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ON THE EIGEN-FUNCTION EXPANSION OF DYADIC GREEN'S FUNCTIONS

Technical Report

By

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ABSTRACT

This work contains a revision of the treatment of the eigen-function expansion of dyadic Green's functions previously discussed by the author in his book [1]. The singular terms which are missing in the previous treatment have been amended. By starting with the differential equation for the dyadic Green's function of the magnetic type only two sets of solenoidal vector eigenfunctions are needed to determine the complete expressions for both the electric and the magnetic type of dyadic Green's function.

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I INTRODUCTION

Professor Per-Olof Brundell of the University of Lund, Sweden, has kindly called the author's attention to an error in the treatment of the eigenfunction expansion of the dyadic Green's functions described in the author's book [1]. In that work only two sets of solenoidal vector wave functions are used in the expansion of the dyadic delta function $\bar{\bar{I}}\,\delta(\bar{R}-\bar{R}')$. Since the latter is non-solenoidal, the use of solenoidal functions to represent such a quantity is not sufficient or complete. As a result of this error the singular behavior of the dyadic Green's functions is not properly formulated in the author's book.

In this work, the correct expressions for various dyadic Green's functions are derived by means of a revised method which removes the shortcomings found in the previous treatment. In the case of a rectangular waveguide the results has been verified by Collin [2] who has independently found the solution for the field in a source region based on the method of potentials.

II GENERAL FORMULATION

For clarity we introduce two types of dyadic Green's functions designated by $\bar{\bar{G}}_e$, the electric type, and $\bar{\bar{G}}_m$, the magnetic type which satisfy the equations

$$\nabla \times \bar{\bar{G}}_e = \bar{\bar{G}}_m \tag{1}$$

$$\nabla \times \bar{\bar{G}}_m = \bar{\bar{I}}\,\delta(\bar{R}-\bar{R}') + k^2 \bar{\bar{G}}_e \tag{2}$$

where $\bar{\bar{I}}$ denotes the idemfactor defined by

$$\bar{\bar{I}} = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z}$$

and

$$k^2 = \omega^2 \mu_o \epsilon_o \quad .$$

These equations are the dyadic version of Maxwell equations as applied to harmonic fields due to infinitesimal current sources. The relationship between $\bar{G}_e$, $\bar{\bar{G}}_m$, $\bar{\bar{I}}\,\delta(\bar{R}-\bar{R}')$ and $\bar{E}$, $\bar{H}$, $\bar{J}$ are

$$\bar{\bar{G}}_e = \bar{E}^{(x)}\hat{x} + \bar{E}^{(y)}\hat{y} + \bar{E}^{(z)}\hat{z} \tag{3}$$

$$\bar{\bar{G}}_m = i\omega\mu_o(\bar{H}^{(x)}\hat{x} + \bar{H}^{(y)}\hat{y} + \bar{H}^{(z)}\hat{z}) \tag{4}$$

$$\bar{\bar{I}}\,\delta(\bar{R}-\bar{R}') = i\omega\mu_o\left[\bar{J}^{(x)}\hat{x} + \bar{J}^{(y)}\hat{y} + \bar{J}^{(z)}\hat{z}\right] \tag{5}$$

where $\bar{E}^{(x)}$ and $\bar{H}^{(x)}$ represent the electric and the magnetic fields due to an infinitesimal current source with a current density $\bar{J}^{(x)} = \frac{1}{i\omega\mu_o}\delta(\bar{R}-\bar{R}')\hat{x}$ and similarly for the other triads. By eliminating $\bar{\bar{G}}_m$ or $\bar{\bar{G}}_e$ between (1) and (2) we obtain

$$\nabla \times \nabla \times \bar{\bar{G}}_e - k^2 \bar{\bar{G}}_e = \bar{\bar{I}}\,\delta(\bar{R}-\bar{R}') \tag{6}$$

$$\nabla \times \nabla \times \bar{\bar{G}}_m - k^2 \bar{\bar{G}}_m = \nabla \times \left[\bar{\bar{I}}\,\delta(\bar{R}-\bar{R}')\right] \quad . \tag{7}$$

Equations (6) and (7) differ from each other in the inhomogeneous term. Furthermore, we have

$$\nabla \cdot \bar{\bar{G}}_e = -\frac{1}{k^2} \nabla \cdot \left[\bar{\bar{I}} \delta (\bar{R} - \bar{R}') \right] = -\frac{1}{k^2} \nabla \delta (\bar{R} - \bar{R}')$$

$$\nabla \cdot \bar{\bar{G}}_m = 0$$

thus $\bar{\bar{G}}_e$ is nonsolenoidal while $\bar{\bar{G}}_m$ is solenoidal.

The dyadic Green's functions are classified according to the boundary conditions which they must satisfy on an assigned surface. The functions of the first kind satisfy the Dirichlet boundary condition

$$\left.\begin{aligned} \hat{n} \times \bar{\bar{G}}_{e1} &= 0 \\ \hat{n} \times \bar{\bar{G}}_{m1} &= 0 \end{aligned}\right\} \tag{8}$$

and the functions of the second kind satisfy the Neumann condition

$$\left.\begin{aligned} \hat{n} \times \nabla \times \bar{\bar{G}}_{e2} &= 0 \\ \hat{n} \times \nabla \times \bar{\bar{G}}_{m2} &= 0 \end{aligned}\right\} . \tag{9}$$

If the region under consideration is open, it is assumed that the radiation condition prevails at infinity

$$\lim_{R \to \infty} R \left[\nabla \times \bar{\bar{G}}_{e1} - ik\hat{n} \times \bar{\bar{G}}_{e1} \right] = 0 \quad .$$

