THEORETICAL STUDY OF THE DISTRIBUTION OF POLES AND ZEROS OF THIN BICONICAL ANTENNA

FINAL REPORT

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THEORETICAL STUDY OF THE DISTRIBUTION
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1. Introduction

In recent years extensive investigations have been done in the area of transient electromagnetic wave phenomena. The new book edited by Leopole B. Felsen [1] summarizes some of the basic works which have been accomplished so far. The book also contains a very substantial bibliography of articles and reports written before 1976.

One may recall that transient analysis for discrete networks in the hands of Cauer [2] and Guillemin [3] developed into one of the major disciplies in electrical engineering curriculum in the late nineteen-forties. The technique involved in that discipline is based mainly upon the concept of zeros and poles. This approach, however, has not yet been extended too far to distributed network, including transmission lines and antennas. Therefore, if one can formulate and solve some canonical problems involving a distributed network based on this approach, such an endeavor certainly enhance our knowledge in a much wider area. It is for this reason that we chose the biconical antenna as the model in this study because methods are now available to analyze this problem comparable to the ones used for analyzing transients in discrete networks. This report will summarize the formulation, and the result which we have obtained; and at the end, some suggestions are made about the research to be done in the future.

2. Transients on Terminated Line

Before we discuss the biconical antenna problem it is desirable to outline the methods which are available to study the transients on a terminated line. Our work on the transmission line was started as a result of a grant from the National Science Foundation. In the final stage of this work the research was also being supported by this contract. A technical report [4] has been written on this subject and a copy of this report is attached. As indicated at the end of that report, the same methods used for the transmission line analysis are also applicable to the biconical antenna.

We wish to call attention to the fact that the Volterra integral equation method, and the method of singularity expansion based on the Mittag-Leffer theorem are two new formulations which we have obtained in analyzing the transients for this class of problems. The so-called Γ-series method has also been modified to include the remainder in the series expansion. This inclusion removes the non-rigorous approach used by many authors for this problem. The work as a whole, therefore, represents a thorough treatment of the transient phenomena on an arbitrarily terminated line.

3. Input Transient Current of Thin Biconical Antenna

The theory of thin biconical antenna is well known for harminically oscillating field [5, 6]. To study the input current of such an antenna subject to an input transient voltage one can transform the harmonic solution to the Laplace-transform domain and then evaluate the inverse transform. The analysis is most conveniently done by introducing a normalized Laplace-transform variable s and a normalized time τ which are defined as follows:

$$s = \frac{j\omega\ell}{c},$$

$$\tau = \frac{tc}{\ell},$$

ℓ = half-length of the bicone

c = velocity of light in free space.

The Laplace-transform of the input current to a biconical antenna is then defined by

$$I(s) = L\,i(\tau) = \int_0^\infty i(\tau)\,e^{-s\tau}\,d\tau\,, \tag{1}$$

and the inverse transform is given by

$$i(\tau) = L^{-1}I(\tau) = \frac{1}{2\pi j}\int_{s_0-j\infty}^{s_0+j\infty} I(s)\,e^{s\tau}\,ds\,. \tag{2}$$

For a unit-step input voltage applied to the input terminal of a biconical antenna, the expression for the input current in the Laplace-transform domain can be written in the form

$$Z_c I(s) = \frac{1}{s}\left[\frac{1 - \Gamma(s)e^{-2s}}{1 + \Gamma(s)e^{-2s}}\right] \tag{3}$$

where Z_c = characteristic impedance of a biconical antenna

$$= \frac{Z_o}{\pi} \ell n \left(\cot \frac{\theta_o}{2}\right) \text{ ohms },$$

Z_o = free-space wave impedance

$= 120\pi$ ohms ,

θ_o = half-angle of the biconical antenna ,

$$\Gamma(s) = \frac{1 - y(s)}{1 - y(s)} , \tag{4}$$

$y(s)$ = normalized terminal admittance .

For thin biconical antennas ($\theta_o < 5^o$), Z_c is approximately given by

$$Z_c = \frac{Z_o}{\pi} \ell n \frac{2}{\theta_o} \tag{5}$$

and $y(s)$ is represented [5, 6] by

$$y(s) = \frac{Z_o}{4\pi Z_c}\left\{2E(2s) + e^{2s}\left[\ell n 2 + E(2s) - E(4s)\right] + e^{-2s}\left[-\ell n 2 + E(-2s)\right]\right\} \tag{6}$$

where $$E(z) = \int_o^z \frac{1 - e^{-t}}{t}\, dt .$$

The function is the same as the exponential function $Ein(z)$ denoted by Abramoweitz and Stegun [7]. It can be shown that the function $y(s)$ vanishes at $s = 0$ and is unbounded at infinity.

4. Poles of I(s)

In order to find $i(\tau)$ by means of the singularity expansion method, our first task is to find the poles of $I(s)$. According to Eq. (3), the poles of $I(s)$ are given by, in addition to $s = 0$, the roots of the equation

$$1 + \Gamma(s)\, e^{-2s} = 0 \,. \tag{7}$$

These roots will be denoted by $s_n^{(\ell)}$. The meaning of the superscript (ℓ) and the subscript n will be explained later when these roots are displayed graphically. The numerical values of these roots were first found by a rather tedious searching method, since then they have been calculated by a contour integration method due to Singaraju, Giri and Baum [8]. In evaluating the function $\Gamma(s)$, we have used the following series for the experimental function $E(z)$ for values of $|z|$ not too large

$$E(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, z^n}{n \cdot n!} \,. \tag{8}$$

For large values of $|z|$ the asymptotic expansion of $E(z)$ is used, namely,

$$E(z) = \ell n \nu \;\; - e^{-z} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(n-1)!}{z^n} \,, \tag{9}$$

where $\ell n \nu$ = Euler's constant = 0.577215664...

The distribution of $s_n^{(\ell)}$ for values of $\theta_o = 0.001^o$ and 0.573^o are shown in Figures 1 and 2. Since these roots are distributed in two distinct layers or branches, we have used the superscript ℓ to distinguish them. Figures 1 and 2 only show the roots situated in the second quadrant, the conjugate ones existed in the third conjugate are not shown. For the real root $s_1^{(2)}$ it turns out to be a simple root as ascertained by the contour integration method. For the case $\theta_o = 0.573^o$, the values of $s_n^{(1)}$ are very close to the ones found by Tesche [9] for a cylindrical antenna with a radius over height ratio equal to 1/100. When a biconical antenna is inscribed in such a cylinder $\theta_o^o = (180/\pi)\, a/h = 1.8/\pi = 0.573^o$. Except the equivalence between the roots for the first layer, we have found no similarity between our $s_n^{(2)}$ and Tesche's $s_n^{(2)}$. In fact, Tesche's calculation based on an integral equation formulation yields more than two layers while we have found only two layers. Based on the contour integration method we are reassured that there are only two layers for a thin biconical antenna. At the present moment we are unable to ascertain the significance of the different layers of these poles of I(s) nor could we explain the difference between the distributions of these poles for a biconical antenna and a cylindrical antenna of comparable dimension.

In Figure 1 and 3 we have also plotted the roots

$$1 - \Gamma(s)\, e^{-2s} = 0 . \tag{10}$$

These roots are denoted by $\hat{s}_n^{(\ell)}$. They correspond to the zeros of I(s). The link between $s_n^{(\ell)}$ and $\hat{s}_n^{(\ell)}$ will be discussed in the section dealing with the theory of receiving antenna.

5. Expansion of $Z_c I(s)$

Once the poles of I(s) are known one can expand $Z_c I(s)$ in terms of a residue series based on Mittag-Leffler theorem [10]. However, the expansion is not unique; there are at least two alternative expansions one can formulate. The first one is to write Eq. (3) in the form

$$Z_c I(s) = \frac{1}{s}\left[1 - \frac{2\Gamma(s)\, e^{-2s}}{1 + \Gamma(s)\, e^{-2s}}\right] . \tag{11}$$

Now the function

$$F(s) = \frac{-\, 2\Gamma(s)}{1 + \Gamma(s)\, e^{-2s}} \tag{12}$$

is finite at $s = 0$ and unbounded at $s \rightarrow \infty$, hence it satisfies the conditions under which Mittag-Leffler theorem holds. According to this theorem

$$F(s) = F(o) + P(s) - P(o) , \tag{13}$$

where

$$P(s) = \frac{1}{2\pi j} \oint \frac{F(t)}{s - t}\, dt$$

$$= \sum_n \frac{A_n}{s - s_n} , \tag{14}$$

where s_n denotes the simple poles of F(s), previously denoted by $s_n^{(\ell)}$, and

$$A_n = \frac{-2\Gamma(s_n)}{\frac{d}{ds}\left[1 + \Gamma(s)\, e^{-2s}\right]_{s = s_n}} = \frac{-2\Gamma(s_n)}{2 + e^{-s_n}\, \Gamma'(s_n)} \quad . \tag{15}$$

In Eq. (15), $\Gamma'(s_n)$ denotes the value of the derivative of $\Gamma(s)$ with respect to s evaluated at $s = s_n$. Thus,

$$\frac{-2\Gamma(s)}{1 + \Gamma(s)\, e^{-2s}} = -1 + \sum_n A_n \left[\frac{1}{s - s_n} + \frac{1}{s_n}\right],$$

and

$$\frac{-2\Gamma(s)}{s\left[1 + \Gamma(s)\, e^{-2s}\right]} = \frac{-1}{s} + \sum_n \frac{A_n}{s_n(s - s_n)} \quad . \tag{16}$$

The last expression results from the identify

$$\frac{1}{s(s - s_n)} = \frac{1}{s_n s} + \frac{1}{s_n(s - s_n)} \quad .$$

The singularity expansion of $Z_c I(s)$ based on Eq. (11) is therefore given by

$$Z_c I(s) = \frac{1}{s} + e^{-2s}\left[-\frac{1}{s} + \sum_n \frac{A_n}{s_n(s - s_n)}\right] \quad , \tag{17}$$

where A_n is given by Eq. (15) and s_n denotes the roots of $1 + \Gamma(s)\, e^{-2s} = 0$. The time domain solution based on Eq. (17) yields

$$Z_c i(\tau) = L^{-1}\left[Z_c I(s)\right] = U(\tau - 0) + U(\tau - 2)\left[-1 + \sum_n \frac{A_n}{s_n}\, e^{s_n(\tau - 2)}\right] \tag{18}$$

It should be recalled that s_n occurs in conjugate pairs, hence Eq. (18) may be written in the form

$$Z_c i(\tau) = U(\tau - 0) + U(\tau - 2)\left[-1 + 2\,\mathrm{Re} \sum_n{}' \frac{A_n}{s_n} e^{s_n(\tau - 2)}\right] . \tag{19}$$

The prime on the summation sign in Eq. (19) means that the sum is taken for these poles lying in the second quadrant in the s-plane. For convenience we shall identify Eq. (18) as Solution (A). This solution shows clearly that the casuality condition is met, namely, for $\tau < 2$ $Z_c i(\tau) = U(\tau - 0)$, corresponding to the initial response of the antenna before the reflected wave from the terminals of the antenna reaches the input end.

