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Integral Equations with Reduced Unknowns for the Simulation of Two-Dimensional Composite Structures

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Abstract

A new set of integral equations with reduced unknowns is derived for the modeling of two-dimensional composite scatterers. In accomplishing this reduction the scatterer is first simulated with thin curvalinear layers of material. The traditional integral equations corresponding to each inhomogeneous layer are then manipulated in a manner allowing the identification of a new equivalent current component to replace two of the traditional ones accross the layer. A major effort in this study was devoted to a moment method implementation of the new compact set of integral equations with special attention given to the analytical evaluation of the diagonal and near diagonal elements of the impedance matrix. Several scattering patterns are presented as computed with the new compact set of integral equations. These are further compared with measured data and computations using alternative analytical techniques.

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Chapter 1

Introduction

Conventional approaches for the numerical computation of the electromagnetic scattered fields by inhomogeneous dieletric scatterers entail the formulation of a system of integral equations for the determination of the induced polarization current densities. These can then be used in the radiation integral for the computation of the scattered field. Assuming an $e^{-i\omega t}$ time convention, the electric ($\vec{J}$) and magnetic ($\vec{J}^*$) polarization current densities are traditionally defined [1] as

$$\vec{J} = -ik_0 Y_0(\varepsilon_r - 1)\vec{E}\ ,\ \vec{J}^* = -ik_0 Z_0(\mu_r - 1)\vec{H} \tag{1.1}$$

where ε_r and μ_r are the relative permittivity and permeability, k_0 is the free space wave number, $Z_0 = 1/Y_0$ is the free space intrinsic impedance and $\vec{E}$ and $\vec{H}$ denote the electric and magnetic fields within the dieletric medium. For the general three dimensional case, (1.1) involves six independent current components which must be determined usually via a numerical solution of the pertinent integral equations. However, when (1.1) are applied to a two dimensional composite scatterer, with TE or TM plane wave incidence, only three unknown current components are required for the complete characterization of the scatterer. Thus, in the numerical implementation of the pertinent integral equations $3N$ unknowns must be determined if N denotes the number of cells comprising the discretized scatterer. As is well known, the required computation time for the solution of a system via a matrix inversion method is proportional to the cubic power of the number of unknowns. Thus, any reduction in the unknowns translates into a substantial improvement in the computational efficiency/economy of the solution algorithm.

In this report we derive a new set of integral equations as applied to two dimensional composite scatterers involving a reduced number of unknowns. Particularly, it is shown that the complete modeling of the scatterer can be accomplished with two equivalent current components over its cross section versus the three current components usually required with the traditional approach [3,4,5,6,7,8]. In accomplishing this reduction we first subdivide the scatterer into thin inhomogeneous curved layers of material. The integral equations corresponding to each layer are then manipulated through various integrations-by-parts, differentiations and rearrangements. This allows the identification of a new equivalent current component to replace two of the traditional polarization currents accross the length of the layer. Namely, the axial and one of the transverse components (to the cylinder) of the polarization current are effectively replaced by a single mathematically equivalent current component. The normal component of the polarization current at the two terminations of the layer, though, cannot be eliminated. However, assuming that the extent of each layer is long, the presence of these last components do not add appreciably to the total system of unknowns. Thus, the presented modeling formulation allows a reduction of the unknown current components from $3N$ to essentially $2N$. It will be seen, though, that this reduction in unknowns is achieved at the expense of complexity in the resulting integral equations.

To demonstrate the validity and applicability of the derived integral equations to cylindrical scatterers of arbitrary cross section and composition, we have considered the numerical implementation of these via the method of moments. In this implementation we employed pulse basis expansion functions and point matching primarily for the purpose of simplifying the derivation of the matrix elements. Special attention is given in this report for the evaluation of the matrix elements and particularly the diagonal (self cell) and near-diagonal ones. As usual, the integrals defining these are associated with integrands involving the green's function which is singular at the self cell and must be evaluated via analytical means. This is accomplished by employing the small argument expansion of the green's function to obtain an integral that can be evaluated analytically. By keeping sufficient terms in this expansion we have attained an extremely accurate evaluation for the diagonal and near diagonal matrix elements. As will be seen, such evaluations are of particular importance in the case of vanishing thin adjacent layers. An example of such a situation is a perfectly conducting surface on a dielectric layer.