The same applies to $\bar{\bar{G}}_{e2}$, $\bar{\bar{G}}_{m1}$ and $\bar{\bar{G}}_{m2}$. If the region has a limited open space, such as in an infinite waveguide other forms of radiation condition are assumed to be existing at the open ends. Because of (1) and (2), the functions of the first kind of the electric type are related to the functions of the second kind of the magnetic type, thus

$$\nabla \times \overline{\overline{G}}_{e1} = \overline{\overline{G}}_{m2} \tag{10}$$

$$\nabla \times \overline{\overline{G}}_{m2} = \overline{\overline{I}}\,\delta(\overline{R}-\overline{R}') + k^2 \overline{\overline{G}}_{e1} \tag{11}$$

and similarly

$$\nabla \times \overline{\overline{G}}_{e2} = \overline{\overline{G}}_{m1} \tag{12}$$

$$\nabla \times \overline{\overline{G}}_{m1} = \overline{\overline{I}}\,\delta(\overline{R}-\overline{R}') + k^2 \overline{\overline{G}}_{e2} \quad . \tag{13}$$

The dyadic Green's functions are introduced mainly to facilitate the integration of the vector wave equations for $\overline{E}$ and $\overline{H}$:

$$\nabla \times \nabla \times \overline{E} - k^2 \overline{E} = i\omega\mu_o \overline{J} \, , \tag{14}$$

$$\nabla \times \nabla \times \overline{H} - k^2 \overline{H} = \nabla \times \overline{J} \quad . \tag{15}$$

The integration can be carried out with the aid of the dyadic Green's identity in the form

$$\iiint \left[\overline{P} \cdot \nabla \times \nabla \times \overline{\overline{Q}} - (\nabla \times \nabla \times \overline{P}) \cdot \overline{\overline{Q}} \right] dv = -\oint\!\!\!\oint \hat{n} \cdot \left[\overline{P} \times \nabla \times \overline{\overline{Q}} + (\nabla \times \overline{P}) \times \overline{\overline{Q}} \right] ds \tag{16}$$

This identity can be derived by superposing three vector Green's identities of the form described by Stratton [3]

$$\iiint \left[\overline{P} \cdot \nabla \times \nabla \times \overline{Q} - (\nabla \times \nabla \times \overline{P}) \cdot \overline{Q} \right] dv = -\oint\!\!\!\oint \hat{n} \cdot \left[\overline{P} \times \nabla \times \overline{Q} + (\nabla \times \overline{P}) \times \overline{Q} \right] ds \tag{17}$$

where we let $\overline{Q}$ be equal to $\overline{Q}^{(x)}$, $\overline{Q}^{(y)}$ and $\overline{Q}^{(z)}$, three distinct vector functions, and then introduce the dyadic function $\overline{\overline{Q}}$ defined by

$$\overline{\overline{Q}} = \overline{Q}^{(x)} \hat{x} + \overline{Q}^{(y)} \hat{y} + \overline{Q}^{(z)} \hat{z} \quad .$$

With the aid of (16) we can integrate the equation for $\overline{E}$, (14), by letting $\overline{P} = \overline{E}$ and

$\overline{\overline{Q}} = \overline{\overline{G}}_{e1}$ where $\overline{\overline{G}}_{e1}$ satisfies Eq. (6) and the boundary condition (9). The result is given by

$$\overline{E}(\overline{R}) = i\omega\mu_0 \iiint \overline{\overline{G}}_{e1}(\overline{R}/\overline{R}') \cdot \overline{J}(\overline{R}')\, dv' - \oint\!\!\!\oint_S \left[\nabla \times \overline{\overline{G}}_{e2}(\overline{R}/\overline{R}') \cdot \hat{n} \times \overline{E}(\overline{R}')\right] ds' . \quad (18)$$

In arriving at this expression we have already made use of the symmetry relations

$$\tilde{\overline{\overline{G}}}_{e1}(\overline{R}'/\overline{R}) = \overline{\overline{G}}_{e2}(\overline{R}/\overline{R}')$$

and

$$\widetilde{\nabla' \times \overline{\overline{G}}_{e1}(\overline{R}'/\overline{R})} = \nabla \times \overline{\overline{G}}_{e2}(\overline{R}/\overline{R}')$$

where the sign '$\sim$' denotes the transpose of a dyadic function. The derivation of these symmetrical relations is found in Ref. [1]. If the surface S corresponds to the site of a perfectly conducting surface where $\hat{n} \times \overline{E} = 0$ only the volume integral in (18) remains, that is,

$$\overline{E}(\overline{R}) = i\omega\mu_0 \iiint \overline{\overline{G}}_{e1}(\overline{R}/\overline{R}') \cdot \overline{J}(\overline{R}')\, dv' \quad . \quad (19)$$

The magnetic field $\overline{H}$ can be obtained either by using $\nabla \times \overline{E} = i\omega\mu_0 \overline{H}$ with $\overline{E}$ given by (18) or by letting $\overline{P} = \overline{H}$ and $\overline{\overline{Q}} = \overline{\overline{G}}_{e2}$ in (16) and the relations (10) to (13). In either case we obtain

$$\overline{H}(\overline{R}) = \iiint \overline{\overline{G}}_{m2}(\overline{R}/\overline{R}') \cdot \overline{J}(\overline{R}')\, dv'$$
$$- \frac{1}{i\omega\mu_0} \oint\!\!\!\oint_S \left[\nabla \times \overline{\overline{G}}_{m1}(\overline{R}/\overline{R}') \cdot \hat{n} \times \overline{E}(\overline{R}')\right] ds' \quad . \quad (20)$$

While most of these formulas have been derived in Ref. [1], the presentation here emphasizes the distinction between the two types of dyadic Green's functions which was not stressed before. In fact, neither the subscript notation 'e' and 'm' nor Eqs. (1) and (2) were introduced previously.