An alternative expansion of $Z_c I(s)$ is to split $Z_c I(s)$, Eq. (3), into two terms of the form

$$Z_c I(s) = \frac{1}{s\left[1 + \Gamma(s) e^{-2s}\right]} - \frac{\Gamma(s) e^{-2s}}{s\left[1 + \Gamma(s) e^{-2s}\right]} . \tag{20}$$

By expanding the function $1/\left[1 + \Gamma(s) e^{-2s}\right]$ as before, one obtains

$$Z_c I(s) = \frac{1}{2s} + \sum_n \frac{b_n}{s_n(s - s_n)} + e^{-2s}\left[\frac{-1}{2s} + \sum_n \frac{c_n}{s_n(s - s_n)}\right], \tag{21}$$

where
$$b_n = \frac{1}{2 + e^{-2s}\;\Gamma'(s_n)} ,$$

$$c_n = \frac{-\;(s_n)}{2 + e^{-2s_n}\;\Gamma'(s_n)} = \frac{1}{2} A_n ,$$

A_n being defined by Eq. (15). The corresponding time domain solution is then given by

$$Z_c i(\tau) = U(\tau - 0)\left[\frac{1}{2} + 2\,\mathrm{Re} \sum_n{}' \frac{b_n}{s_n} e^{s_n \tau}\right]$$

$$+ U(\tau - 2)\left[-\frac{1}{2} + 2\,\mathrm{Re} \sum_n{}' \frac{c_n}{s_n} e^{s_n(\tau - 2)}\right] . \tag{23}$$

We shall identify Eq.(23) as Solution (B). This solution does not offer the immediate impression that the casuality condition is satisfied unless the function within the bracket attached to the unit-step function $U(\tau - 0)$ is numerically equal to unity for $2 > \tau > 0$. However, numerical calculation based on these two alternative solutions seems to support this identity as will be presented graphically later.

Before we conclude this section, another alternative solution should be mentioned. This solution is based on the Γ- series method as discussed in the attached paper on transmission line transients [4]. The only difference is that Γ(s) is a more complicated function of s for the antenna problem. According to this method we can write $Z_c I(s)$ is the form

$$Z_c I(s) = \frac{1}{s}\left[1 - \Gamma(s)\, e^{-2s}\right] \sum_{n=0}^{\infty} \left[- \Gamma(s)\, e^{-2s}\right]^n$$

$$= \frac{1}{s}\left[1 - 2\,\Gamma(s)\, e^{-2s} + 2\Gamma^2(s)\, e^{-4s} + \ldots\right] \quad . \qquad (24)$$

For $\tau < 4$, the time domain solution is given by

$$Z_c i(\tau) = U(\tau - 0) - L^{-1}\left[\frac{2\,\Gamma(s)}{s}\, e^{-2s}\right] \quad . \qquad (25)$$

In view of the integral equation method [4] the higher order solution for $\tau > 4$ can be obtained by a successive integration of the low order solutions. It is therefore sufficient to discuss the solution represented by Eq. (25), which will be designated as Solution (C). To evaluate the inverse Laplace-transform of the term contained in Eq. (25) we will expand the function Γ(s) in the form of a residue series. Since $\Gamma(s) = \frac{1 - y(s)}{1 + y(s)}$ as defined by Eq. (4), the poles of Γ(s) are given by the roots of the equation

$$1 + y(s) = 0 \quad . \qquad (26)$$

These roots will be denoted by s_{ny}. The distribution of these roots is shown in Figure 4, for $\theta_o = 0.001^o$ and 0.573^o. There is only one layer. The roots also appear in conjugate pairs. Only those in the second quadrant are shown in the figure. Comparing to $s_n^{(\ell)}$, they are distributed more or less like $s_n^{(2)}$, but not like $s_n^{(1)}$. The singularity expansion of the function $-2\,\Gamma(s)/s$ contained in Eq. (25) can be obtained by applying the Mittag-Leffler theorem to the function $\Gamma(s)$. The results yields

$$\frac{-2\,\Gamma(s)}{s} = -2\left[\frac{1}{s} + \sum_n \frac{d_n}{s_{ny}(s - s_{ny})}\right], \tag{27}$$

where $d_n = \dfrac{2}{y'(s_{ny})}$. Thus the time-domain solution as described in Eq. (25) is given by

$$Z_c\, i(\tau) = U(\tau - 0) - 2\,U(\tau - 2)\left[1 + 2\,\mathrm{Re}\sum{}' \frac{d_n}{s_{ny}}\, e^{s_{ny}(\tau - 2)}\right] \tag{28}$$

Comparing with Solutions (A) and (B) as represented by Eqs. (19) and (23) we see that Solution (C) involves a completely different set of singularities. This immediately raises the question as to which set is more desirable or preferable? From an analytical point of view it seems the question cannot be answered definitively. In fact, it is even difficult to attach much physical significance to these sets of poles. Facing this dilemma we could only accept these alternative solutions as equally valid. A preference perhaps could be chosen if we could examine more critically the rate of convergence of these series. This work which is related to Prony's method of synehesis using an infinite set of exponential functions needs further investigation in the future.

6. Numerical Calculation

Based on Eqs. (19), (23) and (28), which will be designated as $Z_c i_A$, $Z_c i_B$ and $Z_c i_C$, we have computed these solutions for a thin biconical antenna with $\theta_o = 0.573^o$. The results are shown in Figure 5 for $\tau > 2$. In general, the wave forms based on these different representatives are comparable. It appears that $Z_c i_C$ departs considerably from $Z_c i_A$ and $Z_c i_B$ for near 2 and 4. Since the high order poles in this representation have a real part which is smaller than that of the low order poles, it is very likely that we need more terms in the series to obtain an accurate solution. For $Z_c i_B$, Eq. (23), we have also computed its values for the time interval $2 > \tau > 0$. The result is shown in Figure 6. The values are very close to unity except for τ near 0 and 2, where a phenomenon similar to the Gibbs' phenomenon in Fourier series analysis appears to be existing. This is another area where further research is needed to ascertain the degree of overshooting.

7. Biconical Antenna as a Receiving Antenna

In this section we give a brief discussion of the formulation using thin biconical antenna as a receiving antenna.

According to the equivalent circuit of a receiving antenna the terminal current of a receiving antenna in the Laplace-transform domain can be written in the form

$$I_L(s) = \frac{[\bar{E}^i(s) \cdot \bar{h}(s)]}{Z_i(s) + Z_L(s)} , \tag{29}$$

where

$\bar{E}^i(s)$ = incident electric field,

$\bar{h}(s)$ = effective height of the receiving antenna,

$Z_i(s)$ = input impedance of the receiving antenna,

$Z_L(s)$ = load impedance.

All these parameters are defined in the Laplace-transform domain. For a thin biconical antenna we can express $Z_i(s)$ in terms of $\Gamma(s)$, i.e.,

$$Z_i(s) = \frac{1 + \Gamma(s) e^{-2s}}{1 - \Gamma(s) e^{-2s}} Z_c . \tag{30}$$

The poles of $I_L(s)$, therefore, are found by setting $Z_i(s) + Z_L(s) = 0$, or

$$\frac{1 + \Gamma(s) e^{-2s}}{1 - \Gamma(s) e^{-2s}} + z_L(s) = 0, \tag{31}$$

where $z_L(s)$ denotes the normalized load impedance. For a purely resistive load we let $z_L(s) = a$ to be a real positive constant then Eq. (31) can be written in the form

$$\Gamma(s) e^{-2s} = \frac{a+1}{a-1} . \tag{32}$$

We shall denote the roots of Eq. (32) by $\tilde{s}_n^{(\ell)}$.

In the special case when $a = 0$, corresponding to a short circuit terminal, $\tilde{s}_n^{(\ell)}$ becomes the same as $s_n^{(\ell)}$ defined previously. For $a \longrightarrow \infty$, corresponding to an open circuit terminal, $\tilde{s}_n^{(\ell)}$ are identical to the zeros of the input impedance function. Like $s_n^{(\ell)}$, there are two layers of these zeros. Figure 7 shows a plot of $\tilde{s}_n^{(\ell)}$ for a thin biconical antenna with $\theta_o = 0.001^o$ for different values of a . It is interesting to observe that these contours do not cross each other in the complex s domain. For $\tilde{s}_n^{(1)}$, corresponding to the curve at the bottom right corner, the curve splits into two branches when a is larger than 1.145. This is analogous to the condition of critical damping for a series R-L-C circuit.

