = \frac{2G(\gamma)}{\pi^2\sin\gamma}\left[\hat{h}_r\cdot\hat{y}\right]\left[\hat{h}_i\cdot\hat{y}\right]e^{2ika}\mathfrak{J}^*\left(\frac{\pi}{4}\sqrt{\frac{ka}{2}\left(\frac{1}{2}-\sin\gamma\right)}\right)\mathfrak{J}\left(\frac{\pi}{4}\,\sin\gamma\sqrt{\frac{ka}{2}\left(\frac{1}{2}-\sin\gamma\right)^{-1}}\right). \tag{4.26}$$

(ii) $\phi_1^{(2)} = \pi$, $\phi_2^{(2)} = 0$:

$$A^{(2)} = -\frac{1}{2}\,ka\left(\frac{1}{2}+\sin\gamma\right) \qquad < 0$$

$$B^{(2)} = \frac{1}{2}\,ka\,\sin^2\gamma\left(\frac{1}{2}+\sin\gamma\right)^{-1} \qquad > 0$$

implying

$$A^{(2)}\,B^{(2)} = -\left(\frac{1}{2}\,ka\,\sin\gamma\right)^2 ,$$

and

$$f\left(\phi_1^{(2)},\,\phi_2^{(2)}\right) = -\,G(\gamma)\left[\hat{h}_r\cdot\hat{y}\right]\left[\hat{h}_i\cdot\hat{y}\right]$$

$$g\left(\phi_1^{(2)},\,\phi_2^{(2)}\right) = 2$$

as in (i). The corresponding contribution to S is

$$S^{(2)} = \frac{2\,G(\gamma)}{\pi^2 \sin\gamma}\left[\hat{h}_r\cdot\hat{y}\right]\left[\hat{h}_i\cdot\hat{y}\right] e^{2ika}\,\mathfrak{F}^*\left(\frac{\pi}{4}\sqrt{\frac{ka}{2}\left(\frac{1}{2}+\sin\gamma\right)}\right)\mathfrak{F}\left(\frac{\pi}{4}\sin\gamma\sqrt{\frac{ka}{2}\left(\frac{1}{2}+\sin\gamma\right)^{-1}}\right). \tag{4.27}$$

(iii) $\phi_1^{(3)} = \tilde{\phi}$, $\phi_2^{(3)} = 2\pi - \tilde{\phi}$ where $\tilde{\phi}$ is defined in eq. (3.42):

$$A^{(3)} = -\frac{1}{2}ka\left(\frac{1}{2}+\sin^2\gamma\right)\left(1+\sin^2\gamma\right)^{-1/2} < 0$$

$$B^{(3)} = -\frac{1}{2}ka\left(\frac{1}{2}+\sin^2\gamma\right)^{-1}\left(1+\sin^2\gamma\right)^{1/2} < 0$$

implying

$$A^{(3)}\,B^{(3)} = \left(\frac{1}{2}ka\sin\gamma\right)^2 ,$$

and

$$f\left(\phi_1^{(2)}, \phi_2^{(3)}\right) = -E(\gamma)\left(1+\sin^2\gamma\right)^{1/2}\left[\hat{h}_r\cdot(\hat{x}-\hat{y}\sin\gamma)\right]\left[\hat{h}_i\cdot(\hat{x}+\hat{y}\sin\gamma)\right]$$

$$g\left(\phi_1^{(3)}, \phi_2^{(3)}\right) = 2\left(1+\sin^2\gamma\right)^{1/2}$$

where E is defined in eq. (3.43). The contribution to S is

$$S^{(3)} = \frac{2\,E(\gamma)}{\pi^2\sin\gamma}\left(1+\sin^2\gamma\right)^{1/2}\left[\hat{h}_r\cdot(\hat{x}-\hat{y}\sin\gamma)\right]\left[\hat{h}_i\cdot(\hat{x}+\hat{y}\sin\gamma)\right] e^{2ika(1+\sin^2\gamma)^{1/2}}$$

$$\cdot\,\mathfrak{F}^*\left(\frac{\pi}{4}\sqrt{\frac{ka}{2}\,\frac{\frac{1}{2}+\sin^2\gamma}{(1+\sin^2\gamma)^{1/2}}}\right)\mathfrak{F}^*\left(\frac{\pi}{4}\sin\gamma\sqrt{\frac{ka}{2}\,\frac{(1+\sin^2\gamma)^{1/2}}{\frac{1}{2}+\sin^2\gamma}}\right). \tag{4.28}$$

(iv) $\phi_1^{(4)} = 2\pi - \tilde{\phi}, \quad \phi_2^{(4)} = \tilde{\phi}$ where $\tilde{\phi}$ is as before:

All quantities are the same as in (iii) except for f, which now takes the form

$$f(\phi_1^{(4)}, \phi_2^{(4)}) = -E(\gamma)(1+\sin^2\gamma)^{1/2}\left[\hat{h}_r\cdot(\hat{x}+\hat{y}\sin\gamma)\right]\left[\hat{h}_i\cdot(\hat{x}-\hat{y}\sin\gamma)\right] \quad .$$

The expression for $S^{(4)}$ is therefore trivially deducible from eq. (4.28) and the combined contribution is

$$S^{(3)} + S^{(4)} = \frac{4E(\gamma)}{\pi^2 \sin\gamma}(1+\sin^2\gamma)^{1/2}\left[(\hat{h}_r\cdot\hat{x})(\hat{h}_i\cdot\hat{x}) - \sin^2\gamma\,(\hat{h}_r\cdot\hat{y})(\hat{h}_i\cdot\hat{y})\right]$$

$$\cdot\; e^{2ika(1+\sin^2\gamma)^{1/2}}\,\mathcal{F}^*\left(\frac{\pi}{4}\sqrt{\frac{ka}{2}\,\frac{\frac{1}{2}+\sin^2\gamma}{(1+\sin^2\gamma)^{1/2}}}\right)\mathcal{F}^*\left(\frac{\pi}{4}\sin\gamma\sqrt{\frac{ka}{2}\,\frac{(1+\sin^2\gamma)^{1/2}}{\frac{1}{2}+\sin^2\gamma}}\right) . \tag{4.29}$$

The complete second order contribution to the backscattering amplitude is now

$$S = S^{(1)} + S^{(2)} + (S^{(3)} + S^{(4)}) \tag{4.30}$$

where the individual terms are given by eqs. (4.26), (4.27) and (4.29). This result has been obtained by a quasi-analytic evaluation of the integral expression (4.7) that is in part asymptotic and closely parallels the derivation of the first order formula (A.7).

