011764-505-M

MEMO TO: T. B. A. Senior

FROM: Sharad R. Laxpati

SUBJECT: Derivation of integral equations for a two dimensional scatterer with impedance boundary condition and partially clad by an absorptive sheet.

The possibility of reduction in radar return from a conducting scatterer by means of cladding it with highly absorptive material has generated a considerable amount of both experimental and theoretical literature in this area. Oshiro, et al. (1966, 1971) have developed integral equations for a two dimensional conducting body (fully and partially) clad by a resistive shell of finite and infinitesimally small thickness. They have confined their study to E-polarized incident field. Numerical solutions to the integral equations have been obtained for several different shapes of the scatterers.

Knott et al. (1973 a, b) have considered the scattering from a two-dimensional scatterer with an impedance boundary condition. This form of the boundary condition has an advantage, that the two polarizations of the incident field do not have to be considered separately. Knott et al. have obtained numerical results for a variety of conducting scatterers clad by an impedance surface; the magnitude of the surface impedance varying as a function of position. Their results indicate that the non-specular return (for example, from an edge) can be reduced by as much as 13 dB by proper choice of impedance surface cladding.

This idealized version of the boundary condition, an impedance boundary condition, has a drawback of not being related to the physical parameters of absorptive materials prevalent in experimental work. It is then of considerable

DISTRIBUTION

Hiatt/File Liepa
Knott Senior
Laxpati

11764-505-M = RL-2243

interest to investigate the problem of cladding by absorptive materials whose permittivity and permeability are different from those for free-space. In this memo, the integral equations for a two-dimensional scatterer with impedance boundary condition and partially clad by an absorptive sheet of infinitesimally small thickness are derived. The formulation is a scalar one, and consideres an E-polarized incident wave. The integral equations are derived for the unknowns; the electric surface current on the impedance surface and the electric and magnetic polarization currents in the absorptive material.

Geometry and the boundary conditions

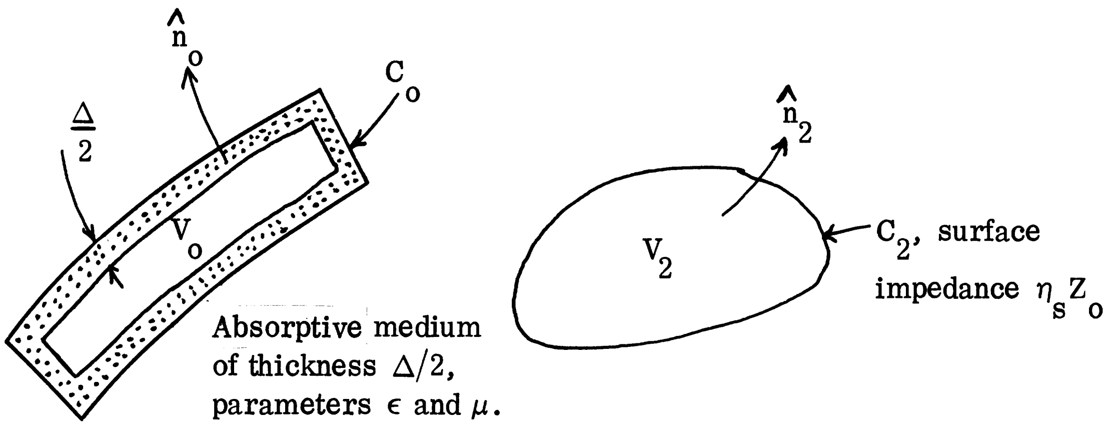


Figure 1.

Let a scalar $\psi$ represent the z-component of the electric field. Figure 1 shows the geometry of the problem. $C_2$ represents the surface of an impedance scatterer and encloses volume $V_2$. $C_o$ is the boundary of the absorptive material of small thickness $\frac{\Delta}{2}$ and encloses volume $V_o$. Note that $C_o$ is a closed contour, since this is necessary for the application of scalar Green's theorem. In the next section the case of an open contour $C_o$ is treated as a limiting case of a closed contour.

