011764-504-M

11 September 1973

MEMO TO: T. B. A. Senior

FROM: Sharad R. Laxpati

SUBJECT: Integral equations of scattering by a thin sheet: Impedance boundary condition on a sheet vs. an impedance sheet.

The formulation of the boundary value problem associated with a thin sheet employing the impedance boundary condition has some definite advantages. In particular, it provides a simultaneous treatment of E and H polarization through duality. However, the physical materials are described through their permeability and permittivity. It would be of considerable interest, from a practical standpoint, to relate these two types of boundary conditions and the associated integral equations. This would then provide a relatively simple means of arriving at the diffraction coefficient for a resistive (and, in general, an impedance) sheet from that derived from an impedance boundary condition.

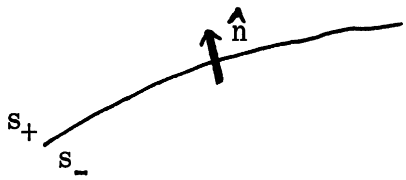
In the following, the characteristics of the two formulations are briefly discussed. Based on this it is apparent that even for two dimensional problems there exists no simple relationships for the general impedance sheet (with both permeability and permittivity different from free space). However, the striking similarity between the integral equations for the electric current for scattering by a two dimensional scatterer, based on these two forms of boundary conditions, does warrant closer examination. It is shown that for edge on incidence, the diffraction coefficient for impedance boundary condition and a resistive sheet, of semi-infinite extent, are related.

DISTRIBUTION

Hiatt/File Knott
Laxpati Senior

11764-504-M = RL-2242

I. Impedance boundary condition on a sheet



Note: Throughout this memo, $\hat{n}$ is a unit vector, directed from s_- to s_+ surface.

(1) Let $\bar{\bar{\eta}}_+$ a (2×2) matrix represent normalized surface impedance on s_+,

$\bar{\bar{\eta}}_-$ a (2×2) matrix represent normalized surface impedance on s_-.

Leontovich form of the boundary condition is

$$\underline{E} - (\hat{n}\cdot\underline{E})\hat{n} = \bar{\bar{\eta}}_{\pm}\, z_o \cdot \hat{n} \times \underline{H} \quad . \tag{1}$$

(2) Both tangential electric and magnetic fields are discontinuous across the sheet.

$$\text{Let} \quad \overline{K}_m = \hat{n} \times (\underline{E}^+ - \underline{E}^-)$$
$$\overline{K}_e = \hat{n} \times (\underline{H}^+ - \underline{H}^-) \tag{2}$$

$\underline{K}$ - surface currents

superscripts $\pm$ represents values on $s_{\pm}$.

(3) Through the boundary condition equation (1), $\underline{K}_m$ and $\underline{K}_e$ are related.

<u>Assuming</u> $\bar{\bar{\eta}}_+ = \bar{\bar{\eta}}_- = \bar{\bar{\eta}}$,

$$\overline{K}_m = z_o\, \hat{n} \times (\bar{\bar{\eta}} \cdot \overline{K}_e) \quad . \tag{3}$$

(4) The integral equation for the electric current for the case of a two dimensional scatterer with E_z^{inc} incident field is (Knott and Senior, 1973)

$$Y_o E_z^{inc}(s) = \frac{1}{2}\eta_s K_z(s) + \frac{1}{4}\int_C K_z(s')\, H_o^{(1)}(kr)\, d(ks') +$$
$$+ \frac{i}{4}\int \eta_s(s') K_z(s') (\hat{n}\cdot\hat{r})\, H_1^{(1)}(kr)\, d(ks') \tag{4a}$$

where

$$\underline{r} = \underline{\rho} - \underline{\rho}'$$

and

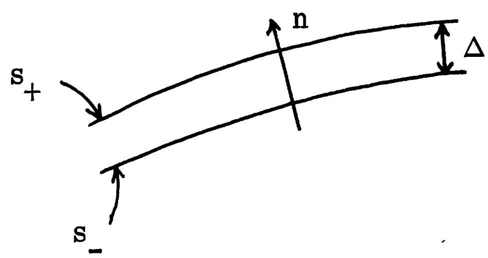
$$\overline{\overline{\eta}} = \eta_s \overline{\overline{1}} \; .$$

Equation (4a) can be rewritten as follows, if the contribution of $s = s'$ point is included in the second integral.

$$Y_o E_z^{inc}(s) = \eta_s K_z(s) + \frac{1}{4}\int_C K_z(s')\, H_o^{(1)}(kr)\, d(ks') +$$

$$+ \frac{i}{4}\int_C \eta_s(s')\, K_z(s')\, (\hat{n}\cdot\hat{r})\, H_1^{(1)}(kr)\, d(ks') \quad . \qquad (4b)$$

II. Impedance Sheet



The material of thickness Δ is described through

x_e — electric susceptibility (volume)

and

x_m — magnetic susceptibility (volume).

