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TRANSIENTS ON LOSSY TRANSMISSION LINES

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ABSTRACT

The solution to the problem of transients on lossy transmission lines in the form of infinite series has been discussed and it has been shown that the works of Jeffreys (1927) and Kuznetsov (1947) are equivalent. The explicit solution for RL terminated line which had been obtained by Kuznetsov for special combination of R and L has been extended to any arbitrary combination of R and L. Also the GC terminated line has been discussed. An approximate solution for low-loss lines has been proposed and some numerical work has been done to compare the exact and approximate solutions.

Introduction

The problem of transients on lossy transmission lines has been discussed by several authors including H. Jeffreys (1927), N. Levinson (1934), S. Carslaw (1940), and P. Kuznetsov (1947). The solution obtained by Levinson and Carslaw are in integral form while those of Jeffreys and Kuznetsov are in the form of infinite series. The principal ideas behind the methods used by Jeffreys and Kuznetsov are the same. Jeffreys, on the other hand, has discussed an infinite line while Kuznetsov has formulated the problem for a general linear termination. In his solution, Kuznetsov has identified the infinite series involved as the Lommel functions of two imaginary independent arguments. As an example, he has obtained an explicit solution for an RL terminated line.

The approximate method we proposed here can not be justified mathematically but for low loss lines it gives a very good approximation to exact solution.

Comparison Between the Method Used by Jeffreys and the One Used by Kuznetsov

Consider the propagation of electromagnetic waves along a semi-infinite lossy line. The problem can be described by the following system of equations

$$- \frac{\partial v}{\partial x} = Ri + L \frac{\partial i}{\partial t}$$

$$- \frac{\partial i}{\partial x} = Gv + C \frac{\partial v}{\partial t}$$

subject to initial and boundary conditions,

$$v(x,0) = 0, \quad i(x,0) = 0, \quad v(0,t) = 1 \quad (t > 0)$$

where v(x,t) and i(x,t) denote the voltage and current in the wire; the constants R, L, G, C are the line parameters. Assuming that the second derivatives exist for v(x,t) and i(x,t) one can obtain the so called telegrapher's equation,

$$\frac{\partial^2 v}{\partial x^2} = LC \frac{\partial^2 v}{\partial t^2} + (RC + LG) \frac{\partial v}{\partial t} + RC\ v$$

Now if we make use of the Laplace transform, we obtain

$$\frac{d^2V}{dx^2} - \mu^2 V = 0$$

where

$$V(x,s) \triangleq \mathcal{L}\,[v(x,t)]$$

$$\mu = \frac{1}{c}\,[(s + 2\alpha)(s + 2\beta)]^{1/2}$$

$$c = \frac{1}{\sqrt{LC}}, \quad \alpha = \frac{R}{2L}, \quad \beta = \frac{G}{2C}$$

The same equation holds for the current. Using the boundary conditions we can find the solution in Laplace transform domain for a semi-infinite line

$$V(x,s) = \frac{e^{-\mu x}}{s}$$

Therefore the solution in time domain can be obtained by applying inverse Laplace transform

$$v(x,t) = \frac{1}{2\pi j}\int_{a-j\infty}^{a+j\infty} e^{\lambda t-\mu x}\,\frac{d\lambda}{\lambda} \qquad (1)$$

To obtain v(x,t) Kuznetsov uses the following transformation

$$u = (\frac{\lambda+2\alpha}{\lambda+2\beta})^{1/2}, \quad y = \frac{u-1}{u+1}, \quad y = \varepsilon w, \quad (\varepsilon = (\frac{\xi-t}{\xi+t})^{1/2}) \qquad (2)$$

Also instead of integrating along a straight line parallel to the imaginary axis, he uses a closed contour, γ, which encircles all singularities of the integrand. Moreover, since the integrand has two branch points $\lambda_1 = -2\alpha$, $\lambda_2 = -2\beta$, and a pole of order one at $\lambda = 0$, using the notation used by Watson v(x,t) can be written as follows:

$$v(x,t) = \frac{1}{2\pi j}\int^{(\infty-)} e^{\lambda t-\mu x}\,\frac{d\lambda}{\lambda}$$

which after applying the above transformation can be written in the form

$$v(x,t) = \frac{1}{2\pi j}\int^{(0-)} (-\frac{dw}{w} + \frac{dw}{w-w_1} + \frac{dw}{w-w_2})\exp[-\rho t + \frac{\sigma}{2}\sqrt{\xi^2-t^2}\ (w - \frac{1}{w})]$$

where

$$m = \sqrt{\alpha} + \sqrt{\beta},\quad n = \sqrt{\alpha} - \sqrt{\beta},\quad \rho = \alpha + \beta$$

$$\sigma = \alpha - \beta,\quad \xi = \frac{x}{c},\quad w_1 = \frac{n}{m\varepsilon},\ w_2 = \frac{m}{n\varepsilon}$$

It can be shown that these integrals can be expressed in terms of zeroth order modified Bessel function and Lommel functions of two independent imaginary argument, where the latter is defined as

$$Y_n(w,z) = \sum_{m=0}^{\infty} (\frac{w}{z})^{n+2m}\ I_{n+2m}(z)$$

with $I_n(z)$ to be modified Bessel function of order n. After carrying out the required manipulation the final expression obtained by Kuznetsov is given by

$$v(x,t) = e^{-\rho t}\{I_o(\sigma\sqrt{t^2-\xi^2}) + Y_1[m^2(t-\xi),\sigma\sqrt{t^2-\xi^2}] \qquad (3)$$

$$+ Y_2[m^2(t-\xi),\ \sigma\sqrt{t^2-\xi^2}] + Y_1[n^2(t-\xi),\sigma\sqrt{t^2-\xi^2}]$$

$$+ Y_2[n^2(t-\xi),\ \sigma\sqrt{t^2-\xi^2}]\}\, U(t-\xi)$$

Now we discuss the solution obtained by Jeffreys briefly and show that it is equivalent to (3). Let the path of integration in (1) as explained above be changed to closed path γ. Making the change of variable

$$\lambda = p - \rho$$

one finds

$$v(x,t) = \frac{e^{-\rho t}}{2\pi j}\int_{\gamma} \exp\left[pt - \frac{x}{c}\,(p^2 - \sigma^2)^{1/2}\right] \frac{dp}{p-\rho}$$

The integrand has $p = \pm\sigma$ as branch points and $p = \rho$ as a pole of order one.

Let $$p^2 - \sigma^2 = (2\nu - p)^2$$

in other words we make the transformation

$$p = \nu + \frac{\sigma^2}{4\nu}$$

to transform the branch points into poles. By doing so one can show that

$$v(x,t) = \frac{e^{-\sigma t}}{2\pi j}\int_{\gamma} e^{\nu t_1 + \frac{\sigma^2}{4\nu} t_2} \left(1 + \frac{\nu_1}{\nu - \nu_2} + \frac{\nu_2}{\nu - \nu_2}\right) \frac{d\nu}{\nu}$$

where

$$t_1 = t - \frac{x}{c}, \qquad t_2 = t + \frac{x}{c}$$

ν_1 and ν_2 are the roots of the quadratic

equation $4\nu^2 - 4\rho\nu + \sigma^2 = 0$

Applying the expansion

$$e^{1/2\, z(\lambda + \frac{1}{\lambda})} = \sum_{n=-\infty}^{\infty} \lambda^n I_n(z)$$

it can be shown that

$$v(x,t) = e^{-\rho t}\left\{I_o(\sigma\sqrt{t_1 t_2}) + \sum_{k=1}^{\infty}\left(\frac{2}{\sigma}\sqrt{\frac{t_1}{t_2}}\right)^k (\nu_1^k + \nu_2^k)\, I_k(\sigma\sqrt{t_1 t_2})\right\} \qquad t_1 > 0$$

$$= 0 \qquad \text{otherwise} \qquad (4)$$

This was the expression obtained by Jeffreys. Now it can be shown that this is in fact an alternative representation of (3).

Let

$$\xi = \frac{x}{c}$$

$$2\nu_1 = \rho + \sqrt{\rho^2 - \sigma^2} = (\sqrt{\alpha} + \sqrt{\beta})^2 = m^2$$

$$2\nu_2 = \rho - \sqrt{\rho^2 - \sigma^2} = (\sqrt{\alpha} - \sqrt{\beta})^2 = n^2$$

and substituting these in (4) we obtain

$$v(x,t) = e^{-\rho t}\{I_o(\sigma\sqrt{t^2-\xi^2}) + \sum_{k=1}^{\infty} (\frac{m^2}{\sigma}\sqrt{\frac{t-\xi}{t+\xi}})^k I_k(\sigma\sqrt{t^2-\xi^2})$$

$$+ \sum_{k=1}^{\infty} (\frac{n^2}{\sigma}\sqrt{\frac{t-\xi}{t+\xi}})^k I_k(\sigma\sqrt{t^2-\xi^2})\}$$

we can write

$$\sum_{k=1}^{\infty} (\frac{m^2}{\sigma}\sqrt{\frac{t-\xi}{t+\xi}})^k I_k(\sigma\sqrt{t^2-\xi^2}) = \sum_{i=0}^{\infty} (\frac{m^2}{\sigma}\sqrt{\frac{t-\xi}{t+\xi}})^{2i+1} I_{2i+1}(\sigma\sqrt{t^2-\xi^2})$$

$$+ \sum_{i=0}^{\infty} (\frac{m^2}{\sigma}\sqrt{\frac{t-\xi}{t+\xi}})^{2i+2} I_{2i+2}(\sigma\sqrt{t^2-\xi^2})$$

Hence

$$\sum_{k=1}^{\infty} (\frac{m^2}{\sigma}\sqrt{\frac{t-\xi}{t+\xi}})^k I_k(\sigma\sqrt{t^2-\xi^2}) = Y_1[m^2(t-\xi), \sigma\sqrt{t^2-\xi^2}] + Y_2[m^2(t-\xi), \sigma\sqrt{t^2-\xi^2}]$$

and as a result we obtain

$$v(x,t) = e^{-\rho t}\{I_o(\sigma\sqrt{t^2-\xi^2}) + Y_1[m^2(t-\xi), \sigma\sqrt{t^2-\xi^2}] + Y_2[m^2(t-\xi), \sigma\sqrt{t^2-\xi^2}]$$

$$+ Y_1[n^2(t-\xi), \sigma\sqrt{t^2-\xi^2}] + Y_2[n^2(t-\xi), \sigma\sqrt{t^2-\xi^2}]$$

which is nothing but (3).

It is interesting to note that even the transformation used by Kuznetsov is of the same nature of the one introduced by Jeffreys. To show this it is sufficient to eliminate u in (2), i.e.

$$u = \sqrt{\frac{\lambda + 2\alpha}{\lambda + 2\beta}}$$

$$y = \frac{u - 1}{u + 1}$$

therefore

$$\frac{\lambda + 2\alpha}{\lambda + 2\beta} = \frac{(1 + y)^2}{(1 - y)^2}$$

or

$$2y\lambda = \sigma(1 + y)^2 - 2\rho y$$

If we now let $\lambda = p - \rho$, we obtain

$$p = \frac{\sigma}{2} (y + \frac{1}{y})$$

Making another change of variable, $y = \frac{2}{\sigma} \nu$ one finds

$$p = \nu + \frac{\sigma^2}{4\nu}$$

which is the same as the transformation used by Jeffreys.

It should, however, be mentioned that Kuznetsov has treated the problem in a more systematic and more general way. In addition, he has formulated the problem of a lossy line with an arbitrary (linear) termination. In the next section we derive the solution for RL and GC terminations with a slightly different notation from Kuznetsov's original treatment. In particular we introduce some normalized parameters which prove to be more convenient specially for graphical representation of the solution.

Exact Solution for the Problem of Transients on a Lossy Transmission Line

Here we discuss briefly the case when the line is terminated by the series RL and parallel GC networks. As was mentioned earlier, the voltage and current in the line satisfy the following system of equations

$$- \frac{\partial v}{\partial x} = Ri + L \frac{\partial v}{\partial t}$$

$$- \frac{\partial i}{\partial x} = G v + C \frac{\partial v}{\partial t} \qquad (5)$$

Let $x = \xi l$, $t = \tau \frac{l}{c}$

where

l is the length of the line.

τ is the normalized time.

ξ is the normalized distance measured from the generator.

$c = \frac{1}{\sqrt{LC}}$

We can write the following equations by using the chain rule of differentiation

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} = \frac{1}{l} \frac{\partial v}{\partial \xi}$$

$$\frac{\partial i}{\partial x} = \frac{\partial i}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} = \frac{1}{l} \frac{\partial i}{\partial \xi}$$

$$\frac{\partial v}{\partial t} = \frac{\partial v}{\partial \tau} \cdot \frac{\partial \tau}{\partial t} = \frac{c}{l} \frac{\partial v}{\partial \tau}$$

$$\frac{\partial i}{\partial t} = \frac{\partial i}{\partial \tau} \cdot \frac{\partial \tau}{\partial t} = \frac{c}{l} \frac{\partial i}{\partial \tau}$$

If we substitute these in (5) the following results are obtained

$$\begin{cases} - \frac{\partial v}{\partial \xi} = Rl\ i + cL \frac{\partial i}{\partial \tau} \\ - \frac{\partial i}{\partial \xi} = Gl\ v + cC \frac{\partial v}{\partial \tau} \end{cases} \qquad (6)$$

By applying Laplace transform (better say normalized Laplace transform since we are using the normalized time) to (6), and assuming zero initial conditions one finds

$$\begin{cases} - \frac{dV}{d\xi} = (Rl + cLs)I \\ - \frac{dI}{d\xi} = (Gl + cCs)V \end{cases} \qquad (7)$$

where $V(\xi,s) = \mathcal{L}[v(\xi,\tau)]$

$I(\xi,s) = \mathcal{L}[i(\xi,\tau)]$

and from (7) the following equations for V and I can be obtained

$$\begin{cases} \dfrac{d^2V}{d\xi^2} - \mu^2 V = 0 \\ \\ \dfrac{d^2I}{d\xi^2} - \mu^2 I = 0 \end{cases} \tag{8}$$

where $\mu = [(s + 2\alpha)(s + 2\beta)]^{1/2}$ (principal branch)

$\alpha = \frac{Rl}{2z_o}, \quad \beta = \frac{Gl}{2Y_o}$

$z_o = Y_o^{-1} = \sqrt{\frac{L}{C}}$

The solution to (8) can be obtained and it can be written in the following form

$$V(\xi,s) = A(s)\ e^{-\mu\xi} + B(s)\ e^{\mu\xi}$$

$$I(\xi,s) = \frac{1}{z_c(s)} [A(s)\ e^{-\mu\xi} - B(s)\ e^{\mu\xi}]$$

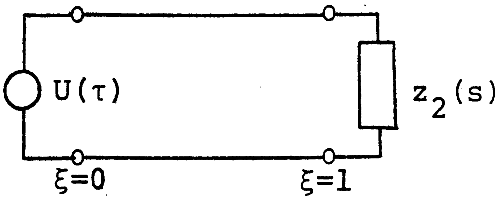
where A and B can be found from the boundary conditions

$$z_c(s) = z_o\ [(s + 2\alpha)/(s + 2\beta)]^{1/2} \quad \text{(principal branch)}$$

Without loss of generality we assume that the generator is a unit step source with zero internal impedance. Let the termination (load) be $z_2(s)$. Then the boundary conditions in s (Laplace transform variable) domain are

$$V(0,s) = \frac{1}{s}$$

$$V(1,s) = z_2(s)I(1,s)$$



Using these boundary conditions, A and B can be determined and they are

$$A(s) = \frac{1}{s[1+\Gamma(s)e^{-2\mu}]}$$

$$B(s) = \frac{\Gamma(s)e^{-2\mu}}{s[1+\Gamma(s)e^{-2\mu}]}$$

where $\Gamma(s) = \frac{Z_2(s) - Z_c(s)}{Z_2(s) + Z_c(s)}$. As a result the expressions for V and I are

$$V(\xi,s) = \frac{e^{-\mu\xi} + \Gamma(s)e^{-\mu(2-\xi)}}{s[1 + \Gamma(s)e^{-2\mu}]}$$

$$I(\xi,s) = \frac{1}{Z_c}\frac{e^{-\mu\xi} - \Gamma(s)e^{-\mu(2-\xi)}}{s[1 + \Gamma(s)e^{-2\mu}]}$$

