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FROM: Jovan Zatkalik

SUBJECT: Non-Specular Radar Cross Section of the Discontinuity in Surface Impedance. Part B: The Case of Real Impedance.

In this memo the far field scattering from the discontinuity in surface impedance is treated for the case of real impedances. The case of reactive impedances was treated in the memo numbered 011764-508-M (Ref. 1). The reason for such a separation lies in the fact that in the case of reactive impedances, the exact evaluation of some integrals can be carried out, while in the other cases the exact solution cannot be obtained except in the convergent infinite series form. Besides this form of solution, it is attempted here to get the approximate solution for the far field in the closed form, along with the estimation of error.

The forms of integrals defining the decomposition functions $F_1^-(-\xi)$ and $F_2^-(\xi)$ (Ref. 1, page 15) depend on the choice of the integration contours and branch cuts. We have tried a lot of contours seeking to find the solution in the simplest form and have finally chosen two contours: one the same as in Fig. 5 of Ref. 1, and the second as in Fig. 1 of this memo. The notation and the meaning of various quantities are the same as in Ref. 1 to which we shall often make reference.

The First Integration Contour

In this case we shall put:

$$\eta_1 = R_1$$

$$\eta_2 = R_2$$

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and for the function $F_1^-(\xi)$, where $\xi = K\cos\theta$, or $\xi = K\cos\theta_0$, we have the following expression (Ref. 1, Eq. (33)):

$$F_1^-(\xi) = -\frac{1}{2\pi i}\int_{i\alpha-\infty}^{i\alpha+\infty} \ell n\left(1-\frac{jKR_1}{\sqrt{\zeta^2-K^2}}\right)\frac{d\zeta}{\zeta-\xi} \quad . \tag{1}$$

It is easy to show that $F_1^-(\xi)$ is real, if $|\xi| \leq K$. Namely, by taking the same contour as in Fig. 5 of Ref. 1 we obtain from Eq. (1):

$$F_1^-(\xi) = -\frac{1}{2\pi i}\int_K^{\infty} \ell n\frac{\sqrt{\sigma^2-K^2}-jKR_1}{\sqrt{\sigma^2-K^2}+jkR_1}\cdot\frac{d\sigma}{\sigma-\xi} \quad . \tag{2}$$

In Eq. (2) the real part of the integral is zero, for:

$$\ell n\left|\frac{a-jb}{a+jb}\right| = \frac{1}{2}\ell n\frac{a^2+b^2}{a^2+b^2} = 0$$

so we have:

$$F_1^-(\xi) = -\frac{1}{2\pi}\int_K^{\infty}\left[\text{arctg}\left(\frac{-KR_1}{\sqrt{\sigma^2-K^2}}\right)-\text{arctg}\left(\frac{KR_1}{\sqrt{\sigma^2-K^2}}\right)\right]\frac{d\sigma}{\sigma-\xi} =$$

$$= \frac{1}{\pi}\int_K^{\infty}\text{arctg}\left(\frac{KR_1}{\sqrt{\sigma^2-K^2}}\right)\frac{d\sigma}{\sigma-\xi} \tag{3}$$

and $F_1^-(\xi)$ is real.

If we put:

$$\sigma = K\,\text{ch}\,x \qquad R_1 = \sin\alpha$$

$$d\sigma = K\,\text{sh}\,x \qquad \frac{\xi}{K} = \cos\theta$$

the function $F_1^-(\xi)$ becomes:

$$F_1^-(K\cos\theta) = -\frac{1}{\pi}\int_0^\infty \operatorname{arctg}\left(\frac{\sin\alpha}{\operatorname{sh}x}\right)\frac{\operatorname{sh}x}{\operatorname{ch}x-\cos\theta}\,dx \quad . \tag{4}$$

Consider the infinite series:

$$\sum_{i=1}^{\infty} p^{2i-1}\cdot\frac{\sin(2i-1)t}{2i-1} = \frac{1}{2}\operatorname{arctg}\left(\frac{2p\sin t}{1-p^2}\right) \quad , \tag{5}$$

given in Ref. 2, page 41, which is convergent for $0 < t < 2\pi$, and $p^2 \leq 1$. If we put:

$$p = e^{-x} \qquad (x \geq 0, \text{ and } p \leq 1 \text{ accordingly})$$

$$t = \alpha$$

we obtain from Eq. (5):

$$\sum_{i=1}^{\infty} e^{-(2i-1)x}\,\frac{\sin(2i-1)\alpha}{2i-1} = \frac{1}{2}\operatorname{arctg}\left(\frac{2e^{-x}\cdot\sin\alpha}{1-e^{-2x}}\right) = \frac{1}{2}\operatorname{arctg}\left(\frac{\sin\alpha}{\operatorname{sh}x}\right)$$

which is precisely (save factor $\frac{1}{2}$) the first factor in the integrand for $F_1^-(K\cos\theta)$. So we have from Eq. (4):

$$F_1^-(K\cos\theta) = -\frac{2}{\pi}\int_0^\infty\left(\sum_{i=1}^{\infty} e^{-(2i-1)x}\,\frac{\sin(2i-1)\alpha}{2i-1}\right)\cdot\frac{\operatorname{sh}x}{\operatorname{ch}x-\cos\theta}\,dx \quad . \tag{5a}$$

