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SUBJECT: Non-Specular Radar Cross Section of the Discontinuity in Surface Impedance. Part A: The Case of Reactive Impedance.

Besides wedges, discontinuity in curvature and creeping waves, there is another source of non-specular reflection -- discontinuity in surface impedance. While the first three have been extensively analyzed, and the diffraction coefficients have been established -- from which non-specular radar cross section of corresponding structures, including combining effects, can be deduced -- the last one has attracted in the past much less attention, at least from the radar cross section point of view. The case of surface reactance discontinuity has been analyzed in connection with surface wave scattering which gives rise to the radiation phenomena and is of great applicability in the field of surface wave antennas (Kay 1957, Trenev 1958). As it was pointed out by Knott et al. (1973), in connection with the analysis of non-specular radar cross section reduction by introducing impedance boundary conditions on a part of a body surface, the discontinuity in surface impedance may cause an undesirable appearance of non-specular scattering. So, it seems to be of some interest to have a quantitative analysis of the scattering phenomena at the discontinuity of surface impedance not only for the surface wave, but for the uniform plane wave and, accordingly, to have some estimation of the associated non-specular radar cross section at such a discontinuity.

For the quantitative analysis of non-specular scattering phenomena at the discontinuity of surface impedance, we shall choose such a diffracting structure in which no other non-specular scattering could appear and in which we can easily discriminate between specular and non-specular reflections. It is evident that these

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conditions are fulfilled if we choose for our structure an infinite plane, half of which has the surface impedance Z_1, and the other half surface impedance Z_2, and upon which a plane electromagnetic wave is incident. In this case we can easily divide between specular and non-specular reflection, the former may be obtained exactly by simple geometrical optic methods. On the other hand, there is obviously no non-specular reflection except one we are interested in. This structure, along with the coordinate system, is represented in Fig. 1.

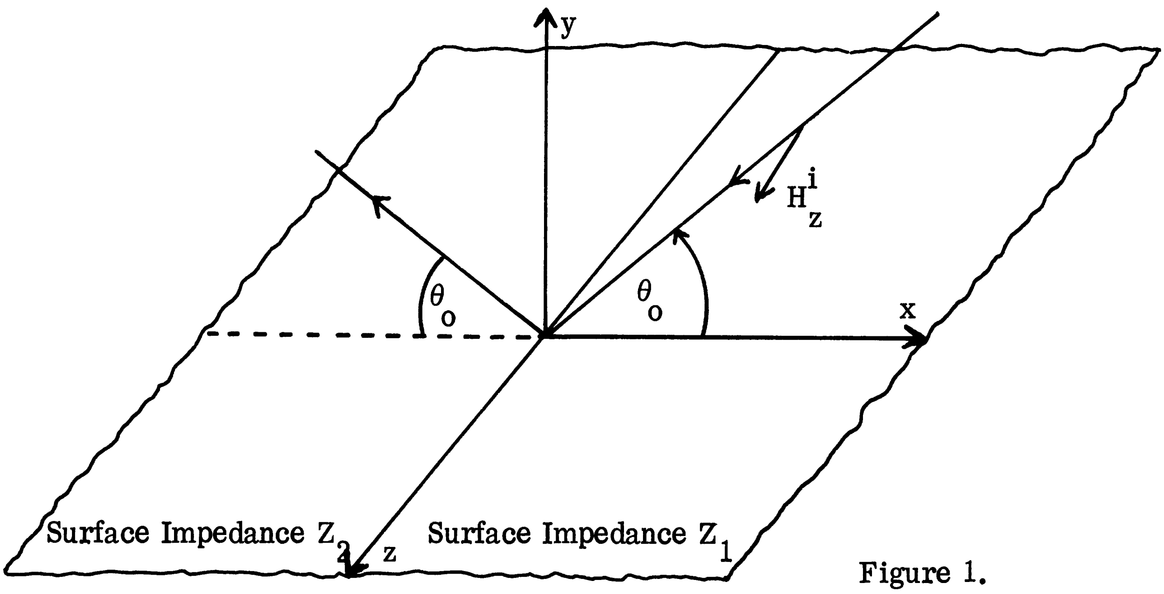

Figure 1.

The z-axis is the dividing line between half planes with the surface impedances Z_1 and Z_2, and the plane wave is incident normally to the z-axis, but at an angle θ_o to the x-axis.

If we imagine that the structure in Fig. 1 is equivalent, from the non-specular scattering point of view, to a cylindrical diffraction structure upon which a unit amplitude plane wave is incident, we can write for the far zone scattered field ϕ_s:

$$\phi_s = P(\theta_o, \theta)\sqrt{\frac{2}{\pi KR}}\, e^{j(KR - \frac{\pi}{4})} \quad . \qquad (1)$$

The radar cross section of such a structure is then:

$$\sigma = \lim_{(R\rightarrow\infty)} 2\pi R \left(\frac{P_s}{P_i}\right) = \lim 2\pi R \left|\phi_s\right|^2 \tag{2}$$

where P_i, P_s are the power densities of the incident and scattered fields respectively at a distance R from the structure. From Eqs. (1) and (2) we get:

$$\frac{\sigma}{\lambda} = \frac{2}{\pi} \left|P(\theta_o, \theta)\right|^2 \quad . \tag{3}$$

In Eqs. (1) and (3) θ is the angle of observation point, θ_o is the angle of plane wave incidence, so Eq. (3) represents the bistatic radar cross section. For monostatic radar cross section we shall put $\theta = \theta_o$.

As was pointed out at the beginning, the problem of scattering of a given surface wave (supported by the structure) by the discontinuity in surface reactance has been treated by Kay and Trenev by application of essentially the same technique: representation of the scattered field by a spectrum of plane waves which leads to the dual integral equation for the field amplitude. According to the definition of radar cross section, we need the solution not for incident surface wave, but for incident uniform plane wave. Although this problem could be treated in the same way, we chose for our analysis the so-called "Jones's method" which consists of application of the Laplace transform directly to the wave equation and solving for the transformed field by Wiener-Hopf technique. However, instead of the Laplace transform we shall apply the Fourier transform in the complex domain, which is extensively used by Noble (1958). After the solution for the transformed field has been obtained, the inverse transform should give the real near field.

