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4 December 1973

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FROM: T. B. A. Senior

SUBJECT: Surface field on a constant impedance half plane

To assist in understanding the behavior of impedance and resistive sheets, it is of interest to examine the field on the surface of a half plane of constant surface impedance when illuminated by a plane wave. Since an impedance sheet is equivalent to a resistive one only for edge-on incidence, we restrict our attention to this case.

For a half plane of constant surface impedance η illuminated by an E-polarised plane wave, the exact solution has been obtained by Senior (1952). For edge-on incidence $(\alpha = \pi)$

$$E_z = E_z^i - \frac{1}{2\pi i} \int_{C'} \frac{K_+(k)}{(\cos\beta - 1)\, K_-(k\cos\beta)}\, e^{ik\rho\cos(\beta - \theta)}\, d\beta$$

and when the observation point is on the upper surface $(\theta = 0)$ of the sheet and at large distances from the edge, the integral can be evaluated by steepest descents to give (see eqs. 38 and 40 of Senior, 1952 with $\alpha = \pi$):

$$E_z \sim \frac{e^{ikx + i\pi/4}}{\sqrt{2\pi kx}} \lim_{\theta \to 0} \left\{ \frac{1}{\cos\theta - 1} \left(\frac{K_+(k)}{K_-(k\cos\theta)} - 2\sin\frac{\theta}{2} \right) \right\} .$$

The limit can be found using l'Hôpital's rule and is 2η, but a somewhat simpler approach which avoids the limiting operation is to consider the total

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current*

$$I_2(x) = H_x(x, +0) - H_x(x, -0).$$

From Senior (1952, eq. 25, with $\alpha = \pi$),

$$I_2(x) = -\frac{iY_o}{\pi}\int_{C'} \frac{K_+(k)}{K_-(k\cos\beta)} \cos\frac{\beta}{2}\, e^{ikx\cos\beta}\, d\beta$$

$$\underset{kx\gg 1}{\sim} -iY_o\sqrt{\frac{2}{\pi kx}}\, e^{ikx - i\pi/4} \lim_{\beta\to 0}\left\{\frac{K_+(k)}{K_-(k\cos\beta)}\cot\frac{\beta}{2}\right\}.$$

But

$$\frac{1}{K_-(k\cos\beta)} = \frac{\sin\beta}{1+\eta\sin\beta}\cdot\frac{1}{K_+(k\cos\beta)}$$

so that

$$\frac{K_+(k)}{K_-(k\cos\beta)}\cos\frac{\beta}{2} \xrightarrow[\beta\to 0]{} 2\ ,$$

giving

$$I_2(x) \sim -2Y_o\sqrt{\frac{2}{\pi kx}}\, e^{ikx+i\pi/4} \quad .$$

Hence, from symmetry,

$$H_x(x, \pm 0) \sim \mp Y_o \frac{2}{\pi kx} e^{ikx+i\pi/4}$$

which is precisely the same result as for a perfectly conducting half plane, but since

$$E_z(x, \pm 0) = \mp \eta Z_o H_x(x, \pm 0)$$

on the upper and lower surfaces respectively, we have

$$E_z(x, \pm 0) \sim \eta \frac{2}{\pi kx} e^{ikx+i\pi/4} \quad .$$

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*Note that $I_2(x)$ is the negative of the total current $J(x)$ used in the electrically resistive sheet program.

This is valid for $kx \gg 1$, but we can also determine the behavior for $kx \ll 1$ from the properties of the Fourier transform $\bar{I}_2(\xi)$ of $I_2(x)$ for large $|\xi|$. Since

$$\bar{I}_2(\xi) = iY_o\sqrt{\frac{2}{\pi}}\,\frac{K_+(k)}{(\xi - k)\,K_-(\xi)}$$

and $K_-(\xi) = O\,(\text{const.})$ for large ξ, $\eta \neq 0$, it follows that

$$\bar{I}_2(\xi) = O\,(|\xi|^{-1})$$

and hence

$$I_2(x) = O\,(\text{const.}), \qquad kx \ll 1.$$

It is somewhat difficult to determine the constant value of $I_2(0)$. Nevertheless from an examination of the 'split' functions, it appears that

$$\frac{K_+(k)}{K_+(\xi)} \longrightarrow 1$$

as $|\xi| \rightarrow \infty$, and if this is true

$$\bar{I}_2(\xi) = iY_o\sqrt{\frac{2}{\pi}}\,\frac{1}{\xi - k}\,\frac{\sqrt{k^2 - \xi^2}}{k + \eta\sqrt{k^2 - \xi^2}}\,\frac{K_+(k)}{K_+(\xi)}$$

$$\sim \frac{iY_o}{\eta\xi}\sqrt{\frac{2}{\pi}} \qquad \text{as} \qquad |\xi| \rightarrow \infty\,,$$

implying

$$I_2(0) = -2\frac{Y_o}{\eta}\ .$$

Hence, from symmetry,

$$H_x(0, \pm 0) = \mp\frac{Y_o}{\eta}$$

and

$$E_z(0, \pm 0) = 1\ .$$

This is in excellent agreement with computed data for electrically resistive sheets with a parabolic variation of resistance. For $R_{max} = 4$ (at the front), it is found that $J(0) = 0.240 e^{i 9 \cdot 9^{o}}$ for sheets of length ℓ, $0.7 \leq \ell/\lambda \leq 1.5$, and since $E_z = RJ$, it follows that

$$E_z(0, \pm 0) = 0.960 e^{i 9 \cdot 9^{o}} \quad .$$

The 4 percent difference in amplitude is no more than expected due to the variable resistivity, and though there is no obvious explanation for the non-zero phase, we note that it is equivalent to a displacement of the 'effective' end of the sheet by $\frac{\lambda}{36}$.

Reference

Senior, T. B. A. (1952) "Diffraction by a semi infinite metallic sheet", Proc. Roy. Soc. (London) 213A, 436-458.