It is easy to prove that in Eq. (5a) the conditions for changing the order of integration and summation are fulfilled, so we have:

$$F_1^-(K\cos\theta) = -\frac{2}{\pi}\sum_{i=1}^{\infty}\left[\frac{\sin(2i-1)\alpha}{2i-1}\int_0^{\infty} e^{-(2i-1)x}\,\frac{\operatorname{sh} x}{\operatorname{ch} x-\cos\theta}\,dx\right]. \tag{6}$$

The integral in the brackets we shall evaluate on the basis of integral:

$$\int_0^{\infty}\frac{e^{-\mu x}}{\operatorname{ch} x-\cos t}\,dx = \frac{2}{\sin t}\sum_{n=1}^{\infty}\frac{\sin nt}{\mu+n}$$

(see Ref. 2, page 357; no. 3545.2). After some manipulation we obtain:

$$\int_0^{\infty} e^{-(2i-1)x}\,\frac{\operatorname{sh} x}{\operatorname{ch} x-\cos\theta}\,dx = \frac{1}{\sin\theta}\sum_{n=1}^{\infty}\left(\frac{\sin n\theta}{2i-2+n}-\frac{\sin n\theta}{2i+n}\right) =$$

$$= \frac{1}{\sin\theta}\left(\frac{\sin\theta}{2i-1}+\frac{\sin 2\theta}{2i}\right)+\frac{1}{\sin\theta}\left(\sum_{n=3}^{\infty}\frac{\sin n\theta}{2i-2+n}-\sum_{n=1}^{\infty}\frac{\sin n\theta}{2i+n}\right). \tag{7}$$

The terms in the brackets, which we shall designate by $F(\theta, i)$, can be rearranged as follows:

$$F(\theta, i) = \sum_{n=1}^{\infty}\frac{\sin(2+n)\theta-\sin n\theta}{2i+n} = 2\sin\theta\sum_{n=1}^{\infty}\frac{\cos(n+1)\theta}{2i+n} =$$

$$= 2\sin\theta\,\sin(2i-1)\theta\sum_{m=2i+1}^{\infty}\frac{\sin m\theta}{m} + 2\sin\theta\cos(2i-1)\theta\sum_{m=2i+1}^{\infty}\frac{\cos m\theta}{m}$$

where we have put: $m = 2i+n$.

We know that:

$$\sum_{K=1}^{\infty}\frac{\sin Kx}{K} = \frac{\pi-x}{2}$$

$$\sum_{K=1}^{\infty} \frac{\cos Kx}{K} = -\frac{1}{2} \ln \sin \frac{x}{2}$$

(see, for example, Ref. 2, page 38) so we have after some manipulation:

$$F(\theta, i) = 2 \sin\theta \sin(2i-1)\theta \left(\frac{\pi-\theta}{2} - \sum_{m=1}^{2i} \frac{\sin m\theta}{m} \right) -$$

$$- 2 \sin\theta \cos(2i-1)\theta \left(\ln \sin \frac{\theta}{2} + \sum_{m=1}^{2i} \frac{\cos m\theta}{m} \right) . \qquad (8)$$

With Eqs. (6), (7) and (8) the function $F_1^-(K \cos\theta)$ becomes:

$$F_1^-(K\cos\theta) = -\frac{2}{\pi} \sum_{i=1}^{\infty} \frac{\sin(2i-\alpha)}{2i-1} \left[\frac{1}{2i-1} + \frac{\cos\theta}{i} + 2 f_1(\theta, i) \sin(2i-1)\theta = \right.$$

$$\left. - 2 f_2(\theta, i) \cos(2i-\theta) \right] =$$

$$= -\frac{2}{\pi} \sum_{i=1}^{\infty} \frac{\sin(2i-\alpha)}{(2i-1)^2} - \frac{2\cos\theta}{\pi} \sum_{i=1}^{\infty} \frac{\sin(2i-\alpha)}{i(2i-\alpha)} +$$

$$+ \frac{2}{\pi} \sum_{i=1}^{\infty} \left\{ \frac{\cos[(2i-1)(\alpha+\theta)]}{2i-1} - \frac{\cos[(2i-1)(\alpha-\theta)]}{2i-1} \right\} f_1(\theta, i) +$$

$$+ \frac{2}{\pi} \sum_{i=1}^{\infty} \left\{ \frac{\sin[(2i-1)(\alpha+\theta)]}{2i-1} + \frac{\sin[(2i-1)(\alpha-\theta)]}{2i-1} \right\} f_2(\theta, i) \qquad (9)$$

where:

$$f_1(\theta, i) = \frac{\pi - \theta}{2} - \sum_{m=1}^{2i} \frac{\sin m\theta}{m}$$

$$f_2(\theta, i) = \ell n \sin \frac{\theta}{2} + \sum_{m=1}^{2i} \frac{\cos m\theta}{m} \quad .$$