Let $\psi(\underline{\rho})$ be the electric field at an arbitrary point $\underline{\rho}$ in V, where V is the infinite volume less $V_o$ and $V_2$. Let $\psi_o(\underline{\rho})$ be the corresponding scalar field in $V_o$. We shall derive the integral equation assuming an incident field $\psi^{inc}(\underline{\rho})$ in V.

The problem will be formulated by the application of the scalar Green's theorem to $\psi$ in V and $\psi_o$ in $V_o$. The application of the boundary conditions and the evaluation of $\psi(\underline{\rho})$ for $\underline{\rho}$ in $C_2$ and $C_o$ and $\psi_o(\underline{\rho})$ for $\underline{\rho}$ in $C_o$ provides the desired integral equations. Since the thickness of the absorptive medium is very small, the volume polarization currents may be approximated as surface currents. These surface currents are defined in terms of $\psi$ and $\frac{\partial\psi}{\partial n}$ on $C_o$ as follows:

$$K_z(\underline{\rho}_o) = -\frac{ik}{Z_o}\frac{\Delta\chi_e}{2}\psi(\underline{\rho}_o)$$

and (1)

$$K_s^*(\underline{\rho}_o) = \frac{\Delta\chi_m}{2}\frac{\partial\psi(\underline{\rho}_o)}{\partial n_o}$$

where $K_z$ and $K_s^*$ are the electric and magnetic surface currents respectively. $\underline{\rho}_o$ is in $C_o$, k is the free-space wave number and $Z_o$ the free-space impedance. $\chi_e$ and $\chi_m$ are the electric and magnetic susceptibilities of the absorber. Using the above defined equivalent surface currents, the boundary conditions on $C_o$ are

$$\psi(\underline{\rho}_o) - \psi_o(\underline{\rho}_o) = K_s^*(\underline{\rho}_o)$$

and (2)

$$\frac{\partial\psi(\underline{\rho}_o)}{\partial n_o} - \frac{\partial\psi_o(\underline{\rho}_o)}{\partial n_o} = -ikZ_oK_z(\underline{\rho}_o) \quad .$$

Note that the unit vectors $\hat{n}_o$ and $\hat{n}_2$ are always directed into volume V; the observation point $\underline{\rho}$ is located in this region.

Derivation of the integral equations

Application of the scalar Green's theorem to $\psi(\underline{\rho})$ in V, and noting that the contribution of the surface at infinity leads to the incident value of $\psi$, we write, for $\underline{\rho}$ in V,

$$\psi(\underline{\rho}) = \psi^{inc}(\underline{\rho}) - \int_{C_o} \left\{ G(\underline{\rho}, \underline{\rho}_o') \frac{\partial \psi(\underline{\rho}_o')}{\partial n_o'} - \psi(\underline{\rho}_o') \frac{\partial G(\underline{\rho}, \underline{\rho}_o')}{\partial n_o'} \right\} ds' -$$

$$- \int_{C_2} \left\{ G(\underline{\rho}, \rho_2') \frac{\partial \psi(\underline{\rho}_2')}{\partial n_2'} - \psi(\underline{\rho}_2') \frac{\partial G(\rho, \rho_2')}{\partial n_2'} \right\} ds' \tag{4}$$

where $G(\underline{\rho}, \underline{\rho}')$ is the free-space Green's function for the two-dimensional geometry and $e^{-i\omega t}$ harmonic variation, viz.,

$$G(\underline{\rho}, \underline{\rho}') = \frac{i}{4} H_o^{(1)}(k|\underline{\rho} - \underline{\rho}'|) \quad . \tag{5}$$

Application of the scalar Green's theorem to $\psi_o$ in $V_o$ leads to, for $\underline{\rho}$ in $V_o$,

$$\psi_o(\underline{\rho}) = \int_{C_o} \left\{ \psi_o(\underline{\rho}_o') \frac{\partial G(\underline{\rho}, \underline{\rho}_o')}{\partial n_o'} - G(\underline{\rho}, \underline{\rho}_o') \frac{\partial \psi_o(\underline{\rho}_o')}{\partial n_o'} \right\} ds' \quad . \tag{6}$$

Boundary conditions (2) and (3) are now used in RHS of equations (4) and (5). The $\psi_o$ terms so introduced can be eliminated from the properties of scalar Green's theorem. The basic property of interest here is the nature of the discontinuity in $\psi$ in its representation through the Green's theorem. These operations and some algebraic manipulations lead to the two integral expressions for $\psi$ and $\psi_o$ in their appropriate domains.