As far as the time domain solution is concerned, the response, of course, depends on our knowledge of $\overline{h}(s)$ and the given functional form of $\overline{E}^i(s)$. If we assume $\overline{E}^i(s) \cdot \overline{h}(s)$ to be equal to $1/s$, a unit-step voltage excitation, then the response can be found in a similar manner as the transmitting case. From the distribution of $\tilde{s}_n^{(\ell)}$, it seems reasonable to predict that the transient response would be weak when the load impedance is approximately matched to the characteristic impedance of the antenna. This corresponds to the value of 'a' in the neighborhood of 1.02.

8. Conclusion

In this report we have discussed the transient input current of a thin biconical antenna based on three distinct methods: (i) the method of Γ-series, (ii) the Volterra integral equation method and (iii) the method of singularity expansion (SEM). In the case of the last method, there are two sets of poles one can use to formulate the problem. Both formulations yield comparable results. There is no simple criterion for us to judge which set of poles are physically more meaningful or mathematically more convenient. Convergence rate of the series involved may be used as a criterion. This property still needs further investigation.

From the point of view of Prony's method of synthesis, our work shows that the representation of a function by an infinite set of exponential functions is certainly not unique. The appearance of a Gibbs' phenomena for a non-periodic function also deserves investigation. We hope that these topics will be considered in our future work dealing with the transient response of antennas.

9. Acknowledgement

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10. References

1. Felsen, L. B., Editor, Transient Electromagnetic Fields, Springer-Verlag, Berlin, (1976).

2. Cauer, W., Synthesis of Linear Communication Networks, Vols. I and II, McGraw-Hill, (1958).

3. Guillemin, E.A., Synthesis of Passive Networks, John Wiley, (1957).

4. Tai, C.T., "Transients on Lossless Terminated Lines", A technical report issued by the Radiation Laboratory, University of Michigan, (1976).

5. Schelkunoff, S.A., "Theory of Antennas of Arbitrary Size and Shape", Proc. I.R.E., 29, p. 493, (1941).

6. Tai, C.T., "Theory of Biconical Antennas", J. of Appl. Phys., 19, p. 1155, (1948); see also, King, R. W. P., Theory of Linear Antennas, Harvard University Press, Chap. VIII, (1956).

7. Abramowitz, M. and I.A. Stegun, Handbook of Mathematical Functions, U.S. Government Printing Office, (1964).

8. Singaraju, B.K., D.V. Giri and C.E. Baum, "Further Developments in the Application of Contour Integration to the Evaluation of the Zeros of Analytic Functions and Relevant Computer Programs", Mathematics Notes 42, Air Force Weapons Laboratory, Kirtland Air Force Base, New Mexico, (1976).

9. Tesche, F.M., "On the Singularity Expansion Method as Applied to Electromagnetic Scattering from Thin Wires", IEEE Trans. A-P 21, p. 53, (1973).

10. Whittaker, E.T. and G.N. Watson, Modern Analysis, Cambridge University Press, p. 143, (1943).

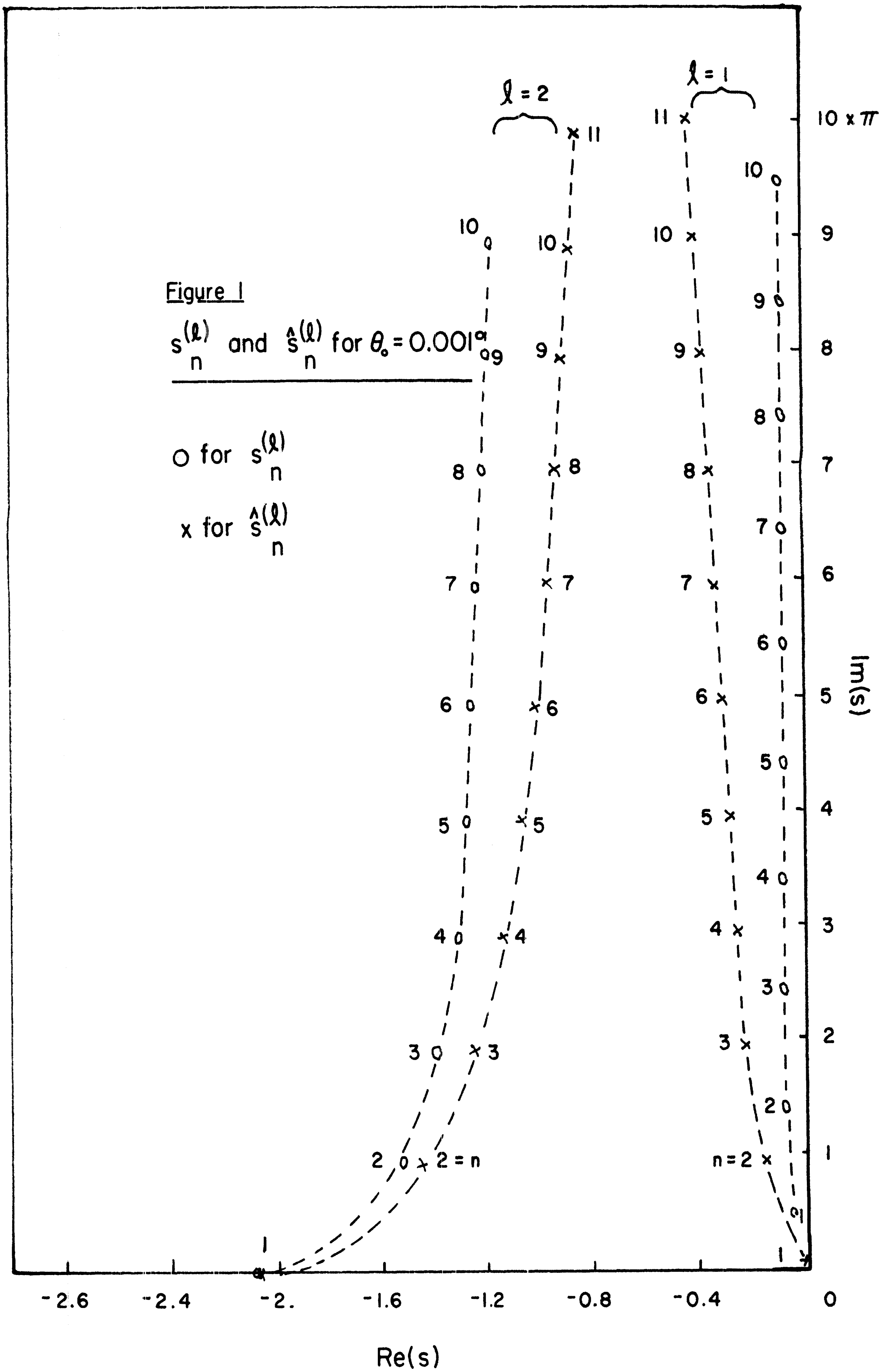
Figure 1
s(ℓ)n and ŝ(ℓ)n for θ₀ = 0.001
o for s(ℓ)n
x for ŝ(ℓ)n
ℓ = 2
ℓ = 1
n = 2
2 = n
10 x π
Im(s)
Re(s)
-2.6
-2.4
-2.
-1.6
-1.2
-0.8
-0.4
0