For $\gamma \geq 0$ (as we have assumed) and $ka >> 1$, the first Fresnel integral in the expression (4.27) for $S^{(2)}$ can be replaced by the leading term of its asymptotic expansion for large argument. Similarly for the first Fresnel integral in (4.29), but we can do the same in (4.26) only if $\sin\gamma$ is bounded away from 1/2. If this assumption is made,

$$S \simeq \frac{G(\gamma)}{\pi^{3/2}\sin\gamma}\left[\hat{h}_r\cdot\hat{y}\right]\left[\hat{h}_i\cdot\hat{y}\right] e^{2ika - i\pi/4}\left\{\mathcal{F}\left(\frac{\pi}{4}\sin\gamma\sqrt{\frac{ka}{2}\left(\frac{1}{2}-\sin\gamma\right)^{-1}}\right)\right.$$

$$\left.+\mathcal{F}\left(\frac{\pi}{4}\sin\gamma\sqrt{\frac{ka}{2}\left(\frac{1}{2}+\sin\gamma\right)^{-1}}\right)\right\}$$

$$+\frac{2E(\gamma)}{\pi^{3/2}\sin\gamma}\left(1+\sin^2\gamma\right)^{1/2}\left[(\hat{h}_r\cdot\hat{x})(\hat{h}_i\cdot\hat{x}) - \sin\gamma\,(\hat{h}_r\cdot\hat{y})(\hat{h}_i\cdot\hat{y})\right]$$

$$\cdot\, e^{2ika(1+\sin^2\gamma)^{1/2} - i\pi/4}\,\mathcal{F}^*\left(\frac{\pi}{4}\sin\gamma\sqrt{\frac{ka}{2}\frac{(1+\sin^2\gamma)^{1/2}}{\frac{1}{2}+\sin^2\gamma}}\right). \tag{4.31}$$

If $\sqrt{ka}\,\sin\gamma >> 1$, the remaining Fresnel integrals in (4.31) can be replaced by their asymptotic expansions and we recover the wide angle expression. But if $ka >> 1$ yet $\sqrt{ka}\,\sin\gamma << 1$, implying $\sin\gamma << 1$ a fortiori, each integral can be approximated by

$$\frac{\pi}{4}\sqrt{ka}\,\sin\,\gamma$$

and when this is substituted into eq. (4.31), we obtain

$$S \simeq \frac{1}{2}\sqrt{\frac{ka}{\pi}}\, e^{-i\pi/4}\left\{G(\gamma)\left[\hat{h}_r\cdot\hat{y}\right]\left[\hat{h}_i\cdot\hat{y}\right] e^{2ika}\right.$$

$$\left.+E(\gamma)\left(1+\sin^2\gamma\right)^{1/2}\left[(\hat{h}_r\cdot\hat{x})(\hat{h}_i\cdot\hat{x}) - \sin^2\gamma\,(\hat{h}_r\cdot\hat{y})(\hat{h}_i\cdot\hat{y})\right]e^{2ika\,(1+\sin^2\gamma)^{1/2}}\right\}.$$

When $\gamma = 0$,

$$S = \frac{1}{2}\sqrt{\frac{ka}{\pi}}\, e^{-i\pi/4}\, G(0) \left[(\hat{h}_r \cdot \hat{y})(\hat{h}_i \cdot \hat{y}) + (\hat{h}_r \cdot \hat{x})(\hat{h}_i \cdot \hat{x}) \right] e^{2ika} \tag{4.32}$$

where we have used the property $E(0) = G(0)$. Equation (4.32) is in agreement with the corrected (Keller) expression for the second order backscattering contribution from a finite cone at axial incidence (Senior and Uslenghi, 1973).

The fact that eq. (4.7) reproduces the known results for both axial and wide angle backscattering is a striking demonstration of the power of the equivalent current approach. This power is combined with a certain simplicity of derivation and from the analytical results we have obtained we can also deduce the caustic matching functions appropriate in second order GTD. In bistatic scattering these are the functions $h(\phi_1, \phi_2)$ given in eq. (4.22), but in the special case of backscattering with $\sin\gamma << \frac{1}{2}$, a somewhat simpler form is possible. Thus, from eqs. (4.26) and (4.27), we have

$$S^{(1)}, S^{(2)} \simeq \frac{1}{4}\sqrt{\frac{ka}{\pi}}\, G(\gamma) \left[\hat{h}_r \cdot \hat{y}\right]\left[\hat{h}_i \cdot \hat{y}\right] e^{2ika - i\pi/4} f_H(\gamma) \tag{4.33}$$

$$f_H(\gamma) = \frac{1}{\Lambda}\, \mathfrak{F}(\Lambda) \tag{4.34}$$

with

$$\Lambda = \frac{\pi}{4}\sqrt{ka}\, \sin\gamma , \tag{4.35}$$

and from eq. (4.29)

$$S^{(3)} + S^{(4)} \simeq \frac{1}{2}\sqrt{\frac{ka}{\pi}}\, E(\gamma)\,(1+\sin^2\gamma)^{1/2} \left[(\hat{h}_r \cdot \hat{x})(\hat{h}_i \cdot \hat{x}) - \sin^2\gamma\, (\hat{h}_r \cdot \hat{y})(\hat{h}_i \cdot \hat{y}) \right]$$

$$\cdot\, e^{2ika(1+\sin^2\gamma)^{1/2} - i\pi/4}\, f_E(\gamma) \tag{4.36}$$

where

$$f_E(\gamma) = f^*_H(\gamma) \quad . \tag{4.37}$$

The asterisk again denotes the complex conjugate. The simpler matching functions $f_H(\gamma)$ and $f_E(\gamma)$ which are here obtained are in contrast to the forms involving modified Fresnel integrals postulated by Senior and Uslenghi (1971).

V

NUMERICAL COMPUTATIONS

The complementary Fresnel integrals correctly reproduce the known wide angle and on-axis scattering and provide a smooth transition between these two extremes. In the particular case of backscattering the caustic matching functions are the functions $f_H(\gamma)$ and $f_E(\gamma)$ defined in eqs. (4.34) and (4.37), and to see how effective these are for intermediate values of Λ, we shall now compare the resulting values of the far field scattering amplitude S with data obtained from a numerical evaluation of the double integral expression (4.7).

It is convenient to consider separately the two principal polarizations. For backscattering ($\gamma_i = \gamma_s = \gamma$, $\phi_s = 0$) with E polarization,

$$\hat{e}_i = \hat{e}_r = \hat{y}$$

implying

$$\hat{h}_i \cdot \hat{y} = \hat{h}_r \cdot \hat{y} = 0, \quad \hat{h}_i \cdot \hat{x} = - \hat{h}_r \cdot \hat{x} = \cos\gamma$$

and the integral expression (4.7) then gives

$$S_E = \frac{ka \cos^2\gamma}{4\pi^2} \int\int_0^{2\pi} \sin\phi \sin\phi' \, \omega(\beta) \, XX' \, e^{ika\left\{2\left|\sin\frac{\phi-\phi'}{2}\right| - \sin\gamma(\cos\phi + \cos\phi')\right\}} \, d\phi d\phi' \tag{5.1}$$

where

$$\omega(\beta) = \frac{\sin\beta}{\sin^2\beta_1 \sin^2\beta_2} \tag{5.2}$$

with

$$\sin\beta = \left|\sin\frac{\phi - \phi'}{2}\right| \quad . \tag{5.3}$$

Similarly, for H polarization

$$\hat{h}_i = \hat{h}_r = \hat{y}$$

and the integral (4.7) becomes

$$S_H = -\frac{ka}{4\pi^2}\int_0^{2\pi}\!\!\int \cos\phi\cos\phi'\,\omega(\beta)\,XX'\,e^{ika\left\{2\left|\sin\frac{\phi-\phi'}{2}\right|-\sin\gamma(\cos\phi+\cos\phi')\right\}}\,d\phi\,d\phi' \tag{5.4}$$