Several scattering patterns are presented using the developed code, described above, for a variety of composite structures. These are also compared with measured data and computations via alternate numerical and analytical methods.

Chapter 2

Review of Standard Integral Expressions

In this chapter, we derive the standard integral expressions for the fields generated by the presence of two dimensional electric and magnetic currents. From Stratton [9], the most general expressions for the electric and magnetic fields are (in terms of Hertz potentials)

$$\vec{E}^s = \nabla\nabla\cdot\vec{\Pi} - \mu\varepsilon\frac{d^2}{dt^2}\vec{\Pi} - \mu\nabla\times\frac{d}{dt}\vec{\Pi}^* \tag{2.1}$$

$$\vec{H}^s = \nabla\nabla\cdot\vec{\Pi}^* - \mu\varepsilon\frac{d^2}{dt^2}\vec{\Pi}^* + \varepsilon\nabla\times\frac{d}{dt}\vec{\Pi}. \tag{2.2}$$

In the above equations, $\varepsilon = \varepsilon_r\varepsilon_0$, $\mu = \mu_r\mu_0$ and the Hertz potentials $\vec{\Pi}$ and $\vec{\Pi}^*$ are given by

$$\vec{\Pi} = \frac{iZ}{k}\int_{V'}\vec{J}G^{3d}(\vec{r},\vec{r}')\,dv', \tag{2.3}$$

$$\vec{\Pi}^* = \frac{iY}{k}\int_{V'}\vec{J}^*G^{3d}(\vec{r},\vec{r}')\,dv', \tag{2.4}$$

where $\vec{J}$ and $\vec{J}^*$ denote the electric and magnetic currents, respectively. In addition, $G^{3d}(\vec{r},\vec{r}')$ is the three dimensional free space green's function and V' is the volume occupied by the currents. Employing the e^{-iwt} time convention, (2.1) and (2.2) become:

$$\vec{E}^s = \nabla\nabla\cdot\vec{\Pi} + k^2\vec{\Pi} + ikZ\nabla\times\vec{\Pi}^* \tag{2.5}$$

$$\vec{H}^s = \nabla\nabla\cdot\vec{\Pi}^* + k^2\vec{\Pi}^* - ikY\nabla\times\vec{\Pi} \tag{2.6}$$

with $k^2 = \omega^2\mu\varepsilon$, $kZ = \omega\mu$, and $kY = \omega\varepsilon$.

If we are interested in the field scattered by a generalized volumetric material in free space (see figure2.1), Maxwell's equations within the material take the form:

$$\nabla X\vec{E} = i\omega\mu\vec{H} = i\omega\mu_0 - \vec{J}^* \tag{2.7}$$

$$\nabla X\vec{H} = -i\omega\varepsilon\vec{E} = i\omega\varepsilon_0 + \vec{J} \tag{2.8}$$

where $\vec{J}^* = -i\omega\mu_0\left(\mu_r - 1\right)\vec{E}$ and $\vec{J} = -i\omega\varepsilon_0\left(\varepsilon_r - 1\right)\vec{H}$ are the equivalent electric and magnetic currents. Thus, the volumetric material can be replaced by equivalent currents acting in free space. This allows the field expressions to be written solely in terms of free space parameters,

$$\begin{aligned}
\vec{E}^s &= \nabla\nabla\cdot\vec{\Pi} + k_0^2\vec{\Pi} + ik_0Z_0\nabla\times\vec{\Pi}^* \\
\vec{H}^s &= \nabla\nabla\cdot\vec{\Pi}^* + k_0^2\vec{\Pi}^* - ik_0Y_0\nabla\times\vec{\Pi} \\
\vec{\Pi} &= \frac{iZ_0}{k_0}\int_{V'}\vec{J}G^{3d}\left(\vec{r},\vec{r}'\right)dv' \\
\vec{\Pi}^* &= \frac{iY_0}{k_0}\int_{V'}\vec{J}^*G^{3d}\left(\vec{r},\vec{r}'\right)dv'.
\end{aligned} \tag{2.9}$$