In order to evaluate the inverse Laplace transform of V and I, we use the following representation

$$\frac{1}{1 + \Gamma(s)e^{-2\mu}} = \sum_{i=0}^{N} (-\Gamma e^{-2\mu})^i + \text{remainder}$$

$$= 1 - \Gamma e^{-2\mu} + \Gamma^2 e^{-4\mu} - \ldots$$

substituting this in V and I, we obtain

$$V(\xi,s) = \frac{e^{-\mu\xi}}{s} + \frac{\Gamma}{s} e^{-\mu(2-\xi)} - \frac{\Gamma}{s} e^{-\mu(2+\xi)} - \ldots$$

$$I(\xi,s) = \frac{e^{-\mu\xi}}{sZ_c} - \frac{\Gamma}{sZ_c} e^{-\mu(2-\xi)} - \frac{\Gamma}{sZ_c} e^{-\mu(2+\xi)} + \ldots$$

$v(\xi,\tau)$ and $i(\xi,\tau)$ can now be found by applying the inversion transform

$$v(\xi,\tau) = v_0(\xi,\tau) + v_1(2-\xi,\tau) - v_2(2+\xi,\tau) - \ldots$$

$$i(\xi,\tau) = i_0(\xi,\tau) - i_1(2-\xi,\tau) - i_2(2-\xi,\tau) + \ldots$$

where

$$v_0(\xi,\tau) = \frac{1}{2\pi j}\int_\gamma e^{s\tau-\mu\xi} \frac{ds}{s}$$

$$v_1(2-\xi,\tau) = \frac{1}{2\pi j}\int_\gamma \Gamma e^{s\tau-\mu(2-\xi)} \frac{ds}{s}$$

$$\vdots$$

$$i_0(\xi,\tau) = \frac{1}{2\pi j}\int_\gamma \frac{1}{Z_c} e^{s\tau-\mu\xi} \frac{ds}{s}$$

$$i_1(2-\xi,\tau) = \frac{1}{2\pi j}\int_\gamma \frac{\Gamma}{Z_c} e^{s\tau-\mu(2-\xi)} \frac{ds}{s}$$

$$\vdots$$

It is clear from the expressions for v_0 and i_0 that they do not depend on the termination and physically they correspond to the direct waves. As we showed briefly earlier v_0 and i_0 can be found to be

$$v_0(\xi,\tau) = e^{-\rho\tau}[I_0(\zeta_0) + Y_1(\eta_{10},\zeta_0) + Y_2(\eta_{10},\zeta_0) + Y_1(\eta_{20},\zeta_0) + Y_2(\eta_{20},\zeta_0)]U(\tau-\xi)$$

$$z_0 i_0(\xi,\tau) = \sqrt{\frac{\beta}{\alpha}}\, e^{-\rho\tau} \left[\sqrt{\frac{\alpha}{\beta}}\, I_0(\zeta_0) + Y_1(\eta_{10},\zeta_0) + Y_2(\eta_{10},\zeta_0) - Y_1(\eta_{20},\zeta_0) - Y_2(\eta_{20},\zeta_0)\right]U(\tau-\xi)$$

where

$$\zeta_0 = \sigma\sqrt{\tau^2 - \xi^2}$$

$$\eta_{10} = m^2(\tau-\xi), \qquad \eta_{20} = n^2(\tau-\xi)$$

$$m = \sqrt{\alpha} + \sqrt{\beta}, \qquad n = \sqrt{\alpha} - \sqrt{\beta}$$

$$\rho = \alpha + \beta, \qquad \sigma = \alpha - \beta$$

Series RL Terminated Line

To obtain $v(\xi,\tau)$ and $i(\xi,\tau)$ along the line we need to know v_1, v_2,... and i_1, i_2,... in addition to v_o and i_0 which has already been obtained. Here we only find v_1, v_2, i_1, and i_2. The rest can be obtaine similarily. In fact for a moderately lossy line higher order reflected waves do not contribute much in the numerical results and by neglecting them we do not commit an appreciable error. Furthermore it is sufficient to find $v_1(2-\xi,\tau)$ and $i_1(2-\xi,\tau)$ only, because v_2 and i_2 can be obtained by substituting (ξ) for $(-\ \xi)$ in the expressions for v_1 and i_1.

As we showed earlier

$$v_1(2-\xi,\tau) = \frac{1}{2\pi j}\int_{\gamma} \Gamma(s) e^{s\tau-\mu(2-\xi)} \frac{ds}{s}$$

$$i_1(2-\xi,\tau) = \frac{1}{2\pi j}\int_{\gamma} \frac{\Gamma(s)}{Z_c(s)} e^{s\tau-\mu(2-\xi)} \frac{ds}{s}$$

where

$$\Gamma(s) = \frac{Z_2(s) - Z_c(s)}{Z_2(s) + Z_c(s)}$$

$$Z_2(s) = R_2 + s \frac{c}{l} L_2$$

Using the transformations

$$u = \sqrt{\frac{s + 2\alpha}{s + 2\beta}}, \qquad \gamma = \frac{u - 1}{u + 1}$$

one finds that

$$v_1(2-\xi,\tau) = \frac{1}{2\pi j}\int^{(0-)} [- \frac{1}{\gamma} + \frac{1}{\gamma-\gamma_1} + \frac{1}{\gamma-\gamma_2} + \frac{a_o(\gamma^2-1)(1+\gamma)}{(\gamma-\gamma_1)(\gamma-\gamma_2)(\gamma^3+a_1\gamma^2+a_2\gamma-1)}]$$

$$.\exp[-\rho\tau + \frac{\sigma}{2}\gamma(\tau+2-\xi) + \frac{\sigma}{2}\frac{1}{\gamma}(\tau-2+\xi)]d\gamma$$

where

$$a_o = \frac{4}{h\sigma}$$

$$a_1 = \frac{2}{h\sigma}(r-1) - (\frac{2\rho}{\sigma}+1)$$

$$a_2 = -\frac{2}{h\sigma}(r+1) + (\frac{2\rho}{\sigma}+1)$$

$$r = \frac{R_2}{Z_0}$$

$$h = \frac{L_2}{L\ell}$$

Now let

$$\Delta = (\frac{2a_1^3}{27} - \frac{a_1a_2}{3} - 1)^2 + 4(\frac{a_2}{3} - \frac{a_1^2}{9})^3$$

be the discreminant of the 3rd order algebraic equation

$$\gamma^3 + a_1\gamma^2 + a_2\gamma - 1 = 0$$

with γ_3, γ_4, and γ_5 as its roots. Two cases exist:

(i) $\Delta \leq 0$, all roots are real.

(ii) $\Delta > 0$, one root is real and the other two are complex.

Kuznetsov has discussed only the first case. We have investigated the second case as well. We will give the solution in case (ii) and also its numerical results.

Since a_1 and a_2 are related to the line parameters and termination, i.e. R_2 and L_2, limiting ourselves to case (i) puts some limitations on R_2 and L_2 which is not desirable.

In case (i) let

$$\gamma = \varepsilon w, \qquad \varepsilon = \sqrt{\frac{2-\xi-\tau}{2-\xi+\tau}}$$

then

$$v_1(2-\xi,\tau) = \frac{1}{2\pi j}\int^{(0-)}[-\frac{dw}{w} + \frac{dw}{w-w_1} + \frac{dw}{w-w_2} + \sum_{k=1}^{5}\frac{b_k dw}{w-w_k}] \tag{10}$$

$$.\exp[-\rho\tau + \frac{\sigma}{2}\sqrt{(2-\xi)^2-\tau^2}(w-\frac{1}{w})]$$

where $b_k = \frac{a_0(\gamma_k^2-1)(1+\gamma_k)}{u_k}$, $u_k = \prod_{j=1}^{j=5} (\gamma_k-\gamma_j) \quad k \neq j$

Finally $v_1(2-\xi,\tau)$ can be obtained which reads

$$v_1(s-\xi,\tau) = e^{-\rho\tau}[I_0(\zeta_1) + Y_1(\eta_{11},\zeta_1) + Y_2(\eta_{11},\zeta_1) + Y_1(\eta_{21},\zeta_1) + Y_2(\eta_{21},\zeta_1)$$

$$+ \sum_{k=1}^{k=5} b_k[Y_1(\eta_{k1},\zeta_1) + Y_2(\eta_{k1},\zeta_1)] \, U(\tau-2+\xi)$$

where $\eta_{k1} = \frac{\sigma}{\gamma_k}(\tau-2+\xi)$

$$\zeta_1 = \sigma\sqrt{\tau^2 - (2-\xi)^2}$$

For case (ii) let γ_3 be the real root and γ_4 and γ_5 the complex roots. Since a_1 and a_2 are real we have

$$\gamma_4 = \gamma_5^* = \gamma_r + j\gamma_i$$

where * stands for conjugate.

γ_r and γ_i are real numbers.

$$j = \sqrt{-1}$$

Then it can be shown that (10) still holds except that b_k's are defined as follows

$$\left.\begin{aligned} b_k &= \frac{a_0(\gamma_k^2-1)(1+\gamma_k)}{d_k} \\ \text{where} \quad d_k &= (\gamma_k^2 - 2\gamma_r\gamma_k + \gamma_r^2 + \gamma_i^2)\prod_{m=1}^{3}(\gamma_k - \gamma_m) \end{aligned}\right\} \quad \begin{aligned} &k = 1, 2, 3 \\ &k \neq m \end{aligned}$$

and $b_4 = b_5^* = b_r + jb_i$

$$b_r = \mathrm{Re}[\frac{a_0(\gamma_4^2-1)(1+\gamma_4)}{u_4}]$$

$$b_i = I_m[\frac{a_0(\gamma_4^2-1)(1+\gamma_4)}{u_4}]$$

where

$$u_4 = \prod_{m=1}^{m=5}(\gamma_4-\gamma_m), \quad m \neq 4$$

Also let

$$\eta_{k1} = \frac{\sigma}{\gamma_{k1}}(\tau-2+\xi), \quad k = 1, 2, 3$$

$$\eta_{41} = \eta_{51}^{*}(\tau-2+\xi) = \eta_c e^{j\theta}$$

Then it can be shown that

$$v_1(2-\xi,\tau) = e^{-\rho\tau}\{I_0(\zeta_1) + Y_1(\eta_{11},\zeta_1) + Y_2(\eta_{11},\zeta_1) + Y_1(\eta_{21},\zeta_1) + Y_2(\eta_{21},\zeta_1) + \sum_{k=1}^{k=3} b_k[Y_1(\eta_{k1},\zeta_1) + Y_2(\eta_{k1},\zeta_1) + 2[b_rY_r(\eta_c,\theta,\zeta_1) + b_iY_i(\eta_c,\theta,\zeta_1)]\}U(\tau-2+\xi)$$

where

$$Y_r(\eta_c,\theta,\zeta_1) = \sum_{n=1}^{\infty}\eta_c^n \cos n\theta \; I_n(\zeta_1)$$

$$Y_i(\eta_c,\theta,\zeta_1) = \sum_{n=1}^{\infty}\eta_c^n \sin n\theta \; I_n(\zeta_1)$$

$i_1(2-\xi,\tau)$ can be found in a similar way. The results are as follows:

For $\Delta < 0$

$$z_0i_1(2-\xi,\tau) = \sqrt{\frac{\beta}{\alpha}}\, e^{-\rho\tau}\{\sqrt{\frac{\alpha}{\beta}}\, I_0(\zeta_1) + Y_1(\eta_{11},\zeta_1) + Y_2(\eta_{11},\zeta_1) - Y_1(\eta_{21},\zeta_1) - Y_2(\eta_{21},\zeta_1)$$

$$+ \sum_{k=1}^{k=3} c_k [Y_1(\eta_{k1},\zeta_1) + Y_2(\eta_{k1},\zeta_1)]$$

$$+ 2[c_r Y_r(\eta_c,\theta,\zeta_1) + c_i Y_i(\eta_c,\theta,\zeta_1)]\}$$

$$.U(\tau-2+\xi)$$

where

$$c_r = \mathrm{Re}[\sqrt{\frac{\alpha}{\beta}} \frac{a_0}{u_4} (\gamma_4^2 - 1)(1 - \gamma_4)]$$

$$c_i = \mathrm{Im}[\sqrt{\frac{\alpha}{\beta}} \frac{a_0}{u_4} (\gamma_4^2 - 1)(1 - \gamma_4)]$$

$$c_k = \sqrt{\frac{\alpha}{\beta}} \frac{a_0}{d_k} (\gamma_k^2 - 1)(1 - \gamma_k), \qquad k = 1, 2, 3$$

d_k and u_4 are as defined previously.

Parallel GC Terminated Line

The solution in this case is quite similar to that of RL termination. Again depends on the discreminant of the following 3rd order algebraic equation we distinguish two different cases.

$$\gamma^3 + a_1\gamma^2 + a_2\gamma + 1 = 0 \qquad (11)$$

where a_1 and a_2 are different from before and they are given by

$$a_1 = \frac{2}{k\sigma}(g-1) + 1 - \frac{2\rho}{\sigma}$$

$$a_2 = \frac{2}{k\sigma}(g+1) + 1 - \frac{2\rho}{\sigma}$$

$$a_0 = \frac{4}{k\sigma}$$

$$g = \frac{G_2}{Y_0}$$

$$k = \frac{C_2}{lC}$$

As before we define the discriminant as follows

$$\Delta = (1 - \frac{a_1a_2}{3} + \frac{2}{27} a_1^3)^2 + 4(\frac{a_2}{3} - \frac{a_1^2}{a})^3$$