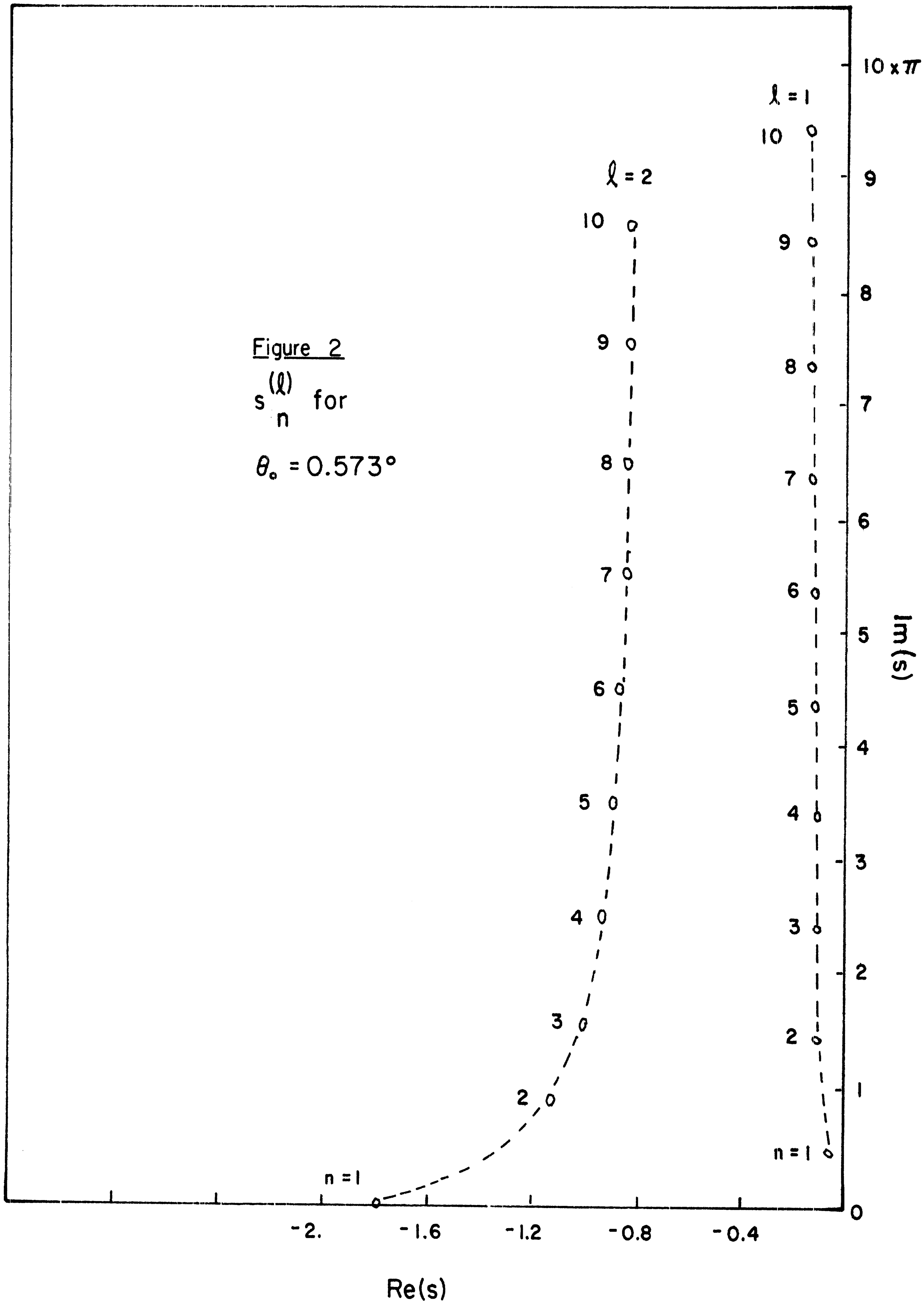

Figure 2

$s_n^{(\ell)}$ for

$\theta_0 = 0.573°$

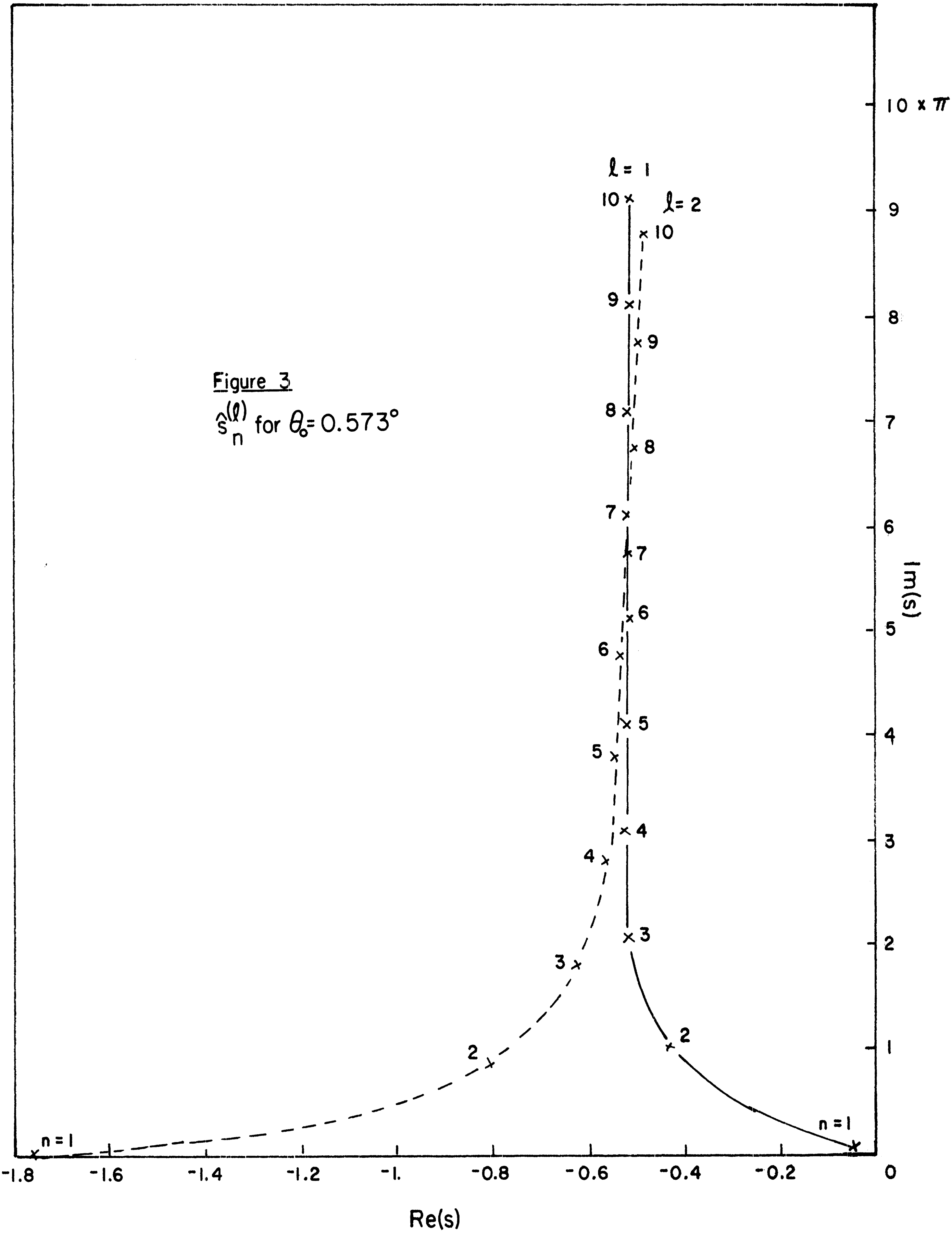

Figure 3

$\hat{s}_n^{(\ell)}$ for $\theta_0 = 0.573°$

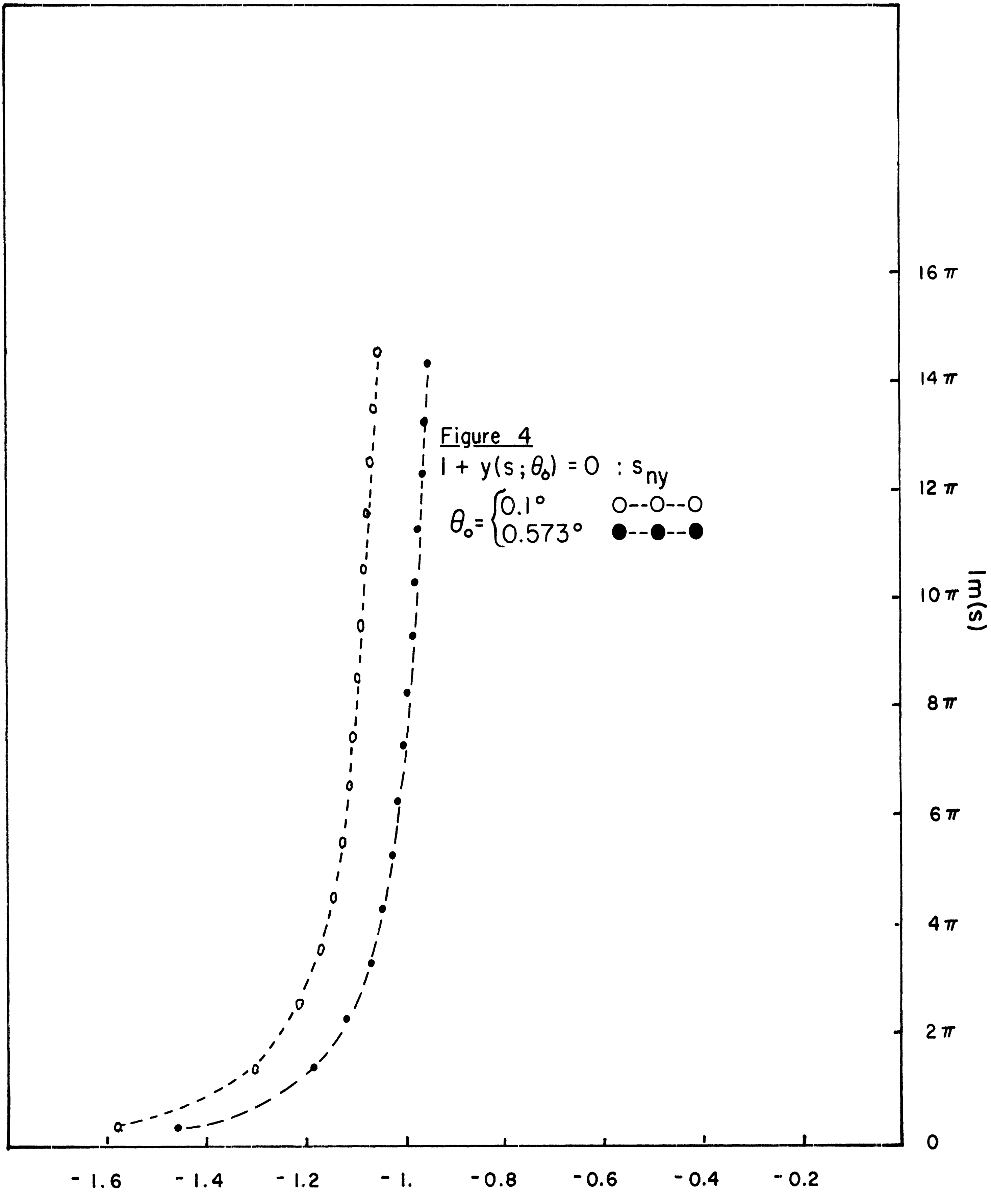
Figure 4
1 + y(s; θ₀) = 0 : s_ny
θ₀ = 0.1° o--o--o
0.573° ●--●--●
16 π
14 π
12 π
10 π
8 π
6 π
4 π
2 π
0
Im(s)
- 1.6
- 1.4
- 1.2
- 1.
-0.8
-0.6
-0.4
-0.2
Re(s)