Using relations:

$$\sum_{i=1}^{\infty} \frac{\cos(2i-1)x}{2i-1} = \frac{1}{2} \ell n \, \mathrm{ctg} \frac{x}{2}$$

$$\sum_{i=1}^{\infty} \frac{\sin(2i-1)x}{2i-1} = \frac{\pi}{4}$$

we can transform Eq. (9) into:

$$F_1^-(K \cos\theta) = f_2(\theta) + f_1(\theta) \, \ell n \left| \frac{R_1 - \sin\theta}{R_1 + \sin\theta} \right| + f_3(\alpha) \cos\theta + f_4(\alpha) \qquad (10)$$

where:

$$f_1(\theta) = \frac{1}{\pi} \sum_{i=1}^{\infty} f_1(\theta, i) = \frac{\pi - \theta}{2\pi} - \frac{1}{\pi} \sum_{i=1}^{\infty} \sum_{m=1}^{2i} \frac{\sin m\theta}{m}$$

$$f_2(\theta) = \frac{1}{\pi} \sum_{i=1}^{\infty} f_2(\theta, i) = \frac{1}{2\pi} \ell n \, \mathrm{ctg} \frac{\theta}{2} - \frac{1}{\pi} \sum_{i=1}^{\infty} \sum_{m=1}^{2i} \frac{\cos m\theta}{m}$$

$$f_3(\alpha) = -\frac{2}{\pi} \sum_{i=1}^{\infty} \frac{\sin(2i-1)\alpha}{(2i-1)^2}$$

$$f_4(\alpha) = -\frac{2}{\pi}\sum_{i=1}^{\infty} \frac{\sin(2i-1)\alpha}{(2i-1)i} \tag{11}$$

with $\alpha = \arcsin R_1$.

In this way, all functions f_i in the above expression are the functions of one variable only (θ or α).

The corresponding expressions for $F_1^-(K\cos\theta_o)$, $F_2^-(K\cos\theta)$ and $F_2(K\cos\theta_o)$ can be obtained from Eq. (10) by simply changing $\theta \rightarrow \theta_o$ and $R_1 \rightarrow R_2$. With the aid of Eq. (39) in Ref. 1 we can obtain the desired expression for radar cross section σ.

The Second Integration Contour

As it was pointed out in the beginning, the form of the integral to be evaluated is strongly dependent on the contour chosen. Here we are going to take the integration contour and branch cut as shown in Fig. 1. This contour represents the proper deformation of the basic integration contour from Fig. 4 in Ref. 1.

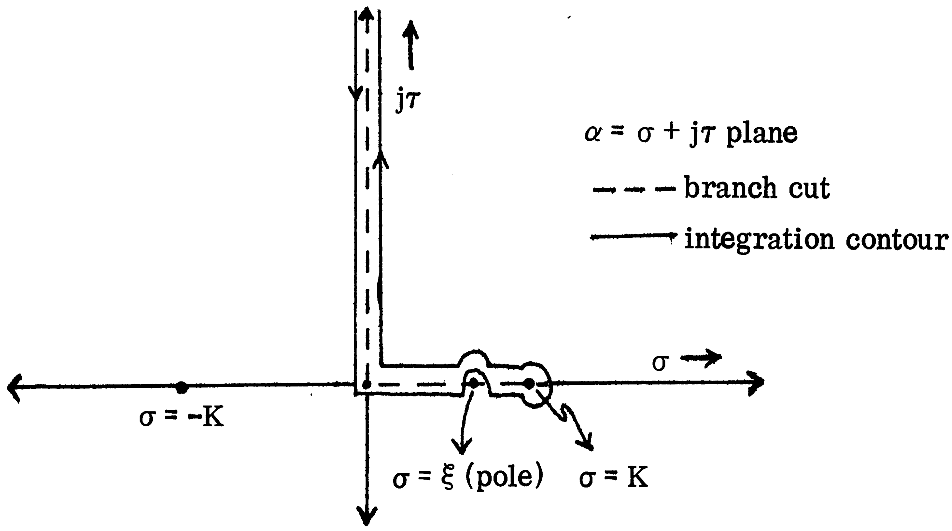

Fig. 1

We now have the pole on the contour at $\sigma = \xi$, so we introduced the indentation above it. Along the right and left-hand sides of the $j\tau$ axis, the function γ has the values:

$$\gamma = +\sqrt{(j\tau)^2 - K^2} = +j\sqrt{\tau^2 + K^2}$$

$$\gamma = -\sqrt{(j\tau)^2 - K^2} = -j\sqrt{\tau^2 + K^2}$$

respectively, while along the upper and lower side of the σ axis, γ is, respectively:

$$\gamma = j\sqrt{K^2 - \sigma^2}$$

$$\gamma = -j\sqrt{K^2 - \sigma^2} \quad .$$

So we have from Eq. (1) and Fig. 1:

$$F_1^-(\xi) = -\frac{1}{2\pi i}\int_0^{\infty} \ell n \frac{\sqrt{\tau^2 + K^2} - KR_1}{\sqrt{\tau^2 + K^2} + KR_1} \cdot \frac{j d\tau}{j\tau - \xi} +$$

$$+\frac{1}{2}\ell n \frac{\sqrt{K^2 - \xi^2} - KR_1}{\sqrt{K^2 - \xi^2} + KR_1} + \frac{1}{2\pi i} P \int_0^{K} \ell n \frac{\sqrt{K^2 - \sigma^2} - KR_1}{\sqrt{K^2 - \sigma^2} + KR_1} \frac{d\sigma}{\sigma - \xi} \tag{12}$$

where we used the well known formula:

$$\int_C \frac{f(\alpha)}{\alpha - \xi} d\alpha = j\pi f(\xi) + P\int_a^b \frac{f(\sigma)}{\sigma - \xi} d\sigma$$

in which C is the part of σ axis between a and b, with indentation above pole $\sigma = \xi$, and P stands for the Cauchy principle value of the integral.

From Eq. (12) we obtain, after simple manipulation:

$$F_1^-(\xi) = \frac{1}{2}\,\ell n \frac{\sqrt{K^2-\xi^2-KR_1}}{\sqrt{K^2-\xi^2+KR_1}} - \frac{\xi}{\pi}\int_0^\infty \ell n \frac{\sqrt{\tau^2+K^2-KR_1}}{\sqrt{\tau^2+K^2+KR_1}} \cdot \frac{d\tau}{\tau^2+\xi^2} -$$

$$-\frac{i}{2\pi}\int_0^\infty \ell n \frac{\sqrt{\tau^2+K^2-KR_1}}{\sqrt{\tau^2+K^2+KR_1}}\, \frac{\tau\, d\tau}{\tau^2+\xi^2} + \frac{1}{2\pi i}\, P\int_0^K \ell n \frac{\sqrt{K^2-\sigma^2-KR_1}}{\sqrt{K^2-\sigma^2+KR_1}} \cdot \frac{d\sigma}{\sigma-\xi}\,. \quad (13)$$

The first and second integrals in Eq. (13) are real. Namely, τ in them goes from 0 to ∞, and we take $R_1 \leq 1$, so the factor under log cannot be negative. The third integral is real from $\sigma = 0$ to $\sigma = K\sqrt{1-R_1^2}$, and from $K\sqrt{1-R_1^2}$ to K it is complex because the factor under log is negative. However, from Eq. (3) we know that $F_1^-(\xi)$ must be real, so the imaginary parts in Eq. (13) have to cancel each other and we have:

$$F^-(\xi) = \frac{1}{2}\,\ell n \left| \frac{\sqrt{K^2-\xi^2-KR_1}}{\sqrt{K^2-\xi^2-KR_1}} \right| - \frac{\xi}{2\pi}\int_0^\infty \ell n \frac{\sqrt{\tau^2+K^2-KR_1}}{\sqrt{\tau^2+K^2+KR_1}} \cdot \frac{d\tau}{\tau^2+\xi^2} +$$

$$+\frac{1}{2\pi}\, P\int_{K\sqrt{1-R_1^2}}^{K} \mathrm{Im}\,\log\left(-\frac{KR_1-\sqrt{K^2-\sigma^2}}{KR_1+\sqrt{K^2-\sigma^2}}\right)\frac{d\sigma}{\sigma-\xi} =$$

$$= \frac{1}{2}\,\ell n \left| \frac{\sqrt{K^2-\xi^2-KR_1}}{\sqrt{K^2-\xi^2+KR_1}} \right| - \frac{\xi}{2\pi}\, T_1 + \frac{1}{2\pi}\, T_2 \quad , \quad (14)$$

where T_1 and T_2 represent corresponding integrals in Eq. (14). Let us evaluate integral T_2 first. We have:

$$T_2 = \pi P \int_{K\sqrt{1-R_1^2}}^{K} \frac{d\sigma}{\sigma-\xi} = \pi \lim_{\epsilon\to 0}\left[\int_{K\sqrt{1-R_1^2}}^{\xi-\epsilon}\frac{d\sigma}{\sigma-\xi} + \int_{\xi+\epsilon}^{K}\frac{d\sigma}{\sigma-\xi}\right] =$$

$$= \pi \ell n \left| \frac{K - \xi}{K\sqrt{1 - R_1^2} - \xi} \right| . \qquad (15)$$