(i) $\Delta \overset{-}{<} 0$: all three roots of (11), i.e. γ_3, γ_4, γ_5 are real. Then it can be shown that

$$v_1(2-\xi,\tau) = -e^{-\rho\tau}\{I_0(\zeta_1) + Y_1(\eta_{11},\zeta_1) + Y_2(\eta_{11},\zeta_1) + Y_1(\eta_{21},\zeta_1) + Y_2(\eta_{21},\zeta_1)$$

$$+ \sum_{k=1}^{k=5} b_k[Y_1(\eta_{k1},\zeta_1) + Y_2(\eta_{k1},\zeta_1)]U(\tau-2+\xi)$$

$$z_oi_1(2-\xi,\tau) = -\sqrt{\frac{\beta}{\alpha}}\, e^{-\rho\tau}\{\sqrt{\frac{\alpha}{\beta}}\, I_0(\zeta_1) + Y_1(\eta_{11},\zeta_1) + Y_2(\eta_{11},\zeta_1) - Y_1(\eta_{21},\zeta_1) - Y_2(\eta_{21},\zeta_1)$$

$$+ \sum_{k=1}^{k=5} c_k[Y_1(\eta_{k1},\zeta_1) + Y_2(\eta_{k1},\zeta_1)]\}U(\tau-2+\xi)$$

where

$$b_k = \frac{a_0}{u_k}(\gamma_k^2 - 1)(\gamma_k - 1)$$

$$c_k = \sqrt{\frac{\alpha}{\beta}}\, \frac{a_0}{u_k}(1 - \gamma_k)^3$$

$$u_k = \prod_{i=1}^{i=5}(\gamma_k - \gamma_i), \quad k \neq i$$

$$\eta_{k1} = \frac{\sigma}{\gamma_k}(\tau-2+\xi)$$

$$\zeta_1 = \sigma\sqrt{\tau^2-(2-\xi)^2}$$

$k = 1$ to 5

(ii) $\Delta > 0$: (11) has one real root which we call γ_3 and two complex roots denoted by γ_4 and γ_5. Since a_1 and a_2 are real it is obvious that

$$\gamma_4 = \gamma_5^* = \gamma_r + j\gamma_i$$

The solution for $v_1(2-\xi,\tau)$ and $i_1(2-\xi,\tau)$ are

$$\begin{aligned} v_1(2-\xi,\tau) = -e^{-\rho\tau}\{ & I_0(\zeta_1) + Y_1(\eta_{11},\zeta_1) + Y_2(\eta_{11},\zeta_1) \\ & + Y_1(\eta_{21},\zeta_1) + Y_2(\eta_{21},\zeta_1) \\ & + \sum_{k=1}^{k=1} b_k[Y_1(\eta_{k1},\zeta_1) + Y_2(\eta_{k1},\zeta_1)] \\ & +2[b_r Y_r(\eta_c,\theta,\zeta_1) + b_i Y_i(\eta_c,\theta,\zeta_1)]]\}U(\tau-2+\xi) \end{aligned}$$

$$\begin{aligned} i(2-\xi,\tau) = -\sqrt{\frac{\beta}{\alpha}}\, e^{-\rho\tau}[& \sqrt{\frac{\alpha}{\beta}}\, I_0(\zeta_1) + Y_1(\eta_{11},\zeta_1) + Y_2(\eta_{11},\zeta_1) \\ & - Y_1(\eta_{21},\zeta_1) - Y_2(\eta_{21},\zeta_1) \\ & + \sum_{k=1}^{k=3} c_k[Y_1(\eta_{k1},\zeta_1) + Y_2(\eta_{k1},\zeta_1)] \\ & + 2[c_r Y_r(\eta_c,\theta,\zeta_1) + c_i(\eta_c,\theta,\zeta_1)]]\}U(\tau-2+\xi) \end{aligned}$$

where

$$b_k = \frac{a_0}{d_k}(\gamma_k^2 - 1)(\gamma_k - 1)$$

$$k = 1,\ 2,\ 3$$

$$c_k = \sqrt{\frac{\alpha}{\beta}}\,\frac{a_0}{d_k}(1 - \gamma_k)^3$$

$$d_k = (\gamma_k^2 - 2\gamma_r\gamma_k + \gamma_r^2 + \gamma_i^2)\prod_{m=1}^{m=3}(\gamma_k - \gamma_m), \quad k \neq m$$

$$b_r = \mathrm{Re}[\frac{a_0}{u_4}(\gamma_4^2 - 1)(\gamma_4 - 1)]$$

$$b_i = \mathrm{Im}[\frac{a_0}{u_4}(\gamma_4^2 - 1)(\gamma_4 - 1)]$$

$$c_r = \mathrm{Re}[\sqrt{\frac{\alpha}{\beta}}\, \frac{a_0}{u_4}\, (1 - \gamma_k)^3]$$

$$c_i = \mathrm{Im}[\sqrt{\frac{\alpha}{\beta}}\, \frac{a_0}{u_k}\, (1 - \gamma_k)^3]$$

$$u_4 = \prod_{m=1}^{m=5} (\gamma_4 - \gamma_m), \qquad m \neq 4$$

Approximate Solution for the Problem of Transients on a Lossy Transmission Line

Recall $V(\xi,s)$ and $I(\xi,s)$ from the previous section

$$\begin{aligned} V(\xi,s) &= \frac{e^{-\mu\xi}}{s} + \frac{\Gamma}{s}\, e^{-\mu(2-\xi)} - \frac{\Gamma}{s}\, e^{-\ (2+\xi)} - \dots \\ I(\xi,s) &= \frac{e^{-\mu\xi}}{sz_c} - \frac{\Gamma}{sz_c}\, e^{-\mu(2-\xi)} - \frac{\Gamma}{sz_c}\, e^{-\mu(2+\xi)} + \dots \end{aligned} \qquad (12)$$

Assuming that the losses of the line (R and G) are small we want to try some approximations on μ and z_c so that $v(\xi,\tau)$ and $i(\tau,\xi)$ can be found from (12) by simply using the tables of Laplace transform without applying the inversion theorem. It should be noted that we cannot justify this approach rigorously since we are applying perturbation in Laplace transform domain which could lead to errors. However, heuristically this approach is not unreasonable and we will show numerically that as for low-loss lines the solution obtained by the approximate method will approach the exact one. In fact, when the losses go to zero the present method (approximate) is exact.

Series RL Terminated Line

If R and G are small then α and β are small and $\alpha\beta$ is negligible. Using normalized parameters we can write

$$\mu = \sqrt{(s+2\alpha)(s+2\beta)} \simeq \sqrt{s^2+2(\alpha+\beta)s} \simeq s(1+\frac{\alpha+\beta}{s}) = s+\rho$$

Also $$Z_c = Z_0\sqrt{\frac{s+2\alpha}{s+2\beta}} \simeq Z_0(1+\frac{\alpha-\beta}{s}) = Z_0(1+\frac{\sigma}{s})$$

where α, β, ρ, and σ are as introduced before.

Since we are considering the direct wave and two reflected waves only, it is desirable to write $\frac{\Gamma}{s}$ and $\frac{\Gamma}{sZ_c}$ (12) in the following forms

$$\frac{\Gamma}{s} \simeq \frac{1}{s}\,\frac{Z_L - Z_c}{Z_L + Z_c} = \frac{1}{s}\,\frac{s^2 + \frac{r-1}{h}s - \frac{\sigma}{h}}{s^2 + \frac{r+1}{h}s + \frac{\sigma}{h}} = -\frac{1}{s} + \frac{1+a}{s-s_1} + \frac{1-a}{s-s_2}$$

$$\frac{\Gamma}{sZ_c} \simeq \frac{1}{Z_0(s+\sigma)}\,\frac{s^2 + \frac{r-1}{h}s - \frac{\sigma}{h}}{s^2 + \frac{r+1}{h}s + \frac{\sigma}{h}} = \frac{1}{Z_0}[\frac{1}{s+\sigma} + \frac{1+a}{1+hs_2}\,\frac{1}{s-s_{.}} + \frac{1-a}{1+hs_1}]$$

where r, h, and σ were defined previously.

$$s_{1,2} = \frac{r+1}{2h} \pm \sqrt{(\frac{r+1}{2h})^2 - \frac{\sigma}{h}}\,,$$

$$a = \frac{r-1}{h}\,\frac{1}{s_1-s_2}$$

Therefore V(ξ,s) and I(ξ,s) read

$$V(\xi,s) \simeq e^{-\rho\xi}\,\frac{e^{-s\xi}}{s} + e^{-\rho(2-\xi)}[-\frac{1}{s} + \frac{1+a}{s-s_1} + \frac{1-a}{s-s_2}]\,e^{-s(2-\xi)}$$

$$- e^{-\rho(2+\xi)}[-\frac{1}{s} + \frac{1-a}{s-s_1} + \frac{1-a}{s-s_2}]\,e^{-s(2+\xi)} - \ldots$$

$$Z_0I(\xi,s) \simeq e^{-\rho\xi}\,\frac{e^{-s\xi}}{s+\sigma} - e^{-\rho(2-\xi)}[\frac{1}{s+\sigma} + \frac{1+a}{1+hs_2}\,\frac{1}{s-s_1} + \frac{1-a}{1+hs_1}\,\frac{1}{s-s_2}]\,e^{-s(2-\xi)}$$

$$- e^{-\rho(2+\xi)}[\frac{1}{s+\sigma} + \frac{1+a}{1+hs_2}\,\frac{1}{s-s_1} + \frac{1-a}{1+hs_1}\,\frac{1}{s-s_1}]\,e^{-s(2+\xi)} + \ldots$$

Now v(ξ,τ) and i(ξ,τ) can readily be found and they are given by

$$v(\xi,\tau) = e^{-\rho\xi}U(\tau-\xi) + e^{-\rho(2-\xi)}[-1+(1+a)e^{s_1(\tau-2+\xi)} + (1-a)e^{s_2(\tau-2+\xi)}]$$

$$U(\tau-2+\xi)$$

$$-e^{-\rho(2+\xi)}[-1+(1+a)e^{s_1(\tau-2-\xi)}+(1-a)e^{s_2(\tau-2-\xi)}]U(\tau-2-\xi)-\ldots$$

$$Z_0 i(\xi,\tau) = e^{-2\beta\xi-\sigma\tau}U(\tau-\xi) - e^{-\rho(2-\xi)}[e^{-\sigma(\tau-2+\xi)}+\frac{1+a}{1+hs_2}e^{s_1(\tau-2+\xi)}$$

$$+\frac{1-a}{1+hs_1}e^{s_2(\tau-2+\xi)}]U(\tau-2+\xi)-e^{-\ (2+\xi)}[e^{-\rho(\tau-2-\xi)}+\frac{1+a}{1+hs_2}e^{s_1(\tau-2-\xi)}$$

$$+\frac{1-a}{1+hs_1}e^{s_2(\tau-2-\xi)}]U(\tau-2-\xi)$$

$$+\ldots$$

Parallel GC Terminated Line

This case is quite similar to the previous one. Here we have

$$Y_c = Y_0\sqrt{\frac{s+2\beta}{s+2\alpha}} \simeq Y_0(1-\frac{\sigma}{s})$$

Then

$$\frac{\Gamma}{s} = \frac{1}{s} - \frac{1+a_1}{s-s_1} - \frac{1-a_1}{s-s_2}$$

$$\frac{\Gamma}{sZ_c} \simeq -\frac{1}{Z_0(s+\sigma)}\frac{s^2+\frac{g-1}{k}s+\frac{\sigma}{k}}{s^2+\frac{g-1}{k}s-\frac{\sigma}{k}} = -\frac{1}{Z_0}[\frac{a_2}{s+\sigma}+\frac{1+a_1}{1-ks_2}\frac{1}{s-s_1}+\frac{1-a_1}{1-ks_1}\frac{1}{s-s_2}]$$

where

$$a_1 = \frac{g-1}{k(s_1-s_2)}$$

$$a_2 = 1+\frac{4}{k\sigma-g-2}$$

g, k, and σ are the same as before.

Substituting $\frac{\Gamma}{s}$, $\frac{\Gamma}{sZ_c}$, μ and Z_c in (12) one finds

$$V(\xi,s) = e^{-\rho\xi}\frac{e^{-s\xi}}{s} + e^{-\rho(2-\xi)}[\frac{1}{s}-\frac{1+a_1}{s-s_1}-\frac{1-a_1}{s-s_2}]e^{-s(2-\xi)}$$

$$-e^{-\ (2+\xi)}[\frac{1}{s}-\frac{1+a_1}{s-s_1}-\frac{1-a_1}{s-s_2}]e^{-s(2+\xi)}-\ldots$$

$$Z_0 I(\xi,s) = e^{-\rho\xi}\frac{e^{-s\xi}}{s+\sigma} + e^{-\rho(2-\xi)}\left[\frac{a_2}{s+\sigma} + \frac{1+a_1}{1-ks_2}\,\frac{1}{s-s_1} + \frac{1-a_1}{1-ks_1}\,\frac{1}{s-s_2}\right] e^{-s(2-\xi)}$$

$$+ e^{-\rho(2+\xi)}\left[\frac{a_2}{s+\sigma} + \frac{1+a_1}{1-ks_2}\,\frac{1}{s-s_1} + \frac{1-a_1}{1-ks_1}\,\frac{1}{s-s_2}\right] e^{-s(2+\xi)} + \dots$$

and after taking the inverse Laplace transform we obtain

$$v(\xi,\tau) = e^{-\rho\xi}U(\tau-\xi) + e^{-\rho(2-\xi)}[1-(1+a_1)e^{s_1(\tau-2+\xi)} - (1-a_1)e^{s_2(\tau-2+\xi)}]\,U(\tau-2+\xi)$$

$$- e^{-\rho(2+\xi)}[1-(1+a_1)e^{s_1(\tau-2-\xi)} - (1-a_1)e^{s_2(\tau-2-\xi)}]U(\tau-2-\xi) - \dots$$

$$Z_0 i(\xi,\tau) = e^{-2\beta\xi-\sigma\tau}U(\tau-\xi) + e^{-\rho(2-\xi)}[a_2 e^{-\sigma(\tau-2+\xi)} + \frac{1+a_1}{1-ks_2}\,e^{s_1(\tau-2+\xi)}$$

$$+ \frac{1-a_1}{1-ks_1}\,e^{s_2(\tau-2+\xi)}]U(\tau-2+\xi) + e^{-\rho(2+\xi)}[a_2 e^{-\sigma(\tau-2-\xi)}$$

$$+ \frac{1+a_1}{1-ks_2}\,e^{s_1(\tau-2-\xi)} + \frac{1-a_1}{1-ks_1}\,e^{s_2(\tau-2-\xi)}]U(\tau-2-\xi) + \dots$$

Numerical Results

We have obtained and plotted the numerical results for the transient current at the beginning of the line ($\xi=0$) for several different cases with the aid of a computer. Lommel functions have been written in the form of a subroutine using the existing subroutine for modified Bessel functions. The results for the exact solution have been obtained using double precision because as the time increases the Lommel functions involved in the formulation of the problem become very large and the expressions like $Y_1(\eta_{11},\zeta_1) - Y_1(\eta_{21},\zeta_1)$ cannot be calculated accurately unless a double precision is used. Even with double precision sometimes we encountered difficulties. Therefore some of the numerical results have been obtained by changing the formulation of the problem to a form which gives a more accurate numerical result rather than finding the Lommel functions explicitly from the subroutine.

Figures 1 to 12 show the transient current at the beginning of the line (generator), $i(0,\tau)$, considering only the direct wave and two reflected waves (up to $\tau=4$) for different values of the losses, length, and terminations. Results based on both the exact and the approximate methods are shown. In the following we give a brief explanation of those plots.

We consider a line with the following parameters as our standard line and later when we change the values of some of the parameters for comparison. Also, whenever we say "transient" from now on we mean $Z_0 i(0,\tau)$.

$R = o.736$ Ω/km, $L = 23.8 \times 10^{-3}$ h/km

$G = .05 \times 10^{-6}$ ℧/km, $C = 11.3 \times 10^{-9}$ f/km

Length = 400 km

Figure 1 shows the transient in the standard line terminated by a series RL network with

$R_2 = 0$ ohm, $L_2 = 1$ henry

It is seen that the agreement between the exact and the approximate solutions is reasonably good.

As we mentioned earlier we anticipate that our approximate solution gets closer to the exact solution as the losses are decreased. Figure 2 shows that this is in fact true. There we decreased the losses by a factor of 10 and kept the other parameters constant. The agreement between the two solutions is very good.

Now if we decrease the losses even more, it is obvious that the two solutions will be closer. But since we do not want to deviate from the realistic and practical situations we do not decrease the losses by decreasing R and G. Rather we decrease the total loss of the line by decreasing its length. It should be noted that by doing so the shape of the transients will change. But it is not of our concern for the time being. In Figure 3 and Figure 4 the losses are decreased by a factor of 10. Lengths are also decreased to 40 kilometers and 4 kilometers, respectively. The agreement between the two solutions is too good to require distinct plots.

In Figure 5 we decreased the losses to zero. In this case it is easy to show analytically that the two solutions are the same.

So far the examples we had were dealing with the case $\Delta < 0$. In Figure 6 and Figure 7 we used

$$R_2 = 10 \text{ ohms}, \qquad L_2 = 1000 \text{ henries}$$

which correspond to $\Delta > 0$. Although $L_2 = 1000$ henries is not a realistic value for self-inductance, we chose it merely for theoretical interest. It should be possible to find other combinations of R_2 and L_2 which give rise to $\Delta > 0$. Figure 6 corresponds to the line with a length of 400 kilometers and Figure 7 to a 40 kilometer long line. In both cases R, L, G, and C are the same as the standard line. In the latter case the agreement between the two solutions is very good.