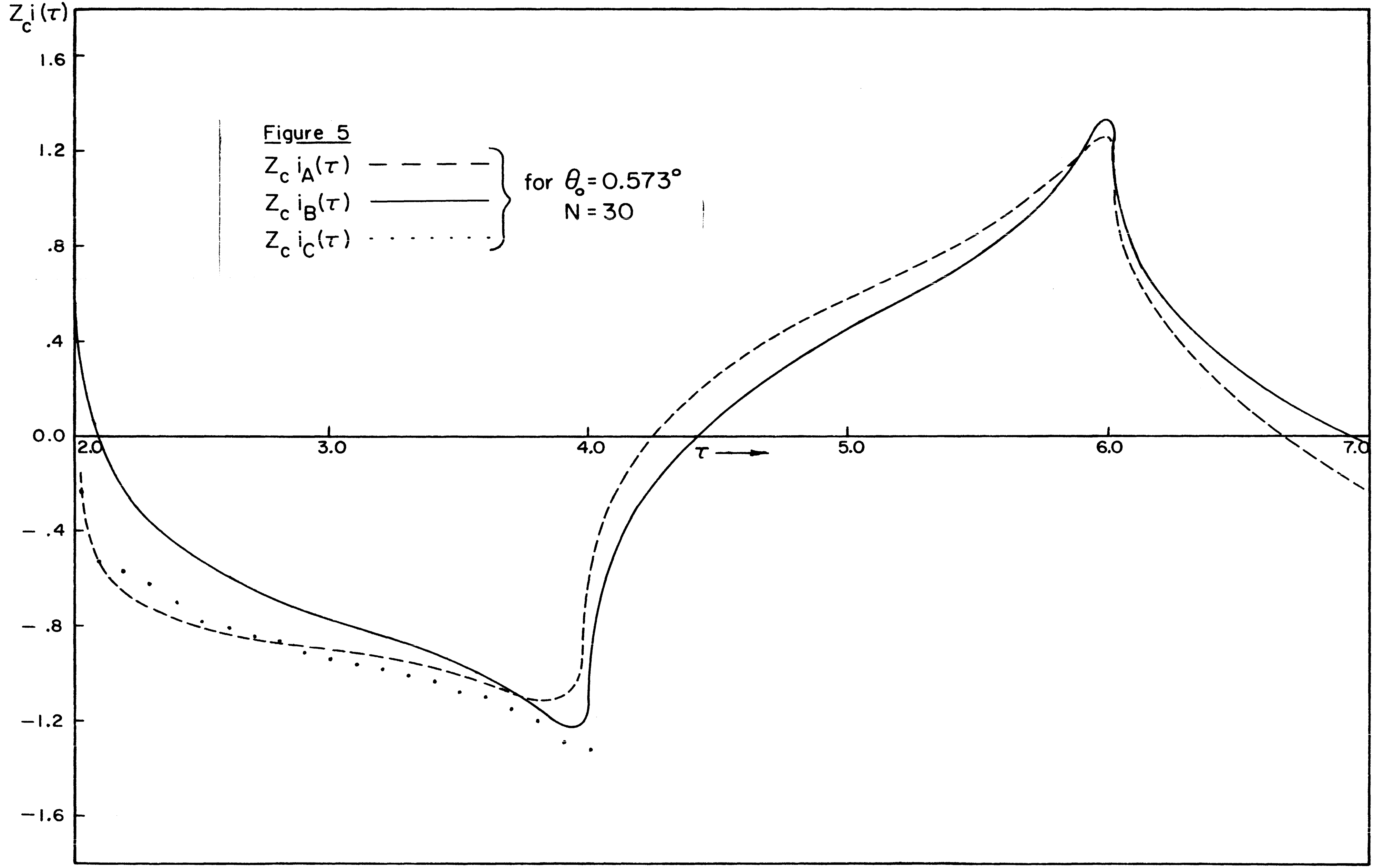
Figure 5
Zc iA(τ)
Zc iB(τ)
Zc iC(τ)
for θo = 0.573°
N = 30
Zc i(τ)
1.6
1.2
.8
.4
0.0
− .4
− .8
−1.2
−1.6
2.0
3.0
4.0
5.0
6.0
7.0
τ