The β's that occur in the expression (5.2) for $\omega(\beta)$ are due in part to the form of the equivalent currents and it must be remembered that these were derived by requiring that a stationary phase evaluation of the radiation integral produce exactly the far field, wide-angle GTD result. Because GTD is a specular theory, the incident and diffracted ray paths subtend the same angle with respect to the local tangent vector and there is no distinction between them. With the equivalent current approach, however, such a distinction can be drawn and the choice that is made affects the value of the integral. Thus, the factor $\sin^2\beta_1$ in the denominator of (5.2) is associated with the primary diffraction at P_1 (see Fig. 2-1) and can be interpreted in any one of three ways:

$$\sin^2\beta_i = \begin{cases}\sin^2\beta_i\\ \sin\beta_i\sin\beta_s\\ \sin^2\beta_s\end{cases}$$

where β_i is the angle made by the incident direction, i.e., $\beta_i = \cos^{-1}(\hat{i}\cdot\hat{t})$ and β_s is the corresponding angle made by the scattering direction, i.e., $\beta_s = \cos^{-1}(\hat{d}\cdot\hat{t})$. Similarly,

$$\sin^2\beta_2 = \begin{cases}\sin^2\beta_i'\\ \sin\beta_i'\sin\beta_s'\\ \sin^2\beta_s'\end{cases}$$

where $\beta_i' = \cos^{-1}(\hat{d}\cdot\hat{t}')$ and $\beta_s' = \cos^{-1}(\hat{s}\cdot\hat{t})$. From the geometry it is evident that $\beta_i' = \beta_s = \beta$. There are now nine possible expressions for $\omega(\beta)$ and of these the following four are important:

$$\omega(\beta) = \begin{cases} \dfrac{\sin\beta}{\sin^2\beta_i \sin^2\beta_s'} \\ \dfrac{1}{\sin\beta\sin\beta_i\sin\beta_s'} \\ \dfrac{1}{\sin^2\beta_i\sin\beta_s'} \\ \dfrac{1}{\sin\beta_i\sin^2\beta_s'} \end{cases}$$

In order to decide which form is the most appropriate, we numerically evaluated the integral expression (5.1) or (5.4) for the particular case of axial backscattering by a 15-degree half-angle cone as a function of frequency using each of the above choices. Since $\sin\beta_i = \sin\beta_s' = 1$ on axis, the three forms that were actually used for the evaluation were

$$\omega(\beta) = \begin{cases} \dfrac{1}{\sin\beta} \\ 1 \\ \sin\beta \end{cases} .$$

The results are plotted in Fig. 5-1 and we note that the first two produce an oscillatory variation as a function of frequency, while the third shows a monotonic behavior. On axis all incident rays arrive at the ring singularity with the same phase, and all second order diffracted rays leave with the same phase. There is consequently no reason to expect an oscillatory behavior and

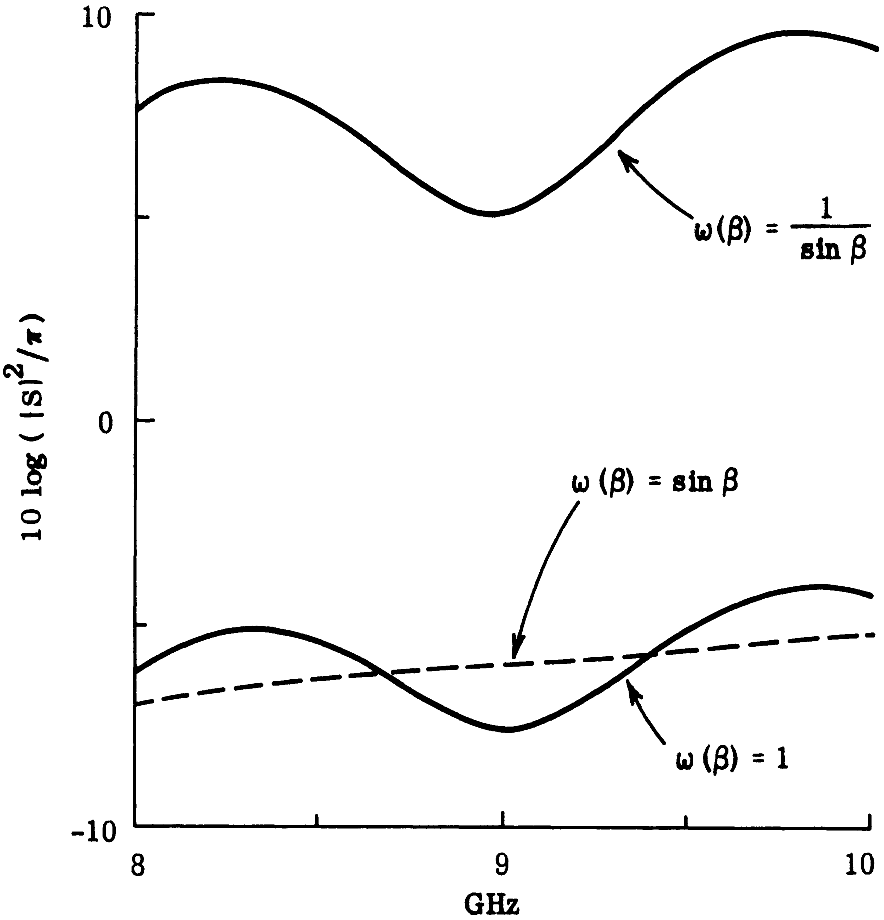

FIG. 5-1: Comparison of the 3 forms of ω (β) for axial backscattering from a 15° half angle cone with a base radius of 4.997 cm.

on this basis the third of the three choices (5.5) was selected as the most likely prescription for $\omega(\beta)$. The resulting interpretation of $\omega(\beta)$ for arbitrary angles of incidence is therefore

$$\omega(\beta) = \frac{\sin\beta}{\sin^2\beta_i \sin^2\beta_s'} \tag{5.6}$$

where $\sin\beta$ is defined in eq. (5.3) and

$$\sin^2\beta_i = 1 - \sin^2\gamma \sin^2\phi ,$$

$$\sin^2\beta_s' = 1 - \sin^2\gamma \sin^2\phi' \quad .$$

With $\omega(\beta)$ specified in this manner, the double integral expressions (5.1) and (5.4) were evaluated numerically as functions of γ for 15^o and 40^o half-angle cones and the results are displayed in Figs. 5-2 and 5-3. The corresponding formulae provided by the quasi analytic evaluation and incorporating the caustic matching functions $f_E(\gamma)$ and $f_H(\gamma)$ can be obtained from eqs. (4.30), (4.33) and (4.34). They are

$$S_E = -\frac{1}{2}\sqrt{\frac{ka}{\pi}}\, E(\gamma)\,(1+\sin^2\gamma)^{1/2} \cos^2\gamma\, e^{2ika(1+\sin^2\gamma)^{1/2} - i\pi/4} f_E(\gamma) \tag{5.8}$$

$$S_H = \frac{1}{2}\sqrt{\frac{ka}{\pi}}\left\{G(\gamma)\, e^{2ika - i\pi/4} f_H(\gamma) - E(\gamma)(1+\sin^2\gamma)^{1/2}\sin^2\gamma\, e^{2ika(1+\sin^2\gamma)^{1/2} - i\pi/4} f_E(\gamma)\right\}$$

c.f. eqs. (1) and (2) of Senior and Uslenghi (1973). Data computed using these expressions are included in Figs. 5-2 and 5-3. The agreement with the results of the numerical evaluation is particularly good for E polarization, the difference