It is important to point out that the structure in Fig. 1 is a special case of a wedge with different side impedances when the angle between the sides is equal to 180^o. This general case is treated by Malyuzhinets (1958) which introduced into the solution some special functions which are impractical for our purposes. Treating this special case of a wedge separately leads to the solution in which only elementary functions exist.

The case of H-polarized field is treated in detail. The solution of the E-polarized field can be obtained from the former by appropriate transformation as will be shown later.

Mathematical Formulation of the Problem

Suppose that a uniform plane electromagnetic wave with z-directed H field is incident from infinity to the structure in Fig. 1. The diffraction problem is essentially a scalar one, so we can put for the incident field:

$$H_{zi} = e^{-jKx\cos\theta_o - jKy\sin\theta_o} = \phi^i(x, y)$$

$$E_x^{(i)} = \frac{j}{\omega\epsilon} \cdot \frac{\partial\phi^i(x,y)}{\partial y} = \frac{K}{\omega\epsilon}\sin\theta_o\,\phi^i(x,y) = Z_o\sin\theta_o\,\phi^i(x, y) \tag{4}$$

$$E_y^{(i)} = -\frac{j}{\omega\epsilon}\frac{\partial\phi^i(x,y)}{\partial x} = -\frac{K}{\omega\epsilon}\cos\theta_o\,\phi^i(x, y) = -Z_o\cos\theta_o\,\phi^i(x, y)$$

where:

the convention $e^{-j\omega t}$ is used,

and K is the propagation constant and Z_o is the intrinsic impedance of the homogeneous medium above the plane $y = 0$,

We are, of course, interested in "pure" non-specularly scattered field which represents diffracted field (i.e., the total field in the presence of a structure) minus the incident field minus the specularly reflected field from both half planes. Several attempts have been made to achieve such a separation at the outset, in which case the solution would have contained the desired "pure" non-specularly scattered field. Unfortunately, all of these attempts failed due to improper domains of regularity of some functions when the Wiener-Hopf technique was hoped to be used. We discovered that only separation which gives the proper domains of regularity is the following: for the case of incidence as in Fig. 1 (i.e.,

θ_o is between 0 and $\pi/2$) we can, at the outset, subtract only the specularly reflected field from the right half plane, so the solution for the scattered field will contain the specularly reflected field from the left half plane. For the case of incidence $\pi/2 < \theta_o < \pi$ the role of both planes are interchanged. However, we can detect the presence of the non-specularly scattered field in the solution. Moreover, if we are sufficiently away from the line $\pi - \theta_o$, where specular reflection takes place, we can take the obtained solution for the field as purely non-specular and the obtained radar cross section as non-specular, especially in the backscattering direction.

We can now formulate our diffraction problem in the following way: Upon the structure in Fig. 1 a field ϕ_1 is incident which consists of the primary incident field ϕ^i and the field ϕ_r reflected from the right half plane $y = 0$, $x > 0$. Find the far scattered field.

We have:

$$\phi_1(x, y) = \phi^i(x, y) + \phi^r(x, y)$$

where ϕ^i is given by Eq. (4), whereas ϕ^r is:

$$\phi^r(x, y) = R \cdot e^{-jKx\cos\theta_o + jKy\sin\theta_o}$$

with:

$$R = \frac{Z_o \sin\theta - Z_1}{Z_o \sin\theta + Z_1} = \frac{\sin\theta_o - \eta_1}{\sin\theta_o + \eta_1}$$

where η_1 is normalized surface impedance. After simple manipulation we get:

$$\phi_1(x, y) = \frac{2}{\sin\theta_o + \eta_1}\left[\sin\theta_o \cos(Ky\sin\theta_o) - j\eta_1 \sin(Ky\sin\theta_o)\right] e^{-jKx\cos\theta}. \qquad (5)$$

The total field is:

$$\phi^t(x, y) = \phi_1(x, y) + \phi(x, y) \qquad (6)$$

where $\phi(x, y)$ is the desired scattered field, for which we can write:

$$\frac{\partial^2 \phi(x,y)}{\partial x^2} + \frac{\partial^2 \phi(x,y)}{\partial y^2} - K^2 \phi(x,y) = 0 \quad . \qquad (7)$$

With the following boundary condition in terms of the total fields:

$$\underline{E}^t - (\hat{n} \cdot \underline{E}^t) \cdot \hat{n} = \begin{cases} \eta_1 Z_o (\hat{n}_\wedge \underline{H}^t) & \text{for } x > 0 \\ \eta_2 Z_o (\hat{n}_\wedge \underline{H}^t) & \text{for } x < 0 \end{cases} \quad \text{for } y = 0 \qquad (8)$$

where η_1 and η_2 are normalized surface impedances.

In our case $\hat{n} = \hat{y}$ and $\underline{H}^t = \hat{z}_o \phi^t(x, y)$, so forming the scalar product of Eq. (8) with $\hat{x}$ we get:

$$E_x^t = \begin{cases} \eta_1 Z_o \phi^t(x,y) & \text{for } x > 0 \\ \eta_2 Z_o \phi^t(x,y) & \text{for } x < 0 \end{cases} \quad \text{for } y = 0 \quad . \qquad (9)$$

Using the relation:

$$E_x^t = \frac{j}{\omega \epsilon} \cdot \frac{\partial \phi^t(x,y)}{\partial y} \qquad (10)$$

we can get from Eqs. (5), (6) and (10) after simple manipulations the following boundary conditions for $\phi(x, y)$:

$$\left. \frac{\partial \phi(x,y)}{\partial y} \right|_{y=0} + jK\eta_1 \phi(x,0) = 0 \quad \text{for x } x \geq 0 \qquad (11a)$$

$$\left. \frac{\partial \phi(x,y)}{\partial y} \right|_{y=0} + jK\eta_2 \phi(x,0) = \frac{j2K \sin\theta_o (\eta_1 - \eta_2)}{\sin\theta_o + \eta_1} e^{-jKx\cos\theta_o}$$

$$\text{for } x \leq 0 \quad . \qquad (11b)$$

So our problem can be formulated as follows: Find the formal solution of the wave equation (7) with the mixed boundary condition (11), and from this formal solution find the exact expression for the magnitude of the far field.