The rest of the Figures pertain to parallel GC terminated lines and all of them correspond to the case $\Delta > 0$. Figure 8 is for the standard line with

$$G_2 = .001 \text{ mho}, \qquad C_2 = 10 \ \mu\text{f}$$

as termination. The agreement between the solutions is poor. In Figure 9 the length of the line is decreased to 40 kilometers. The agreement is improved but the approximate solution is not yet acceptable.

In Figures 10, 11, and 12 losses are decreased from that of the standard line by a factor of 10; they correspond to the lengths of 400 km, 200 km, and 40 km, respectively. Also

$$G_2 = .001 \text{ mho}, \qquad C_2 = 100 \ \mu f$$

has been used as termination. The agreement between the exact and the approximate solutions for the case Length = 40 km is quite good.

Conclusion

In this work we have discussed the solution to the problem of the transients on a lossy transmission line. We have shown that the work done by Kuznetsov is principally the same as the work done by Jeffreys some twenty years earlier. We have also formulated the solution of a line terminated by a parallel GC network.

The limitation on the values of R and L in the solution of the problem obtained by Kuznetsov has been removed.

An approximate method to solve the problem has been proposed and the exact and approximate solutions have been compared numerically. It is observed that if the losses (R and G) are small the two solutions are in very good agreement. We are unable to establish an exact criterion for the approximate solution. It appears that the agreement between the two solutions is generally better for RL terminated lines than for GC terminated lines. The reason is not yet clear to us. Other approximate methods might exist which give better results and can be the subject of further studies.

REFERENCES

1. H.S. Carslaw and J.C. Jaeger, Operational Methods in Applied Mathematics. Oxford University Press, 1941.
2. H. Jeffreys, Operational Methods in Mathematical Physics. Cambridge University Press, First Edition, 1927.
3. P.I. Kuznetsov and R.L. Stratonovich, The Propagation of Electromagentic Waves in Multiconductor Transmission Lines. The Macmillan Company, New York, 1964.
4. N. Levinson, "The Fourier Transform Solution of Ordinary and Partial Differential Equations", Journal of Mathematics and Physics, Vol. 14, pp 195-227, 1935.
5. C.T. Tai, Transients on Lossless Terminated Transmission Lines, Radiation Laboratory, The University of Michigan, Ann Arbor.
6. G.N. Watson, A Treatise on the Theory of Bessel Functions. Cambridge University Press, 1966.
7. E.B. Wedmore (Director of Research), Surge Phenomena, Seven Years' Research for the Central Electricity Board (1933-1940), The British Electrical and Allied Industries Research Association.

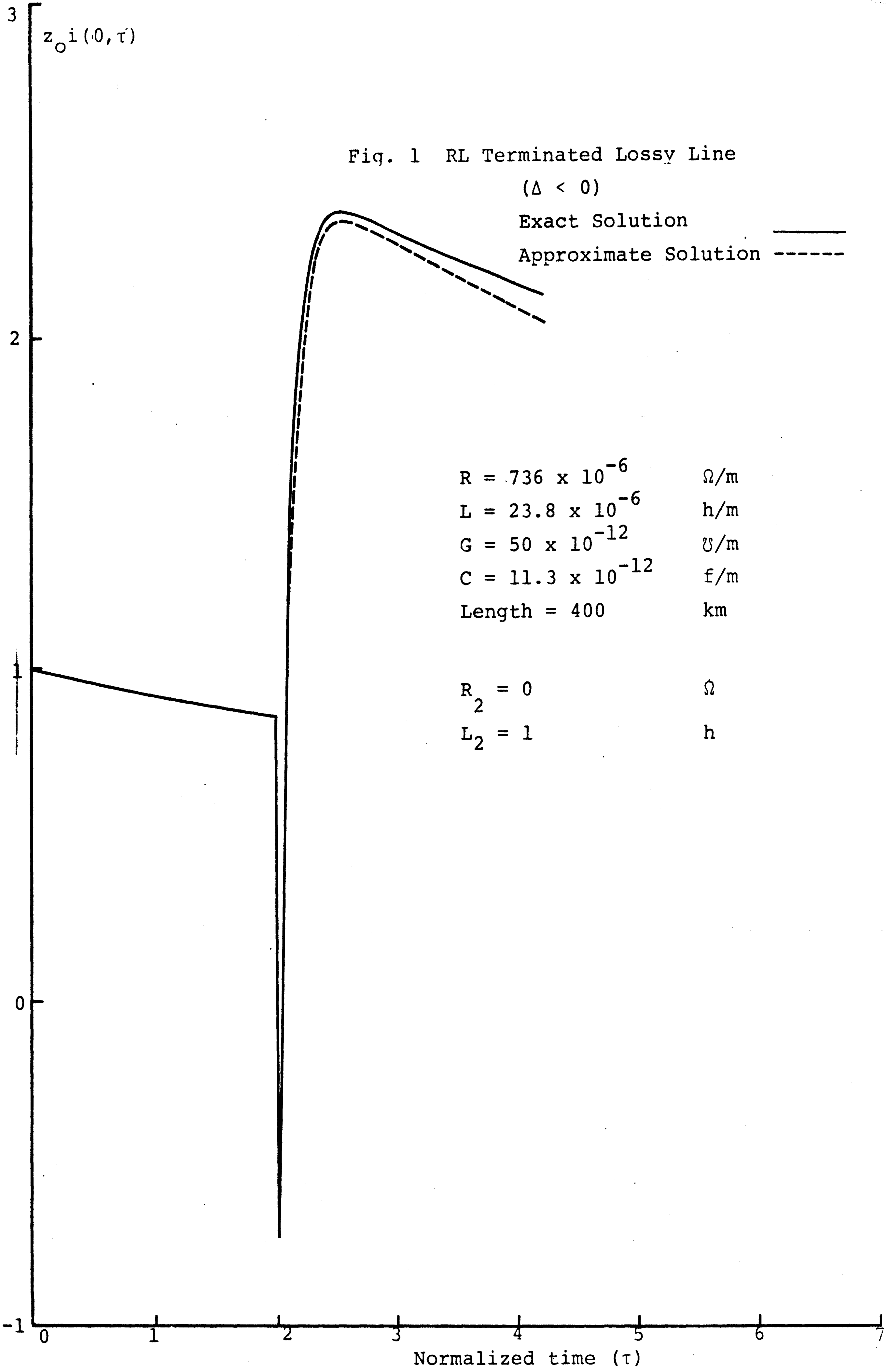


Fig. 1 RL Terminated Lossy Line

$(\Delta < 0)$

Exact Solution ———

Approximate Solution ------

$R = 736 \times 10^{-6}$ Ω/m

$L = 23.8 \times 10^{-6}$ h/m

$G = 50 \times 10^{-12}$ ℧/m

$C = 11.3 \times 10^{-12}$ f/m

Length = 400 km

$R_2 = 0$ Ω

$L_2 = 1$ h

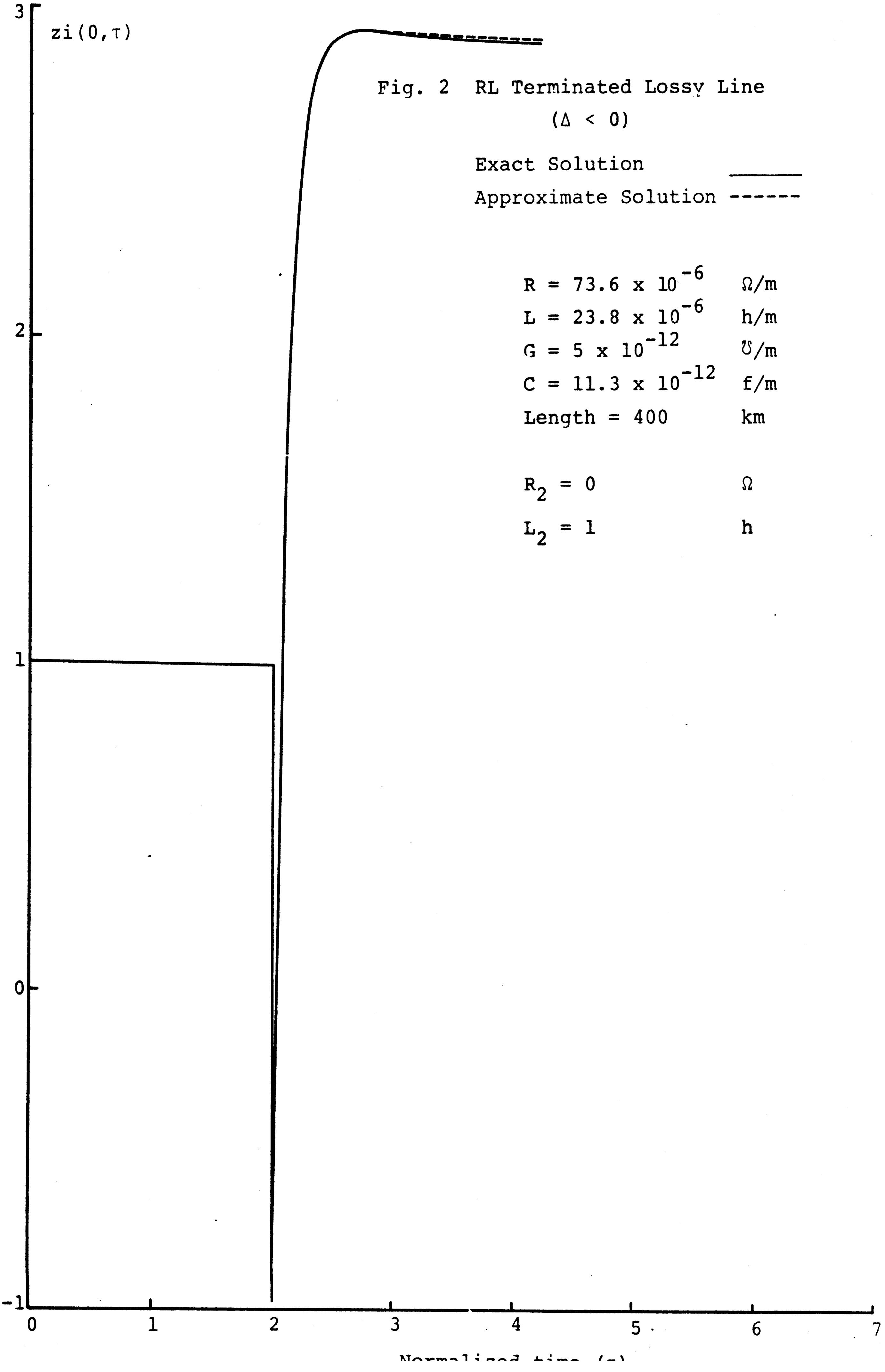


Fig. 2 RL Terminated Lossy Line
($\Delta < 0$)

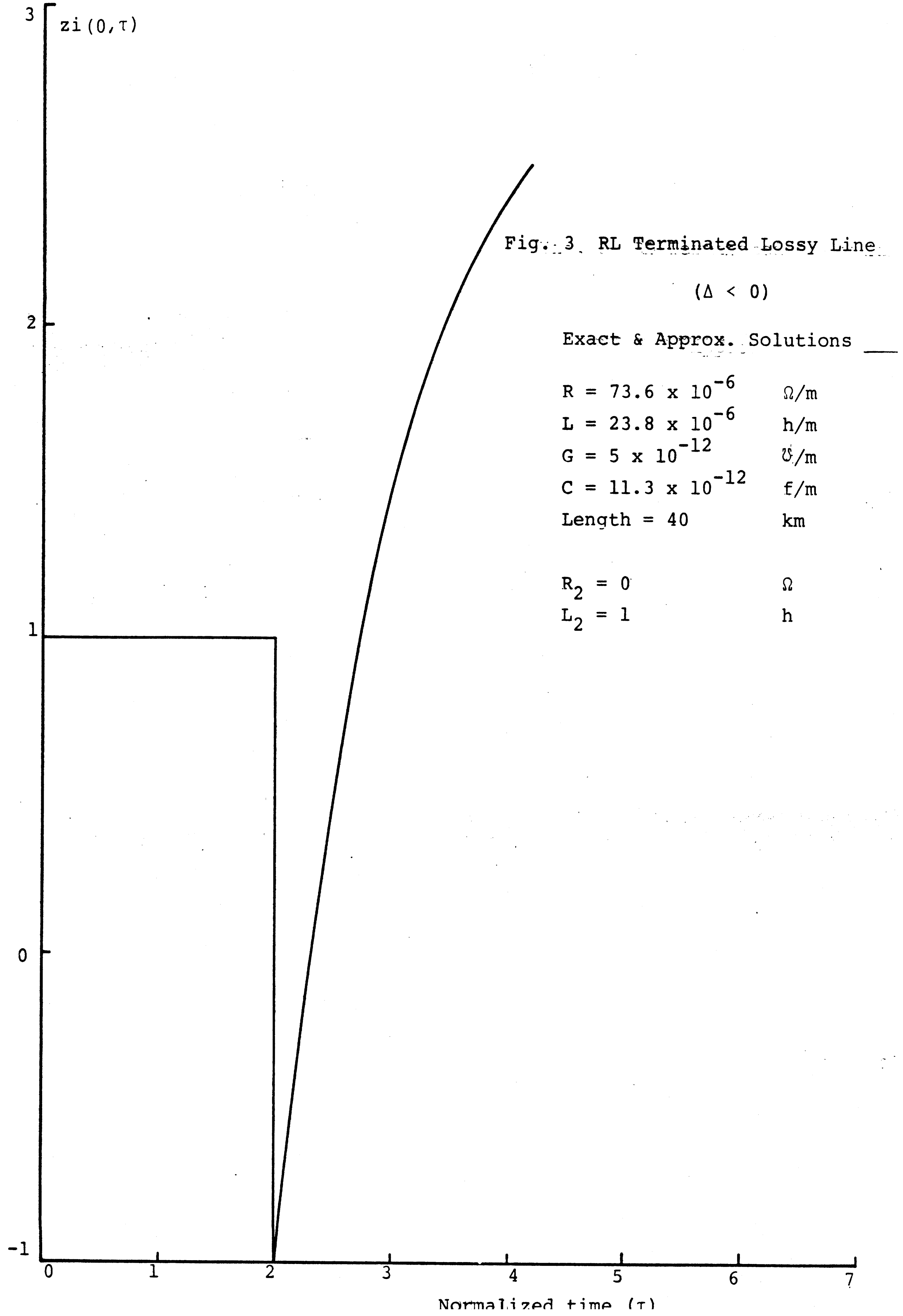


Fig. 3 RL Terminated Lossy Line

($\Delta < 0$)

Exact & Approx. Solutions ___

$R = 73.6 \times 10^{-6}$ Ω/m
$L = 23.8 \times 10^{-6}$ h/m
$G = 5 \times 10^{-12}$ ℧/m
$C = 11.3 \times 10^{-12}$ f/m
Length = 40 km