Let $\quad \Sigma_e = \Delta x_e$ — surface electric susceptibility

$\Sigma_m = \Delta x_m$ — surface magnetic susceptibility.

Before considering the general case, consider the two special cases: (a) electric resistive sheet and (b) magnetic resistive sheet.

(a) Electric resistive sheet: $x_e \neq 0,\ x_m = 0$.

(1) Let $\underline{K}_e^{\pm} = -\frac{ik}{z_o}\frac{\Delta}{2} x_e \underline{\underline{E}}_{tan}^{\pm}$

$$= -\frac{ik}{z_o}\frac{\Delta}{2} x_e \left[\underline{E}^{\pm} - (\hat{n}\cdot\underline{E}^{\pm})\hat{n}\right] . \tag{5}$$

(2) The boundary conditions are:

$$\begin{aligned} &\hat{n} \times (\underline{E}^{+} - \underline{E}^{-}) = 0 \\ &\hat{n} \times (\underline{H}^{+} - \underline{H}^{-}) = (\underline{K}_e^{+} + \underline{K}_e^{-}) \end{aligned} \tag{6}$$

(3) From (5) and (6) we have

Since $\hat{n} \times \underline{E}^{+} = \hat{n} \times \underline{E}^{-}$, $\underline{K}_e^{+} = \underline{K}_e^{-}$,

let $\underline{E} = \underline{E}^{+} = \underline{E}^{-}$, then

$$-\frac{ik}{z_o}\Delta x_e \left[\underline{E} - (\hat{n}\cdot\underline{E})\hat{n}\right] = \hat{n} \times (\underline{H}^{+} - \underline{H}^{-}) . \tag{7}$$

(4) An equivalent of (3) above, viz. a relationship between the electric and magnetic currents, is not possible. However, we note that $\underline{K}_m = 0$, hence, (1) and (3) are satisfied if

$$\overline{\overline{\eta}} \equiv 0 . \tag{8}$$

(5) The integral equation for the electric current for the case of a two dimensional scatterer with E_z^{inc} incident field is (Oshiro and Cross, 1966)

$$Y_o E_z^{inc}(s) = Y_o R_s K_z(s) + \frac{1}{4}\int_C K_z(s') H_o^{(1)}(kr)\, d(ks') \tag{9}$$

$$R_s = -\frac{z_o}{ik\Delta x_e} = \frac{1}{-i\omega\epsilon_o \Delta x_e} \tag{10}$$

(b) <u>Magnetic resistive sheet</u>: $x_e = 0$, $x_m \neq 0$.

(1) Let $\underline{K}_m^{\pm} = -ikz_o \frac{\Delta}{2} x_m \underline{H}_{tan}^{\pm}$

$$= -ikz_o \frac{\Delta x_m}{2} \left[\underline{H}^{\pm} - (\hat{n} \cdot \underline{H}^{\pm})\, \hat{n}\right] \quad . \tag{11}$$

(2) The boundary conditions are

$$\begin{aligned} \hat{n} \times (\underline{E}^{+} - \underline{E}^{-}) &= (\underline{K}_m^{+} + \underline{K}_m^{-}) \\ \hat{n} \times (\underline{H}^{+} - \underline{H}^{-}) &= 0 \quad . \end{aligned} \tag{12}$$

(3) From (11) and (12) we have,

Since $\hat{n} \times \underline{H}^{+} = \hat{n} \times \underline{H}^{-}$, $\underline{K}_m^{+} = \underline{K}_m^{-}$,

$$-ikz_o \Delta x_m \left[\underline{H} - (\hat{n} \cdot \underline{H})\, \hat{n}\right] = \hat{n} \times (\underline{E}^{+} - \underline{E}^{-}) \quad . \tag{13}$$

(4) Once again, an attempt to develop an equivalent of (3) leads to some difficulty. However, we note,

$$\underline{K}_e \equiv 0 \ ,$$

hence (3) is satisfied if $\bar{\bar{\eta}} \rightarrow \infty$.

(5) A dual of integral equation (9) may be derived for a magnetic resistive sheet, if H_z-polarized incident field is assumed.

(c) Now we consider the general case of an impedance sheet.

(1) Let

$$\begin{aligned} \underline{K}_e^{\pm} &= -\frac{1}{2} \frac{ik}{z_o} \Delta x_e \left\{\underline{E}^{\pm} - (\hat{n} \cdot \underline{E}^{\pm})\, \hat{n}\right\} \ , \\ \underline{K}_m^{\pm} &= -\frac{1}{2} ikz_o \Delta x_m \left\{\underline{H}^{\pm} - (\hat{n} \cdot \underline{H}^{\pm})\, \hat{n}\right\} \ . \end{aligned} \tag{14}$$

(2) Both tangential electric and magnetic fields are discontinuous across both surfaces $s_{\pm}$. We assume the fields <u>inside</u> the material are non-zero and

equal to $\underline{E}_i$ and $\underline{H}_i$.