Expanding the above field terms we find that

$$\begin{aligned}
\vec{E}^s &= \left(k_0^2\Pi_x + \frac{d^2}{dx^2}\Pi_x + \frac{d^2}{dxdy}\Pi_y + \frac{d^2}{dxdz}\Pi_z + ik_0Z_0\frac{d}{dy}\Pi_z^* - ik_0Z_0\frac{d}{dz}\Pi_y^*\right)\hat{x} \\
&+ \left(\frac{d^2}{dydx}\Pi_x + k_0^2\Pi_y + \frac{d^2}{dy^2}\Pi_y + \frac{d^2}{dydz}\Pi_z + ik_0Z_0\frac{d}{dz}\Pi_x^* - ik_0Z_0\frac{d}{dx}\Pi_z^*\right)\hat{y}
\end{aligned}$$

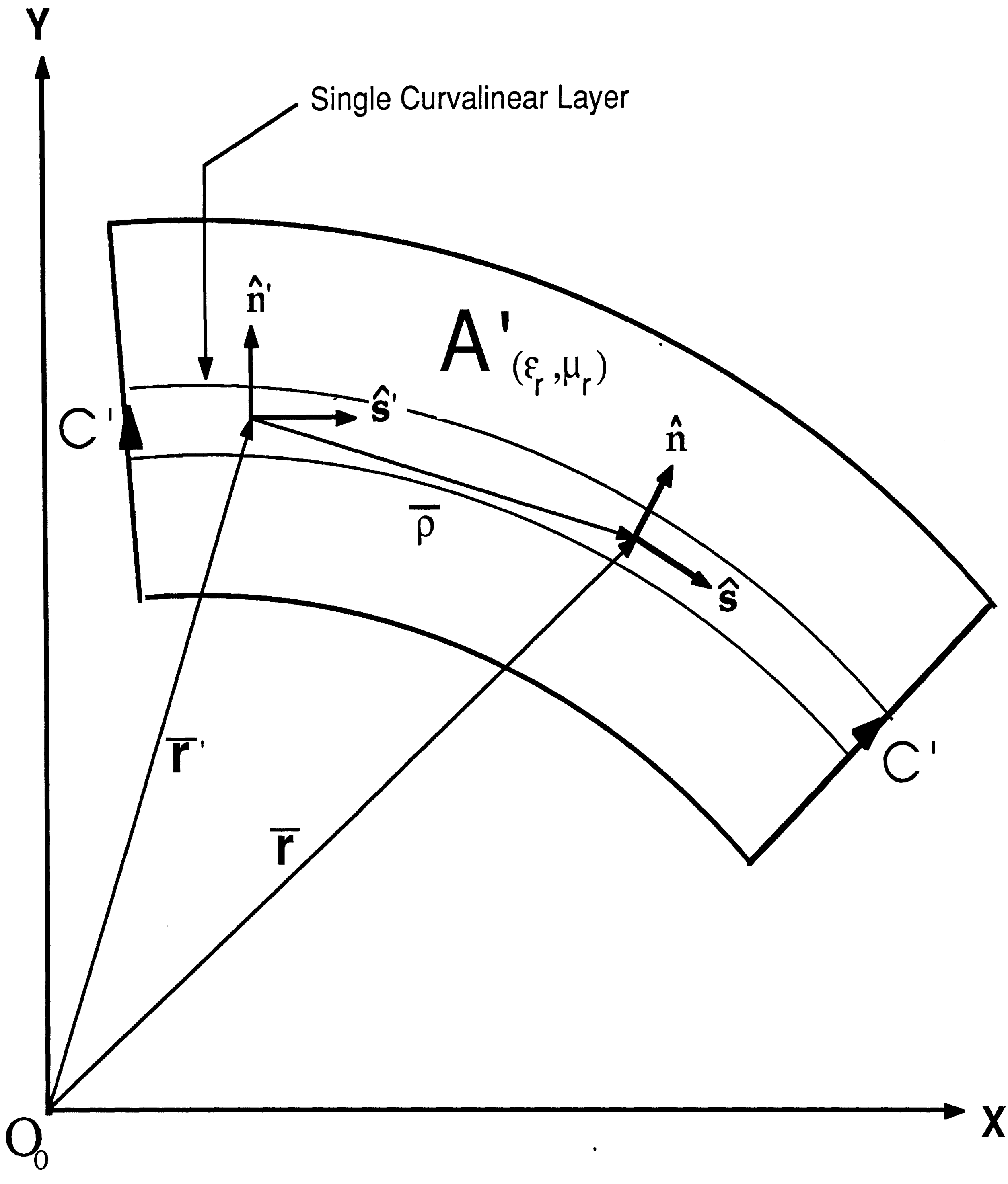

Figure 2.1: Arbitrary cylindrical structure in free space.