$R_2 = 0$ Ω
$L_2 = 1$ h

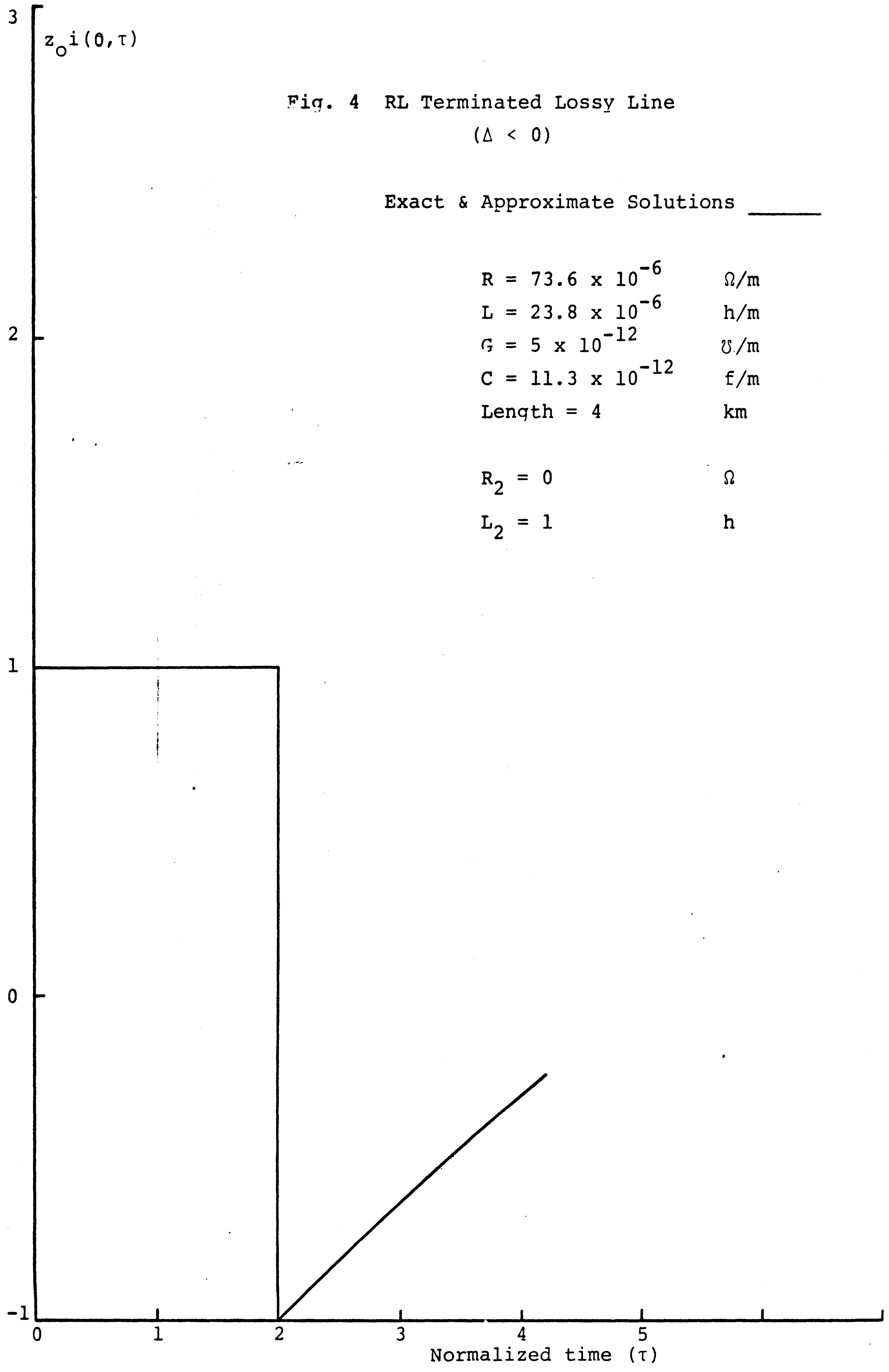


Fig. 4 RL Terminated Lossy Line
($\Delta < 0$)

Exact & Approximate Solutions ______

$R = 73.6 \times 10^{-6}$ Ω/m
$L = 23.8 \times 10^{-6}$ h/m
$G = 5 \times 10^{-12}$ ℧/m
$C = 11.3 \times 10^{-12}$ f/m
Length = 4 km

$R_2 = 0$ Ω
$L_2 = 1$ h

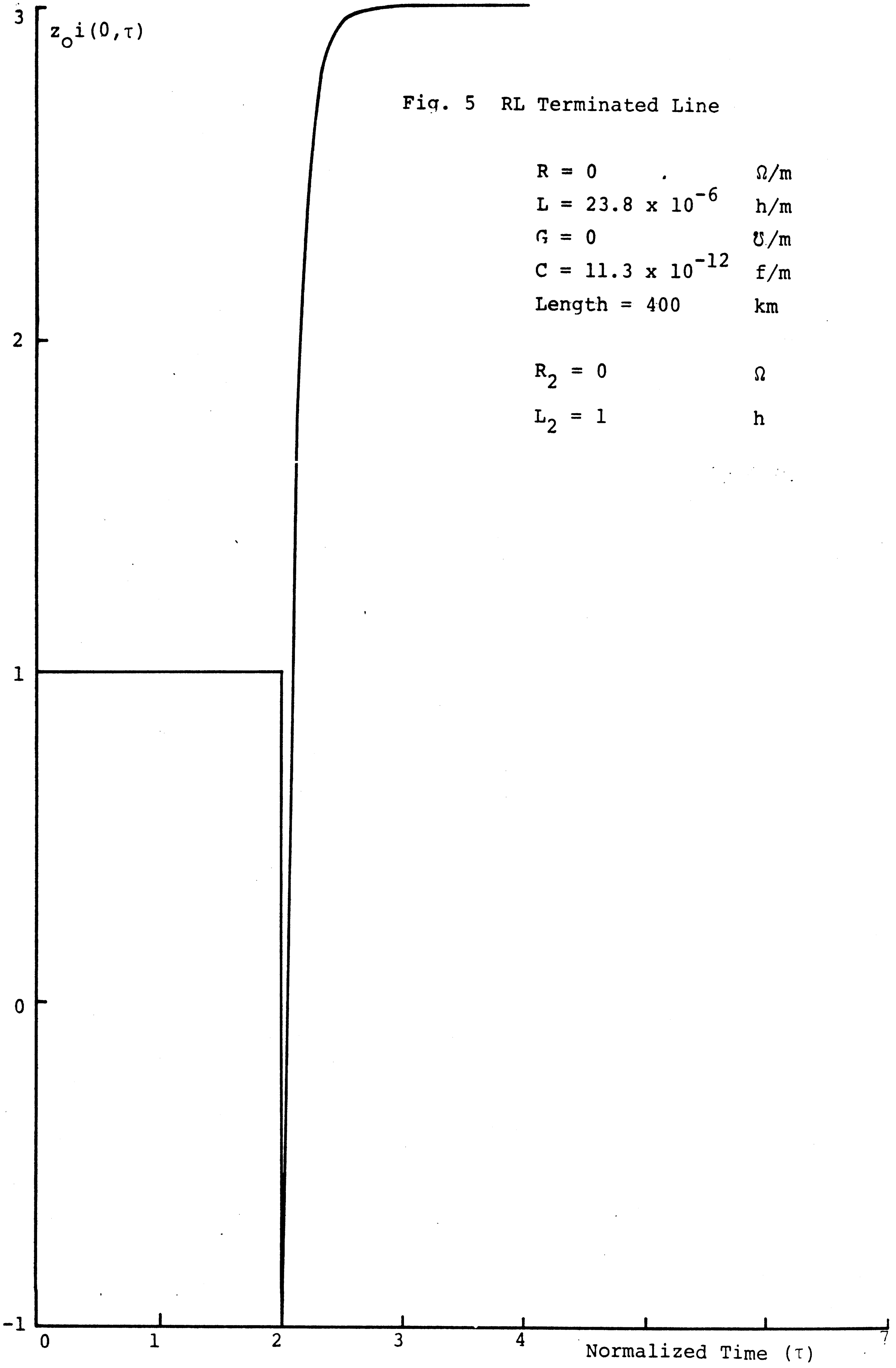


Fig. 5 RL Terminated Line

$R = 0$ Ω/m
$L = 23.8 \times 10^{-6}$ h/m
$G = 0$ ℧/m
$C = 11.3 \times 10^{-12}$ f/m
Length = 400 km

$R_2 = 0$ Ω
$L_2 = 1$ h

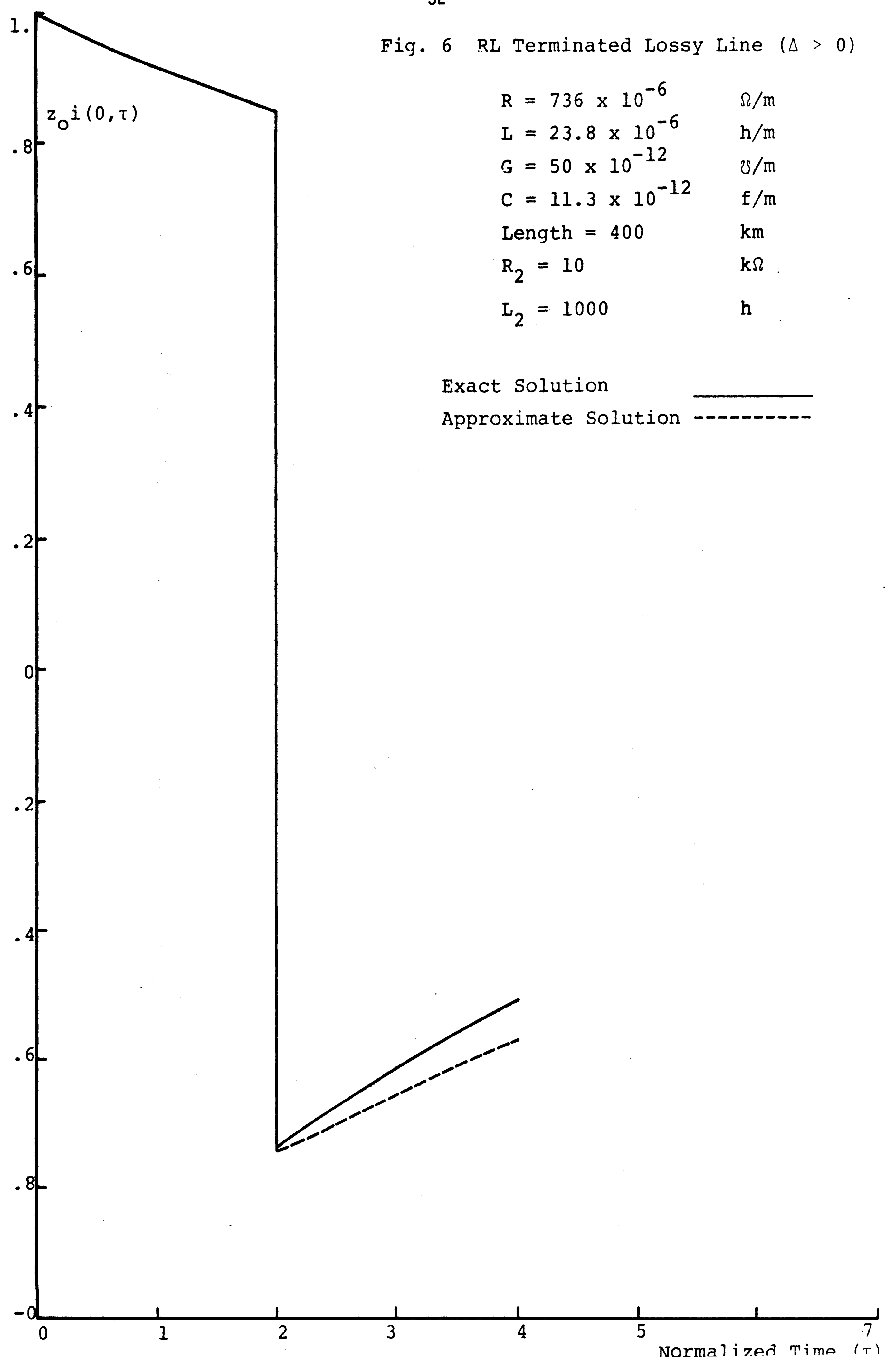


Fig. 6 RL Terminated Lossy Line ($\Delta > 0$)

$R = 736 \times 10^{-6}$ Ω/m

$L = 23.8 \times 10^{-6}$ h/m

$G = 50 \times 10^{-12}$ ℧/m

$C = 11.3 \times 10^{-12}$ f/m

Length = 400 km

$R_2 = 10$ kΩ

$L_2 = 1000$ h

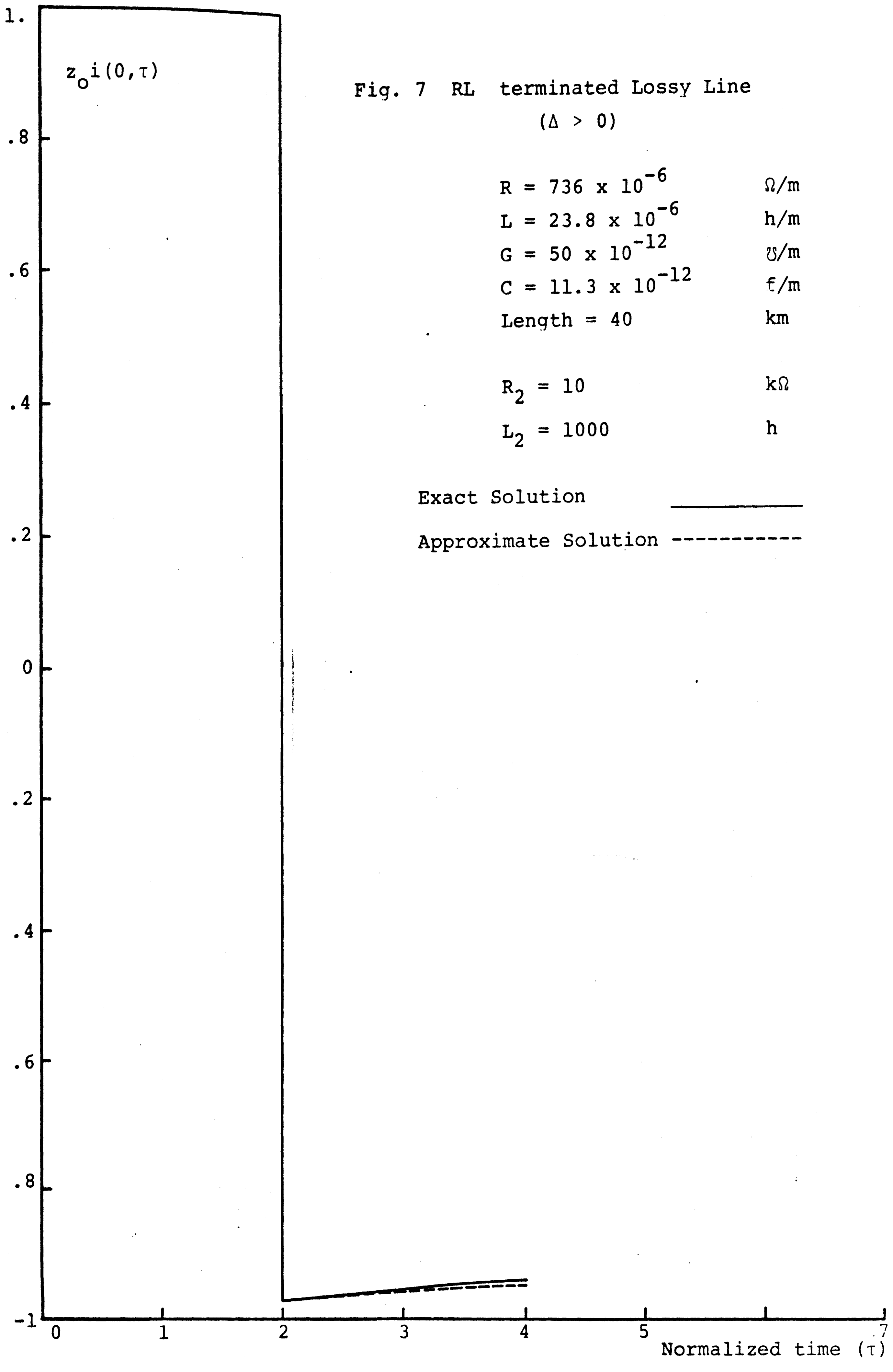


Fig. 7 RL terminated Lossy Line
($\Delta > 0$)