Let
$$\underline{K}_m^{\pm} = \pm \hat{n} \times (\underline{E}^{\pm} - \underline{E}_i^{\pm})$$
and
$$\underline{K}_e^{\pm} = \pm \hat{n} \times (\underline{H}^{\pm} - \underline{H}_i^{\pm}) \quad . \tag{15}$$

(3) From (15), since for the case of $\Delta \to 0$, $\underline{E}_i^{+} \to \underline{E}_i^{-}$ and $\underline{H}_i^{+} \to \underline{H}_i^{-}$, we have

$$\hat{n} \times (\underline{E}^{+} - \underline{E}^{-}) = \underline{K}_m^{+} + \underline{K}_m^{-}$$
and
$$\hat{n} \times (\underline{H}^{+} - \underline{H}^{-}) = \underline{K}_e^{+} + \underline{K}_e^{-} \tag{16}$$

using (14) in (16) above.

$$-\frac{1}{2}\frac{ik}{z_o}\Delta x_e \left[\left\{ \underline{E}^{+} - (\hat{n} \cdot \underline{E}^{+})\hat{n} \right\} + \left\{ \underline{E}^{-} - (\hat{n} \cdot \underline{E}^{-})\hat{n} \right\} \right] = \hat{n} \times (\underline{H}^{+} - \underline{H}^{-})$$
and
$$-\frac{1}{2}\, ikz_o \Delta x_m \left[\left\{ \underline{H}^{+} - (\hat{n} \cdot \underline{H}^{+})\hat{n} \right\} + \left\{ \underline{H}^{-} - (\hat{n} \cdot \underline{H}^{-})\hat{n} \right\} \right] = \hat{n} \times (\underline{E}^{+} - \underline{E}^{-}) \quad . \tag{17}$$

(4) In general, the quantities on the $s_{\pm}$ surfaces are not equal and are not necessarily related through a constant multiplier. A relationship between the electric and magnetic currents can not be obtained.

(5) Integral equations for the currents may be obtained as a special case of the more general problem discussed in a separate memo (Laxpati).

III. Conclusion

(a) The inability to arrive at an equation similar to equation (3) for the case of an impedance sheet demonstrates that, in general, the diffraction coefficients for the case of an impedance boundary condition can not be related to those for an impedance sheet. This lack of similarity arises due to the fundamental difference in the 'physical' property of the two surfaces. In the case of an impedance surface, there exists a direct coupling between the tangential components of the electric and magnetic fields (see eq. (1)). The discontinuity of the magnetic field is defined through the boundary conditions as the electric current on the surface. In direct contrast, in the case of an impedance sheet, the tangential electric and magnetic fields determine the electric and magnetic currents, respectively (see eq. (14)). There is no direct coupling between the electric and magnetic fields through the boundary conditions.

(b) Although the physics is dissimilar, the similarity between the integral equations (4) and (9) deserves a second look. If the second integral in equation (4b), viz.

$$\int \eta_s(s')\, K_z(s')\, (\hat{n}\cdot\hat{r})\, H_1^{(1)}(kr)\, d(ks') = 0 \tag{18}$$

then equation (9) will be identical to equation (4b) with η_s replaced by $Y_o R_s$. This identity is a mathematical one only.

For the case of magnetic resistance sheet, the dual of that for electric resistance sheet will hold. For the impedance sheet, it should be possible to arrive at a similar conclusion. However, two complexities occur. The first one is that for a scalar formulation, the appropriate integral equations for impedance boundary conditions must be those for an anisotropic surface impedance. The second difficulty may occur since a vector formulation of the problem may be necessary. This particular aspect will be dealt with at a later time.

(c) The implications of the above conclusion (b) are quite important for the scattering by a semi-infinite sheet. Senior (1952) has derived analytical expressions for the diffracted field employing the impedance boundary condition. Equations (32) and (33) define the scattered field for E-polarization. The dependence of the second term in the expression for $F(\tau)$ (equation 33) on the angle of incidence α shows that for $\alpha = \pi$ (this corresponds to edge-on incidence), the contribution of this term (specifically this corresponds to the left hand side of our equation (18)) is zero. Hence, we conclude:

For edge-on incidence ($\alpha = \pi$), for the case of a semi-infinite sheet with impedance boundary conditions, the diffraction coefficient for E-polarization is identical to that for the case of a semi-infinite resistive sheet if one replaces η by $Y_o R_s$.

A dual of this will hold for a magnetic resistive sheet.

References:

E. F. Knott and T. B. A. Senior, "Non-specular radar cross-section study," Tech. Rpt. AFAL-TR-73-2, February 1973.

F. K. Oshiro and R. C. Cross, "A source distribution technique for solution of two dimensional scattering problems," Phase III-Report, Northrop Corporation Report No. NOR-66-74, March 1966.

T. B. A. Senior, "Diffraction by a semi-infinite metallic sheet," Proc. Roy. Soc. (London), Series A, 213, 436-457, July 1952.