Formal Solution of the Wave Equation

For the medium above the plane $y > 0$ we shall eventually adopt free space with ϵ_o, μ_o, but for the present we shall assume that there is a slight conductivity σ_o in space, find the solution, and then by the limiting process $\sigma_o \rightarrow 0$ come to the final expression. This is a standard procedure which makes possible certain integrals to be convergent. It is easy to prove that adopting time convention $e^{-j\omega t}$, we have, with finite σ_o:

$$K = K_1 + jK_2$$

where:

$$K_1 > 0$$
$$K_2 > 0$$

which is important to note.

Before proceeding further it is necessary to examine the asymptotic behavior of $\phi(x, y)$. We shall take two different regions in the half space $y > 0$, one defined by $0 < \theta < \pi - \theta_o$, and the other $\pi - \theta_o < 0 < \pi$. In the first region ϕ by definition consists of only non-specularly scattered field with the asymptotic behavior (because of the cylindrical structure):

$$\phi \underset{r \to \infty}{\sim} A_1 \cdot H_o^{(1)}(kr) \sim A_2 \sqrt{\frac{1}{r}}\, e^{jK_1 r}\, e^{-K_2 r}$$

where:

$$r = \sqrt{x^2 + y^2}$$ and A_1 and A_2 are constants.

So we have in the first region:

$$|\phi| < A_3 e^{-K_2\sqrt{x^2+y^2}} \quad . \tag{12}$$

In the second region ϕ includes the reflected waves so we have:

$$|\phi| < A_4\, e^{K_2 x \cos\theta_o - K_2 y \sin\theta_o} \quad . \tag{13}$$

As the x in the second region is negative, while y is positive, ϕ has the proper asymptotic behavior in both regions, which makes possible the + and - Fourier transforms of the function $\phi(x, y)$ be regular in appropriate domains. By definition we have:

$$\Phi_+(\alpha, y) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \phi(x, y)\, e^{j\alpha x} dx$$

$$\Phi_-(\alpha, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \phi(x, y)\, e^{j\alpha x} dx \tag{14}$$

$$\Phi(\alpha, y) = \Phi_+(\alpha, y) + \Phi_-(\alpha, y)$$

where $\alpha = \sigma + j\tau$, and Φ represents Fourier transform in x of $\phi(x, y)$. From Eqs. (12) and (13) it is evident that Φ_+ is regular in the upper half plane: $\tau > -K_2$, and Φ_- is regular in the lower half plane $\tau < K_2 \cos\theta_0$, while the Φ is regular in the strip $-K_2 < \tau < K_2 \cos\theta_0$ (see Fig. 2).

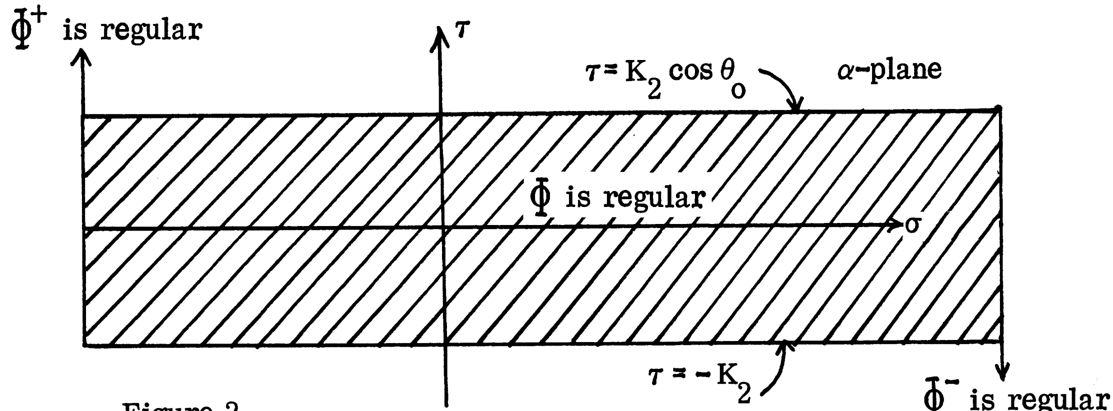

Figure 2.

By inversion we can get $\phi(x, y)$ from the known $\Phi(\alpha, y)$:

$$\phi(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty + j\tau_0}^{\infty + j\tau_0} \Phi(\alpha, y) \cdot e^{-j\alpha x} d\alpha \tag{15}$$

with τ_o in the strip $-K_2 < \tau < K_2 \cos\theta_o$.

Apply now the Fourier transform in x to the wave equation (7). It is easy to find that the transformed equation is of the form:

$$\frac{d^2 \Phi(\alpha, y)}{dy^2} - \gamma^2 \Phi(\alpha, y) = 0$$

where: $\gamma^2 = \alpha^2 - K^2$

with the solution:

$$\Phi(\alpha, y) = A(y)\, e^{\gamma y} + B(y)\, e^{-\gamma y}$$

which contains two branch points: $\alpha_1 = +K$, and $\alpha_2 = -K$, and we have to choose the appropriate branch cuts in such a way that $\Phi(\alpha, y)$ remains bounded in the strip $-K_2 < \tau < K_2 \cos\theta_o$ as $y \to +\infty$, so we can apply Eq. (16) to get $\phi(x, y)$. If we choose the branch of γ such that $\gamma \to +\sigma$ and $\alpha \to +\infty$ for α in the strip $-K_2 < \sigma < K_2$, then to assure that $\gamma \to |\sigma|$ as $\sigma \to -\infty$ for α in the same strip, no branch cut can pass through the strip, so the branch cut must go from $+K_2$ to ∞ in the upper half plane and from $-K_2$ to ∞ in the lower half plane, as is represented in Fig. 3.