For $\underline{\rho}$ in V,

$$\psi^{inc}(\underline{\rho}) = \psi(\underline{\rho}) - ikZ_o \int_{C_o} K_z(\underline{\rho}_o') G(\underline{\rho}, \underline{\rho}_o') ds' - \int_{C_o} K_s^*(\underline{\rho}_o') \frac{\partial G(\underline{\rho}, \underline{\rho}_o')}{\partial n_o'} ds' -$$

$$- ikZ_o \int_{C_2} K_{z2}(\underline{\rho}_2') G(\underline{\rho}, \underline{\rho}_2') ds' - Z_o \int_{C_2} \eta K_{z2}(\underline{\rho}_2') \frac{\partial G(\underline{\rho}, \underline{\rho}_2')}{\partial n_2'} ds' \tag{7}$$

and, for $\underline{\rho}$ in $V_o$,

$$\psi^{inc}(\underline{\rho}) = \psi_o(\underline{\rho}) + \int_{C_o} K_s^*(\underline{\rho}_o') \frac{\partial G(\underline{\rho}, \underline{\rho}_o')}{\partial n_o'} ds' + ikZ_o \int_{C_o} K_z(\underline{\rho}_o) G(\underline{\rho}, \underline{\rho}_o') ds'$$

$$-ikZ_o \int_{C_2} K_{z2}(\underline{\rho}_2') G(\underline{\rho}, \underline{\rho}_2') ds' - Z_o \int_{C_2} \eta K_{z2}(\underline{\rho}_2') \frac{G(\underline{\rho}, \underline{\rho}_2')}{\partial n_2'} ds'. \quad (8)$$

Equations (7) and (8) are used to derive the integral equations. First, evaluate (7) at $\underline{\rho} = \underline{\rho}_o$. This leads to equation (9). If we substitute $\underline{\rho} = \underline{\rho}_2$ in equation (7) we obtain equation (11). Third equation is obtained by evaluation of equation (8) at $\underline{\rho} = \underline{\rho}_o$. This is equation (10) below. In the following equations we have also used the definition of the Green's function from equation (5).

$$\psi^{inc}(\underline{\rho}_o) = \psi(\underline{\rho}_o) + \frac{kZ_o}{4} \int_{C_o} K_z(\underline{\rho}_o') H_o^{(1)}(kr_{oo}) ds' -$$

$$- \frac{ik}{4} \int_{C_o} K_s^*(\underline{\rho}_o')(\hat{n}_o' \cdot \hat{r}_{oo}) H_1^{(1)}(kr_{oo}) ds' + \frac{kZ_o}{4} \int_{C_2} K_{z2}(\underline{\rho}_2') H_o^{(1)}(kr_{02}) ds' -$$

$$- \frac{ikZ_o}{4} \int_{C_2} \eta(\underline{\rho}_2') K_{z2}(\underline{\rho}_2')(\hat{n}_2' \cdot \hat{r}_{02}) H_1^{(1)}(kr_{02}) ds' \quad (9)$$

$$\psi^{inc}(\underline{\rho}_o) = \psi_o(\underline{\rho}_o) - \frac{kZ_o}{4} \int_{C_o} K_z(\underline{\rho}_o') H_o^{(1)}(kr_{oo}) ds' +$$

$$+ \frac{ik}{4} \int_{C_o} K_s^*(\underline{\rho}_o')(\hat{n}_o' \cdot \hat{r}_{oo}) H_1^{(1)}(kr_{oo}) ds' + \frac{kZ_o}{4} \int_{C_2} K_{z2}(\underline{\rho}_2') H_o^{(1)}(kr_{02}) ds' -$$