$R = 736 \times 10^{-6}$ Ω/m
$L = 23.8 \times 10^{-6}$ h/m
$G = 50 \times 10^{-12}$ ℧/m
$C = 11.3 \times 10^{-12}$ f/m
Length = 40 km

$R_2 = 10$ kΩ
$L_2 = 1000$ h

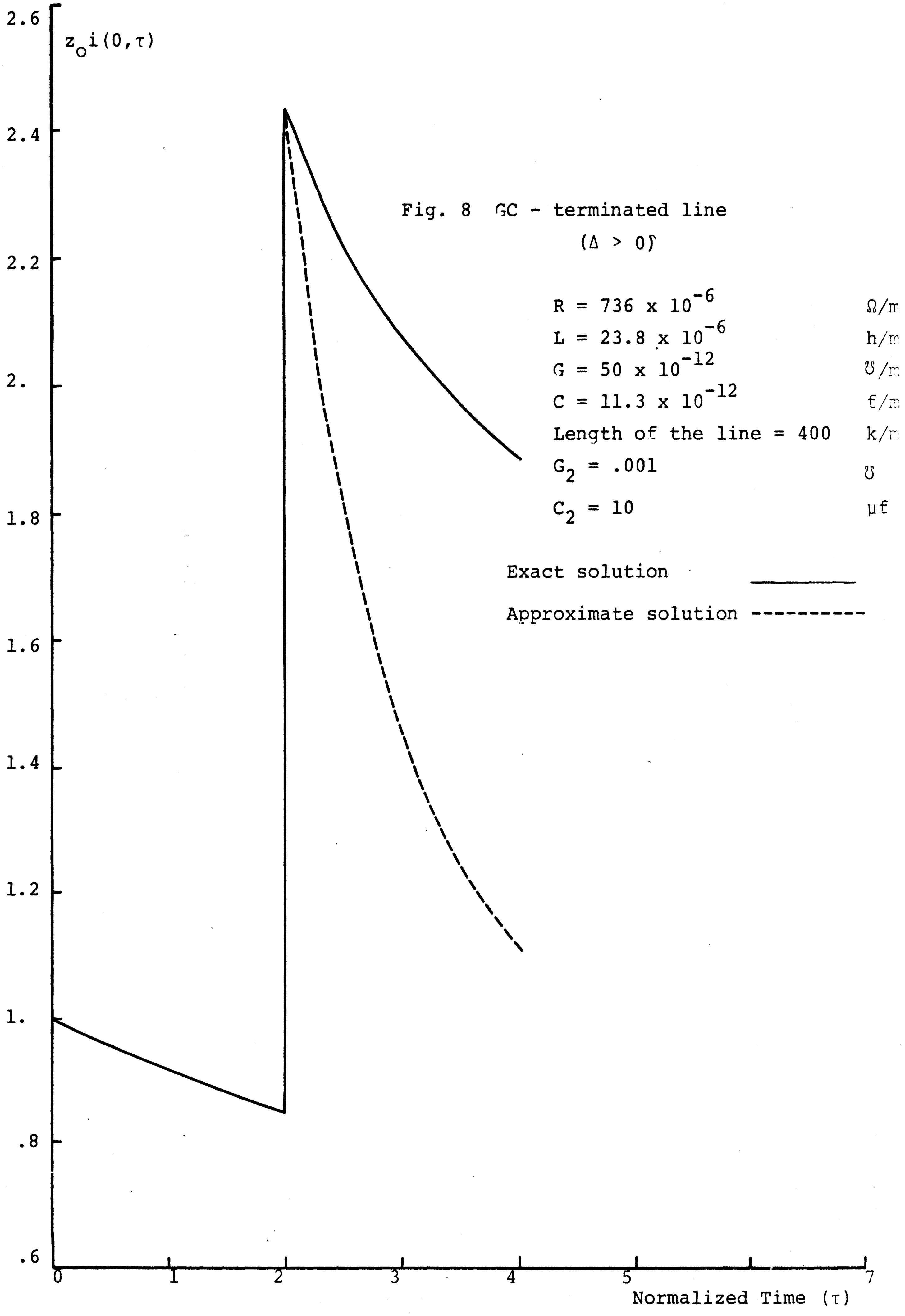


Fig. 8 GC - terminated line ($\Delta > 0$)

$R = 736 \times 10^{-6}$ Ω/m
$L = 23.8 \times 10^{-6}$ h/m
$G = 50 \times 10^{-12}$ ℧/m
$C = 11.3 \times 10^{-12}$ f/m
Length of the line = 400 k/m
$G_2 = .001$ ℧
$C_2 = 10$ μf

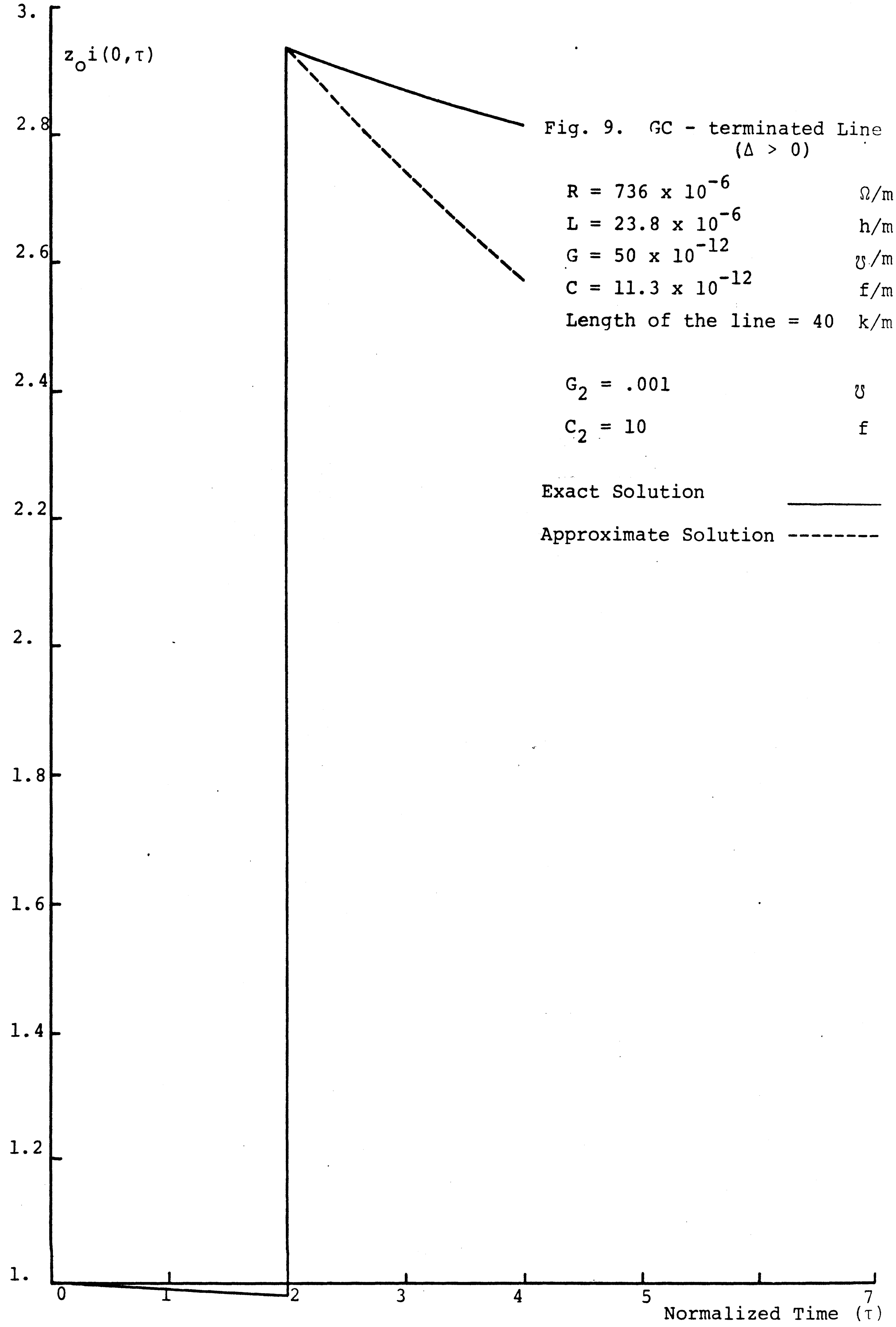


Fig. 9. GC - terminated Line
($\Delta > 0$)

$R = 736 \times 10^{-6}$ Ω/m
$L = 23.8 \times 10^{-6}$ h/m
$G = 50 \times 10^{-12}$ ℧/m
$C = 11.3 \times 10^{-12}$ f/m
Length of the line = 40 k/m

$G_2 = .001$ ℧
$C_2 = 10$ f

Exact Solution ————
Approximate Solution --------

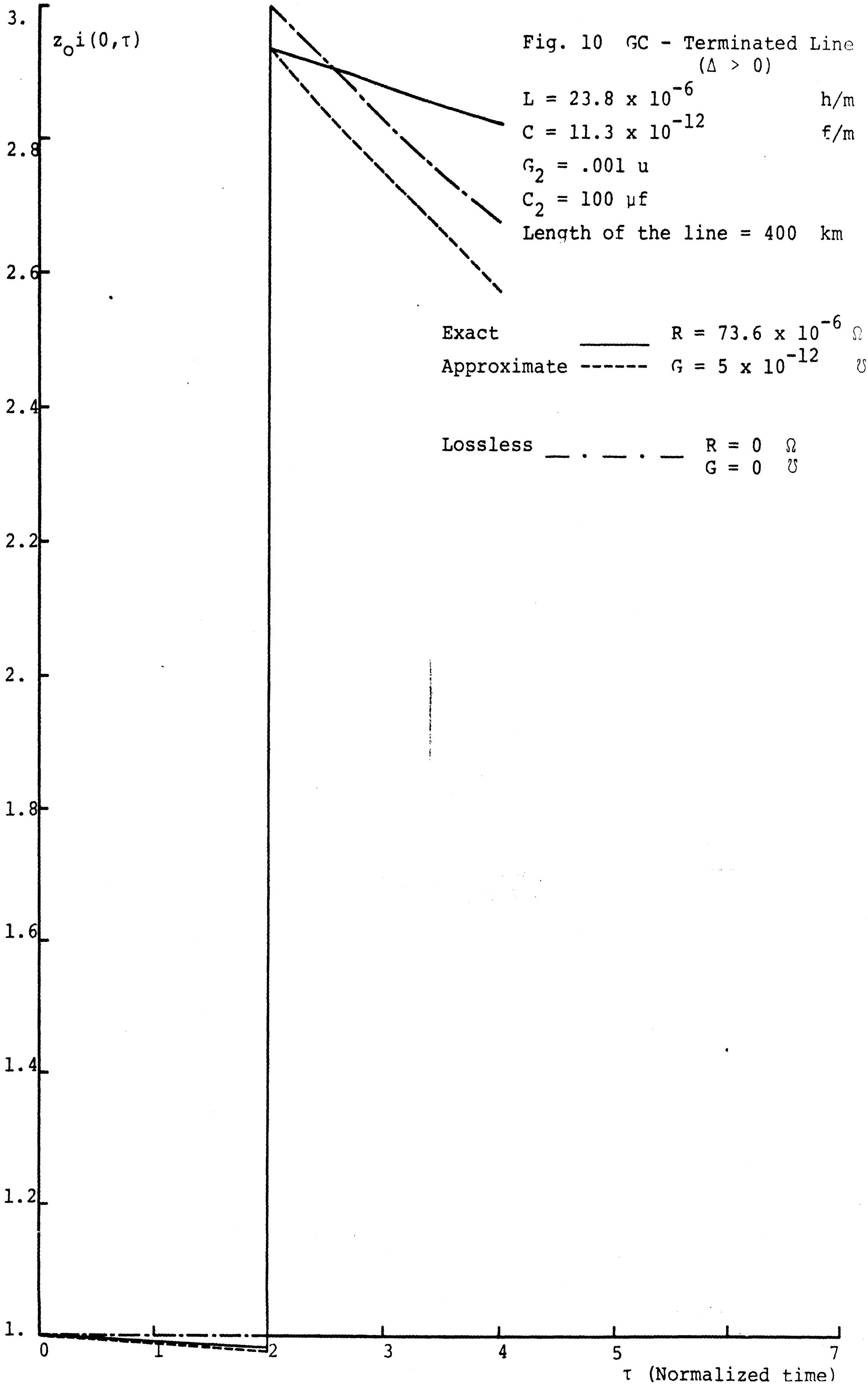


Fig. 10 GC - Terminated Line ($\Delta > 0$)

$L = 23.8 \times 10^{-6}$ h/m

$C = 11.3 \times 10^{-12}$ f/m

$G_2 = .001$ u

$C_2 = 100$ μf

Length of the line = 400 km

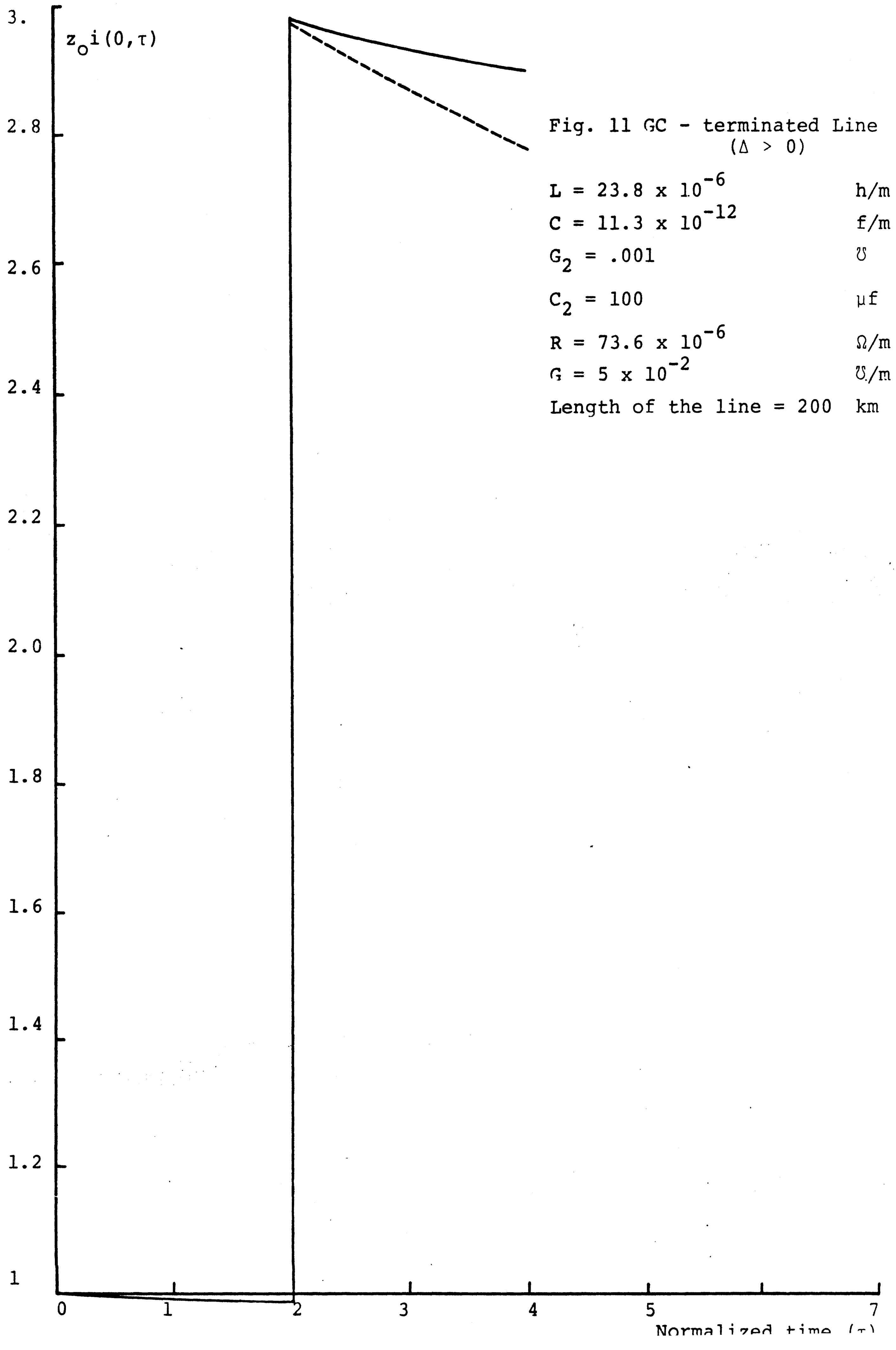


Fig. 11 GC - terminated Line ($\Delta > 0$)