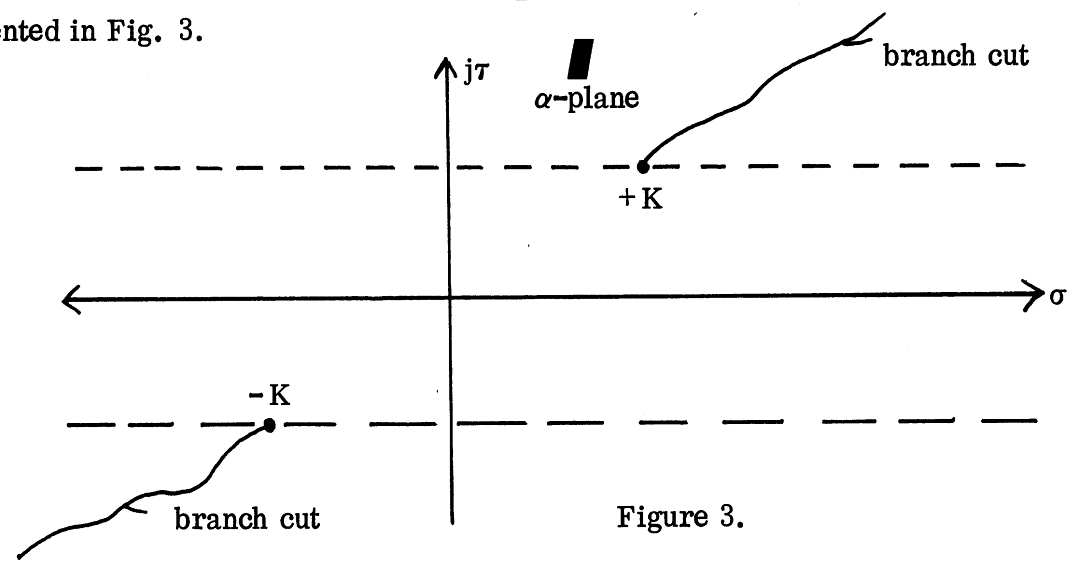

Figure 3.

In an α plane so cut it is evident that the function $\Phi(\alpha, y)$ has to have the form:

$$\Phi(\alpha, y) = B(\alpha) e^{-\gamma y} \tag{16}$$

i.e., we are obliged to put: $A = 0$. The constant $B(\alpha)$ remains to be determined from the boundary conditions.

By applying + transform to the boundary condition (11a) and − transform to the boundary condition (11b) we get:

$$\Phi'_+(\alpha, 0) + jK\eta_1 \Phi_+(\alpha, 0) = 0 \tag{17a}$$

$$\Phi'_-(\alpha, 0) + jK\eta_2 \Phi_-(\alpha, 0) = \frac{2K \sin\theta_o (\eta_1 - \eta_2)}{\sqrt{2\pi}\ (\sin\theta_o + \eta_1)} \frac{1}{\alpha - K\cos\theta_o} = f(\alpha) \tag{17b}$$

where:

$$\Phi'_+(\alpha, 0) = \left.\frac{d\Phi_+(\alpha, y)}{dy}\right|_{y=0}$$

$$\Phi'_-(\alpha, 0) = \left.\frac{d\Phi_-(\alpha, y)}{dy}\right|_{y=0}$$

and Φ'_+ is regular in the upper half plane: $\tau > -K_2$, while Φ'_- and $f(\alpha)$ are regular in the lower half plane $\tau < K_2 \cos\theta_o$.

From Eq. (16) we have:

$$\Phi_+(\alpha, 0) + \Phi_-(\alpha, 0) = \Phi(\alpha, 0) = B(\alpha) \quad , \tag{18a}$$

$$\Phi'_+(\alpha, 0) + \Phi'_-(\alpha, 0) = -\gamma \cdot B(\alpha) \quad . \tag{18b}$$

The four equations (17) and (18) are sufficient to form a standard Wiener-Hopf equation of the type:

$$C(\alpha)\psi_+(\alpha) + D(\alpha)\psi_-(\alpha) + E(\alpha) = 0$$

where:

$\psi_+(\alpha)$ is regular in the upper half plane,

$\psi_-(\alpha)$ is regular in the lower half plane,

C, D, E are regular in the common strip of regularity of ψ_+ and ψ_-.

Multiplying Eq. (18a) by $jK\eta_1$ and adding (18b), we get, with the help of (17a):

$$B(\alpha)\ (jK\eta_1-\gamma) = \psi_-(\alpha) \tag{19}$$

where:

$$\psi_-(\alpha) = \Phi'_-(\alpha, 0) + jK\eta_1\, \Phi_-(\alpha, 0) \quad . \tag{20}$$

By multiplying Eq. (18a) by $jK\eta_2$ and adding (18b), we, get with the help of (17b):

$$B(\alpha)\ (jK\eta_2-\gamma) = \psi_+(\alpha) + f(\alpha) \tag{21}$$

where:

$$\psi_+(\alpha) = j\eta K_2\, \Phi_+(\alpha, 0) + \Phi'_+(\alpha, 0) \quad . \tag{22}$$

Eliminating $B(\alpha)$ from Eqs. (19) and (21) we get the functional equation:

$$(1-j\frac{K\eta_1}{\gamma})\,\psi+(\alpha) - (1-\frac{jK\eta_2}{\gamma})\,\psi_-(\alpha) + (1-\frac{jK\eta_1}{\gamma})\,f(\alpha) = 0 \tag{23}$$

which is of the proper Wiener-Hopf type and holds in the strip $-K_2 < \tau < K_2 \cos\theta_o$.