$$- \frac{ikZ_o}{4} \int_{C_2} \eta(\rho_2') K_{z2}(\rho_2')(\hat{n}_2' \cdot \hat{r}_{02}) H_1^{(1)}(kr_{02}) \, ds' \qquad (10)$$

$$\psi^{inc}(\rho_2) = \eta(\rho_2) Z_o K_{z2}(\rho_2) + \frac{kZ_o}{4} \int_{C_o} K_z(\rho_o') H_o^{(1)}(kr_{20}) ds' -$$

$$- \frac{ik}{4} \int_{C_o} K_s^*(\rho_o')(\hat{n}_o' \cdot \hat{r}_{20}) H_1^{(1)}(kr_{20}) \, ds' + \frac{kZ_o}{4} \int_{C_2} K_{z2}(\rho_2') H_o^{(1)}(kr_{22}) \, ds' -$$

$$- \frac{ikZ_o}{4} \int_{C_2} \eta(\rho_2') K_{z2}(\rho_2')(\hat{n}_2' \hat{r}_{02}) H_1^{(1)}(kr_{22}) \, ds' \qquad (11)$$

where $\underline{r}_{ij} = \rho_i - \rho_j' \; ; \qquad i, j = 0, 2.$

Equations (9) through (11) are the required integral equations. They can be readily transformed into the usual form by means of the boundary condition equations (2) and (3) along with the definition of the currents from equation (1). For numerical purposes, it will be found advantageous to simplify equation (10) by use of equation (9). This explicit form of the integral equations is not shown.

Integral equations for a partially clad impedance surface

The previously derived integral equations, equations (9) through (11) are valid only for a closed boundary $C_o$. Since the practical cladding is mostly partial, it is necessary to develop the integral equations for this case. The technique employed here is to start with the closed boundary $C_o$ and consider the limiting case of the region enclosed by $C_o$ approaching 0. Figure 2(a) shows the contour $C_o$ along with the identification of the sections of $C_o$.

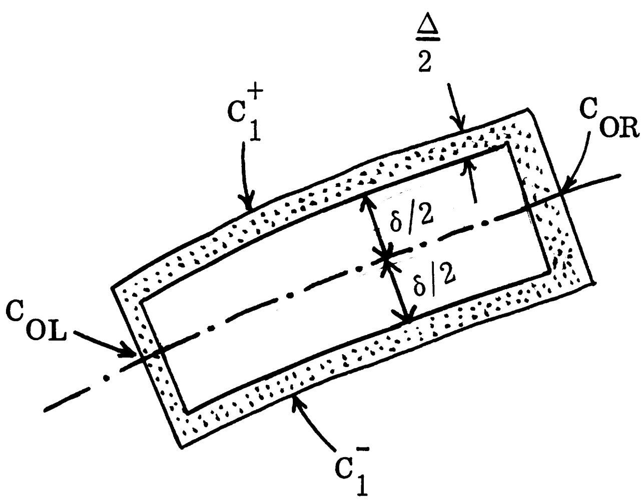


Figure 2(a)

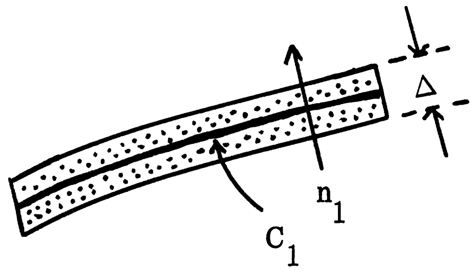


Figure 2(b)

Figure 2(b) shows the result of the limit $\delta \rightarrow 0$ on the geometry of Figure 2(a).