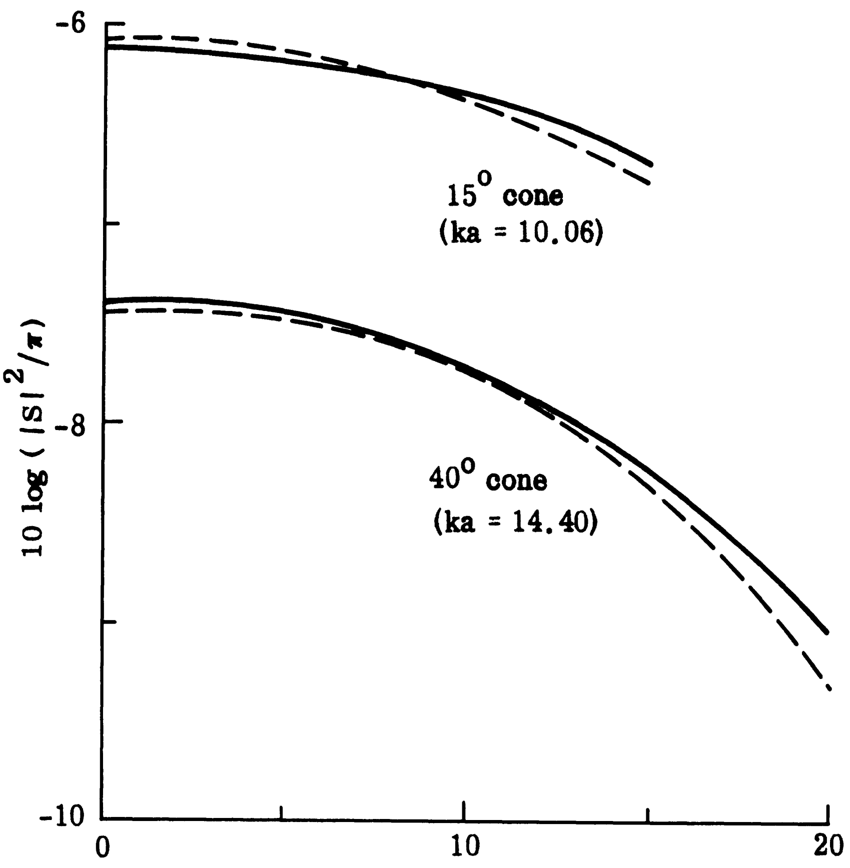

FIG. 5-2: Comparison of numerical (—) and analytic (---) evaluation of the integral for E-polarization for two different cones.

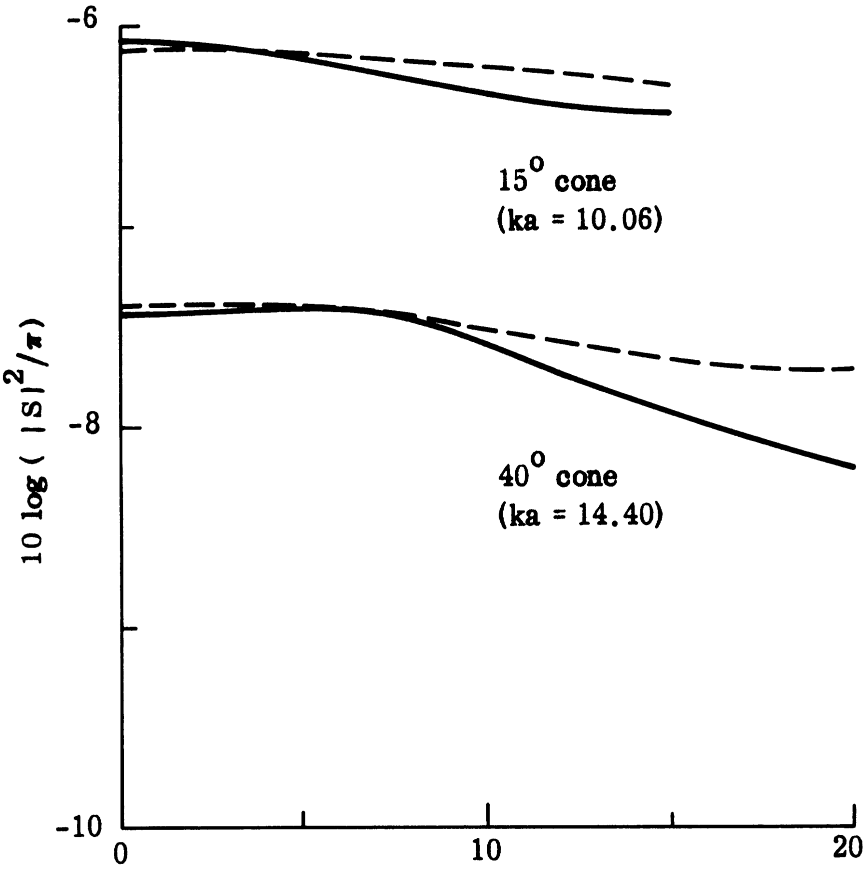

FIG. 5-3: Comparison of numerical (—) and analytic (----) evaluation of the integral for H-polarization for two different cones.

being less than 0.3dB over the entire range of angles considered, and though the H polarization curves do differ by as much as 0.5dB for the 40^{o} cone at 20^{o}, it should be remembered that the precise form of the matching functions $f_{E,H}(\gamma)$ was based on the assumption $\sin\gamma << 1$. Moreover, for all practical purposes the accuracy of the results (5.8) appears adequate.

The caustic matching functions in bistatic scattering are also complementary Fresnel integrals as indicated in eq. (4.22), but any attempt to compute the scattered field in this manner would require a knowledge of the ray paths, just as in the case of wide angle scattering. As we saw in Chapter II, the determination of these paths is a difficult matter involving the solution of a quartic equation, and the expression for the field contribution of each path is also rather complicated (see eq. 3.36 or equivalently, 4.21). In contrast, the equivalent current method leads to a relatively compact expression for the bistatic field (see eq. 4.7), and though the numerical evaluation of such a double integral is not a task which is lightly undertaken, it is well to note that it does not require a knowledge of the ray paths nor does it involve the computation of divergence factors. Moreover, it is a continuous function of the incidence and scattering angles, and since it is finite even in the direction of the axial caustic, it provides a means for determining the caustic matching functions in second order diffraction. In the particular case of backscattering, these functions are $f_E(\gamma)$ and $f_H(\gamma)$ and are similar in form, but not in detail, to the functions postulated by Senior and Uslenghi (1971) based only on physical reasoning.