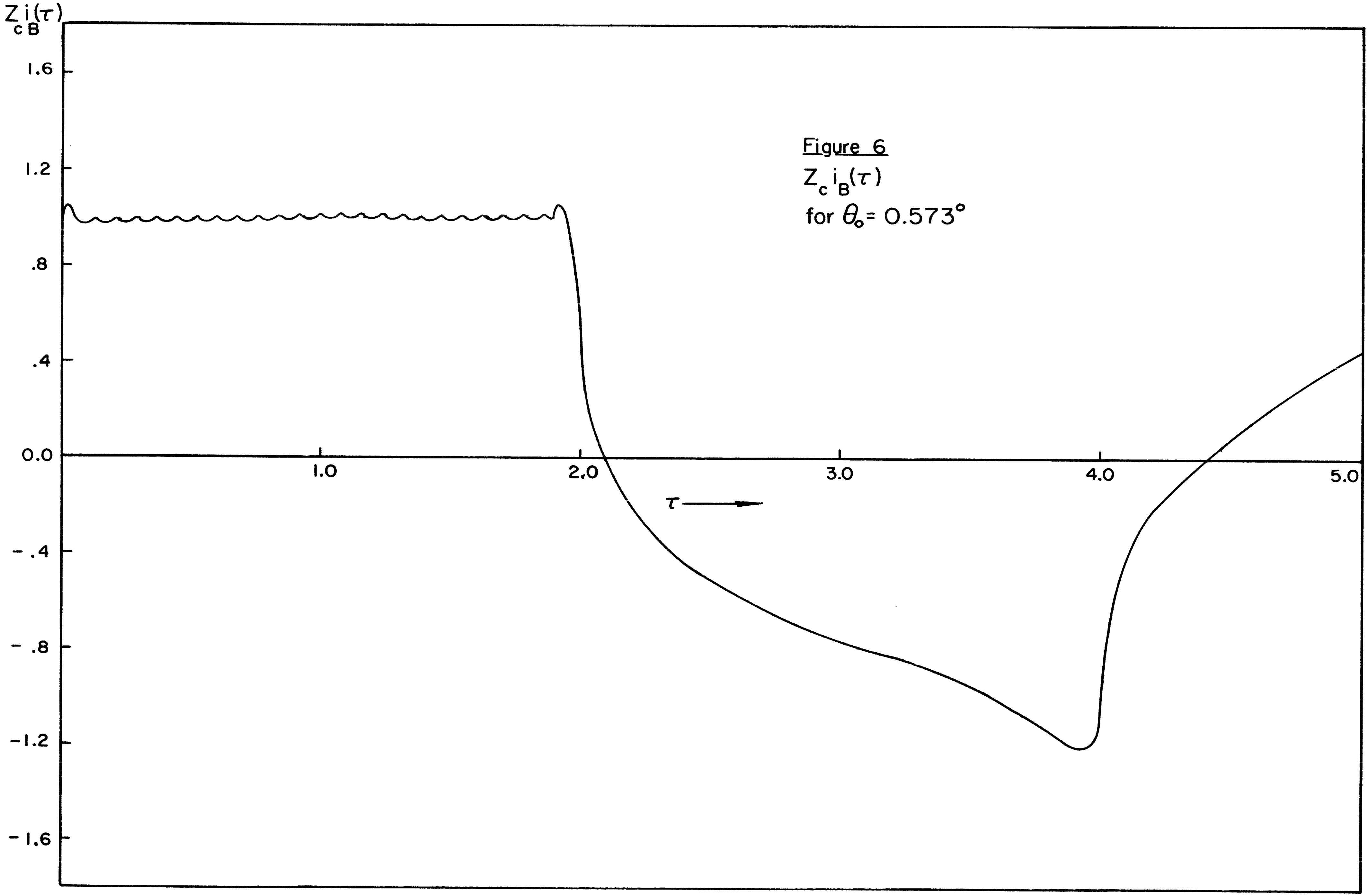

Figure 6
$Z_c i_B(\tau)$
for $\theta_o = 0.573°$

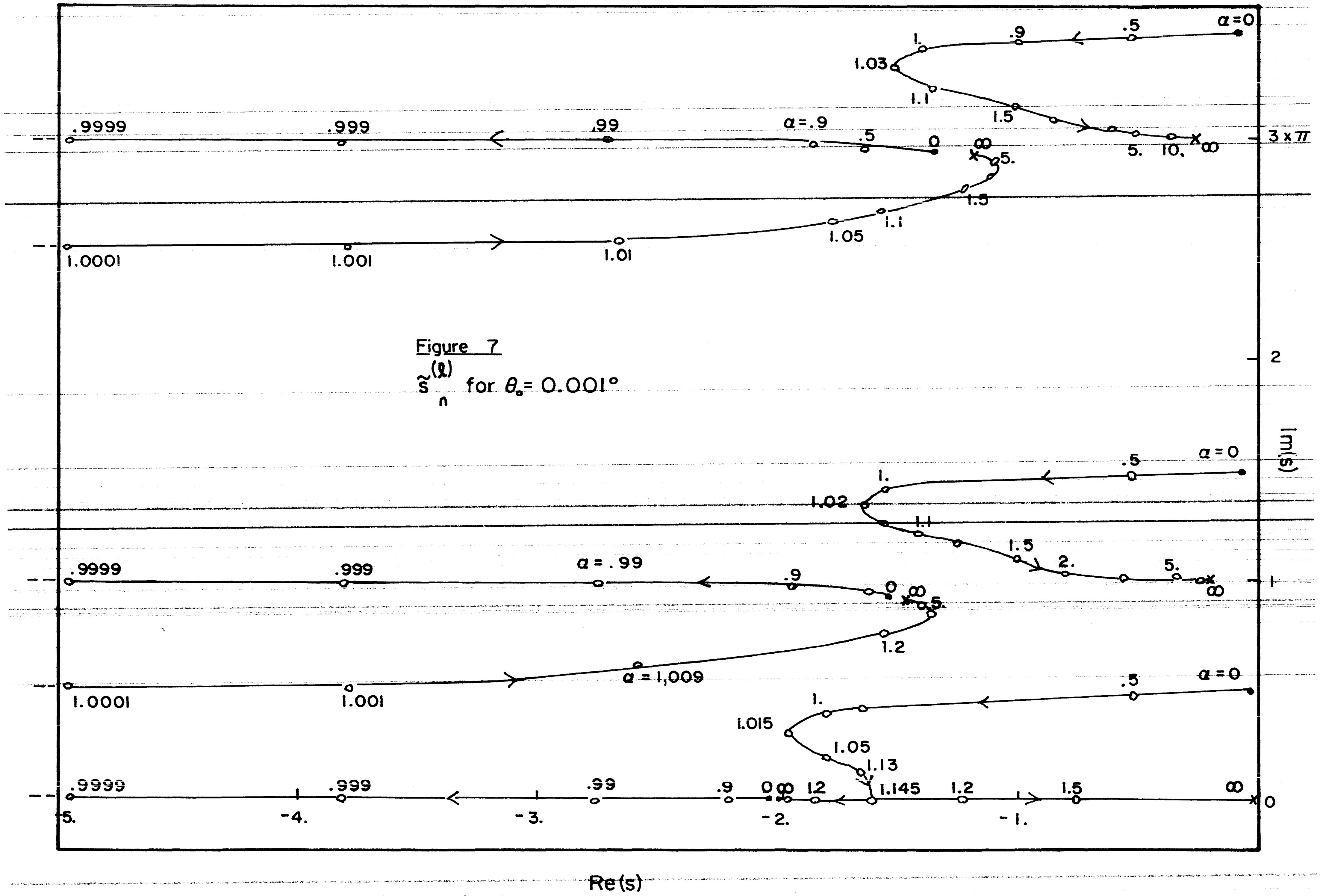

Figure 7

$\tilde{s}_n^{(\ell)}$ for $\theta_0 = 0.001°$

Transients on Lossless Terminated Transmission Lines*

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Abstract

This work contains a general investigation of the methods in analyzing the transients on lossless terminated transmission lines. After reviewing the conventional method we present two alternative methods one in the form of a Volterra integral equation and another corresponding to the so-called singularity expansion method. Based on these methods several specific problems are treated in detail. For the case of a short-circuited termination it is shown that all the solutions are equivalent although they appear different in analytical form. For a resistively terminated line we have observed a Gibbs' phenomenon associated with the series solution obtained by the singularity expansion method although the series is by no means an ordinary Fourier series. Application of these methods to study transients on biconical antennas is briefly outlined.

Introduction

The excitation and the propagation of transients on transmission lines have been studied by several authors. In particular we like to mention the work by Levinson [1], Bewley [2], Weber [3], Kuzmetsov and Stratonovich [4]. While the formulation for lossy line terminated by an arbitrary load is known, a general solution is not available because of the difficulty in evaluating some of the inverse Laplace transforms. When the line is lossless the situation in quite different. However, no detailed treatment seems to be available for arbitrary terminations except for the case of a resistive load. It is therefore desirable to present a general treatment by which one can solve the problem for an arbitrary termination in a systematic way. The research conducted here is

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partly motivated by our desire to investigate the transient phenomena on biconical antennas which can be interpreted as a pair of biconical transmission lines terminated by a distributed load [5,6]. Before we present the general methods let us review first the conventional treatment for a pair of lossless lines terminated by a resistive load.

Conventional Method of Treating a Lossless Line Terminated by a Resistive Load

We consider a pair of lossless lines terminated by an impedance load Z shown in Figure 1. The lines are assumed to be excited by an unit step voltage at the input end.

For convenience we introduce several normalized variables defined as follows:

ξ = x/ℓ = normalized distance

τ = tc/ℓ = normalized time

where ℓ = length of the line

c = velocity of propagation on the lossless line being equal to $1/(L'C')^{1/2}$

L', C' = inductive and capacitive line constants of the line

s = $\sigma\ell/c$ = normalized Laplace tranform variable

σ = the ordinary or conventional Laplace transform variable being equal to $j\omega$ where ω denotes the complex angular frequency

In terms of these normalized variables we deonte

$v(\xi,\tau)$ = instantaneous line voltage

$i(\xi,\tau)$ = instantaneous line current

$V(\xi,s)$ = Laplace transform of $v(\xi,\tau)$

$$= \mathcal{L}[v(\xi,\tau)] = \int_0^\infty v(\xi,\tau)e^{-s\tau}d\tau$$

$I(\xi,s)$ = Laplace transform of $i(\xi,\tau)$

$$= \mathcal{L}[i(\xi,\tau)] = \int_0^\infty i(\xi,\tau)e^{-s\tau}d\tau$$

For a unit step voltage applied at the input end we have

$$v(0,\tau) = U(\tau - 0)$$

hence

$$V(0,s) = \int_0^\infty U(\tau - 0)e^{-st} = \frac{1}{s}$$

In terms of these normalized variables and V(0,s) the line voltage and the line current in the Laplace transform domain can be written in the form

$$V(\xi,s) = \frac{e^{-\xi s} + \Gamma(s)e^{-(2 - \xi)s}}{s[1 + \Gamma(s)e^{-2s}]} \tag{1}$$

$$Z_c I(\xi,s) = \frac{e^{-\xi s} - \Gamma(s)e^{-(2 - \xi)s}}{s[1 + \Gamma(s)e^{-2s}]} \tag{2}$$

where Z_c denotes the characteristic impedance of the line, being equal to $(L'/C')^{1/2}$, and $\Gamma(s)$ the voltage reflection coefficient defined in the s-domain at the output end.

The conventional method of determining $v(\xi,\tau)$ or $i(\xi,\tau)$ is to express (1) or (2) in a series using the expansion

$$\frac{1}{1 + \Gamma(s)e^{-2s}} = \sum_{n=0}^{\infty} [-\Gamma(s)e^{-2s}]^n \tag{3}$$

Substituting (3) into (1) we have

$$V(\xi,s) = \frac{1}{s}[e^{-\xi s} + \Gamma(s)e^{-(2 - \xi)s}] \sum_{n=0}^{\infty} [-\Gamma(s)e^{-2s}]^n \tag{4}$$

For a resistive load $\Gamma(s)$ is a real constant which will be denoted by Γ and its value is given by

$$\Gamma = \frac{r - 1}{r + 1}$$

where

$r = R/Z_c$

R = load resistance

Z_c = characteristic impedance of the line

The inverse Laplace transform of (4) with $\Gamma(s) = \Gamma$ yields

$$v(\xi,\tau) = \sum_{n=0}^{\infty} [(-\Gamma)^n U(\tau - 2n + \xi) - (-\Gamma)^{n+1} U(\tau - 2n - 2 + \xi)] \quad (5)$$

where $U(\tau-\tau_n)$ denotes an unit step function commencing at $\tau = \tau_n$. The solution can be displayed most conveniently by a characteristic diagram as shown in Figure 2.

Although (5) is known to be a valid solution by physical reasoning its derivation is considered to be unsatisfactory from the mathematical point of view because expansion (3) holds true only if $|\Gamma(s)e^{-2s}| < 1$, and in executing the inverse Laplace transform the contour of integration lies in the left-half plane where $|\Gamma(s)e^{-2s}|$ could exceed unity. This conventional method of finding $v(\xi,\tau)$, however, has been adopted by many authors including Levinson [1] and Weber [2].

One way of removing this weak step is to expand the same function in terms of a finite series with a remainder, i.e.,

$$\frac{1}{1 + \Gamma(s)e^{-2s}} = \sum_{n=0}^{N} [-\Gamma(s)e^{-2s}]^n - \frac{[-\Gamma(s)e^{-2s}]^{N+1}}{1 + \Gamma(s)e^{-2s}} \quad (6)$$

when substituting (6) into (1) the remainder would yield a term of the form

$$\frac{[-\Gamma(s)]^{N+2}}{s[1 + \Gamma(s)e^{-2s}]} e^{-s[2(N + 2) - \xi]} \quad (7)$$

Because of the shifting theorem the inverse Laplace transform of (7) vanishes when $\tau < 2(N+2) - \xi$. In other words if one evaluates the series (5) up to $\tau<2(N+2) - \xi$ the remaining terms do not enter the picture. From this point of view (5) is an exact solution since N can be fixed any value. From now on we will designate the solution based on (4) as the Γ - series solution. In addition to the Γ - series method there are two alternative methods to formulate the problem and to find the solution. One is an integral equation method and another is the so-called singularity expansion method or the method of residues.

Integral Equation Method

We consider the general case where $\Gamma(s)$ is a function of s. If (1) is multiplied by $1 + \Gamma(s)e^{-2s}$ the following equation results

$$V(\xi,s) + \Gamma(s)e^{-2s}V(\xi,s) = \frac{1}{s}[e^{-\xi s} + \Gamma(s)e^{-(2-\xi)s}] \quad (8)$$

By taking the inverse Laplace transform of (8) we obtain

$$v(\xi,\tau) = \mathscr{L}^{-1}[-\Gamma(s)e^{-2s}V(\xi,s)] + v_o(\xi,\tau) \quad (9)$$

where

$$v_o(\xi,\tau) = \mathscr{L}^{-1}\frac{1}{s}[e^{-\xi s} + \Gamma(s)e^{-(2-\xi)s}]$$

$$= v_{of}(\xi,\tau) + v_{ob}(\xi,\tau) \quad (10)$$

with

$$v_{of}(\xi,\tau) = \mathscr{L}^{-1}[\frac{e^{-\xi s}}{s}] = U(\tau - \xi) \quad (11)$$

$$v_{ob}(\xi,\tau) = \mathscr{L}^{-1}[\frac{\Gamma(s)}{s}e^{-(2-\xi)s}] \quad (12)$$

$v_{of}(\xi,\tau)$ represents the initial forward wave propagating on the line and $v_{ob}(\xi,\tau)$ represents the first reflected wave or backward wave from the termination. For a given $\Gamma(s)$ we assume (12) can be evaluated. Thus $v_o(\xi,\tau)$ is a known function. On account of the convolution theorem in the theory of Laplace transform (9) can be written in the form

$$v(\xi,\tau) = \int_0^\tau k\,(\tau - \tau')v(\xi,\tau')d\tau' + v_o(\xi,\tau) \tag{13}$$

where

$$k(\tau) = \mathscr{L}^{-1}[-\Gamma(s)e^{-2s}] \tag{14}$$

Equation (13) with $v(\xi,\tau)$ as the unknown function corresponds to the Volterra integral equation of the second kind. Its solution is given by Picards' series [7], namely,

$$v(\xi,\tau) = \sum_{n=0}^{\infty} v_n(\xi,\tau) \tag{15}$$

where

$$v_n(\xi,\tau) = \int_0^\tau k(\tau - \tau')v_{n-1}(\xi,\tau')d_\tau' \tag{16}$$

$$n = 1,2,...$$

In the case $\Gamma(s)$ is a real constant, previously denoted by Γ, we obtain, from (12),

$$v_{ob}(\xi,\tau) = \Gamma U(\tau - 2 + \xi)$$

hence

$$v_o(\xi,\tau) = U(\tau - \xi) + \Gamma U(\tau - 2 + \xi) \tag{17}$$

and from (14) we have

$$k(\tau) = -\Gamma\delta(\tau - 2) \tag{18}$$

where $\delta(\tau - 2)$ denotes the delta function defined at $\tau = 2$. Substituting (17) and (18) into (16) we obtain

$$v_1(\xi,\tau) = -\Gamma U(\tau - 2 - \xi) - \Gamma^2 U(\tau - 4 + \xi)$$

$$v_2(\xi,\tau) = \Gamma^2 U(\tau - 4 + \xi) + \Gamma^3 U(\tau - 6 + \xi)$$

...