The contour $C_o$ consists of $C_1^+$, $C_1^-$, $C_{OR}$ and $C_{OL}$. Note that as $\delta \rightarrow 0$, the length of contours $C_{OR}$ and $C_{OL} \rightarrow 0$. We rewrite our definition of the currents and the boundary conditions as follows:

$$K_z(\rho_o^{\pm}) = -\frac{ik}{Z_o}\frac{\Delta\chi_e}{2}\psi(\rho_o^{\pm})$$

$$K_s^*(\rho_o^{\pm}) = \frac{\Delta\chi_m}{2}\frac{\partial\psi(\rho_o^{\pm})}{\partial n_o^{\pm}} \tag{12}$$

$$\psi(\rho_o^{\pm}) - \psi_o(\rho_o^{\pm}) = K_s^*(\rho_o^{\pm})$$

$$\frac{\partial\psi(\rho_o^{\pm})}{\partial n_o^{\pm}} - \frac{\partial\psi_o(\rho_o^{\pm})}{\partial n_o^{\pm}} = -ikZ_o K_z(\rho_o^{\pm}) \tag{13}$$

Note that the superscripts $\pm$ are associated with the contours $C_1^{\pm}$. Under the limit $\delta \rightarrow 0$, we observe that

(1) Contour $C_1^+ \longrightarrow C_1^- \longrightarrow C_1$

(2) $\psi_o(\underline{\rho}_o^+) \longrightarrow \psi_o(\underline{\rho}_o^-)$

(3) Contributions from the integral over $C_{OR}$ and $C_{OL}$ will both approach zero since: (a) the length of the contours are proportional to $\delta$ and (b) for infinitesimally small thickness $\Delta$, the currents tangential to $C_{OR}$ and $C_{OL}$ must approach zero.

In order to arrive at the necessary integral equations, first evaluate (9) at $\underline{\rho}_o = \underline{\rho}_1^+$ and $\underline{\rho}_1^-$ and add the two. We also replace integrals over $C_o$ by $C_1$ with appropriate changes in the integrand. Furthermore, we define

$$\begin{aligned} K_z(\underline{\rho}_1) &= K_z(\underline{\rho}_1^+) + K_z(\underline{\rho}_1^-) \\ K_s^*(\underline{\rho}_1) &= K_s^*(\underline{\rho}_1^+) + K_s^*(\underline{\rho}_1^-) \end{aligned} \tag{14}$$

and use the definition of currents through equation (12). We obtain

$$\begin{aligned} \psi^{inc}(\underline{\rho}_1) = \frac{iZ_o}{k\Delta\chi_e} K_z(\underline{\rho}_1) + \frac{kZ_o}{4}\int_{C_1} K_z(\underline{\rho}_1')\, H_o^{(1)}(kr_{11})\, ds' - \\ - \frac{ik}{4}\int_{C_1} K_s^*(\underline{\rho}_1')(\hat{n}_1 \cdot \hat{r}_{11})\, H_1^{(1)}(kr_{11})\, ds' + \frac{kZ_o}{4}\int_{C_2} K_{z2}(\underline{\rho}_2')\, H_o^{(1)}(kr_{12})\, ds' - \\ - \frac{ikZ_o}{4}\int_{C_2} \eta(\underline{\rho}_2') K_{z2}(\underline{\rho}_2')(\hat{n}_2 \cdot \hat{r}_{12})\, H_1^{(1)}(kr_{12})\, ds' \quad . \end{aligned} \tag{15}$$

An analogous manipulation of equation (10) and use of the boundary condition (13) to eliminate $\psi_o$ leads to the following integral equation:

$$\psi^{inc}(\underline{\rho}_1) = \frac{iZ_o}{k\Delta\chi_e} K_z(\underline{\rho}_1) - \frac{1}{2} K_s^*(\underline{\rho}_1) + \frac{kZ_o}{4}\int_{C_1} K_z(\underline{\rho}_1')\, H_o^{(1)}(kr_{11})\, ds' +$$

$$-\frac{ik}{4}\int_{C_1} K_s^*(\underline{\rho}_1')(\hat{n}_1'\cdot\hat{r}_{11})H_1^{(1)}(kr_{11})\,ds' + \frac{kZ_o}{4}\int_{C_2} K_{z2}(\underline{\rho}_2')H_o^{(1)}(kr_{12})\,ds' -$$

$$-\frac{ikZ_o}{4}\int_{C_2}\eta(\underline{\rho}_2')K_{z2}(\underline{\rho}_2')(\hat{n}_2'\cdot\hat{r}_{12})H_1^{(1)}(kr_{12})\,ds' \quad . \tag{16}$$