According to the standard Wiener-Hopf procedure, we have to factorize the following quantities:

$$\begin{aligned} 1-j\frac{K\eta_1}{\gamma} &= K_{1+}(\alpha)\cdot K_{1-}(\alpha) \\ 1-j\frac{K\eta_2}{\gamma} &= K_{2+}(\alpha)\cdot K_{2-}(\alpha) \end{aligned} \tag{24}$$

where K_i^+ and K_i^- ($i = 1, 2$) are regular and non-zero in upper and lower half planes respectively. With Eqs. (23) and (24) we obtain:

$$\frac{K_1^+(\alpha)}{K_2^+(\alpha)}\psi_+(\alpha) - \frac{K_2^-(\alpha)}{K_1^-(\alpha)}\psi_-(\alpha) + \frac{K_1^+(\alpha)}{K_2^+(\alpha)} \cdot f(\alpha) = 0 \quad . \tag{25}$$

The function $f(\alpha)$ according to (17b) is regular in the lower half plane $\tau < K_2 \cos\theta_o$ but has only one simple pole at $\alpha = K\cos\theta_o$ in the upper half plane $\tau > -K_2$, so if we add and subtract in Eq. (25) the function:

$$f(\alpha) \cdot \frac{K_1^+(K\cos\theta_o)}{K_2^+(K\cos\theta_o)}$$

we can write Eq. (25) as:

$$\frac{K_1^+(\alpha)}{K_2^+(\alpha)}\psi_+(\alpha) + f(\alpha)\left[\frac{K_1^+(\alpha)}{K_2^+(\alpha)} - \frac{K_1^+(K\cos\theta_o)}{K_2^+(K\cos\theta_o)}\right] =$$

$$= \frac{K_2^-(\alpha)}{K_1^-(\alpha)}\psi_-(\alpha) - f(\alpha) \cdot \frac{K_1^+(K\cos\theta_o)}{K_2^+(K\cos\theta_o)} \quad . \tag{26}$$

The left hand side of Eq. (26) is regular in the upper half plane $\tau > -K_2$, and the right hand side is regular in the lower half plane $\tau < K_2 \cos\theta_o$, with the common strip of regularity $-K_2 < \tau < K_2 \cos\theta_o$. By the principle of analytical continuation both sides must be equal to a function $y(\alpha)$ regular in the whole α-plane. The nature of $y(\alpha)$ can be determined from the asymptotic behavior, say, the left hand side of Eq. (26) as $\alpha \to \infty$ along a path in the upper half plane. As the functions $K_1^+(\alpha)$ and $K_2^+(\alpha)$ are quite similar (see Eq. (24)), their quotient tends to a constant as $\alpha \quad \infty$. From Eq. (22) we can conclude that $\psi_+(\alpha)$ has the same behavior as the + Fourier transform of the diffraction field $\phi(x, 0)$, and $\left.\frac{\partial \phi(x,y)}{\partial y}\right|_{y=0}$ i.e., as H_z and E_x fields. It is shown for example in Noble (1958) that if some function $f(x)$ has the behavior:

$$f(x) \sim A \cdot x^{\eta} \quad (x \to 0)$$

where $-1 < \eta < 0$, then for the + Fourier transform $F_+(\alpha)$ we can write (Abelian theorem):

$$F_+(\alpha) \sim A(2\pi)^{-1/2}\,\Gamma(\eta+1)\, e^{\frac{1}{2}\pi i(\eta+1)}\, \alpha^{-\eta-1} \quad (\alpha \to \infty).$$

It can be proved that near the origin $y = 0$, $x \to +0$ the fields E_x and H_z have the behavior

$$E_x,\ H_z \sim C x^{-\frac{1}{2}} \qquad x \to +0 \ .$$

The justification for this assumption lies in the fact that we can consider the sources of the diffracted field as a collection of equivalent charges accumulated along line z, and that for small x the field behaves as a static field. Now we have

$$\psi_+(\alpha) \le C_1 \alpha^{-\frac{3}{2}} \quad .$$

From Eq. (17b) it is clear that as $\alpha \to \infty$:

$$f(\alpha) \sim C_2 \alpha^{-1}$$

so we have:

$$y(\alpha) \to 0\ , \quad \text{as} \quad \alpha \to \infty$$

i.e.,

$$y(\alpha) \equiv 0$$

by Lionville's theorem. From Eq. (26) we now have:

$$\psi_-(\alpha) = f(\alpha)\,\frac{K_1^-(\alpha)}{K_2^-(\alpha)} \cdot \frac{K_1^+(K\cos\theta_o)}{K_2^+(K\cos\theta_o)}$$

and from Eq. (19):

$$B(\alpha) = -\frac{f(\alpha)}{\gamma - jK\eta_1} \cdot \frac{K_1^-(\alpha)}{K_2^-(\alpha)} \cdot \frac{K_1^+(K\cos\theta_o)}{K_2^+(K\cos\theta_o)} \qquad (27)$$

while from Eq. (16) we get:

$$\Phi(\alpha, y) = B(\alpha) \cdot e^{-\gamma y} \quad . \tag{28}$$

Applying the inverse Fourier transform (15) to Eq. (28) we obtain:

$$\phi(x, y) = -\frac{1}{\sqrt{2\pi}} \frac{K_1^+(K\cos\theta_o)}{K_2^+(K\cos\theta_o)} \int_{-\infty+j\tau_o}^{\infty+j\tau_o} \frac{f(\alpha)}{\gamma - jK\eta_1} \cdot \frac{K_1^-(\alpha)}{K_2^-(\alpha)} \cdot e^{-\gamma y - j\alpha x} d\alpha \tag{29}$$

with the τ_o in the strip $-K_2 < \tau < K_2 \cos\theta_o$.

Equation (29) represents the formal solution of the wave equation (5) with the boundary conditions (11).