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APPENDIX

QUASI-ANALYTICAL INTEGRATION OF FIRST ORDER EQUIVALENT CURRENTS

The first order equivalent ring currents deduced by Knott and Senior (1973) are

$$\underline{I}_e = -2\hat{t}(\underline{E}^i\cdot\hat{t})(X-Y)/ikZ_0\sin^2\beta \ ,$$

$$\underline{I}_m = -2\hat{t}(\underline{H}^i\cdot\hat{t})(X+Y)/ikY_0\sin^2\beta$$

where $\hat{t}$ is a unit vector tangent to the edge of the ring, $\underline{E}^i$ and $\underline{H}^i$ are the incident electric and magnetic field intensities, X and Y are diffraction coefficients whose form shall be examined in a moment, $Z_0 = 1/Y_0$ is the impedance of free space and β is the angle subtended by $\hat{t}$ and the direction of incidence. We denote by unit vectors $\hat{e}_i$ and $\hat{h}_i$ the electric and magnetic polarizations of the incident wave, and by $\hat{e}_r$ and $\hat{h}_r$ the polarization orientation of a remote, linearly polarized receiver, as shown in Fig. A-1. If we let $\hat{i}$ and $\hat{s}$ be the directions of propagation of the incident and scattered waves, respectively, then the signal detected by the receiver is proportional to

$$S = \frac{ka}{2\pi}\int_0^{2\pi} \frac{e^{ika\hat{n}\cdot(\hat{i}-\hat{s})}}{\sin^2\beta}\left\{e_{it}e_{rt}(X-Y)+h_{it}h_{rt}(X+Y)\ d\phi\right\}, \qquad (A.1)$$

where S is a far field scattering coefficient, a is the radius of the ring, $\hat{n}$ is a unit position vector directed away from the ring axis and the subscript t signifies the tangential component.

If equation (A. 1) is evaluated by means of the method of stationary phase, the wide-angle GTD result is recovered. A more accurate evaluation of the integral can be carried out if the diffraction coefficients are represented in terms of their Fourier series expansions about the angle $\phi_1-\phi$, where ϕ_1 is the

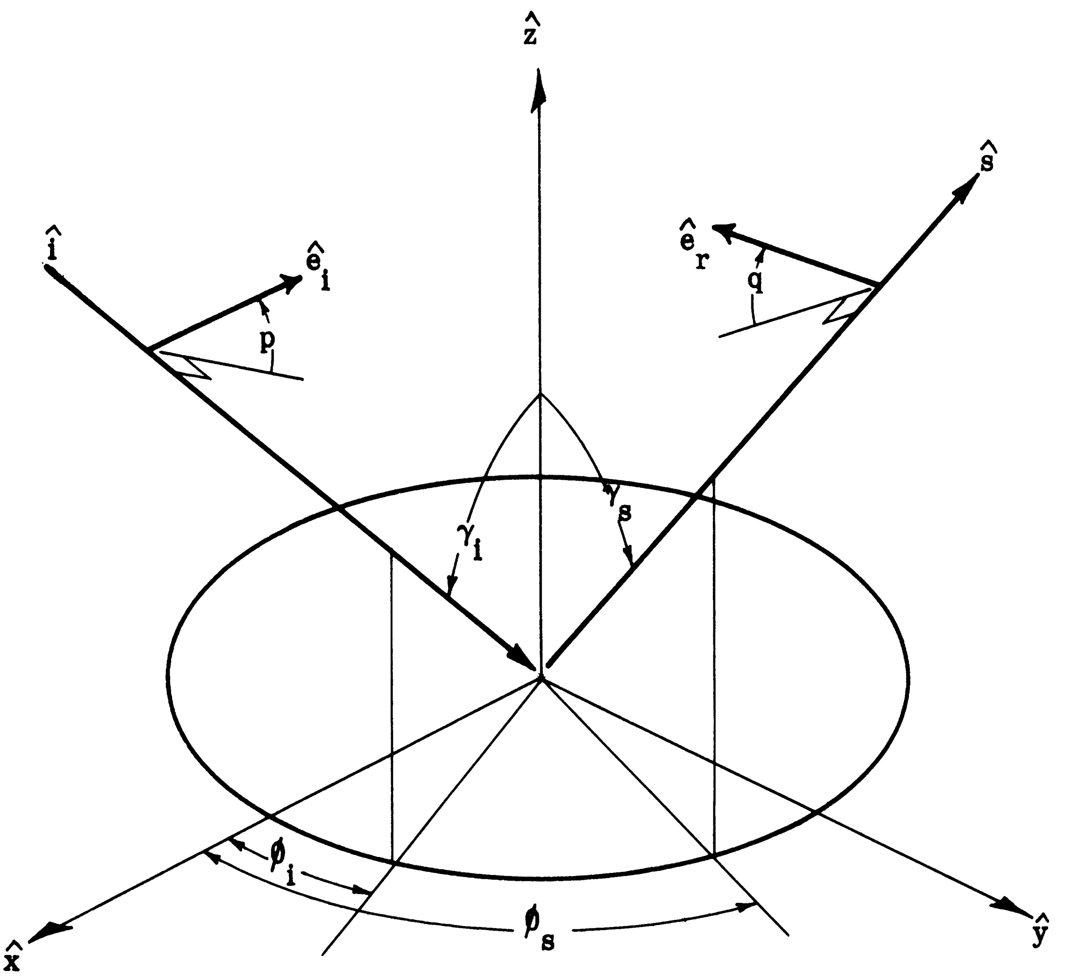

FIG. A-1: Geometry of incident and scattering directions, and incident and receiver polarizations.

the angular location of a flash point as given by wide-angle GTD. In the analysis below, the denominator of (A. 1) is considered constant and equal to the value it takes on at a flash point and may thus be removed from the integral. Numerical studies of the integral have shown that this is justifiable, but since $\sin^2\beta$ is not permitted to vary with ϕ, we regard the analysis below as "quasi-analytical".

The diffraction coefficients are

$$X = \frac{\frac{1}{n}\sin\frac{\pi}{n}}{\cos\frac{\pi}{n} - \cos\frac{\psi-\psi_0}{n}}$$

$$Y = \frac{\frac{1}{n}\sin\frac{\pi}{n}}{\cos\frac{\pi}{n} - \cos\frac{\psi+\psi_0}{n}}$$

where $n\pi$ is the exterior wedge angle of the singularity and ψ_0 and ψ are the angles between one side of the local wedge and the projections of $\hat{i}$ and $\hat{s}$ onto the plane perpendicular to the local wedge axis (i.e., perpendicular to $\hat{t}$). The variations of X and Y around a ring are plotted in Figure A-2 for an arbitrary set of parameters involving mixed polarizations and bistatic directions not in the plane of incidence. It can be seen that X has but a small sinusoidal variation that undergoes two complete cycles in one circuit of the ring, while Y exhibits a much stronger variation. Figure A-3 shows that $Y = C+W+Z$ has essentially only the two sinusoidal components

$$W = D\cos(\phi_1-\phi) \qquad \text{and} \qquad Z = E\cos 2(\phi_1-\phi) \ .$$

in addition to the constant coefficient C. Thus quite good approximations of the coefficients are

$$X = A+B\sin 2(\phi_1-\phi) \ , \qquad \text{(A. 2)}$$

$$Y = C+D\cos(\phi_1-\phi)+E\cos 2(\phi_1-\phi) \ . \qquad \text{(A. 3)}$$

As required by (A. 1), these coefficients must be multiplied by the polarization factors $e_{it}e_{rt}$ and $h_{it}h_{rt}$, and then combined. In order to obtain explicit