In the remaining sections we shall present the eigen-function expansions of various dyadic Green's functions. The topics will be arranged in the same order as they appeared in the author's book. Some of the basic formulas, such as orthogonal properties of various vector wave functions, the circulation theorem involving the product of Bessel functions and many other mathematical theorems will not be reviewed here.

III RECTANGULAR WAVEGUIDE

We start with the equation

$$\nabla \times \nabla \times \overline{\overline{G}}_{m2} - k^2 \overline{\overline{G}}_{m2} = \nabla \times \left[\overline{\overline{I}} \delta (\overline{R} - \overline{R}') \right] \tag{21}$$

for the magnetic dyadic Green's function of the second kind $\overline{\overline{G}}_{m2}$ which satisfies the boundary condition

$$\hat{n} \times \nabla \times \overline{\overline{G}}_{m2} = 0$$

at the walls of a rectangular waveguide corresponding to $x = 0$ and a; $y = 0$ and b. The function also satisfies the radiation condition

$$\nabla \times \overline{\overline{G}}_{m2} = \alpha \, \hat{z} \times \overline{\overline{G}}_{m2}$$

or

$$\overline{\overline{G}}_{m2} = \beta \, \hat{z} \times \nabla \times \overline{\overline{G}}_{m2} \tag{22}$$

at $z = \pm\infty$. These conditions correspond to the radiation condition of the TE or TM modes in a rectangular waveguide where α and β are two sets of constants.

Since $\nabla \cdot \nabla \times \left[\overline{\overline{I}} \delta (\overline{R} - \overline{R}') \right] = 0$, the generalized function $\nabla \times \left[\overline{\overline{I}} \delta (\overline{R} - \overline{R}') \right]$ can be expressed in terms of the vector wave functions $\overline{M}_{omn}(h)$ and $\overline{N}_{emn}(h)$ defined by

$$\overline{M}_{omn}(h) = \nabla \times \left[\psi_{omn}(h) \, \hat{z} \right]$$

$$\overline{N}_{emn}(h) = \frac{1}{K} \nabla \times \nabla \times \left[\psi_{emn}(h) \, \hat{z} \right]$$

where

$$\psi_{\substack{e \\ o}mn}(h) = \left\{ \begin{array}{c} \cos \frac{m\pi}{a} x \cos \frac{n\pi}{b} y \\ \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \end{array} \right\} e^{ihz}$$

$$K^2 = h^2 + k_c^2$$

$$k_c^2 = \left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 .$$

Applying the Ohm-Rayleigh method we let

$$\nabla \times \left[\overline{\overline{I}}\,\delta(\overline{R}-\overline{R}') \right] = \int_{-\infty}^{\infty} \sum_{m,n} \left[\overline{M}_{omn}(h)\,\overline{A}_{omn}(h) + \overline{N}_{emn}(h)\,\overline{B}_{emn}(h) \right] dh \tag{23}$$

where $\overline{A}$, $\overline{B}$ are two sets of unknown vector coefficients to be determined. By taking the anterior scalar product of (23) with $\overline{M}_{om'n'}(-h')$ and integrating through the entire volume of the guide we obtain, as a result of the orthogonal property of the vector wave functions

$$\nabla' \times \overline{M}'_{om'n'}(-h') = \frac{(1+\delta_o)\,\pi\,ab\,k'^2_c}{2}\,\overline{A}_{om'n'}(h')$$

where
$$k'^2_c = \left(\frac{m'\pi}{a}\right)^2 + \left(\frac{n'\pi}{b}\right)^2$$

and
$$\delta_o = \begin{cases} 1, & m' \text{ or } n' = 0 \\ 0, & m' \text{ and } n' \neq 0 \end{cases} ,$$

Hence

$$\overline{A}_{omn}(h) = \frac{(2-\delta_o)}{\pi ab k_c^2}\,\nabla' \times \overline{M}'_{omn}(-h) = \frac{(2-\delta_o)K}{\pi ab k_c^2}\,\overline{N}'_{omn}(-h) \tag{24}$$

where the primed function is defined with respect to the primed variables x', y', z' pertaining to $\overline{R}'$.

Similarly, by taking the scalar product of (23) with $\overline{N}_{em'n'}(-h')$ and performing the integration, we obtain

$$\overline{B}_{emn}(h) = \frac{2-\delta_o}{\pi ab k_c^2}\,\nabla' \times \overline{N}'_{emn}(-h) = \frac{(2-\delta_o)K}{\pi ab k_c^2}\,\overline{M}'_{emn}(-h) \quad , \tag{25}$$

thus

$$\nabla \times \left[\overline{\overline{I}}\,\delta(\overline{R}-\overline{R}')\right] = \int_{-\infty}^{\infty} \sum_{m,n} C_{mn} K \left[\overline{M}_{omn}(h)\,\overline{N}'_{omn}(-h) + \right.$$

$$\left. + \overline{N}_{emn}(h)\,\overline{M}'_{emn}(-h)\right] dh \qquad (26)$$

where $$C_{mn} = \frac{(2-\delta_o)}{\pi a b k_c^2} \quad .$$

To determine $\overline{\overline{G}}_{m2}(\overline{R}/\overline{R}')$ we let

$$\overline{\overline{G}}_{m2}(\overline{R}/\overline{R}') = \int_{-\infty}^{\infty} \sum_{m,n} C_{mn} K \left[a_{omn}\overline{M}_{omn}(h)\,\overline{N}'_{omn}(-h) + \right.$$

$$\left. + b_{emn}\overline{N}_{emn}(h)\,\overline{M}'_{emn}(-h)\right] dh \quad . \qquad (27)$$