Calculation of the Far Field Magnitude

Let us take in Eq. (29) $\tau_o = 0$ and perform integration along the σ-axis. Putting:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Eq. (29) is transformed into:

$$\phi(x, y) = -\frac{1}{\sqrt{2\pi}} \frac{K_1^+(K\cos\theta_o)}{K_2^+(K\cos\theta_o)} \int_{-\infty}^{\infty} \frac{f(\sigma)}{\sqrt{\sigma^2 - K^2} - jK\eta_1} \cdot \frac{K_1^-(\sigma)}{K_2^-(\sigma)} e^{-r(\sqrt{\sigma^2 - K^2}\sin\theta + j\sigma\cos\theta)} d\sigma \quad . \tag{30}$$

Evaluating integral (30) asymptotically by the method of steepest descent, the saddle point being:

$$\sigma_s = -K\cos\theta$$

we come, after simple manipulation, to the solution:

$$\phi(x, y) = \sqrt{\frac{2}{\pi K r}} \cdot e^{jKr - j\frac{\pi}{4}} \cdot \frac{\sin\theta_o}{\sin\theta_o + \eta_1} \cdot \frac{K_1^+(K\cos\theta_o)}{K_2^+(K\cos\theta_o)} \cdot \frac{K_1^-(K\cos\theta)}{K_2^-(-K\cos\theta)} \cdot \frac{\sin\theta}{(\cos\theta + \cos\theta_o)(\sin\theta - \eta_1)} \quad . \tag{31}$$

The solution (31) is still formal, for the function $K_i^+(\alpha)$ and $K_i^-(\alpha)$ are unknown, although we know that they exist. It we put:

$$F_1(\alpha) = \ell n K_1(\alpha) = \ell n\left(1 - \frac{jK\eta_1}{\sqrt{\alpha^2 - K^2}}\right)$$

$$F_2(\alpha) = \ell n K_2(\alpha) = \ell n\left(1 - \frac{jK\eta_2}{\sqrt{\alpha^2 - K^2}}\right)$$

we have by definition:

$$F_i^+(\alpha) = \ell n K_i^+(\alpha) = \frac{1}{2\pi i} \int_{ic-\infty}^{ic+\infty} \frac{\ell n\left(1 - \frac{jK\eta_i}{\sqrt{\zeta^2 - K^2}}\right)}{\zeta - \alpha} d\zeta \tag{32}$$

$$F_i^-(\alpha) = -\frac{1}{2\pi i} \int_{id-\infty}^{id+\infty} \ell n\left(1 - \frac{jK\eta_i}{\sqrt{\zeta^2 - K^2}}\right) \frac{d\zeta}{\zeta - \alpha} \tag{33}$$

where: $-K_2 < C < \tau < d < K_2$.

Now we have:

$$K_i^+(\alpha) = e^{F_+(\alpha)}$$

$$K_i^-(\alpha) = e^{F_-(\alpha)} \tag{34}$$

$$K_i^+(\alpha)\ K_i^-(\alpha) = e^{F_i^+(\alpha) + F_-(\alpha)} = e^{F(\alpha)} = K(\alpha) \quad . \tag{35}$$

From Eqs. (32) and (33) one can easily conclude that:

$$F_i^+(\alpha) = F_i^-(-\alpha) \tag{36}$$

so we need only, say, $F^-(\alpha)$, or $K^-(\alpha)$.

Introducing Eq. (34) into Eq. (31) we get, with $\alpha = K\cos\theta$ or $K\cos\theta_o$:

$$\phi(x, y) = \sqrt{\frac{2}{\pi K r}}\, e^{j(Kr - \pi/4)} \frac{\sin\theta_o}{\sin\theta_o + \eta_1} \frac{(\eta_2 - \eta_1)\sin\theta}{(\cos\theta + \cos\theta_o)(\sin\theta + \eta_1)}\, e^{F(\theta, \theta_o)} \tag{37}$$

where:

$$F(\theta, \theta_o) = F_1^-(-K\cos\theta_o) + F_1^-(-K\cos\theta) - F_2^-(-K\cos\theta_o) - F_2^-(-K\cos\theta) . \quad (38)$$

Comparing equations (37) and (1), we can conclude that we have obtained the desired form of the field in the far zone.

According to formula (3), for the calculation of radar cross section we need not a complete expression for the far field but only for the function $P(\theta_o, \theta)^2$ which can be easily obtained from Eq. (37), taking the square of magnitude. We come finally to the expression:

$$\left|P(\theta, \phi)\right|^2 = \frac{\sin^2\theta_o}{\left|\sin\theta_o + \eta_1\right|^2} \cdot \frac{\left|\eta_2 - \eta_1\right|^2 \sin^2\theta}{(\cos\theta + \cos\theta_o)^2 \left|\sin\theta + \eta_1\right|^2} e^{2 \cdot \mathrm{Real}\, F(\theta, \theta_o)} \quad (39)$$

where Real $F(\theta, \theta_o)$ means real part of $F(\theta, \theta_o)$, and $F(\theta, \theta_o)$ is given by the expression (38).

So, the problem of finding the exact expression for $\left|P(\theta, \phi)\right|^2$ is reduced to the problem of finding the real part of the function $F_i^-(\alpha)$. If we take now, by the limiting process, the imaginary part of K to be zero, as it is in free space, we have to solve the following problem: find the real part of the function $F_i^-(\alpha)$ for real $\alpha = \xi = K\cos\theta_o$, or $\xi = K\cos\theta$. Taking K as a real number means that the branch points +K and -K lie on the real axis, which must be carefully taken into account along with the proper rule of choosing branch cuts (see Fig. 3). For example, with the cuts from (-K to $-\infty$) and (+K, $+\infty$) which is the limiting case of proper branch cuts from Fig. 3, the path of integration along the real axis must go just above the real axis in the interval ($-\infty$, -K), and just under the real axis in the interval (+K, $+\infty$). On the other hand, taking α as real means that we introduce a pole in the integrand (32) and (33), if the path of integration goes along the real axis in the interval 0 - K. This example is shown in Fig. 4.