$$+ \quad \left(\frac{d^2}{dzdx}\Pi_x + \frac{d^2}{dzdy}\Pi_y + k_0^2\Pi_z + \frac{d^2}{dz^2}\Pi_z + ik_0 Z_0\frac{d}{dx}\Pi_y^* - ik_0 Z_0\frac{d}{dy}\Pi_x^*\right)\hat{z} \tag{2.10}$$

and

$$\begin{aligned}
\vec{H}^s \; = \; & \left(k_0^2\Pi_x^* + \frac{d^2}{dx^2}\Pi_x^* + \frac{d^2}{dxdy}\Pi_y^* + \frac{d^2}{dxdz}\Pi_z^* - ik_0 Z_0\frac{d}{dy}\Pi_z + ik_0 Z_0\frac{d}{dz}\Pi_y\right)\hat{x} \\
+ \; & \left(\frac{d^2}{dydx}\Pi_x^* + k_0^2\Pi_y^* + \frac{d^2}{dy^2}\Pi_y^* + \frac{d^2}{dydz}\Pi_z^* - ik_0 Z_0\frac{d}{dz}\Pi_x + ik_0 Z_0\frac{d}{dx}\Pi_z\right)\hat{y} \\
+ \; & \left(\frac{d^2}{dzdx}\Pi_x^* + \frac{d^2}{dzdy}\Pi_y^* + k_0^2\Pi_z^* + \frac{d^2}{dz^2}\Pi_z^* - ik_0 Z_0\frac{d}{dx}\Pi_y + ik_0 Z_0\frac{d}{dy}\Pi_x\right)\hat{z}.
\end{aligned} \tag{2.11}$$

Substituting the full expressions for the Hertz potentials, we further obtain

$$\begin{aligned}
E_x^s &= \frac{iZ_0}{k_0}\int_{V'}\left(J_x\left[k_0^2 + \frac{d^2}{dx^2}\right] + J_y\frac{d^2}{dxdy} + J_z\frac{d^2}{dxdz}\right)G^{3d}(\vec{r},\vec{r}')\,dv' - \int_{V'}\left(J_z^*\frac{d}{dy} - J_y^*\frac{d}{dz}\right)G^{3d}(\vec{r},\vec{r}')\,dv' \\
E_y^s &= \frac{iZ_0}{k_0}\int_{V'}\left(J_x\frac{d^2}{dydx} + J_y\left[k_0^2 + \frac{d^2}{dy^2}\right] + J_z\frac{d^2}{dydz}\right)G^{3d}(\vec{r},\vec{r}')\,dv' - \int_{V'}\left(J_x^*\frac{d}{dz} - J_z^*\frac{d}{dx}\right)G^{3d}(\vec{r},\vec{r}')\,dv' \\
E_z^s &= \frac{iZ_0}{k_0}\int_{V'}\left(J_x\frac{d^2}{dzdx} + J_y\frac{d^2}{dzdy} + J_z\left[k_0^2 + \frac{d^2}{dz^2}\right]\right)G^{3d}(\vec{r},\vec{r}')\,dv' - \int_{V'}\left(J_y^*\frac{d}{dx} - J_x^*\frac{d}{dy}\right)G^{3d}(\vec{r},\vec{r}')\,dv' \\
H_x^s &= \frac{iY_0}{k_0}\int_{V'}\left(J_x^*\left[k_0^2 + \frac{d^2}{dx^2}\right] + J_y^*\frac{d^2}{dxdy} + J_z^*\frac{d^2}{dxdz}\right)G^{3d}(\vec{r},\vec{r}')\,dv' + \int_{V'}\left(J_z\frac{d}{dy} - J_y\frac{d}{dz}\right)G^{3d}(\vec{r},\vec{r}')\,dv' \\
H_y^s &= \frac{iY_0}{k_0}\int_{V'}\left(J_x^*\frac{d^2}{dydx} + J_y^*\left[k_0^2 + \frac{d^2}{dy^2}\right] + J_z^*\frac{d^2}{dydz}\right)G^{3d}(\vec{r},\vec{r}')\,dv' + \int_{V'}\left(J_x\frac{d}{dz} - J_z\frac{d}{dx}\right)G^{3d}(\vec{r},\vec{r}')\,dv' \\
H_z^s &= \frac{iY_0}{k_0}\int_{V'}\left(J_x^*\frac{d^2}{dzdx} + J_y^*\frac{d^2}{dzdy} + J_z^*\left[k_0^2 + \frac{d^2}{dz^2}\right]\right)G^{3d}(\vec{r},\vec{r}')\,dv' + \int_{V'}\left(J_y\frac{d}{dx} - J_x\frac{d}{dy}\right)G^{3d}(\vec{r},\vec{r}')\,dv'.
\end{aligned} \quad (2.1$$