Substituting (26) and (27) into (21) we find

$$a_{omn} = b_{emn} = \frac{1}{K^2 - k^2} \quad .$$

Having obtained the eigen-function expansion of $\overline{\overline{G}}_{m2}$ we can determine $\overline{\overline{G}}_{e1}$ by means of (10). The term $\nabla \times \overline{\overline{G}}_{m2}$ is given by, in view of (27),

$$\nabla \times \overline{G}_{m2} = \int_{-\infty}^{\infty} \sum_{m,n} C_{mn} \frac{K^2}{K^2 - k^2} \left[\overline{N}_{omn}(h)\,\overline{N}'_{omn}(-h) + \right.$$

$$\left. + \overline{M}_{emn}(h)\,\overline{M}'_{emn}(-h)\right] dh \qquad (28)$$

where we have made use of the relations

$$\nabla \times \overline{M}_{omn}(h) = K\,\overline{N}_{omn}(h)$$

and $$\nabla \times \overline{N}_{emn}(h) = K\,\overline{M}_{emn}(h) \quad .$$

The expression for $\nabla \times \overline{\overline{G}}_{m2}$ as given by (28) has a singular term which can be extracted from the expression. For that reason we split (28) into two parts

$$\nabla \times \overline{\overline{G}}_{m2} = \int_{-\infty}^{\infty} \sum_{m,n} C_{mn} \left[\overline{M}_e \overline{M}'_e + \frac{K^2}{h^2} \overline{N}_{ot} \overline{N}'_{ot} \right] dh +$$

$$+ \int_{-\infty}^{\infty} \sum_{m,n} C_{mn} \left[\frac{k^2}{K^2 - k^2} \overline{M}_e \overline{M}'_e + \frac{K^2}{K^2 - k^2} \left(\frac{k^2 - k_c^2}{h^2} \overline{N}_{ot} \overline{N}'_{ot} + \right. \right.$$

$$\left. \left. + \overline{N}_{ot} \overline{N}'_{oz} + \overline{N}_{oz} \overline{N}'_{ot} + \overline{N}_{oz} \overline{N}'_{oz} \right) \right] dh \qquad (29)$$

where $\overline{N}_{ot}$ and $\overline{N}_{oz}$ denote, respectively, the transversal part and the longitudinal part of $\overline{N}_o$. It can be shown that

$$\overline{\overline{I}}_t\, \delta(\overline{R} - \overline{R}') = \int_{-\infty}^{\infty} \sum_{m,n} C_{mn} \left[\overline{M}_e \overline{M}'_e + \frac{K^2}{h^2} \overline{N}_{ot} \overline{N}'_{ot} \right] dh$$

and the second integral in (29) can be closed at infinity in the h-plane which yields a residue series. The final result can be written in the form

$$\nabla \times \overline{\overline{G}}_{m2} = \overline{\overline{I}}_t\, \delta(\overline{R} - \overline{R}') + k^2 \overline{\overline{S}}(\overline{R}/\overline{R}') \qquad (30)$$

where

$$\overline{\overline{S}}(\overline{R}/\overline{R}') = \frac{i}{ab} \sum_{m,n} \frac{2 - \delta_o}{k_g k_c^2} \left[\overline{M}_{emn}(\pm k_g) \overline{M}'_{emn}(\mp k_g) + \right.$$

$$\left. + \overline{N}_{omn}(\pm k_g) \overline{N}'_{omn}(\mp k_g) \right] \quad , \; z \gtrless z' \, .$$

Substituting (30) into (11), we obtain

$$\overline{\overline{G}}_{e1}(\overline{R}/\overline{R}') = \overline{\overline{S}}(\overline{R}/\overline{R}') - \frac{\hat{z}\hat{z}\,\delta(\overline{R}-\overline{R}')}{k^2} \quad . \tag{31}$$

The singular term $-\hat{z}\hat{z}\,\delta(\overline{R}-\overline{R}')/k^2$ is missing in the old expression for $\overline{\overline{G}}_{e1}(\overline{R}/\overline{R}')$ discussed in Ref. [1]. The residue series $\overline{\overline{S}}(\overline{R}/\overline{R}')$ is the same as the one defined by Eq. (8), p. 79 of that reference. The singular term vanishes when $\overline{R} \neq \overline{R}'$. When the point of observation lies inside the source region the singular term must be included in evaluating the electric field for an arbitrary current source; that is,

$$\overline{E}(\overline{R}) = i\omega\mu_0 \iiint \overline{\overline{G}}_{e1}(\overline{R}/\overline{R}') \cdot \overline{J}(\overline{R}')\, dv' \tag{32}$$

The method described in this section can be applied to all other dyadic Green's functions. Omitting the details we list below the complete expressions for these functions together with some of the essential formulas.