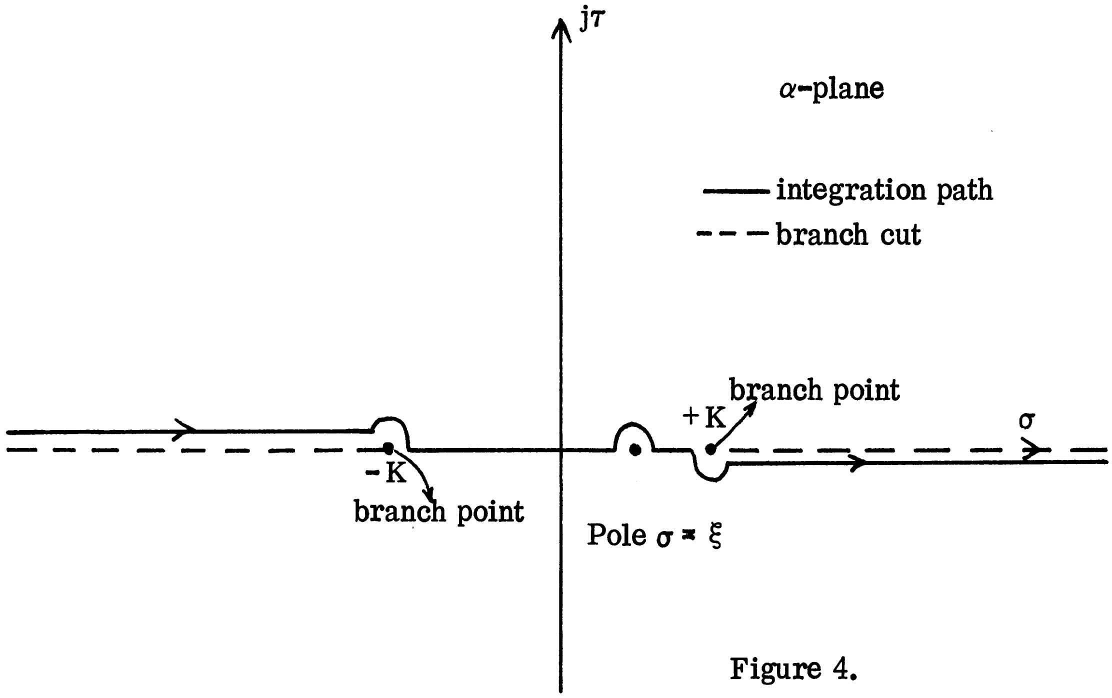

Figure 4.

By a different choice of branch cut and by suitable deformation of the basic integration path of Fig. 4 we can transform the contour integrals (32) and (33) into various kinds of definite integrals (with, possibly, principal values).

We shall examine three cases of surface impedances η_1, η_2 separately:

(a) The case where η_1, η_2 are purely imaginary,

(b) The case where η_1, η_2 are purely real,

(c) The case where η_1, η_2 are complex.

The reason for such separation is that each solution has its own peculiarity. For example, in case (a) we can get the exact solution, while in the other cases it is impossible.

In this case the plane $y = 0$ is a reactive plane with $\eta_1 = jX_1$ for $x > 0$, and $\eta_2 = jX_2$ for $x < 0$ where X_1 and X_2 are real, and we shall take them to be positive (inductive reactance). For this case we shall deform the path of integration in the upper plane. The path is shown in Fig. 5 and we have no pole on the contour.

With $\eta_1 = jX_1$ we have, for $K_1(\alpha)$:

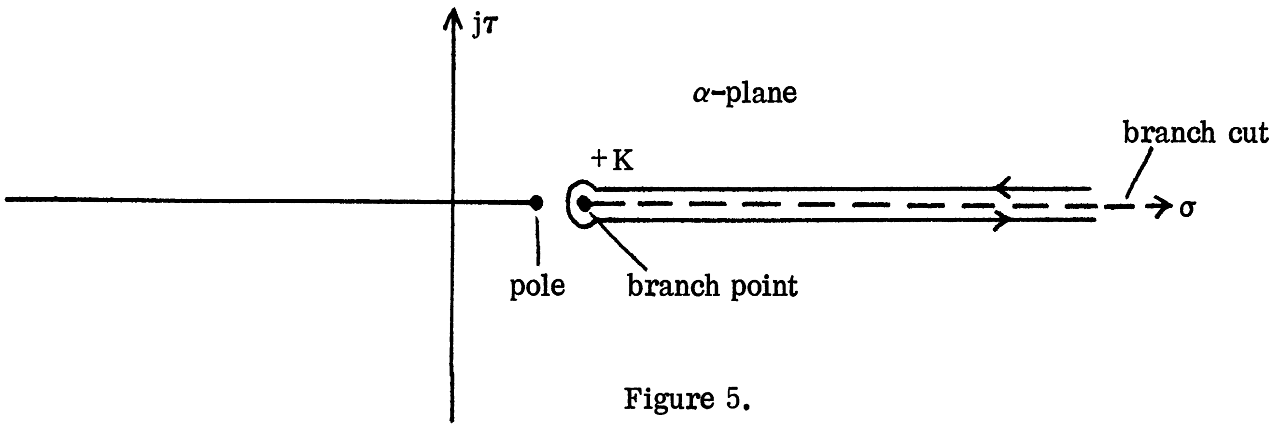

Figure 5.

$$K_1(\alpha) = 1 + \frac{KX_1}{\sqrt{\alpha^2 - K^2}} \quad . \tag{40}$$

Along the integration path, and with the cut as shown, the function $\sqrt{\alpha^2 - K^2}$ has the value:

$$\sqrt{\alpha^2 - K^2} = -\sqrt{\sigma^2 - K^2} \quad \text{on the upper side}$$
$$\sqrt{\alpha^2 - K^2} = +\sqrt{\sigma^2 - K^2} \quad \text{on the lower side}$$

with $\sigma > K$.

So according to Eq. (33):

$$F_1^-(\xi) = -\frac{1}{2\pi i}\int_{\infty}^{K} \ell n\left(1 - \frac{KX_1}{\sqrt{\sigma^2 - K^2}}\right)\frac{d\sigma}{\sigma - \xi} -$$

$$-\frac{1}{2\pi i}\int_{K}^{\infty} \ell n\left(1 + \frac{KX_1}{\sqrt{\sigma^2 - K^2}}\right)\frac{d\sigma}{\sigma - \xi} =$$

$$= \frac{1}{2\pi i}\int_{K}^{\infty} \ell n \frac{\sqrt{\sigma^2 - K^2} - KX_1}{\sqrt{\sigma^2 - K^2} + KX_1}\,\frac{d\sigma}{\sigma - \xi} \quad .$$