In order to specialize these equations to the two-dimensional case, all derivatives with respect to z must be equated to zero since in the two dimensional case there is no field variation in the $\hat{z}$ direction. After doing so, we obtain

$$\begin{aligned}
E_x^s &= \frac{iZ_0}{k_0}\int_{V'}\left(J_x\left[k_0^2+\frac{d^2}{dx^2}\right]+J_y\frac{d^2}{dxdy}\right)G^{3d}(\vec{r},\vec{r}')\,dv' - \int_{V'} J_z^*\frac{d}{dy}G^{3d}(\vec{r},\vec{r}')\,dv' \\
E_y^s &= \frac{iZ_0}{k_0}\int_{V'}\left(J_x\frac{d^2}{dydx}+J_y\left[k_0^2+\frac{d^2}{dy^2}\right]\right)G^{3d}(\vec{r},\vec{r}')\,dv' + \int_{V'} J_z^*\frac{d}{dx}G^{3d}(\vec{r},\vec{r}')\,dv' \\
E_z^s &= iZ_0k_0\int_{V'} J_zG^{3d}(\vec{r},\vec{r}')\,dv' - \int_{V'}\left(J_y^*\frac{d}{dx}-J_x^*\frac{d}{dy}\right)G^{3d}(\vec{r},\vec{r}')\,dv' \\
H_x^s &= \frac{iY_0}{k_0}\int_{V'}\left(J_x^*\left[k_0^2+\frac{d^2}{dx^2}\right]+J_y^*\frac{d^2}{dxdy}\right)G^{3d}(\vec{r},\vec{r}')\,dv' + \int_{V'} J_z\frac{d}{dy}G^{3d}(\vec{r},\vec{r}')\,dv' \\
H_y^s &= \frac{iY_0}{k_0}\int_{V'}\left(J_x^*\frac{d^2}{dydx}+J_y^*\left[k_0^2+\frac{d^2}{dy^2}\right]\right)G^{3d}(\vec{r},\vec{r}')\,dv' - \int_{V'} J_z\frac{d}{dx}G^{3d}(\vec{r},\vec{r}')\,dv' \\
H_z^s &= iY_0k_0\int_{V'} J_z^*G^{3d}(\vec{r},\vec{r}')\,dv' + \int_{V'}\left(J_y\frac{d}{dx}-J_x\frac{d}{dy}\right)G^{3d}(\vec{r},\vec{r}')\,dv'. \qquad (2.13)
\end{aligned}$$

Since $\vec{J}$ and $\vec{J}^*$ are independent of z', the integration over z' can be carried out as

$$\begin{aligned}
\int_{-\infty}^{\infty} G^{3d}(\vec{r},\vec{r}')\,dz' &= \int_{-\infty}^{\infty}\frac{e^{ik_0\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}}{4\pi\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}dz' \\
&= \int_{-\infty}^{\infty}\frac{e^{ik_0\sqrt{\rho^2+(z-z')^2}}}{4\pi\sqrt{\rho^2+(z-z')^2}} \\
&= \frac{1}{4\pi}\left(i\pi H_0^{(1)}(k_0\rho)\right) \\
&= \frac{i}{4}H_0^{(1)}(k_0\rho) \\
&= G^{2d}(\vec{r},\vec{r}') \qquad (2.14)
\end{aligned}$$