Integral T_1 is much more difficult to evaluate. Moreover, it turned out, after many attempts, that integral T_1 cannot be expressed as a finite combination of elementary functions. So, we shall give here two solutions: an exact one, but in the form of series, and an approximate one which is in closed form, more suitable for our purposes.

We have:

$$T_1 = \int_0^\infty \ell n \frac{\sqrt{\tau^2 + K^2} - KR_1}{\sqrt{\tau^2 + K^2} + KR_1} \cdot \frac{d\tau}{\tau^2 + \xi^2} .$$

If we put:

$$\tau = K \, sh\, x$$

we obtain:

$$T_1 = \int_0^\infty \ell n \frac{Ch\, x - R_1}{Ch\, x + R_1} \cdot \frac{Ch\, x \; dx}{sh^2 x + \left(\frac{\xi}{K}\right)^2} .$$

If we take $0 \leq R \leq 1$, we can put:

$$R_1 = \sin\alpha \qquad (16)$$

where α is real and $0 \leq \alpha \leq \frac{\pi}{2}$. On the other hand, $\left|\frac{\xi}{K}\right| \leq 1$, so we can write:

$$1 - \left(\frac{\xi}{K}\right)^2 = \cos^2\beta \qquad (17)$$

and T_1 becomes:

$$T_1 = \int_0^\infty \ell n \frac{Ch\, x - \sin\alpha}{Ch\, x + \sin\alpha} \cdot \frac{Ch\, x\, dx}{Ch^2 x - \cos^2\beta} \qquad (18)$$

which is the more suitable form for evaluation.

From Bateman's Tables of cosine transform (Ref. 3, p. 36, No. 50) we can write for the first factor under integral in Eq. (18):

$$\ln\left(\frac{\mathrm{Ch}\,x-\sin\alpha}{\mathrm{Ch}\,x+\sin\alpha}\right)=-2\int_0^\infty \frac{\mathrm{sh}\,\alpha y}{y\,\mathrm{Ch}\frac{1}{2}\pi y}\cos xy\,dy\ . \qquad (19)$$

Putting Eq. (19) into Eq. (18) and changing the order of integration (which is obviously permissible here) we obtain:

$$T_1=-2\int_0^\infty\left[\frac{\mathrm{sh}\,\alpha y}{y\,\mathrm{Ch}\frac{1}{2}\pi y}\int_0^\infty \frac{\mathrm{Ch}\,x}{\mathrm{Ch}^2x-\cos^2\beta}\cos xy\,dx\right]dy \qquad (20)$$

The integral

$$T_3=\int_0^\infty \frac{\mathrm{Ch}\,x}{\mathrm{Ch}^2x-\cos^2\beta}\cos xy\,dx$$

can be evaluated exactly on the basis of the integral:

$$T_4=\int_0^\infty \frac{\cos\alpha x\ \mathrm{ch}x}{\mathrm{Ch}\,x-\cos\beta}\,dx=-\pi\ \mathrm{ctg}\ b\ \frac{\mathrm{sh}\,\alpha\beta}{\mathrm{sh}\,\alpha\pi}$$

given in Ref. 2, p. 506, No. 3.984.2. After simple manipulation we obtain:

$$T_3=\frac{\pi}{2\sin\beta}\left[\frac{\mathrm{sh}\beta y+\mathrm{sh}(\pi-\beta)y}{\mathrm{sh}\,\pi y}\right]=\frac{\pi}{2\sin\beta}\ \frac{\mathrm{Ch}(\beta-\pi/2)y}{\mathrm{Ch}\frac{\pi}{2}y}$$

which, after introducing in Eq. (20), gives:

$$T_1=-\frac{\pi}{\sin\beta}\int_0^\infty \frac{\mathrm{sh}\,\alpha y\cdot\mathrm{Ch}(\beta-\frac{\pi}{2})y}{y\,\mathrm{Ch}^2\frac{\pi}{2}y}\,dy=-\frac{\pi}{\sin\beta}\int_0^\infty \frac{\mathrm{sh}\frac{2\alpha t}{\pi}\cdot\mathrm{Ch}(\frac{2\beta}{\pi}-1)t\,dt}{t\cdot\mathrm{Ch}^2t}$$