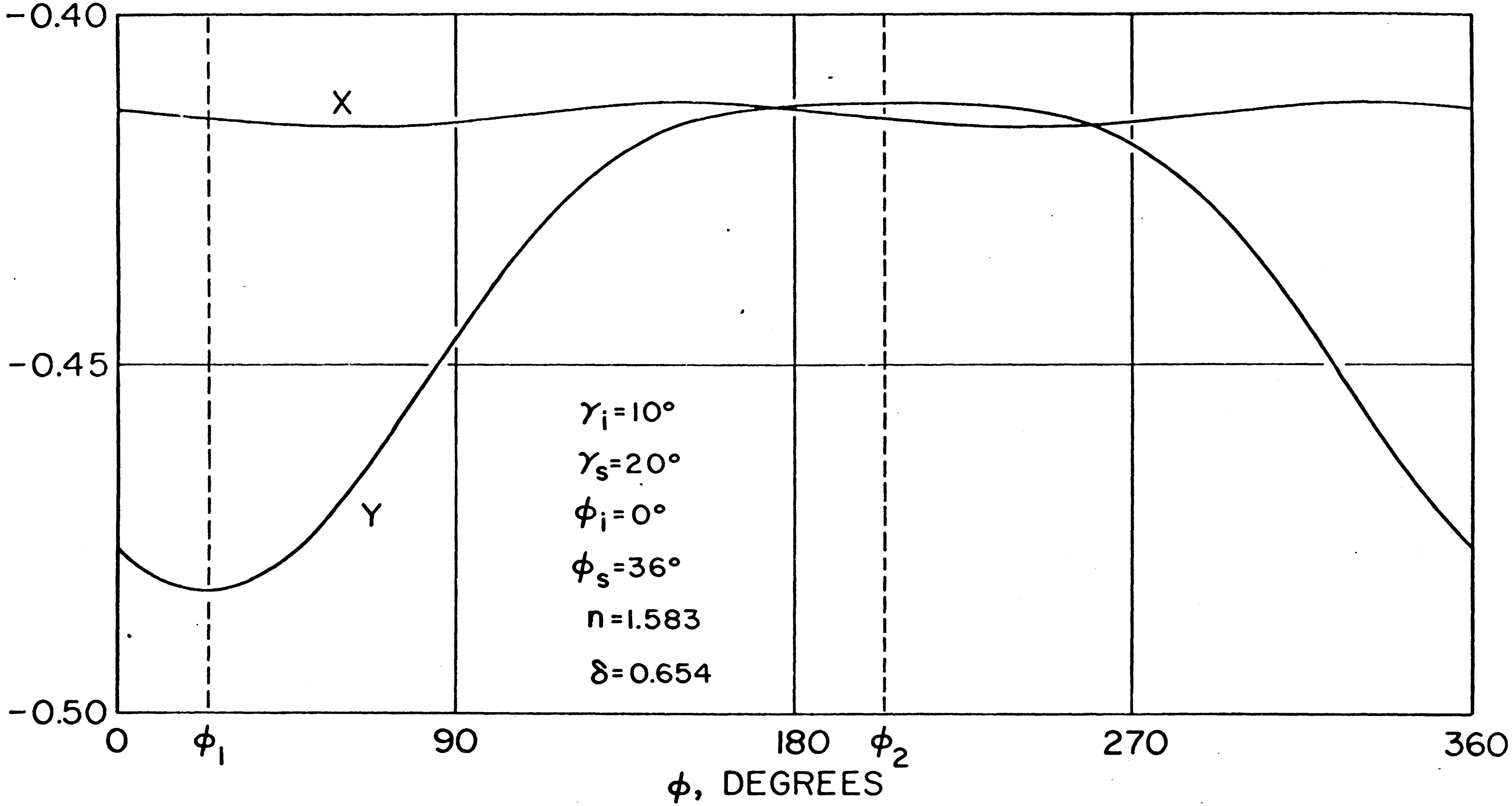

FIG. A-2: Typical variation of X and Y around a ring discontinuity.

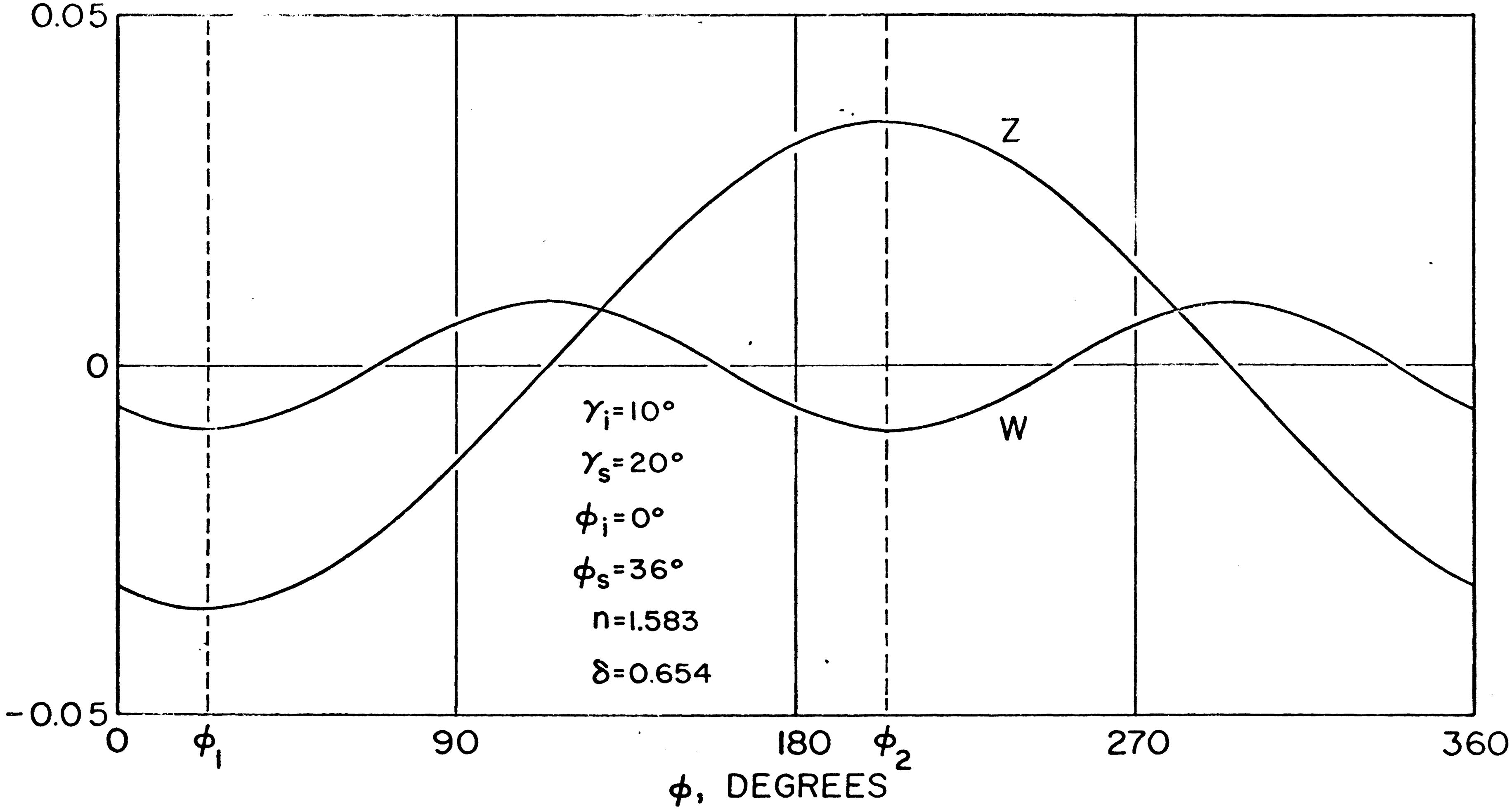

FIG. A-3: Y has essentially only two sinusoidal components.