$L = 23.8 \times 10^{-6}$ h/m

$C = 11.3 \times 10^{-12}$ f/m

$G_2 = .001$ ℧

$C_2 = 100$ μf

$R = 73.6 \times 10^{-6}$ Ω/m

$G = 5 \times 10^{-2}$ ℧/m

Length of the line = 200 km

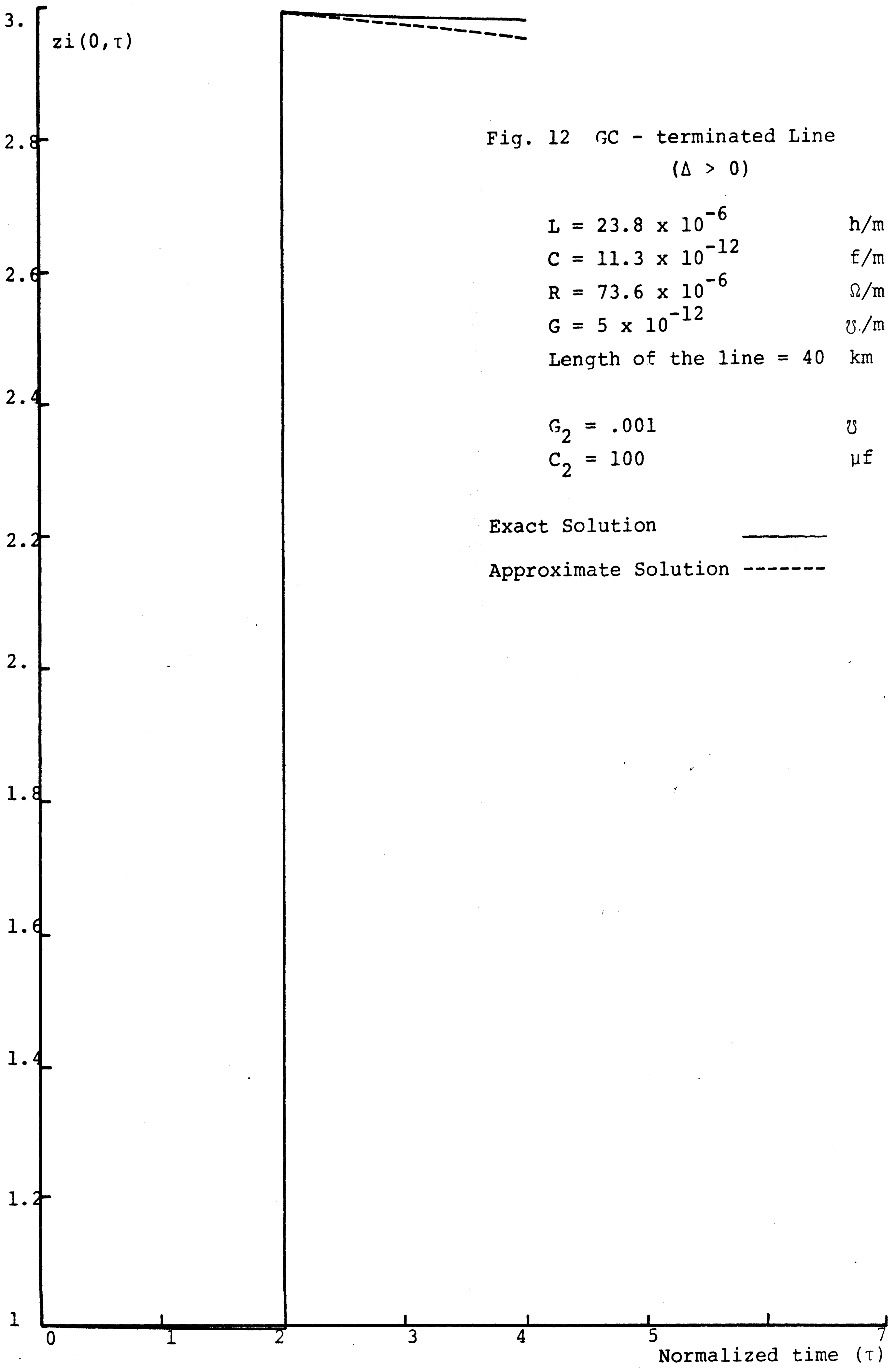


Fig. 12 GC - terminated Line

($\Delta > 0$)

$L = 23.8 \times 10^{-6}$ h/m

$C = 11.3 \times 10^{-12}$ f/m

$R = 73.6 \times 10^{-6}$ Ω/m

$G = 5 \times 10^{-12}$ ℧/m

Length of the line = 40 km

$G_2 = .001$ ℧

$C_2 = 100$ μf

APPENDIX A

Transients on Lossless Terminated Transmission Lines

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Abstract

This work contains a general exposition of the methods which are available in analyzing the transients on a lossless terminated line. After reviewing the well known method based on the Γ-series expansion we present two alternative methods, one in the form of a Volterra integral equation and another corresponding to the so-called singularity expansion method. For a resistively terminated line we have proved the identity between the Γ-series solution and the one obtained by the singularity expansion method. The application of these methods to more complicated terminations is illustrated by the case of a series R-L termination. Weber's solution for a short-circuited line is compared with our solution. The importance of injecting the causality condition in our formulation for this class of problems is emphasized. The application of these methods to the treatment of the input current response of thin biconical antenna is briefly outlined.

Introduction

Transient on transmission lines is a classical problem in linear system analysis. Many authors have contributed significantly to the study of this problem. We would like to mention particularly the work of Levinson [1], Bewley [2], Weber [3], Kuznetsov and Stratonovich [4]. Although the formulation for lossy lines terminated by an arbitrary load is known, a general solution seems to be not available because of the difficulty in evaluating some of the inverse Laplace transforms. For a lossy line terminated by a series R-L load, the exact solution was found by Kuznetsov [4] with the aid of Lommel functions. When the line is lossless the analysis is considerably simpler. Even then, no detailed treatment seems to be available for arbitrary terminations except for the case of a resistive load which is discussed in many standard books. It is therefore desirable to present a general treatment by which one can solve the problem for arbitrary termination in a systematic way. The work reported here is partly motivated by our desire to investigate the transient phenomena on biconical antennas which can be interpreted as a pair of biconical transmission lines terminated by a distributed load admittance [5,6]. Whatever method we may use for the transmission line problem is then equally applicable to analyze the transient response of biconical antenna. Before the general methods are presented, a review of the conventional treatment for a pair of lossless lines terminated by a resistive load is in order.

Conventional Method of Treating a Lossless Line Terminated by a Resistive Load

We consider a pair of lossless lines terminated by an impedance load Z. The lines are assumed to be excited by a unit step voltage at the input end.

For convenience we introduce several normalized variables defined as follows:

$\xi = x/\ell$ = normalized distance; $1 \geq \xi \geq 0$

$\tau = tc/\ell$ = normalized time

where ℓ = length of the line

c = velocity of propagation on the lossless line, being equal to $1/(L'C')^{1/2}$

L', C' = inductive and capacitive constants per unit length of the line

s = $\sigma\ell/c$ = normalized Laplace transform variable

σ = the ordinary or conventional Laplace transform variable being equal to $j\omega$ where ω denotes the complex angular frequency

In terms of these normalized variables we denote

$v(\xi,\tau)$ = instantaneous line voltage

$i(\xi\ \tau)$ = instantaneous line current

$V(\xi,s)$ = Laplace transform of $v(\xi,\tau)$

$$= \mathscr{L}[v(\xi,\tau] = \int_0^\infty v(\xi,\tau)e^{-s\tau}d\tau$$

$I(\xi,s)$ = Laplace transform of $i(\xi,\tau)$

$$= \mathscr{L}[i(\xi,\tau)] = \int_0^\infty i(\xi,\tau)e^{-s\tau}d\tau$$

For a unit step voltage applied at the input end we have

$$v(0,\tau) = U(\tau - 0)$$

hence

$$V(0,s) = \int_0^\infty U(\tau - 0)e^{-s\tau}\,dt = \frac{1}{s}$$

In terms of normalized variables ξ,s, the line voltage and the line current in the Laplace transform domain can be written in the form

$$V(\xi,s) = \frac{e^{-\xi s} + \Gamma(s)e^{-(2-\xi)s}}{s[1 + \Gamma(s)e^{-2s}]} \tag{1}$$

$$Z_c I(\xi,s) = \frac{e^{-\xi s} - \Gamma(s)e^{-(2-\xi)s}}{s[1 + \Gamma(s)e^{-2s}]} \tag{2}$$

where $\Gamma(s)$ denotes the voltage reflection coefficient defined in the s-domain at the output end of the line, $\xi = 1$, and Z_c denotes the characteristic impedance of the line, being equal to $(L'/C')^{1/2}$. For convenience, we assume Z_c to be equal to unity in the subsequent analysis.

The conventional method of determining $v(\xi,\tau)$ or $i(\xi,\tau)$ is to express (1) or (2) in a series using the expression

$$\frac{1}{1+\Gamma(s)e^{-2s}} = \sum_{n=o}^{\infty} [-\Gamma(s)e^{-2s}]^n \tag{3}$$

Substituting (3) into (2), with $Z_c = 1$, we have

$$I(\xi,s) = \frac{1}{s}[e^{-\xi s}-\Gamma(s)e^{-(2-\xi)s}] \sum_{n=o}^{\infty} [-\Gamma(s)e^{-2s}]^n \tag{4}$$

For a resistive load $\Gamma(s)$ is a real constant which will be denoted by Γ and its value is given by

$$\Gamma = \frac{r-1}{r+1}$$

where r denotes the normalized terminal resistance. The inverse Laplace transform of (4) with $\Gamma(s) = \Gamma$ yields

$$i(\xi,\tau) = \sum_{n=o}^{\infty} [(-\Gamma)^n U(\tau-2n-\xi) + (-\Gamma)^{n+1} U(\tau-2n-2+\xi)] \tag{5}$$

where $U(\tau-\tau_n)$ denotes a unit step function commencing at $\tau = \tau_n$.

Although (5) is known to be a valid solution by physical reasoning its derivation is considered to be unsatisfactory from the mathematical point of view because expansion (3) holds true only if $|\Gamma(R)e^{-2s}| < 1$, and in executing the inverse Laplace transform the contour of integration lies in the left-half plane where $|\Gamma(s)e^{-2s}|$ could exceed unity. This presentation is found in many books without justification. One way of removing this weak step is to expand the same function in terms of a finite sum with a remainder instead of as an infinite series. Thus, we write

$$\frac{1}{1+\Gamma(s)e^{-2s}} = \sum_{n=o}^{N} [-\Gamma(s)e^{-2s}]^n + \frac{[-\Gamma(s)e^{-2s}]^{N+1}}{1+\Gamma(s)e^{-2s}} \qquad (6)$$

when substituting (6) into (2) the remainder would yield a term of the form

$$\frac{[-\Gamma(s)]^{N+2}}{s[1+\Gamma(s)e^{-2s}]} e^{-s[2(N+2)-\xi]} \qquad (7)$$

Because of the shifting theorem and the causality condition the inverse Laplace transform of (7) vanishes when $\tau < 2(N+2)-\xi$. In other words, if one evaluates the series (5) up to $\tau < 2(N+2)-\xi$ the remaining terms vanish identically. The importance of this remark is that (6) applies not only to resistive termination but to any termination. From now on we will designate the solution based on (6) as the Γ-series solution. In addition to the Γ-series method there are two alternative methods for treating the transients in an arbitrary terminated line. The discussion of these two methods is the main objective of this paper.

Volterra Integral Equation Method

We consider the general case where $\Gamma(s)$ is a function of s. If (2), with $Z_c = 1$, is multiplied by $1 + \Gamma(s)e^{-2s}$ the following equation results

$$I(\xi,s) + \Gamma(s)e^{-2s}I(\xi,s) = \frac{1}{s}[e^{-\xi s} - \Gamma(s)e^{-(2-\xi)s}] \qquad (8)$$

By taking the inverse Laplace transform of (8) we obtain

$$i(\xi,\tau) = i_o(\xi,\tau) + \mathscr{L}^{-1}[-\Gamma(s)e^{-2s}I(s)] \qquad (9)$$

where

$$i_o(\xi,\tau) = i_{of}(\xi,\tau) + i_{ob}(\xi,\tau) \qquad (10)$$

with

$$i_{of}(\xi,\tau) = \mathscr{L}^{-1}[\frac{e^{-\xi s}}{s}] = U(\tau-\xi) \qquad (11)$$

$$i_{ob}(\xi,\tau) = \mathscr{L}^{-1}[-\frac{\Gamma(s)}{s} e^{-(2-\xi)s}] \qquad (12)$$