The solution represented by (15) is obviously the same as (5).

To illustrate the application of the integral equation method to more complicated terminations let us consider the case where the terminal load consists of a series R-L lumped circuit then

$$\Gamma(s) = \frac{z(s) - 1}{z(s) + 1}$$

where

$$z(s) = \frac{1}{Z_c}[R + j\omega L) = \frac{1}{Z_c}[R + s(\frac{c}{\ell})L]$$

$$= r + \alpha s$$

$$r = R/Z_c \text{ , } \alpha = \frac{cL}{Z_c\ell} = \frac{L}{L'\ell}$$

$$L' = \text{inductive line constant}$$

The coefficient α is a measure of the load inductance in terms of the total line inductance. The reflection coefficient $\Gamma(s)$ can now be written in the form

$$\Gamma(s) = \frac{s - s_o}{s - s_1} \tag{19}$$

where

$$s_o = -\frac{(r - 1)}{\alpha}, \quad s_1 = -\frac{(r + 1)}{\alpha}$$

thus

$$\frac{\Gamma(s)}{s} = \frac{1}{s}(\frac{s - s_o}{s - s_1}) = \frac{\rho}{s} + \frac{1 - \rho}{s - s_1} \tag{20}$$

where

$$\rho = \frac{s_o}{s_1} = \frac{r - 1}{r + 1}$$

Using (12) and (14) one finds

$$v_{ob}(\xi,\tau) = U(\tau - 2 + \xi)[\rho + (1 - \rho)e^{s_1(\tau - 2 + \xi)}] \qquad (21)$$

$$k(\tau) = - \delta(\tau - 2) - U(\tau - 2)[(1 - \rho)s_1 e^{s_1(\tau - 2)}] \qquad (22)$$

Knowing $v_{ob}(\xi,\tau)$ and $k(\tau)$ we can find $v_1(\xi,\tau)$, using (16). The result is given below:

$$v_1(\xi,\tau) = - U(\tau - 2 - \xi)[\rho + (1 - \rho)e^{s_1(\tau - 2 - \xi)}]$$

$$+ U(\tau - 4 + \xi)\{\rho^2 + [1 - \rho^2 + (1 - \rho)^2 s_1(\tau - 4 + \xi)e^{s_1(\tau - 4 + \xi)}]\} \qquad (23)$$

The successive terms of $v_n(\xi,\tau)$ for $n \geq 2$ can be found accordingly.

The same result, of course, can also be found by the Γ-series method. If we follow this route then it is necessary to expand the function $[\Gamma(s)]^n/s$ in partial fractions; after that the inverse Laplace transform of (4) can be evaluated.

One unique feature of the integral equation method should be pointed out. It concerns the relationship between $v_{ob}(\xi,\tau)$ and $k(\tau)$, the kernel of the integral equation. Since

$$V_{ob}(\xi,s) = \frac{1}{s}\Gamma(s)e^{-(2 - \xi)s}$$

and

$$K(s) = \mathcal{L}[k(\tau)] = -\Gamma(s)e^{-2s}$$

so

$$K(s) = -sV_{ob}(0,s)$$

thus

$$k(\tau) = -\frac{\partial V_{ob}(0,\tau)}{\partial \tau} \qquad (24)$$

where we interpret the derivative in the generalized sense that for a discontinuous function

$$\frac{\partial}{\partial \tau} U(\tau - 0) = \delta(\tau - 0) \qquad (25)$$

Equation (24) suggests that once the characteristics of $V_{ob}(\xi,\tau)$, the first reflected wave, is known one can determine the kernel of the integral equation, subsequently the complete solution. As an example, from (21) one finds

$$-\frac{\partial v_{ob}(0,\tau)}{\partial\tau} = -\delta(\tau - 2) - U(\tau - 2)[(1 - \rho)s_1 e^{s_1(\tau - 2)}]$$

which is the same as $k(\tau)$ given by (22). This completes our discussion of the integral equation method. Our next topic deals with the singularity expansion method or the residue method.

The Singularity Expansion Method Supplied to a Short-Circuit Line

The terminology of this method was suggested by Baum [8] in his work dealing with the scattering of electromagnetic waves by objects. This method when applied to transmission lines was treated previously by Weber [9] for a short-circuit terminiation. For our purpose we shall present our treatment in a different manner emphasizing the casuality condition which is inherently in our solution. We consider just (1) with $\Gamma(s) = -1$, corresponding to a short-circuit termination, then

$$V(\xi,s) = \frac{e^{-\xi s} - e^{-(2 - \xi)s}}{s(1 - e^{-2s})} \tag{26}$$

Because of the retarded factors $e^{-\xi s}$ and $e^{-(2 - \xi)s}$ we will take advantage of the shifting theorem and treat $V(\xi,s)$ as consisting of two terms with

$$V_1(\xi,s) = \frac{e^{-\xi s}}{s(1 - e^{-2s})} \tag{27}$$

and

$$V_2(\xi,s) = \frac{-e^{-(2 - \xi)s}}{s(1 - e^{-2s})} \tag{28}$$

Observing that both (27) and (28) have a double pole at $s = 0$ and simple poles at $s_n = jn\pi$, $n = \pm1, \pm2, \ldots$ By means of Mittag-Leffler theorem [10] we can derive the following expansion

$$\left(\frac{1}{1 - e^{-2s}} - \frac{1}{2s}\right) = \frac{1}{2}\left[1 + \sum_{n=\pm1}^{\pm\infty}\left(\frac{1}{s - s_n} + \frac{1}{s_n}\right)\right]$$

where

$$s_n = jn\pi,$$

hence

$$\frac{1}{s(1 - e^{-2s})} = \frac{1}{2}\left[\frac{1}{s^2} + \frac{1}{s} + \sum_{n=\pm1}^{\pm\infty}\frac{1}{s_n(s - s_n)}\right]$$

As a result of the shifting theorem we find

$$v_1(\xi,\tau) = \mathscr{L}^{-1}[V_1(\xi,s)]$$

$$= U(\tau - \xi)\left[\frac{1}{2} + \frac{1}{2}(\tau - \xi) + \sum_{n=1}^{\infty}\frac{1}{n\pi}\sin n\pi(\tau - \xi)\right] \qquad (29)$$

similarly,

$$v_2(\xi,\tau) = -U(\tau - 2 + \xi)\left[\frac{1}{2} + \frac{1}{2}(\tau - 2 + \xi) + \sum_{n=1}^{\infty}\frac{1}{n\pi}\sin n\pi(\tau - 2 + \xi)\right] \qquad (30)$$

Except for the negative sign $v_2(\xi,\tau)$ is merely a delayed reproduction of $v_1(\xi,\tau)$. The sum of (29) and (30) gives $v(\xi,\tau)$. Now the Fourier series representation of the periodic function

$$f(\tau) = 1 - \tau, \; 2 \geq \tau \geq 0$$

is given by

$$f(\tau) = \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin n\pi\tau, \qquad (31)$$

the two functions $v_1(\xi,\tau)$ and $v_2(\xi,\tau)$ can therefore be plotted out easily as shown in Figure 3. The sum of the two functions produces the repeated square wave shown in Figure 4. The last figure checks with the characteristic diagram shown in Figure 2 if we let $\Gamma = -1$.

As we mentioned before the same problem was previously treated by Weber who obtained the solution by evaluating the inverse Laplace transform for the function $V(\xi,s)$ without separating into V_1 and V_2. His result is given by

$$v(\xi,\tau) = U(\tau - 0)\left[\sum_{n=1}^{\infty} \frac{2 \sin n\pi\xi}{n\pi} + \sum_{n=1}^{\infty} \frac{\sin n\pi(\tau-\xi)}{n\pi} + \sum_{n=1}^{\infty} \frac{\sin n\pi(\tau+\xi)}{n\pi}\right] \qquad (32)$$

In appearance his formula is quite different from ours, but if we make use of (31) it can be shown that the result is identical to ours in spite of the fact that his formula involves a unit step function starting at $\tau=0$ while ours involves $U(\tau-\xi)$ and $U(\tau-2+\xi)$. These two different approaches have important bearing to more complicated cases. In our treatment the casuality condition is automatically met, namely, $U(\xi,\tau)= 0$ for $\tau<\xi$. If Weber's approach is followed enough terms in the series must be included in the numerical calculation to ensure the vanishing of $v(\xi,\tau)$ for $\tau<\xi$. We have experienced this behavior for the case of a resistive loading with Γ different from -1.