functions for these factors, we designate the polarizations of the incident and scattered electric field by a pair of angles, p and q, as shown in Figure A-1. Angles p and q measure the amount by which the unit electric vectors depart from being perfectly normal to the planes of incidence and scattering, both planes being in turn normal to the ring and containing $\hat{i}$ and $\hat{s}$, respectively. We find that

$$e_{it} = (\cos\gamma_i \cos\phi_i \sin p + \sin\phi_i \cos p) \sin\phi - (\cos\gamma_i \sin\phi_i \sin p - \cos\phi_i \cos p)\cos\phi \ ,$$

$$e_{rt} = (\cos\gamma_s \cos\phi_s \sin q - \sin\phi_s \cos q) \sin\phi - (\cos\gamma_s \sin\phi_s \sin q + \cos\phi_s \cos q)\cos\phi \ ,$$

$$h_{it} = -(\cos\gamma_i \cos\phi_i \cos p - \sin\phi_i \sin p) \sin\phi + (\cos\gamma_i \sin\phi_i \cos p + \cos\phi_i \sin p)\cos\phi \ ,$$

$$h_{rt} = -(\cos\gamma_s \cos\phi_s \cos q + \sin\phi_s \sin q) \sin\phi + (\cos\gamma_s \sin\phi_s \cos q - \cos\phi_s \sin q)\cos\phi \ ,$$

with the angles γ_i, γ_s, ϕ_i, ϕ_s as defined in Figure A-1.

Since the coefficients X and Y are expressed in terms of the angle $\phi_1 - \phi$, the above polarization components must also be represented in terms of this angle. By means of the identities

$$\sin\phi = \sin\phi_1 \cos(\phi_1 - \phi) - \cos\phi_1 \sin(\phi_1 - \phi) \ ,$$

$$\cos\phi = \cos\phi_1 \cos(\phi_1 - \phi) + \sin\phi_1 \sin(\phi_1 - \phi) \ ,$$

the polarization terms become

$$e_{it} e_{rt} = \frac{1}{2}\left\{F_e - G_e \cos 2(\phi_1 - \phi) + H_e \sin 2(\phi_1 - \phi)\right\} \ ,$$

$$h_{it} h_{rt} = \frac{1}{2}\left\{F_h - G_h \cos 2(\phi_1 - \phi) + H_f \sin 2(\phi_1 - \phi)\right\} \ ,$$

where

$$F_e - F_h = -(1 + \cos\gamma_i \cos\gamma_s) \cos(p+q) \cos(\phi_i - \phi_s) + (\cos\gamma_i + \cos\gamma_s) \sin(p+q) \sin(\phi_i - \phi_s)$$

$$F_e + F_h = -(1 - \cos\gamma_i \cos\gamma_s) \cos(p-q) \cos(\phi_i - \phi_s) + (\cos\gamma_i - \cos\gamma_s) \sin(p-q) \sin(\phi_i - \phi_s)$$

$$G_e - G_h = (1 - \cos\gamma_i \cos\gamma_s) \cos(p+q) \cos(2\phi_1 - \phi_i - \phi_s)$$

$$+ (\cos\gamma_i - \cos\gamma_s) \sin(p+q) \sin(2\phi_1 - \phi_i - \phi_s)$$

$$G_e + G_h = (1+\cos\gamma_i \cos\gamma_s)\cos(p-q)\cos(2\phi_1 - \phi_i - \phi_s)$$
$$+ (\cos\gamma_i + \cos\gamma_s)\sin(p-q)\sin(2\phi_1 - \phi_i - \phi_s)$$

$$H_e - H_h = -(1-\cos\gamma_i \cos\gamma_s)\cos(p+q)\sin(2\phi_1 - \phi_i - \phi_s)$$
$$+ (\cos\gamma_i - \cos\gamma_s)\sin(p+q)\cos(2\phi_1 - \phi_i - \phi_s)$$

$$H_e + H_h = -(1+\cos\gamma_i \cos\gamma_s)\cos(p-q)\sin(2\phi_1 - \phi_i - \phi_s)$$
$$+ (\cos\gamma_i + \cos\gamma_s)\sin(p-q)\cos(2\phi_1 - \phi_i - \phi_s)$$

Thus the integral to be evaluated is

$$S = -\frac{ka}{4\pi \sin^2\beta} \int_0^{2\pi} e^{-ikaT\cos(\phi_1 - \phi)} \cdot$$
$$\cdot \Big\{ \Big[A(F_e + F_h) + C(F_h - F_e) + \frac{1}{2} E(G_e - G_h) + \frac{1}{2} B(H_e + H_h) \Big]$$
$$+ \Big[\frac{1}{2} D(H_h - H_e) \Big] \sin(\phi_1 - \phi)$$
$$+ \Big[D(F_h - F_e) + \frac{1}{2} D(G_e - G_h) \Big] \cos(\phi_1 - \phi)$$
$$+ \Big[A(H_e + H_h) + B(F_e + F_h) + C(H_h - H_e) \Big] \sin 2(\phi_1 - \phi)$$
$$+ \Big[E(F_h - F_e) - A(G_e + G_h) + C(G_e - G_h) \Big] \cos 2(\phi_1 - \phi)$$
$$+ \Big[\frac{1}{2} D(H_h - H_e) \Big] \sin 3(\phi_1 - \phi)$$
$$+ \Big[\frac{1}{2} D(G_e - G_h) \Big] \cos 3(\phi_1 - \phi)$$
$$+ \Big[\frac{1}{2} E(H_h - H_e) - \frac{1}{2} B(G_e + G_h) \Big] \sin 4(\phi_1 - \phi)$$
$$+ \Big[\frac{1}{2} E(G_e - G_h) - \frac{1}{2} B(H_e + H_h) \Big] \cos 4(\phi_1 - \phi) \Big\} \, d(\phi_1 - \phi) \; . \qquad (A.4)$$

where $T = \left\{ \sin^2\gamma_i + 2\sin\gamma_i \sin\gamma_s \cos(\phi_s - \phi_i) + \sin^2\gamma_s \right\}^{1/2}$ and is independent of ϕ.

All of the terms in (A.4) can now be evaluated by means of the relations

$$\int_0^{2\pi} \cos n\psi \, e^{iz\cos\psi} d\psi = i^n 2\pi J_n(z)$$

$$\int_0^{2\pi} \sin n\psi \, e^{iz\cos\psi} d\psi = 0 \ ,$$

where $J_n(z)$ is the Bessel function of the first kind of order n. The result is

$$S = -\frac{ka}{4\sin^2\beta}\Big\{2A\left[(F_e+F_h)J_0+(G_e+G_h)J_2\right]-2C\left[(F_e-F_h)J_0+(G_e-G_h)J_2\right]$$
$$+E\left[(G_e-G_h)J_0+2(F_e-F_h)J_2+(G_e-G_h)J_4\right]$$
$$+iD\left[2(F_e-F_h)J_1-(G_e-G_h)J_1+(G_e-G_h)J_3\right]$$
$$+B\left[(H_e+H_h)(J_0-J_4)\right]\Big\} \ , \qquad \text{(A.5)}$$

in which all the Bessel functions share the common argument, kaT, which argument is implicit. The coefficients