<u>Cylindrical Waveguide</u>

$$\nabla \times \left[\overline{\overline{I}}\,\delta(\overline{R}-\overline{R}')\right] = \int_{-\infty}^{\infty} \left[\sum_{n,\lambda} C_\lambda K_\lambda \overline{M}_{{}^{e}_{o}n\lambda}(h)\,\overline{N}_{{}^{e}_{o}n\lambda}(-h) + \sum_{n,\mu} C_\mu K_\mu \overline{N}_{{}^{e}_{o}n\mu}(h)\,\overline{M}_{{}^{e}_{o}n\mu}(-h)\right] dh$$

where

$$\overline{M}_{{}^{e}_{o}n\lambda}(h) = \nabla \times \left[\psi_{{}^{e}_{o}n\lambda}(h)\,\hat{z}\right]$$

$$\overline{N}_{{}^{e}_{o}n\lambda}(h) = \frac{1}{K_\lambda} \nabla \times \nabla \times \left[\psi_{{}^{e}_{o}n\lambda}(h)\,\hat{z}\right]$$

$$\psi_{{}^{e}_{o}n\lambda}(h) = J_n(\lambda r) \begin{matrix}\cos\\ \sin\end{matrix}\, n\phi\, e^{ihz}$$

$$J_n(\lambda r) = 0 \quad \text{at} \quad r = a; \quad K_\lambda^2 = \lambda^2 + h^2$$

$$\overline{M}_{\substack{e\\o}n\mu}(h) = \nabla x \left[\psi_{\substack{e\\o}n\mu}(h)\, e^{ihz} \right]$$

$$\overline{N}_{\substack{e\\o}n\mu}(h) = \frac{1}{K_\mu} \nabla x \nabla x \left[\psi_{\substack{e\\o}n\mu}(h)\, e^{ihz} \right]$$

$$\psi_{\substack{e\\o}n\mu}(h) = J_n(\mu r) \begin{matrix}\cos\\ \sin\end{matrix}\, n\phi\, e^{ihz}$$

$$\frac{\partial J_n(\mu r)}{\partial r} = 0 \quad \text{at} \quad r = a; \; K_\mu^2 = \mu^2 + h^2$$

$$C_\lambda = \frac{2-\delta_o}{4\pi^2 \mu^2 I_\lambda}$$

$$I_\lambda = \int_0^a J_n^2(\lambda r)\, r\, dr = \frac{a^2}{2\lambda^2} \left[\frac{\partial J_n(\lambda r)}{\partial r} \right]^2_{r=a}$$

$$C_\mu = \frac{2-\delta_o}{4\pi^2 \lambda^2 I_\mu}$$

$$I_\mu = \int_0^a J_n^2(\mu r)\, r\, dr = \frac{a^2}{2\mu^2} \left(\mu^2 - \frac{n^2}{a^2}\right) J_n^2(\mu a)$$

The origin of these functions or coefficients is found in Ref. [1].

$$\overline{\overline{G}}_{m2}(\overline{R}/\overline{R}') = \int_{-\infty}^{\infty} \left[\sum_{n,\lambda} \frac{C_\lambda K_\lambda}{K_\lambda^2 - k^2} \overline{M}_{\substack{e\\o}n\lambda}(h)\, \overline{N}'_{\substack{e\\o}n\lambda}(-h) + \sum_{n,\mu} \frac{C_\mu K_\mu}{K_\mu^2 - k^2} \overline{N}_{\substack{e\\o}n\mu}(h)\, \overline{M}'_{\substack{e\\o}n\mu}(-h) \right] dh$$

$$\nabla \times \overline{\overline{G}}_{m2}(\overline{R}/\overline{R}') = \overline{\overline{I}}_t \delta(\overline{R}-\overline{R}') + k^2 \overline{\overline{S}}(\overline{R}/\overline{R}')$$

$$\overline{\overline{G}}_{e1}(\overline{R}/\overline{R}') = \overline{\overline{S}}(\overline{R}/\overline{R}') - \frac{\hat{z}\hat{z}\delta(\overline{R}-\overline{R}')}{k^2} \tag{33}$$

$$\overline{\overline{S}}(\overline{R}/\overline{R}') = \frac{i}{4\pi} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} (2-\delta_0) \cdot$$

$$\cdot \left[\frac{1}{\lambda^2 k_\lambda I_\lambda} \overline{N}_{\substack{e\\o}n\lambda}(\pm k_\lambda) \overline{N}'_{\substack{e\\o}n\lambda}(\mp k_\lambda) + \frac{1}{\mu^2 k_\mu I_\mu} \overline{M}_{\substack{e\\o}n\mu}(\pm k_\mu) \overline{M}'_{\substack{e\\o}n\mu}(\mp k_\mu) \right]$$

$$z \gtrless z'$$

$$k_\lambda = \sqrt{k^2 - \lambda^2}\ , \qquad k_\mu = \sqrt{k^2 - \mu^2}\ .$$

The residue series $\overline{\overline{S}}(\overline{R}/\overline{R}')$ is the same as the one described by Eq. (5), p. 89 of Ref. [1].

Eigen-function Expansion of Free-space Dyadic Green's Functions Using Cylindrical Vector Wave Functions

$$\overline{M}_{\substack{e\\o}n\lambda}(h) = \nabla \times \left[\psi_{\substack{e\\o}n\lambda}(h)\, \hat{z} \right],$$

$$\overline{N}_{\substack{e\\o}n\lambda}(h) = \frac{1}{K} \nabla \times \nabla \times \left[\psi_{\substack{e\\o}n\lambda}(h)\, \hat{z} \right]$$

$K^2 = h^2 + \lambda^2$; h and λ both continuous.

$$\nabla \times \left[\overline{\overline{I}} \delta(\overline{R}-\overline{R}') \right] = \int_{-\infty}^{\infty} dh \int_0^{\infty} d\lambda \sum_n C_\lambda K \cdot$$

$$\cdot \left[\overline{M}_{\substack{e\\o}n\lambda}(h) \overline{N}'_{\substack{e\\o}n\lambda}(-h) + \overline{N}_{\substack{e\\o}n\lambda}(h) \overline{M}'_{\substack{e\\o}n\lambda}(-h) \right]$$

$$C_\lambda = \frac{2-\delta_0}{4\pi^2 \lambda}\ .$$

We denote the free-space dyadic Green's function of the magnetic type by $\bar{\bar{G}}_{mo}(\bar{R}/\bar{R}')$ and the function of the electric type by $\bar{\bar{G}}_{eo}(\bar{R}/\bar{R}')$.