$i_{of}(\xi,\tau)$ represents the initial forward current wave propagating on the line and $i_{ob}(\xi,\tau)$ represents the first reflected wave or backward wave from the termination. For a given $\Gamma(s)$ we assume (12) can be evaluated, thus $i_{ob}(\xi,\tau)$ is considered to be a known function. On account of the convolution theorem in Laplace transform (9) can be written in the form

$$i(\xi,\tau) = i_o(\xi,\tau) + \int_0^\tau k(\tau-\tau')i(\xi,\tau')dt' \qquad (13)$$

where

$$k(\tau) = \mathscr{L}^{-1}[-\Gamma(s)e^{-2s}] \qquad (14)$$

Equation (13) with $i(\xi,\tau)$ as the unknown function corresponds to the Volterra integral equation of the second kind. Its solution is given by Picards' series [7], namely

$$i(\xi,\tau) = \sum_{n=o}^{\infty} i_n(\xi,\tau) \qquad (15)$$

where $i_o(\xi,\tau)$ is given by (10) and

$$i_n(\xi,\tau) = \int_0^\tau k(\tau-\tau')i_{n-1}(\xi,\tau')dt'$$

$$\text{for } n = 1,2,\ldots \qquad (16)$$

In the case $\Gamma(s)$ is a real constant, previously denoted Γ, we obtain from (12)

$$i_{ob}(\xi,\tau) = -\Gamma U(\tau-2+\xi)$$

hence

$$i_o(\xi,\tau) = U(\tau-\xi) - \Gamma U(\tau-2+\xi) \qquad (17)$$

and from (14) we have

$$k(\tau) = -\Gamma\delta(\tau-2) \qquad (18)$$

where $\delta(\tau-2)$ denotes the delta function defined at $\tau = 2$. Substituting (17) and (18) into (16) we obtain the same expression given by (5). Of course, for a resistively terminated live it is entirely unnecessary to formulate the problem by this integral equation method as the method of Γ-series is much simpler. The integral equation method, however, is much more efficient and convenient for more complicated termination. As an illustration we consider a series R-L termination. In this case, we have

$$\Gamma(s) = \frac{z(s) - 1}{z(s) + 1}$$

where

$$z(s) = \frac{1}{Z_c} [R + s(\frac{c}{\ell})L]$$

$$= r + \alpha s$$

$$r = R/Z_c \ , \quad \alpha = \frac{cL}{Z_c \ell} = \frac{L}{L'\ell}$$

L' = inductive line constant

The coefficient α is a measure of the load inductance in terms of the total line inductance . The reflection coefficient $\Gamma(s)$ can now be written in the form

$$\Gamma(s) = \frac{s-s_0}{s-s_1} \tag{19}$$

where $\qquad s_0 = -(\frac{r-1}{\alpha}), \ s_1 = -(\frac{r+1}{\alpha})$

thus

$$\frac{\Gamma(s)}{s} = \frac{1}{s}(\frac{s-s_0}{s-s_1}) = \frac{\rho}{s} + \frac{1-\rho}{s-s_1} \tag{20}$$

where $\qquad \rho = \frac{s_0}{s_1} = \frac{r-1}{r+1}$

using (12) and (14) one finds

$$i_{ob}(\xi,\tau) = -U(\tau-2+\xi)[\rho + (1-\rho)e^{s_1(\tau-2+\xi)}] \tag{21}$$

$$k(\tau) = -\delta(\tau-2) - U(\tau-2)(1-\rho)s_1 e^{s_1(\tau-2)} \qquad (22)$$

$i_{ob}(\xi,\tau)$ and $k(\tau)$ we can find $i_1(\xi,\tau)$ using (16). The
ives

$$i_1(\xi,\tau) = -U(\tau-2-\xi)[\rho + (1-\rho)e^{s_1(\tau-2-\xi)}]$$

$$+U(\tau-4+\xi)\{\rho^2 + [1-\rho^2+(1-\rho)^2 s_1(\tau-4+\xi)]e^{s_1(\tau-4+\xi)}\} \qquad (23)$$

essive terms of $i_n(\xi,\tau)$ for $n \geq 2$ can be found accordingly.
the Γ-series method were used the process is more tedious
one has to expand $[\Gamma(s)]^n/s$ in partial fraction that is
volved as a result of the multiplicity of the poles
d in $[\Gamma(s)]^n$. Another advantage of the Volterra
equation method is that once the first reflected wave
the successive waves can be found based on this infor-
lone. This is due to the fact that the kernel $k(\tau)$
in the integral equation is related to the derivative
irst reflected wave. Since

$$I_{ob}(\xi,s) = -\frac{1}{s}\Gamma(s)e^{-(2-\xi)s}$$

$$K(s) = \mathscr{L}[k(\tau)] = -\Gamma(s)e^{-2s}$$

$$K(s) = sI_{ob}(o,s)$$

s that

$$k(\tau) = \frac{\partial i_{ob}(o,\tau)}{\partial \tau}$$

interpret the derivative in a generalized sense that
continuous unit step function

$$\frac{\partial U(\tau-\tau_n)}{\partial \tau} = \delta(\tau-\tau_n)$$

le, from (21) one finds

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for $\Gamma < 0$, and

$$s_n = \frac{1}{2}(\mathrm{Ln}\Gamma + jn\pi), \quad n = \pm 1, \pm 3, \ldots \qquad (28)$$

for $\Gamma > 0$. Without loss of generality we assume Γ to be negative and not equal to -1 in the following discussion. The case of $\Gamma = -1$ requires a special treatment as going to be discussed later.

The expansion for I(o, s) as given by (25) can now be expanded into a residue series in terms of the poles of that function by means of Mittag Leffler theorem [10]. According to this theorem the function under expansion must be bounded at infinity and be analytic at the origin. We consider the function $(1 + \Gamma e^{-2s})^{-1}$ which satisfies the criteria, then its expansion is given by

$$\frac{1}{1+\Gamma e^{-2s}} = \frac{1}{1+\Gamma} + \sum_{n=0}^{\pm\infty} \frac{1}{2}\left(\frac{1}{s-s_n} + \frac{1}{s_n} \right) \qquad (29)$$

hence

$$\frac{1}{s(1+\Gamma e^{-2s})} = \frac{1}{(1+\Gamma)s} + \sum_{n=0}^{\pm\infty} \frac{1}{2s_n(s-s_n)} \qquad (30)$$

Substituting (30) into (25), we have

$$I(o,s) = \frac{1}{s} - 2\Gamma e^{-2s}\left[\frac{1}{(1+\Gamma)s} + \sum_{n=0}^{\pm\infty} \frac{1}{2s_n(s-s_n)} \right] \qquad (31)$$

The inverse Laplace transform of (31) yields

$$i(o,\tau) = U(\tau-o) - 2\Gamma U(\tau-2)\left[\frac{1}{1+\Gamma} + \sum_{n=0}^{\pm\infty} \frac{1}{2s_n} e^{s_n(\tau-2)} \right] \qquad (32)$$

If we let $s_n = \alpha + j\,\beta_n$

where $\alpha = \frac{1}{2}\,\mathrm{Ln}|\Gamma|$

$\beta_n = n\pi , \quad n = 0, \pm 1, \pm 2, \ldots$

then (32) can be written in the form

$$i(0,\tau) = U(\tau-0)-2\Gamma U(\tau-2)[\frac{1}{1+\Gamma}$$

$$+ e^{\alpha(\tau-2)} \sum_{n=0}^{\infty}(\frac{2-\delta_o}{2}) \frac{\alpha \cos \beta_n(\tau-2) + \beta_n \sin \beta_n(\tau-2)}{\alpha^2 + \beta_n^2}] \quad (33)$$

here δ_o denotes the Kronecker delta.

According to the Γ-series method, for $\xi = o$, (5) reduces to

$$i(0,\tau) = U(\tau-0) - 2\Gamma U(\tau-2) + 2\Gamma^2 U(\tau-4) + . . . \quad (34)$$

Equations (33) and (34) would be equivalent only if the series contained in the summation sign of (33) is proportional to $e^{-\alpha(\tau-2)}$ with the constant of proportionality determined by the time interval in which the series represents. The proof of the identity between (33) and (34) is shown as follows: we recognize that $\cos \beta_n(\tau-2)$ and $\sin \beta_n(\tau-2)$ are two orthogonal sets of functions with a periodicity equal to 2, thus we let

$$e^{-\alpha(\tau-2)} = \sum_{n=0}^{\infty}[a_n \cos \beta_n(\tau-2) + b_n \sin \beta_n(\tau-2)]$$

for $2(m+1) > \tau > 2m$. One finds

$$a_n = -(\frac{2-\delta_o}{2}) \frac{(1+\Gamma)}{(-\Gamma)^m} \frac{\alpha}{\alpha^2+\beta_n^2}$$

$$b_n = - \frac{(1+\Gamma)}{(-\Gamma)^m} \frac{\beta_n}{\alpha^2+\beta_n^2}$$

hence,

$$\sum_{n=0}^{\infty} (\frac{2-\delta_o}{2}) \frac{\alpha \cos \beta_n(\tau-2) + \beta_n \sin \beta_n(\tau-2)}{\alpha^2 + \beta_n^2} = - \frac{(-\Gamma)^m}{1+\Gamma} e^{-\alpha(\tau-2)}$$

$$\text{for } 2(m+1) > \tau > 2m \quad (35)$$

In view of (35), we can write (33) in the form

$$i(0,\tau) = U(\tau-0) - 2\Gamma U(\tau-2)[\frac{1-(-\Gamma)^m}{1+\Gamma}]$$

$$2(m+1) > \Gamma > 2m \quad . \; . \; . \tag{36}$$

If we let m take the successive values 1,2,3 ...(36) indeed is identical to (34). Of course, it is not easy to recognize that the series obtained by the singularity expansion method as given by (33) is an alternative representation of the Γ-series solution without such a detailed analysis. For a non-resistive termination the poles are more complicately distributed. In fact for most of the cases there is no closed form solution for these poles the proof of the identity between the Γ-series solution and the one obtained by the singularity expansion method would be extremely difficult. Based on what we have discussed for the resistively terminated case, we have the confidence that these alternative representations must be equivalent.

Finally, we like to comment on the treatment given by Weber [9] for a short-circuited line ($\Gamma = -1$). The function which Weber analyzed corresponds to the voltage distribution along the line for a step input voltage excitation. In Laplace-transform domain, the function which he considered is

$$V(\xi,s) = \frac{e^{\xi s}-e^{-(2-\xi)s}}{s(1-e^{-2s})} \tag{37}$$

The residue series representation of (37) was obtained by Weber without following the discipline as demanded by Mittag-Leffler theorem. Although his result is correct the procedure which leads to his solution is,strictly speaking, not justified for many irrational functions. The final solution which Weber obtained is of the form

$$v(\xi,\tau) = U(\tau-0)\Big[\sum_{n=1}^{\infty} \frac{2\sin n\pi\xi}{n\pi} + \sum_{n=1}^{\infty}\frac{\sin n\pi(\tau-\xi)}{n\pi} + \sum_{n=1}^{\infty}\frac{\sin n\pi(\tau+\xi)}{n\pi}\Big] \tag{38}$$

The solution clearly represents D'Alembert's solution for the one dimensional wave equation. From the point of view of transient analysis it does not explicitly exhibit the causality condition: $v(\xi,\tau) = 0$ for $\tau < \xi$. Actually (38) is a Fourier series expansion of the periodic wave shown in Fig.1. The function indeed is vanishing for $\xi > \tau > 0$. In contrast to Weber's presentation we treat (37) as consisting of two terms, i.e., we let

$$V(\xi,s) = V_1(\xi,s) + V_2(\xi,s) \tag{39}$$

where

$$V_1(\xi,s) = \frac{e^{-\xi s}}{s(1-e^{-2s})},$$

$$V_2(\xi,s) = -\frac{e^{-(2-\xi)s}}{s(1-e^{-2s})}.$$

By applying Mittag-Leffler theorem to the function

$$\frac{1}{1-e^{-2s}} - \frac{1}{2s}$$

which has no pole at the origin, a condition required by that theorem, we obtain

$$\frac{1}{1-e^{-2s}} - \frac{1}{2s} = \frac{1}{2}\left[\ 1+\sum_{n=\pm 1}^{\pm\infty} \frac{1}{s-s_n}\ \right]$$

where

$$s_n = jn\pi$$

hence

$$\frac{1}{s(1-e^{-2s})} = \frac{1}{2}\left[\ \frac{1}{s^2} + \frac{1}{s} + \sum_{n=\pm 1}^{\pm\infty} \frac{1}{s_n(s-s_n)}\ \right]$$

As a result of the shifting theorem we obtain

$$v_1(\xi,\tau) = U(\tau-\xi)\ [\ \frac{1}{2} + \frac{1}{2}\ (\tau-\xi) + \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin n\pi(\tau-\xi)] \ldots (43)$$

$$v_2(\xi-\tau) = U(\tau-2+\xi)[\ \frac{1}{2} + \frac{1}{2}(\tau-2+\xi) + \sum_{n=1}^{\infty} \sin n\pi(\tau-2+\xi)] \quad (44)$$

Except for the negative sign $v_2(\xi,\tau)$ is merely a delayed reproduction of $v_1(\xi,\tau)$. It is observed that because of the step function $U(\tau-\xi)$ contained in $v_1(\xi,\tau)$ the causality condition is automatically met. It can be shown that our solution is actually equivalent to Weber's because the function $f(\tau) = 1-\tau$, $2 \geq \tau \geq 0$ has a Fourier series representation given by

$$1 - \tau = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin n\pi\ \tau \quad (45)$$

Eqs.(43) and (44) are shown in Fig.2. The sum of the two functions yields again the periodic square wave shown in Fig.1.

Conclusion

In this paper we have examined several distinct methods of analyzing transients on lossless transmission lines with arbitrary terminations. It appears that the integral equation method is potentially more appealing because from the information of the first reflected wave it is possible to construct the kernal of the intergral equation and subsequently to find the complete solution based on quadrature. The singularity-expansion method , on the other hand, does furnish the complete solution without iteration, provided that the singularities of the response function are available. Unfortunately, even for a simple series R-L termination it is necessary to solve a transcendental equation to determine the numerical values of these singularities. For a resistively terminated load we have shown that the solutions obtained by these different methods are analytically equivalent. This establishes the foundation that for an arbitrarily terminated line all these methods are equivalent. The methods discussed here are equally applicable to the transient analysis of smallangle biconical antennas. The only difference is that the terminal impedance or admittance function involves exponential integral functions hence the determination of the singularities becomes more laborious. This work will be reported elsewhere in a separate article.

Acknowledgement

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REFERENCES

1) Norman Levinson. "The Fourier Transform Solution of Ordinary and Partial Differential Equations," Jour of Mathematics and Physics. Vol. 14, pp. 195-227, 1935.

2) L. V. Bewley, Traveling Waves on Transmission Systems, second edition, John Wiley & Sons, Inc., New York 1951.

3) Ernest Weber, Linear Transient Analysis, Vol. II, John Wiley & Sons, Inc., New York 1956.

4) P. I. Kuznetsov and R. L. Stratonovich, The Propagation of Electromagnetic Waves in Multiconductor Transmission Lines, The MacMillan Company, New York, 1964.

5) S. A. Schelkunoff, "Theory of Antennas of Arbitrary Size and Shape," Proc. I.R.E., 29, 493, 1941.

6) C. T. Tai, "On The Theory of Biconical Antennas," J. Appl. Phys., 19,

7) F. G. Tricomi, Integral Equations, Interscience Publishers, Inc., New York, 1955.

8) Carl E. Baum, The Singularity Expansion Method, Chap. 3 of the book, Transient Electromagnetic Fields, edited by L. B. Felson, Springer-Verlag, Berlin, 1976.

9) Ernest Weber, loc. cit., Sec 6.9.

10) E. T. Whittaker and G. N. Watson, Modern Analysis, Cambrodge University Press, p. 134, 1943.

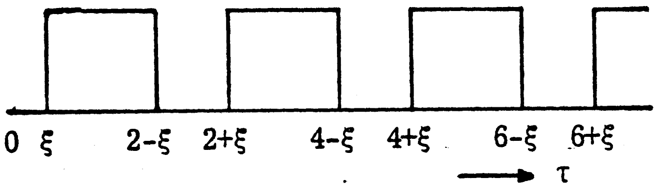

Fig. 1: $v(\xi,\tau)$ for a short-circuit termination.

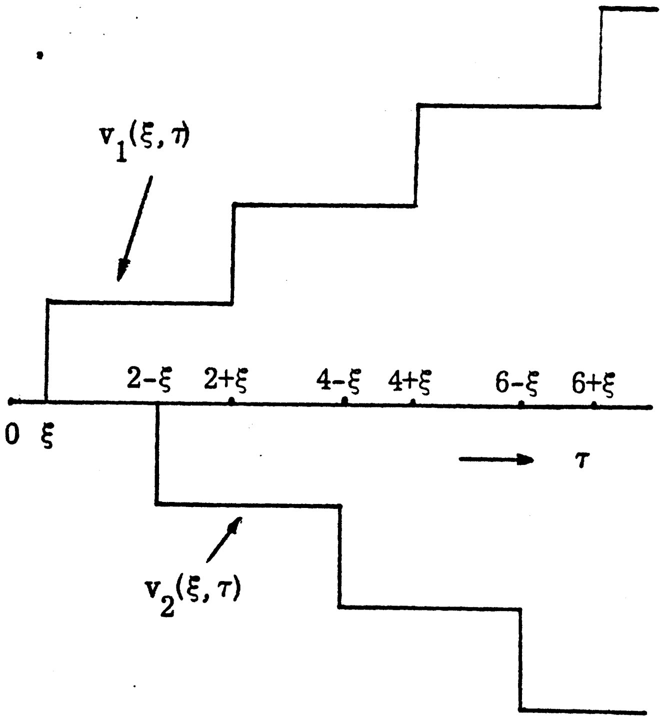

Fig. 2: $v_1(\xi,\tau)$ and $v_2(\xi,\tau)$ for a short-circuit termination.