where we put $\frac{\pi}{2} y = t$. Using the relation:

$$\text{sh}\, a \cdot \text{ch}\, b = \frac{1}{2}\left[\text{sh}(a+b) + \text{sh}(a-b)\right]$$

we can write:

$$T_1 = -\frac{\pi}{2\sin\beta}\left[T_5 + T_6\right] \tag{21}$$

where:

$$T_5 = \int_0^\infty \frac{\text{sh}\, pt}{t\, \text{ch}^2 t}\, dt \tag{22}$$

$$T_6 = \int_0^\infty \frac{\text{sh}\, qt}{t\, \text{ch}^2 t}\, dt \tag{23}$$

with:

$$p = \frac{2(\alpha+\beta)}{\pi} - 1$$

$$q = \frac{2(\alpha-\beta)}{\pi} + 1 \tag{24}$$

Integrals T_5 and T_6 cannot be expressed by finite combination of elementary functions, but they may serve as the basis for approximate evaluation, as will be shown later. The exact expression for T_5 and T_6 can be obtained in the form of convergent power series in the following way. Consider the integrals:

$$T_7 = \frac{\partial T_5}{\partial \alpha} = \frac{2}{\pi}\int_0^\infty \frac{\text{ch}\, pt}{\text{ch}^2 t}\, dt\, ; \qquad T_8 = \frac{\partial T_6}{\partial \alpha} = \frac{2}{\pi}\int_0^\infty \frac{\text{ch}\, qt}{\text{ch}^2 t}\, dt$$

which can be evaluated exactly on the basis of integral No. 3.514.3 given in Ref. 2, p. 345. We have, after simple manipulation,

$$T_7 = \frac{2}{\pi}\,\frac{\frac{\pi}{2}p}{\sin\frac{\pi}{2}p} = \frac{1}{\cos(\alpha+\beta)} - \frac{2}{\pi}\,\frac{\alpha+\beta}{\cos(\alpha+\beta)}$$

$$T_8 = \frac{\frac{\pi}{2}q}{\sin\frac{\pi}{2}q} = \frac{1}{\cos(\alpha-\beta)} + \frac{2}{\pi}\,\frac{\alpha-\beta}{\cos(\alpha-\beta)} \tag{25}$$

It is known (see Ref. 2, p. 190 and Ref. 4, p. 47) that:

$$T_9 = \int \frac{x}{\cos x}dx = \sum_{b=0}^{\infty} \frac{|E_{2i}|x^{2i+2}}{(2i+2)(2i!)} = \frac{x^2}{2} + \frac{x^4}{4\cdot 2!} + \frac{5x^6}{6\cdot 4!} + \frac{61x^8}{8\cdot 6!} + \frac{1385\,x^{10}}{10\cdot 8!} + \ldots$$

where E_n are Euler numbers which are well tabulated in standard Mathematical Tables (for example, Ref. 5, p. 810). On the other hand we have (Ref. 4, p. 40):

$$\int \frac{dx}{\cos x} = \ell n\, tg\,(\pi/4 + x) = \frac{1}{2}\log\frac{1+\sin x}{1-\sin x} \quad .$$

So we have:

$$T_5 + T_6 = \int \left[T_7 + T_8\right] d\alpha = \int \left[\frac{1}{\cos(\alpha+\beta)} + \frac{1}{\cos(\alpha-\beta)}\right] d\alpha -$$

$$-\frac{2}{\pi}\int \left[\frac{\alpha+\beta}{\cos(\alpha+\beta)} - \frac{\alpha-\beta}{\cos(\alpha-\beta)}\right] d\alpha =$$

$$= \frac{1}{2}\ell n \left[\frac{1+\sin\frac{\alpha+\beta}{2}}{1-\sin\frac{\alpha+\beta}{2}} \cdot \frac{1+\sin\frac{\alpha-\beta}{2}}{1-\sin\frac{\alpha-\beta}{2}}\right] -$$

$$-\frac{2}{\pi}\sum_{i=0}^{\infty} \frac{|E_{2i}|}{(2i+2)(2i!)} \left[(\alpha+\beta)^{2i+2} - (\alpha-\beta)^{2i+2}\right] .$$

After some manipulation we obtain, with the aid of Eq. (21):

$$T_1 = -\frac{\pi}{2\sin\beta}\ell n\frac{\sin\frac{\alpha}{2}+\cos\frac{\beta}{2}}{\sin\frac{\alpha}{2}-\cos\frac{\beta}{2}} - \frac{1}{\sin\beta}\sum_{i=0}^{\infty}\frac{\left|E_{2i}\right|}{(2i+2)(2i!)}\left[\sum_{n=0}^{i}\binom{2i+2}{2n+1}\alpha^{2(i-n)+1}\beta^{2n+1}\right] \tag{26}$$