$$\bar{\bar{G}}_{mo}(\bar{R}/\bar{R}') = \int_{-\infty}^{\infty} dh \int_{0}^{\infty} d\lambda \sum_{n} \frac{C_\lambda K}{K^2 - k^2} \cdot$$

$$\cdot \left[\bar{M}_{\substack{e\\o}n\lambda}(h)\, \bar{N}'_{\substack{e\\o}n\lambda}(-h) + \bar{N}_{\substack{e\\o}n\lambda}(h)\, \bar{M}'_{\substack{e\\o}n\lambda}(-h) \right], \tag{35}$$

$$\nabla \times \bar{\bar{G}}_{mo}(\bar{R}/\bar{R}') = \int_{-\infty}^{\infty} dh \int_{0}^{\infty} d\lambda \sum_{n} \frac{C_\lambda K^2}{K^2 - k^2} \cdot$$

$$\cdot \left[\bar{N}_{\substack{e\\o}n\lambda}(h)\, \bar{N}'_{\substack{e\\o}n\lambda}(-h) + \bar{M}_{\substack{e\\o}n\lambda}(h)\, \bar{M}'_{\substack{e\\o}n\lambda}(-h) \right].$$

For cylindrical problems, we remove the λ-integration which yields

$$\nabla \times \bar{\bar{G}}_{mo}(\bar{R}/\bar{R}') = k^2 \bar{\bar{S}}_h(\bar{R}/\bar{R}') + \hat{z}\hat{z}\,\delta(\bar{R} - \bar{R}')$$

where

$$\bar{\bar{S}}_h(\bar{R}/\bar{R}') = \int_{-\infty}^{\infty} dh \sum_{n} \frac{i(2-\delta_o)}{8\pi\eta^2} \cdot$$

$$\cdot \begin{Bmatrix} \bar{N}^{(1)}_{\substack{e\\o}n\eta}(h)\, \bar{N}'_{\substack{e\\o}n\eta}(-h) + \bar{M}^{(1)}_{\substack{e\\o}n\eta}(h)\, \bar{M}'_{\substack{e\\o}n\eta}(-h) \\ \bar{N}_{\substack{e\\o}n\eta}(h)\, \bar{N}'^{(1)}_{\substack{e\\o}n\eta}(-h) + \bar{M}_{\substack{e\\o}n\eta}(h)\, \bar{M}'^{(1)}_{\substack{e\\o}n\eta}(-h) \end{Bmatrix} \quad r \gtrless r' \ . \tag{36}$$

Functions with superscript (1) are defined with respect to the Hankel function of the first kind and $\eta = \sqrt{k^2 - h^2}$.

$$\overline{\overline{G}}_{eo}(\overline{R}/\overline{R}') = \overline{\overline{S}}_h(\overline{R}/\overline{R}') - \frac{\overline{\overline{I}}_t \delta(\overline{R}-\overline{R}')}{k^2} \quad . \tag{37}$$

The integral of the residue series given by (35) is the same as Eq. (5), p. 96 of Ref. [1].

For flat earth problems, we remove the h-integration in (35) which yields

$$\nabla x \overline{\overline{G}}_{mo}(\overline{R}/\overline{R}') = k^2 \overline{\overline{S}}_\lambda(\overline{R}/\overline{R}') + \overline{\overline{I}}_t \delta(\overline{R}-\overline{R}')$$

hence

$$\overline{\overline{G}}_{eo}(\overline{R}/\overline{R}') = \overline{\overline{S}}_\lambda(\overline{R}/\overline{R}') - \frac{\hat{z}\hat{z}\delta(\overline{R}-\overline{R}')}{k^2} \tag{38}$$

where

$$\overline{\overline{S}}_\lambda(\overline{R}/\overline{R}') = \frac{i}{4\pi}\int_0^\infty d\lambda \sum_n \frac{2-\delta_o}{\lambda h_1} \cdot$$

$$\cdot \begin{Bmatrix} \overline{M}_{\substack{e\\o}n\lambda}(h_1)\, \overline{M}'_{\substack{e\\o}n\lambda}(-h_1) + \overline{N}_{\substack{e\\o}n\lambda}(h_1)\, \overline{N}'_{\substack{e\\o}n\lambda}(-h_1) \\ \overline{M}_{\substack{e\\o}n\lambda}(-h_1)\, \overline{M}'_{\substack{e\\o}n\lambda}(h_1) + \overline{N}_{\substack{e\\o}n\lambda}(-h_1)\, \overline{N}'_{\substack{e\\o}n\lambda}(h_1) \end{Bmatrix} \quad z \gtrless z' \tag{39}$$

where $h_1 = \sqrt{k^2 - \lambda^2}$. Equation (39) is the same as Eq. (1), p. 103 found in Ref. [1].

<u>Eigen-function Expansion of Free-space Dyadic Green's Functions Using Elliptical Vector Wave Functions</u>

$$\nabla x \overline{\overline{G}}_{mo}(\overline{R}/\overline{R}') = k^2\, \overline{\overline{S}}(\overline{R}/\overline{R}') + \hat{z}\hat{z}\, \delta(\overline{R}-\overline{R}')$$

$$\overline{\overline{G}}_{eo}(\overline{R}/\overline{R}') = \overline{\overline{S}}(\overline{R}/\overline{R}') - \frac{\overline{\overline{I}}_t \delta(\overline{R}-\overline{R}')}{k^2} \tag{40}$$

$\overline{\overline{S}}(\overline{R}/\overline{R}')$ is the same as the one given by Eq. (3), p. 118 of Ref. [1].