$$(F_e+F_h)-(G_e+G_h) = 2(e_{it1}e_{rt1}+h_{it1}h_{rt1}) \ ,$$

$$(F_e-F_h)-(G_e-G_h) = 2(e_{it1}e_{rt1}-h_{it1}h_{rt1}) \ ,$$

where the subscript "1" signifies that the polarization components are to be evaluated at a flash point. Furthermore, by virtue of the recursion relations for the Bessel functions,

$$J_0(kaT)+J_2(kaT) = \frac{2J_1(kaT)}{kaT} \ ,$$

$$J_1(kaT)+J_3(kaT) = \frac{4J_2(kaT)}{kaT} \ ,$$

$$J_2(kaT) + J_4(kaT) = \frac{6J_3(kaT)}{kaT} \quad ,$$

$$J_0(kaT) - J_4(kaT) = \frac{2J_1(kaT)}{kaT} - \frac{6J_3(kaT)}{kaT} \quad .$$

Thus (A. 5) can be written

$$S = -\frac{ka}{\sin^2\beta}\Big\{A(e_{it1}e_{rt1} + h_{it1}h_{rt1})\,J_2(kaT)$$
$$+ (e_{it1}e_{rt1} - h_{it1}h_{rt1})\Big[(C+E)\,J_0(kaT) - iDJ_1(kaT)\Big]\Big\}$$
$$+ \frac{1}{T\sin^2\beta}\Big\{\Big[A(F_e + F_h) - C(G_e - G_h) + E(F_e - F_h)\Big]J_1(kaT)$$
$$+ \frac{1}{2}\Big[E(G_e - G_h) + B(H_e + H_h)\Big]\Big[J_1(kaT) - 3J_3(kaT)\Big]$$
$$+ iD(G_e - G_h)\,J_2(kaT)\Big\} \quad . \qquad (A.6)$$

For kaT sufficiently large, the second collection of terms may be neglected in comparison with the first and the scattering is given essentially by

$$S \approx -\frac{ka}{\sin^2\beta}\Big\{A(e_{it1}e_{rt1} + h_{it1}h_{rt1})\,J_2(kaT)$$
$$+ (e_{it1}e_{rt1} - h_{it1}h_{rt1})\Big[(C+E)\,J_0(kaT) - iDJ_1(kaT)\Big]\Big\} \quad .$$
$$(A.7)$$

The approximation given by (A. 7) is quite good, even for modest values of kaT, as demonstrated in Figure A-4. The solid trace was obtained from (A. 7), while the dashed trace was computed via a numerical integration of (A. 1), for an arbitrary set of polarization components and bistatic directions. The two curves

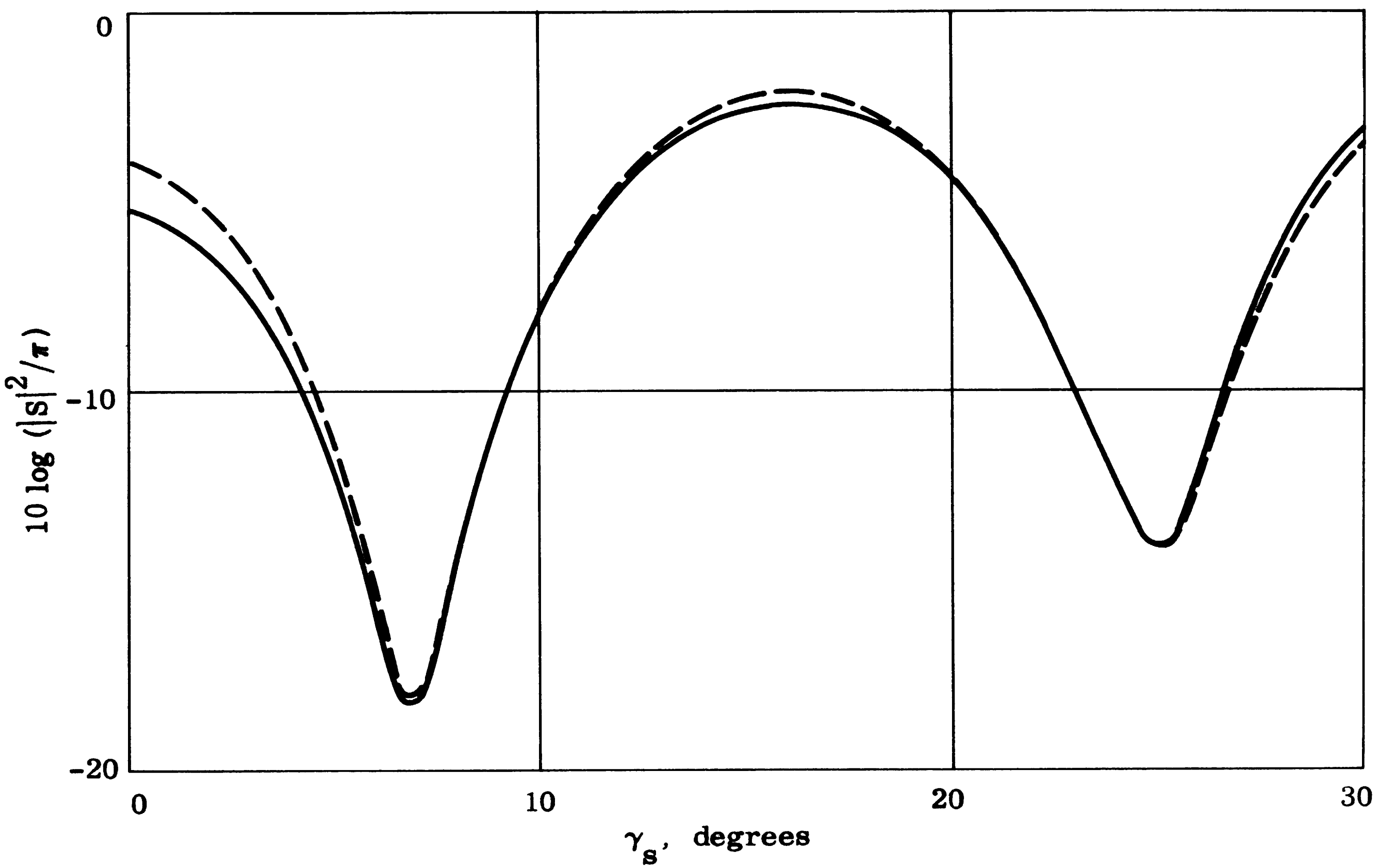

FIG. A-4: Evaluation of the integral (A.1) for ka = 10, $\gamma_i = 30^o$, $\phi_i = 0^o$, $\phi_s = 36^o$, $e_{it}e_{rt} = 0.604$, $h_{it}h_{rt} = 0.262$. Numerical integration (———), analytic (– – – –).

are virtually coincident, with but a slight discrepancy for small values of γ_s. We conclude that the quasi-analytical result of (A.6), and even the approximation of (A.7), are quite good representations of the integral (A.1). By virtue of the form used in (A.2) and (A.3), the coefficients required in (A.7) are simply

$$A = X_1 = X_2 ,$$

$$C+E = \frac{1}{2}(Y_1+Y_2) ,$$

$$D = \frac{1}{2}(Y_1-Y_2) ,$$

where the subscripts denote that the diffraction coefficients must be evaluated at flash points 1 and 2. The above identification is appropriate for a range of bistatic angles spanning the backscattering direction, and an alternative identification is required when the angles embrace the forward direction.