Taking into account Eqs. (14), (15) and (26), we can write for the function $F_1^-(\xi)$:

$$F_1^-(\xi) = \frac{1}{2}\ell n\left|\frac{\sqrt{K-\xi^2}-KR_1}{\sqrt{K-\xi^2}+KR_1}\right| + \frac{1}{4}\ell n\,\frac{\sin\frac{\alpha}{2}+\cos\frac{\beta}{2}}{\sin\frac{\alpha}{2}-\cos\frac{\beta}{2}} + \frac{1}{2}\ell n\left|\frac{K-\xi}{K\sqrt{1-R_1^2}-\xi}\right| -$$

$$-\frac{K}{\xi}\sum_{i=0}^{\infty}\frac{(2i+1)\left|E_{2i}\right|}{2}F_i(\alpha,\beta) \tag{27}$$

where:

$$F_i(\alpha,\beta) = \sum_{n=0}^{i}\frac{\alpha^{2(i-n)+1}\beta^{2n+1}}{(2n+1)!\,(i-n)!} \quad .$$

If we put in Eq. (27):

$$\sin\alpha = R_1$$

$$\sin\beta = \frac{\xi}{K} = \cos\theta$$

$$\beta = \frac{\pi}{2} - \theta \quad .$$

We obtain for the function $F_1^-(-K\cos\theta)$:

$$F_1^-(-K\cos\theta) = \frac{1}{2}\ell n\left|\frac{\sin\theta - R_1}{\sin\theta+R_1}\cdot\frac{\sqrt{1-\sqrt{1-R_1^2}+\sqrt{1+\sin\theta}}}{\sqrt{1-\sqrt{1-R_1^2}-\sqrt{1+\sin\theta}}}\cdot\frac{1+\cos\theta}{\sqrt{1-R_1^2}+\cos\theta}\right| - \Phi(R_1,\theta) \tag{28}$$

where:

$$\Phi(R_1, \theta) = \frac{1}{\cos\theta} \sum_{i=0}^{\infty} \frac{(2i+1)\left|E_{2i}\right|}{2} \cdot \psi_i(R_1, \theta)$$

and:

$$\psi_1(R_1, \theta) = \sum_{n=0}^{i} \frac{(\arcsin R_1)^{2(i-n)+1} (\pi/2 - \theta)^{2n+1}}{(2n+1)!\,(i-n)!} \quad .$$

The remaining functions we need, i.e., $F_1^-(-\cos\theta_0)$, $F_2^-(-\cos\theta)$ and $F_2^-(-\cos\theta_0)$, we can obtain from Eq. (28) by simply putting $\cos\theta_0$ instead of $\cos\theta$ and R_2 instead of R_1. For evaluation of σ we need not directly $F_1(-\cos\theta)$ but $e^{2F_1(-\cos\theta)}$, according to formula (39) in Ref. 1. So we have:

$$e^{2F_1^-(-K\cos\theta)} = \frac{\sin\theta - R_1}{\sin\theta + R_1} \cdot \frac{\sqrt{1 - \sqrt{1-R_1^2} + \sqrt{1+\sin\theta}}}{\sqrt{1 - \sqrt{1-R_1^2} - \sqrt{1+\sin\theta}}} \cdot \frac{1+\cos\theta}{\sqrt{1-R_1^2+\cos\theta}} \cdot e^{-2\Phi(R_1, \theta)} \qquad (29)$$

The obtained formulas are rather cumbersome and are not suitable for practical application, so we are going to find some approximate and simpler ones. We shall start with the integral T_5 and T_6 in Eqs. (22) and (23). For T_5 we have:

$$T_5 = \int_0^{\infty} \frac{\operatorname{sh} pt}{t\,\operatorname{ch}^2 t}\,dt = \int_0^{\infty} \frac{\operatorname{sh} pt\ dt}{t\left(\frac{\operatorname{ch} 2t}{2} + \frac{1}{2}\right)} \quad .$$

For approximate evaluation of T_5 we shall try the following method: find an integral, say T_{10}, for which we can prove that $T_5 < T_{10}$, and a second integral, say T'_{10}, for which we can prove that $T_5 > T'_{10}$. Suppose that we can evaluate T_{10} and T'_{10} exactly and find the difference between them. It may happen that this difference is small in which case the difference between

T_5 and T_{10}, or T_5 and T'_{10} is small also and we can take $T_5 \approx T_{10}$, or $T_5 \approx T'_{10}$ with a good estimation of error.