Perfectly Conducting Wedge

$$\nabla \times \overline{\overline{G}}_{m2}(\overline{R}/\overline{R}') = k^2 \overline{\overline{S}}(\overline{R}/\overline{R}') + \hat{z}\hat{z}\,\delta(\overline{R}-\overline{R}')$$

$$\overline{\overline{G}}_{e1}(\overline{R}/\overline{R}') = \overline{\overline{S}}(\overline{R}/\overline{R}') - \frac{\overline{\overline{I}}_t\,\delta(\overline{R}-\overline{R}')}{k^2} \tag{41}$$

$\overline{\overline{S}}(\overline{R}/\overline{R}')$ is the same as the one given by Eq. (9), p. 123 of Ref. [1].

Eigen-function Expansion of Free-space Dyadic Green's Functions Using Spherical Vector Wave Functions

$$\nabla \times \overline{\overline{G}}_{mo}(\overline{R}/\overline{R}') = \int_0^\infty \sum_{n=1}^{\infty} \sum_{m=0}^{n} C_{mn} \frac{K^4\,dK}{K^2 - k^2} \cdot$$

$$\cdot \left[\overline{M}_{\substack{e\\o}mn}(K)\,\overline{M}'_{\substack{e\\o}mn}(K) + \overline{N}_{\substack{e\\o}mn}(K)\,\overline{N}'_{\substack{e\\o}mn}(K)\right] \tag{42}$$

where

$$\overline{M}_{\substack{e\\o}mn}(K) = \nabla \times \left[\psi_{\substack{e\\o}mn}(K)\,\overline{R}\right]$$

$$\overline{N}_{\substack{e\\o}mn}(K) = \frac{1}{K} \nabla \times \nabla \times \left[\psi_{\substack{e\\o}mn}(K)\,\overline{R}\right]$$

$$\psi_{\substack{e\\o}mn}(K) = j_n(KR) \begin{matrix}\cos\\ \sin\end{matrix}\, m\phi\, P_n^m(\cos\theta)$$

$$C_{mn} = \frac{2-\delta_o}{2\pi^2} \frac{(2n+1)}{n(n+1)} \frac{(n-m)!}{(n+m)!} \qquad \delta_o = \begin{cases} 1, & m = 0 \\ 0, & m \neq 0 \end{cases}.$$

Equation (42) has a singular part in the integration with respect to K represented by

$$\overline{\overline{s}}_1(\overline{R}/\overline{R}') = \int_0^\infty \sum_{n=1}^{\infty} \sum_{m=0}^{n} C_{mn} K^2 \left[\overline{M}_{\substack{e\\o}mn}(K)\,\overline{M}'_{\substack{e\\o}mn}(K)\right] dK =$$

$$= \frac{\pi\delta(R-R')}{2R^2} \sum_{n=1}^{\infty} \sum_{m=0}^{n} C_{mn} \overline{m}_{\substack{e\\o}mn} \overline{m}'_{\substack{e\\o}mn} \tag{43}$$

where

$$\overline{m}_{\substack{e\\o}mn} = \left[\mp \frac{m P_n^m(\cos\theta)}{\sin\theta} \begin{matrix}\sin\\ \cos\end{matrix} n\phi\, \hat{\theta} - \frac{\partial P_n^m(\cos\theta)}{\partial\theta} \begin{matrix}\cos\\ \sin\end{matrix} m\phi\, \hat{\phi} \right] .$$

It should be pointed out that the function $\delta(R-R')$ is a one-dimensional delta function resulting from

$$\int_0^{\infty} K^2 j_n(KR)\, j_n(KR')\, dK = \frac{\pi\,\delta(R-R')}{2R^2} .$$

Having recognized the singular part of (42), we can evalute the remaining part by contour integration. Thus, we obtain

$$\nabla x \overline{\overline{G}}_{mo}(\overline{R}/\overline{R}') = \int_0^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{n} C_{mn} K^2 \left[\overline{M}_{\substack{e\\o}mn}(K)\, \overline{M}'_{\substack{e\\o}mn}(K) \right] dK +$$

$$+ \int_0^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{n} C_{mn} K^2 \left[\frac{k^2}{K^2-k^2} \overline{M}_{\substack{e\\o}mn}(K)\, \overline{M}'_{\substack{e\\o}mn}(K) \right.$$

$$\left. + \frac{K^2}{K^2-k^2} \overline{N}_{\substack{e\\o}mn}(K)\, \overline{N}'_{\substack{e\\o}mn}(K) \right] dK \tag{44}$$

$$\overline{\overline{G}}_{eo}(\overline{R}/\overline{R}') = \overline{\overline{S}}(\overline{R}/\overline{R}') - \frac{1}{k^2} \left[\hat{R}\hat{R}\, \delta(\overline{R}-\overline{R}') + \overline{\overline{s}}_2(\overline{R}/\overline{R}') \right] \tag{45}$$

where

$$\overline{\overline{s}}_2(\overline{R}/\overline{R}') = \frac{\pi\delta(R-R')}{2R^2} \sum_{n=1}^{\infty} \sum_{m=0}^{n} C_{mn} (\hat{R} x \overline{m}_{\substack{e\\o}mn})\, (\hat{R}' x\, \overline{m}'_{\substack{e\\o}mn}) .$$

and $\overline{\overline{S}}(\overline{R}/\overline{R}')$ is the same as Eq. (18), p. 174 of Ref. [1].