So we have:

$$\text{Real}(F_1^-(\xi)) = \frac{1}{2\pi i} I_m \int_K^{\infty} \ell n \frac{\sqrt{\sigma^2 - K^2} - KX_1}{\sqrt{\sigma^2 - K^2} + KX_1} \frac{d\sigma}{\sigma - \xi} = \frac{1}{2\pi i} I_m J_1 + \frac{1}{2\pi i} I_m J_2 \tag{41}$$

where:

$$J_1 = \int_K^{K\sqrt{1+\eta_1^2}} (\qquad) \, d\sigma$$

$$J_2 = \int_{K \; 1+\eta_1^2}^{\infty} (\qquad) \, d\sigma \quad .$$

The integrand in J_2 is always real because the factor under the log is always positive and we have:

$$I_m \gamma_2 \equiv 0 \quad .$$

In J_1, the factor under the log is always negative. We know that:

$$\log(-a) = \log a + j(2n+1)\pi \qquad (n = 0, 1, \ldots)$$

and we have, with $n = 0$ (the principal value of log):

$$I_m J_2 = j\pi \int_K^{K\sqrt{1+\eta_1^2}} \frac{d\sigma}{\sigma - \xi} = j\pi \log \frac{K\sqrt{1 - X_1^2} - \xi}{K - \xi} \quad .$$

Now Eq. (41) becomes:

$$\text{Real } F_1^-(\xi) = \frac{1}{2} \ell n \frac{K\sqrt{1 + X_1^2} - \xi}{K - \xi} \quad .$$

Now we have:

$$\text{Real } F(\theta,\theta_o) = \frac{1}{2}\ln \frac{\cos\theta_o+\sqrt{1+X_1^2}}{1+\cos\theta_o}\cdot\frac{\cos\theta+\sqrt{1+X_1^2}}{1+\cos\theta}\cdot\frac{1+\cos\theta_o}{\cos\theta_o+\sqrt{1+X_2^2}}\cdot\frac{1+\cos\theta}{\cos\theta+\sqrt{1+X_2^2}} =$$

$$= \frac{1}{2}\ln \frac{\cos\theta_o+\sqrt{1+X_1^2}}{\cos\theta_o+\sqrt{1+X_2^2}}\cdot\frac{\cos\theta+\sqrt{1+X_1^2}}{\cos\theta+\sqrt{1+X_2^2}} \quad . \qquad (42)$$

and from Eq. (39) and Eq. (42):

$$\left|P(\theta,\theta_o)\right|^2 = \frac{(X_1-X_2)^2\sin^2\theta_o}{\sin^2\theta_o+X_1^2}\cdot\frac{\cos\theta_o+\sqrt{1+X_1^2}}{\cos\theta_o+\sqrt{1+X_2^2}}\cdot\frac{\sin^2\theta}{(\cos\theta+\cos\theta_o)^2(\sin^2\theta+X_1^2)}\cdot\frac{\cos\theta+\sqrt{1+X_1^2}}{\cos\theta+\sqrt{1+X_2^2}} =$$

$$= \frac{(X_1-X_2)^2\sin^2\theta_o}{\sin^2\theta_o+X_1^2}\cdot\frac{\sqrt{1+X_1^2}+\cos\theta_o}{\sqrt{1+X_2^2}+\cos\theta_o}\cdot\frac{\sin^2\theta}{(\cos\theta+\cos\theta_o)^2(\sqrt{1+X_1^2}-\cos\theta)(\sqrt{1+X_2^2}+\cos\theta)}$$

Radar cross section is now:

$$\frac{\sigma}{\lambda} = \frac{2}{\pi}\left|P(\theta,\theta_o)\right|^2 = \frac{2}{\pi}\cdot\frac{(X_1-X_2)^2\sin^2\theta_o}{\sin^2\theta_o+X_1^2}\cdot\frac{\sqrt{1+X_1^2}+\cos\theta_o}{\sqrt{1+X_2^2}+\cos\theta_o}\cdot\frac{\sin^2\theta}{(\cos\theta+\cos\theta_o)^2(\sqrt{1+X_1^2}-\cos\theta)(\sqrt{1+X_2^2}+\cos\theta)} \qquad (43)$$

One case which is of special importance is when the right half plane is perfectly conducting. With $X_1 = 0$ and $X_2 = X$, we get:

$$\frac{\sigma}{\lambda} = \frac{4}{\pi}\cdot X^2\cdot\frac{1+\cos\theta_o}{\sqrt{1+X^2}+\cos\theta_o}\cdot\frac{\sin^2\frac{\theta}{2}}{(\cos\theta+\cos\theta_o)^2(\sqrt{1+X^2}+\cos\theta)} \quad . \qquad (44)$$

For monostatic cross section we can get from Eqs. (43) and (44) the following expressions:

$$\frac{\sigma}{\lambda} = \frac{1}{2\pi} \frac{(X_1-X_2)^2}{\sin^2\theta_o + X_1^2} \frac{\sqrt{1+X_1^2}+\cos\theta_o}{\sqrt{1+X_1^2}-\cos\theta_o} \frac{\sin^2\theta_o \cdot tg^2\theta_o}{(\sqrt{1+X_2^2}+\cos\theta_o)^2} \tag{45}$$

$$\frac{\sigma}{\lambda} = \frac{1}{2\pi} X^2 \frac{tg^2\theta_o}{tg^2\,\theta_o/2} \cdot \frac{1}{(\sqrt{1+X^2}+\cos\theta_o)^2} = \frac{2}{\pi} X^2 \frac{1}{1-tg^2\,\theta_o/2} \cdot \frac{1}{(\sqrt{1+X^2}+\cos\theta_o)^2} \,. \tag{46}$$

From the above equations one can easily notice the presence of the singularities in the solutions for $\theta = \pi - \theta_o$, and it has to be attributed to the specularly reflected wave from the right half plane, which we could not avoid. This was discussed at the beginning. However, the expressions (43)-(46) give us the quantitative idea of the non-specular reflection and the corresponding cross section if we are sufficiently far away from $\pi - \theta_o$ direction.

(The solutions for cases b and c have been prepared for this memo but the author discovered some errors in them, which will take some time to correct. A future memo will discuss these solutions.)

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