<u>Singularity Expansion Method Applied to a Resistive Load</u>

For a resistive termination with $\Gamma(s) = \Gamma$, a constant,

$$V(\xi,s) = \frac{e^{-\xi s}+\Gamma e^{-(2-\xi)s}}{s(1+\Gamma e^{-2s})} \qquad (33)$$

For definiteness we assume Γ, being equal to $(r-1)/(r+1)$, to be a negative constant corresponding to $r<1$. The poles of (33) then are given by s = 0 and $s = s_n = \frac{1}{2}\text{Ln}|\Gamma| + j\,n\pi$, with n = ±1, ±2,···. Again, by means of Mittag-Leffler theorem one finds

$$\frac{1}{s(1 + \Gamma e^{-2s})} = \frac{1}{(1 + \Gamma)s} + \frac{1}{2}\sum_{n=\pm 1}^{\pm\infty} \frac{1}{s_n(s - s_n)} \tag{34}$$

where

$$s_n = \frac{1}{2}\text{Ln}|\Gamma| + jn\pi \,.$$

By considering

$$V(\xi,s) = V_1(\xi,s) + V_2(\xi,s) \tag{35}$$

with

$$V_1(\xi,s) = \frac{e^{-\xi s}}{s(1+\Gamma e^{-2s})} \tag{36}$$

and

$$V_2(\xi,s) = \frac{e^{-(2-\xi)s}}{s(1+\Gamma e^{-2s})} \tag{37}$$

the inverse Laplace transform of (36) and (37) yields

$$v(\xi,\tau) = U(\tau-\xi)\left[\frac{1}{1+\Gamma} + \sum_{-\infty}^{\infty} \frac{1}{2s_n} e^{s_n(\tau-\xi)}\right]$$

$$+ U(\tau-2+\xi)\left[\frac{\Gamma}{1+\Gamma} + \sum_{-\infty}^{\infty} \frac{\Gamma}{2s_n} e^{s_n(\tau-2+\xi)}\right] \qquad \cdots(38)$$

Since $\Gamma e^{-2s_n} = -1$, (38) can be simplified to

$$v(\tau,\xi) = U(\tau-\xi)\left[\frac{1}{1+\Gamma} + \sum_{-\infty}^{\infty} \frac{1}{2s_n} e^{s_n(\tau-\xi)}\right]$$

$$+ U(\tau-2+\xi)\left[\frac{\Gamma}{1+\Gamma} + \sum_{-\infty}^{\infty} e^{s_n(\tau+\xi)}\right] \cdots(39)$$

For the case $\Gamma = 0$, (39) reduces to $v(\xi,\tau) \doteq U(\tau-\xi)$ which is certainly true. Eq.(38) or (39) does not apply to $\Gamma = -1$, a special case which was treated previously. The solution represented by (39) has been computed and is shown in Figs. 5 and 6 for $\Gamma = -\frac{1}{2}$ and $\xi = \frac{1}{2}$. In Fig. 5, twenty-five terms are used to compute the curve. In Fig. 6, fifty terms are used. The result as a whole agrees with the one obtained by the Γ-series method or the integral equation method except at the points of discontinuity where the curves exhibit the Gibbs' phenomenon commonly encountered in Fourier-series analysis. To the knowledge of this author Gibbs' phenomenon associated with non-periodic discontinous functions have not been examined in the past. The subject matter is related to Prony's method of representing a discontinuous function by exponential functions with complex damping constants. The problem is currently under investigation.

Application to Transient Analysis of Biconical Antennas

The methods which we have discussed apply equal well to the transient analysis of small-angle biconical antennas. The only difference is that the reflection coefficient $\Gamma(s)$ is a transcendental function of s. In fact it is convenient to write $\Gamma(s)$ in the form of $[1-y(s)]/[1+y(s)]$ where $y(s)$ is expressible in terms of exponential integral functions of s. If the integral equation method is used to investigate the input current then we have to find the poles of $\Gamma(s)$ or the zeros of $1+y(s) = 0$ so that $i_{ob}(0,\tau)$ and $k(\tau)$ can be determined. The Picard series can then be evaluated either analytically or simply by numerical integration. If the singular expansion method is used, then the roots of $1+\Gamma(s)e^{-2s}=0$ have to be determined first in order to build the residue series for $I(0,s)$. These methods apply equally well to biconical antennas operating in the receiving mode. This work will be reported elsewhere in a separate article.

Conclusion

In this paper we have investigated various methods of analyzing transients on lossless transmission lines with arbitrary terminations. It is shown that there are three distinct methods to formulate and to solve the general problems, namely the Γ-series method, the Volterra integral equation method and the singularity expansion method, the integral equation method is potentially more appealing because from the information of the first reflected wave it is possible to construct the kernel of the integral equation and thereupon to find the complete solution based on quadrature. In principle all these methods are applicable to study the transient on thin-angle biconical autennas.

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REFERENCES

1) Norman Levinson, "The Fourier Transorm Solution of Ordinary and Partial Differential Equations," Jour of Mathematics and Physics. Vol. 14, pp. 195-227, 1935.

2) L. V. Bewley, "Traveling Waves on Transmission Systems," second edition, John Wiley & Sons, Inc., New York 1951.

3) Ernest Weber, "Linear Transient Analysis," Vol. II, John Wiley & Sons, Inc., New York 1956.

4) P. I. Kuznetsov and R. L. Stratonovich, "The Propagation of Electromagnetic Waves in Multiconductor Transmission Lines," The MacMillan Company, New York, 1964.

5) S. A. Schelkunoff, "Theory of Antennas of Arbitrary Size and Shape," Proc. I.R.E., 29, 493, 1941.

6) C. T. Tai, "On The Theory of Biconical Antennas," J. Appl. Phys., 19, 1155, 1948.

7) F. G. Tricomi, "Integral Equations," Interscience Publishers, Inc., New York, 1955.

8) Carl E. Baum, "The Singular Expansion Method," Chap. 3 of the book, "Transient Electromagnetic Fields," edited by L. B. Felsen, Springer-Verlag, Berlin, 1976.

9) Ernest Weber, loc. cit., Sec 6.9.

10) E. T. Whittaker and G. N. Watson, "Modern Analysis," Cambridge University Press, p. 134, 1943.

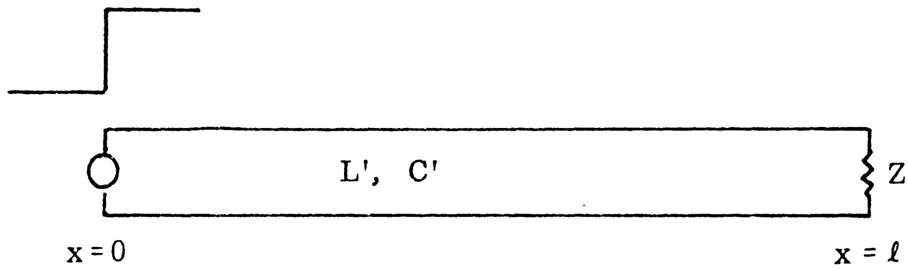

Figure 1: A terminated line excited at the input end by a unit step voltage.

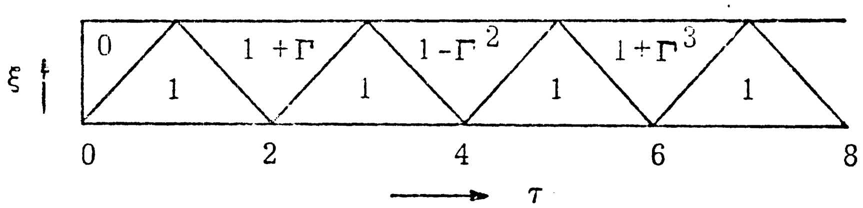

Figure 2: Characteristic diagram displaying the solution for $v(\xi, \tau)$ for a resistively terminated line.

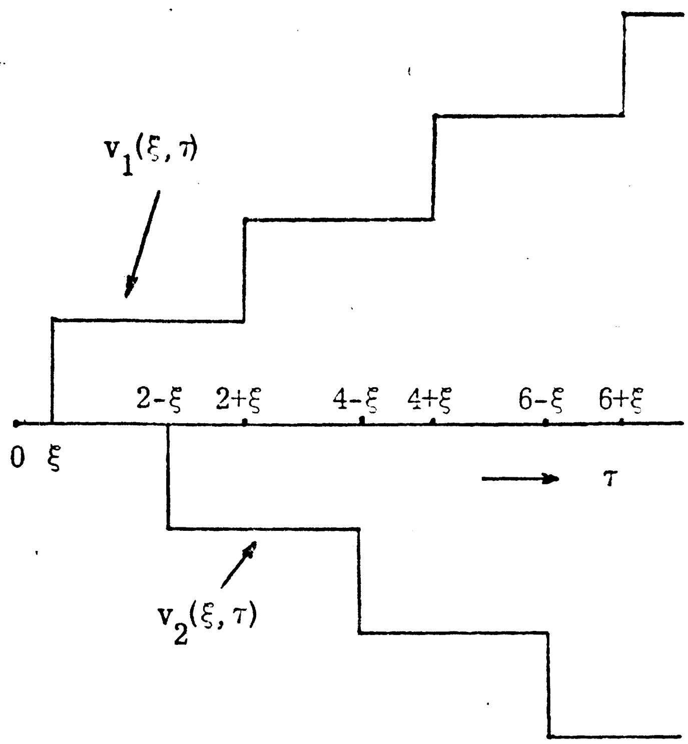

Figure 3: $v_1(\xi, \tau)$ and $v_2(\xi, \tau)$ for a short-circuit termination.

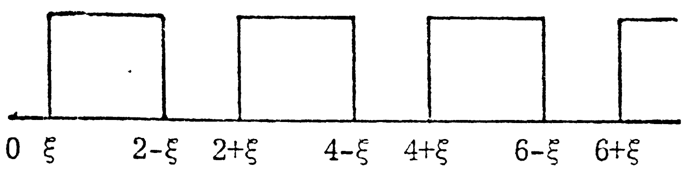

Figure 4: Sum of $v_1(\xi, \tau)$ and $v_2(\xi, \tau)$.