Cone

$$\overline{\overline{G}}_{e1}(\overline{R}/\overline{R}') = \overline{\overline{S}}(\overline{R}/\overline{R}') - \frac{1}{k^2}\Bigg[\hat{R}\hat{R}\,\delta(\overline{R}-\overline{R}') + \frac{\delta(R-R')}{2\pi R^2}\sum_{m,\mu}\frac{2-\delta_o}{\mu(\mu+1)I_\mu}(\hat{R}\times\overline{m}_{\substack{e\\o}m\mu})(\hat{R}'\times\overline{m}'_{\substack{e\\o}m\mu})\Bigg] \quad (46)$$

where

$$\overline{\overline{S}}(\overline{R}/\overline{R}') = \frac{ik}{2\pi}\sum_m (2-\delta_o)\left[\sum_\lambda \frac{1}{\lambda(\lambda+1)I_{m\lambda}}\left\{\begin{matrix}\overline{M}^{(1)}_{\substack{e\\o}m\lambda}(k)\,\overline{M}'_{\substack{e\\o}m\lambda}(k)\\ \overline{M}_{\substack{e\\o}m\lambda}(k)\,\overline{M}'^{(1)}_{\substack{e\\o}m\lambda}(k)\end{matrix}\right\} + \sum \frac{1}{\mu(\mu+1)I_{m\mu}}\left\{\begin{matrix}\overline{N}^{(1)}_{\substack{e\\o}n\mu}(k)\,\overline{N}'_{\substack{e\\o}n\mu}(k)\\ \overline{N}_{\substack{e\\o}n\mu}(k)\,\overline{N}'^{(1)}_{\substack{e\\o}n\mu}(k)\end{matrix}\right\}\right] \quad R \gtrless R'$$

$P^m_\mu(\cos\theta_o) = 0$, characteristic equation for μ

$$I_\mu = \int_{\theta_o}^{\pi} (P^m_\mu)^2 \sin\theta\, d\theta$$

$$\overline{m}_{\substack{e\\o}m\mu} = \mp \frac{m P^m_\mu(\cos\theta)}{\sin\theta}\,\frac{\sin}{\cos}\, m\phi\,\hat{\theta} - \frac{\partial P^m_\mu(\cos\theta)}{\partial\theta}\,\frac{\cos}{\sin}\, m\phi\,\hat{\phi}$$

$$\frac{\partial P^m_\lambda}{\partial\theta} = 0, \quad \text{at } \theta = \theta_o, \text{ characteristic equation for } \lambda$$

$$I_\lambda = \int_{\theta_o}^{\pi} (P^m_\lambda)^2 \sin\theta\, d\theta$$

$$\overline{M}_{\substack{e\\o}m\lambda}(k) = \nabla \times \left[j_\lambda(kR)\, P_\lambda^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix}\, m\phi\, \overline{R} \right]$$

$$\overline{N}_{\substack{e\\o}m\mu}(k) = \frac{1}{k} \nabla \times \nabla \times \left[j_\mu(kR)\, P_\mu^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix}\, m\phi\, \overline{R} \right] .$$

Functions with superscript (1) are defined with respect to spherical Hankel functions of the first kind. The residue series $\overline{\overline{S}}(\overline{R}/\overline{R}')$ is the same as the one given by Eq. (22), p. 191 of Ref. [1].

Rectangular Waveguide with a Moving Isotropic Medium

Because of the incomplete sets of functions used the residue series derived in the book was wrong even where it is applied to regions where there are no current sources. The correct expression for $\overline{\overline{G}}_{e1}(\overline{R}/\overline{R}')$ is found to be

$$\overline{\overline{G}}_{e1}(\overline{R}/\overline{R}') = \overline{\overline{S}}(\overline{R}/\overline{R}') - \frac{1}{k^2} \hat{z}\hat{z}\, \delta(\overline{R}-\overline{R}') \tag{47}$$

where

$$\overline{\overline{S}}(\overline{R}/\overline{R}') = \frac{i a^3}{x_o y_o} \sum_{m,n} \frac{2-\delta_o}{k_g k_c^2} \cdot$$

$$\cdot \left[\overline{N}_{omn}(\pm k_g)\overline{\overline{b}} \cdot \overline{N}'_{omn}(\mp k_g) + \overline{M}_{emn}(\pm k_g)\overline{\overline{b}} \cdot \overline{M}'_{\substack{e\\o}mn}(\mp k_g) \right]$$

$$z \gtrless z' \quad . \tag{48}$$

The terms involving the $\overline{N}$ functions are different from the corresponding terms in the residue series given by Eq. (14), p. 219 of Ref. [1]. The parameters in (48) are defined as before. They are:

$$k_g = \left(a^2 k^2 - a k_c^2 \right)^{1/2}$$

$$a = \frac{1-\beta^2}{1-n^2\beta^2} , \quad \beta = \frac{v}{c} , \quad \overline{\overline{b}} = \frac{1}{a} (\hat{x}\hat{x} + \hat{y}\hat{y}) + \hat{z}\hat{z}$$

$$\overline{N}_{omn}(h) = \frac{1}{Ka} \nabla x \nabla x \left[\psi_{omn}(h) \hat{z} \right]$$

$$\overline{M}_{emn}(h) = \nabla x \left[\psi_{emn}(h) \hat{z} \right]$$

$$\psi_{\substack{e \\ o}mn}(h) = \left\{ \begin{matrix} \cos \frac{m\pi x}{x_o} \cos \frac{n\pi y}{y_o} \\ \\ \sin \frac{m\pi x}{x_o} \sin \frac{n\pi y}{y_o} \end{matrix} \right\} e^{ihz}$$

$$K^2 a^2 = h^2 + a k_c^2 , \quad k_c^2 = \left(\frac{m\pi}{x_o}\right)^2 + \left(\frac{n\pi}{y_o}\right)^2$$

walls of guide: $x = 0$ and x_o, $y = 0$ and y_o .

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