Consider now the integral:

$$T_{10} = \int_0^\infty \frac{\text{sh}\, pt\, dt}{t \frac{\text{Ch}\, 2t}{2}}$$

which differs from T_5 in missing the term $\frac{1}{2}$ in the denominator of T_5. It is clear that:

$$T_5 < T_{10} \quad .$$

On the other hand, from Eq. (25) we know that the exact form of T_5 is:

$$T_5 = \int \frac{\partial T_5}{\partial p} dp = \int \left[\int_0^\infty \frac{\text{sh}\, pt}{\text{ch}^2 t} dt \right] dp =$$

$$= \int \frac{\frac{\pi}{2} p}{\sin \frac{\pi}{2} p} dp = \frac{2}{\pi} \left[(-1)^{K+1} \frac{2(2^{2K-1}-1)}{(2K+1)!} B_{2K} (\frac{\pi}{2} p)^{2K+1} \right] =$$

$$= \frac{2}{\pi} \left[\frac{\pi}{2} p + (\frac{\pi}{2} p)^3 \cdot \frac{1}{3 \cdot 3!} + 7 (\frac{\pi}{2} p)^5 \cdot \frac{1}{3 \cdot 5 \cdot 5!} + 31 (\frac{\pi}{2} p)^7 \frac{1}{3 \cdot 7 \cdot 7!} + \ldots \right] \quad (30)$$

where B_i are Bernoulli numbers, tabulated in standard Mathematical Tables (for example, in Ref. 5, p. 810). If we denote by T'_{10} the sum of the first few terms in the above infinite series, we are sure that:

$$T_5 > T'_{10}$$

because all the terms are positive.

So we have:

$$T'_{10} < T_5 < T_{10} \quad .$$

If we can now prove that the difference Δ between T_{10} and T'_{10} is small for the whole range of p, then the difference Δ_1 between T_{10} and T_5 is still smaller and we can have:

$$T_5 \approx T_{10}$$

which has the exact and relatively simple solution:

$$T_{10} = 2\ln \operatorname{tg}\left(\frac{p\pi}{8} + \frac{\pi}{4}\right) = 2\ln \frac{1 + \operatorname{tg}\frac{p\pi}{8}}{1 - \operatorname{tg}\frac{p\pi}{8}}$$

(see, for example, Ref. 2, p. 351, No. 254.24).

After some computation we found that taking the first four member of series T_5, the maximum value of:

$$\eta = \frac{\Delta}{T_{10}} 100 \text{ percent} = \left(1 - \frac{T'_{10}}{T_{10}}\right) 100 \text{ percent}$$

which represents the maximum relative error in percent, is less than 8 percent, for all values of p between 0 and 1. So we have:

$$T_5 \approx 2\ln \frac{1 + \operatorname{tg}\frac{p\pi}{8}}{+ - \operatorname{tg}\frac{p\pi}{8}} \approx 2\ln \frac{1 + \frac{p\pi}{8}}{1 - \frac{p\pi}{8}}$$

with additional error of about 2 percent, and

$$T_6 \approx 2\ln \frac{1 \pm \frac{q\pi}{8}}{1 - \frac{q\pi}{8}} .$$

The integral T_1 after some manipulation becomes:

$$T_1 \approx -\frac{\pi}{\sin\beta} \ln \frac{4(\alpha+2)^2 - (\beta - \frac{\pi}{2})^2}{4(\alpha-2)^2 - (\beta - \frac{\pi}{2})^2} .$$

The function $F_1^-(-K\cos\theta)$ is now:

$$F_1^-(-\cos\theta) = \frac{1}{2}\ln\left|\frac{\sin\theta - R_1}{\sin\theta + R_1}\cdot\frac{1+\cos\theta}{\sqrt{1-R_1^2}+\cos\theta}\cdot\frac{(\arcsin R_1+2)^2-(\frac{\theta}{2})^2}{(\arcsin R_1-2)^2-(\frac{\theta}{2})^2}\right|$$

while the function $\left|K_1^-(-K\cos\theta)\right|^2$ is:

$$\left|K_1^-(-K\cos\theta)\right|^2 = e^{2F_1^-(-\cos\theta)} = \left|\frac{\sin\theta - R_1}{\sin\theta + R_1}\cdot\frac{1+\cos\theta}{\sqrt{1-R_1^2}+\cos\theta}\cdot\frac{(\arcsin R_1+2)^2-(\frac{\theta}{2})^2}{(\arcsin R_1-2)^2-(\frac{\theta}{2})^2}\right|$$

which is much simpler than Eq. (28) and accurate enough. The corresponding function $K_1^-(-\cos\theta_o)$, $K_2^-(-\cos\theta)$ and $K_2^-(-\cos\theta_o)$ can be obtained from $K_1^-(-\cos\theta)$ by simple change of variables. With the aid of formula (39) in Ref. 1 we can obtain the desired expression for radar cross section σ.

References

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