Transforming the integration over contour $C_o$ in equation (11) to the contour $C_1$ leads to the third integral equation.

$$\psi^{inc}(\underline{\rho}_2) = \eta(\underline{\rho}_2)Z_o K_{z2}(\underline{\rho}_2) + \frac{kZ_o}{4}\int_{C_1} K_z(\underline{\rho}_1')H_o^{(1)}(kr_{21})\,ds' -$$

$$-\frac{ik}{4}\int_{C_1} K_s^*(\underline{\rho}_1')(\hat{n}_1'\cdot\hat{r}_{21})H_1^{(1)}(kr_{21})\,ds' + \frac{kZ_o}{4}\int_{C_2} K_{z2}(\underline{\rho}_2')H_o^{(1)}(kr_{22})\,ds' -$$

$$-\frac{ikZ_o}{4}\int_{C_2}\eta(\underline{\rho}_2')K_{z2}(\underline{\rho}_2')(\hat{n}_2'\cdot\hat{r}_{22})H_1^{(1)}(kr_{22})\,ds' \quad . \tag{17}$$

<u>Note</u>: $\underline{r}_{ij} = \underline{\rho}_i - \underline{\rho}_j'$ , $i, j = 1, 2$.

Equations (15), (16) and (17) are the integral equations for the unknown currents. Once again, for numerical evaluation, equation (16) can be simplified by means of equation (15).

## <u>Scattered Fields</u>

The expression for the scattered component of $\psi$ can be readily obtained from equation (9).

$$\psi^s(\underline{\rho}) = -\frac{kZ_o}{4}\int_{C_1} K_z(\underline{\rho}_1')H_o^{(1)}(kr_{11})ds' + \frac{ik}{4}\int_{C_1} K_s^*(\underline{\rho}_1')(\hat{n}_1'\cdot\hat{r}_{11})H_1^{(1)}(kr_{11})ds' -$$

$$-\frac{kZ_o}{4}\int_{C_2} K_{z2}(\underline{\rho}_2')H_o^{(1)}(kr_{12})ds' + \frac{ikZ_o}{4}\int_{C_2} \eta(\underline{\rho}_2')K_{z2}(\underline{\rho}_2')(\hat{n}_2'\cdot\hat{r}_{12})H_1^{(1)}(kr_{12})ds' . \tag{18}$$

Remarks:

The three integral equations in this work have been derived based on scalar Green's theorem. It does have one shortcoming. In this derivation, although boundary condition on the tangential component of the magnetic field (via the normal derivation of $\psi$) has been used explicitly, no integral equation for the magnetic field is developed. To do so would require a vector formulation and consequently second derivatives of the Green's functions would appear in the integrand. This would cause difficulties in numerical solution of these equations.

The numerical solution may convince us that it is necessary to use an integral equation for the magnetic field. At that time, the necessary equations can be readily developed. The numerical difficulty anticipated in such a case is the evaluation of the contribution of the self cell ($\underline{\rho}\rightarrow\underline{\rho}'$). However, I believe, that this can be circumvented by the use of the theory of distribution functions, and in particular, the Hadamard's principal value technique.

References

E. F. Knott and T. B. A. Senior, "Non-Specular Radar Cross Section Study," University of Michigan Radiation Laboratory Technical Report AFAL-TR-73-2, February 1973.

E. F. Knott, V. V. Liepa and T. B. A. Senior, "Non-Specular Radar Cross Section Study," University of Michigan Radiation Laboratory, Technical Report AFAL-TR-73-70, April 1973.

F. K. Oshiro and R. C. Cross, "A Source Distribution Technique for the Solution of Two Dimensional Scattering Problems," Phase III Report, Northrop Corporation Report No. NOR-66-74, March 1966.

F. K. Oshiro, K. M. Mitzner, et al., "Calculation of Radar Cross Section," Northrop Corporation Report No. NOR-71-14, 1971.