where $H_0^{(1)}(k_0\rho)$ denotes the Hankel function of the first kind and zeroth order and $G^{2d}(\vec{r},\vec{r}')$ is the two dimensional Green's function. Substituting (2.13) into (2.14), we obtain

$$\begin{aligned}
E_x^s &= \frac{iZ_0}{k_0}\int_{A'}\left(J_x\left[k_0^2+\frac{d^2}{dx^2}\right]+J_y\frac{d^2}{dxdy}\right)G^{2d}(\vec{r},\vec{r}')\,dA' - \int_{A'} J_z^*\frac{d}{dy}G^{2d}(\vec{r},\vec{r}')\,dA' \\
E_y^s &= \frac{iZ_0}{k_0}\int_{A'}\left(J_x\frac{d^2}{dydx}+J_y\left[k_0^2+\frac{d^2}{dy^2}\right]\right)G^{2d}(\vec{r},\vec{r}')\,dA' + \int_{A'} J_z^*\frac{d}{dx}G^{2d}(\vec{r},\vec{r}')\,dA'
\end{aligned}$$

$$\begin{aligned}
E_z^s &= iZ_0k_0\int_{A'} J_z G^{2d}(\vec{r},\vec{r'})\,dA' - \int_{A'}\left(J_y^*\frac{d}{dx} - J_x^*\frac{d}{dy}\right)G^{2d}(\vec{r},\vec{r'})\,dA' \\
H_x^s &= \frac{iY_0}{k_0}\int_{A'}\left(J_x^*\left[k_0^2+\frac{d^2}{dx^2}\right] + J_y^*\frac{d^2}{dxdy}\right)G^{2d}(\vec{r},\vec{r'})\,dA' + \int_{A'} J_z\frac{d}{dy}G^{2d}(\vec{r},\vec{r'})\,dA' \\
H_y^s &= \frac{iY_0}{k_0}\int_{A'}\left(J_x^*\frac{d^2}{dydx} + J_y^*\left[k_0^2+\frac{d^2}{dy^2}\right]\right)G^{2d}(\vec{r},\vec{r'})\,dA' - \int_{A'} J_z\frac{d}{dx}G^{2d}(\vec{r},\vec{r'})\,dA' \\
H_z^s &= iY_0k_0\int_{A'} J_z^* G^{2d}(\vec{r},\vec{r'})\,dA' + \int_{A'}\left(J_y\frac{d}{dx} - J_x\frac{d}{dy}\right)G^{2d}(\vec{r},\vec{r'})\,dA'. \qquad (2.15)
\end{aligned}$$

We can now derive integral equations for the solution of $\vec{J}$ and $\vec{J}^*$. It will suffice to consider development of the E_x and E_z integral equations, since the evolution of the remaining integral equations parallels these two cases. Rewriting E_x^s from (2.15) in a slightly different manner, we have

$$\begin{aligned}
E_x^s &= \frac{iZ_0}{k_0}\int_{A'}\left(J_x\left[k_0^2+\frac{d^2}{dx^2}+\frac{d^2}{dy^2}-\frac{d^2}{dy^2}\right] + J_y\frac{d^2}{dxdy}\right)G^{2d}(\vec{r},\vec{r'})\,dA' \\
&\quad - \int_{A'} J_z^*\frac{dG^{2d}(\vec{r},\vec{r'})}{dy}dA'. \qquad (2.16)
\end{aligned}$$

We can subsequently employ the identity

$$\left(k_0^2+\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)G^{2d}(\vec{r},\vec{r'}) = -\delta(\vec{r}-\vec{r'}) \qquad (2.17)$$

and assuming that the field is measured within the volumetric material, (2.16) becomes

$$\begin{aligned}
E_x^s &= -\frac{iZ_0}{k_0}\int_{A'} J_x\delta(\vec{r}-\vec{r'})dA' + \frac{iZ_0}{k_0}\int_{A'}\left(-J_x\frac{d^2}{dy^2} + J_y\frac{d^2}{dxdy}\right)G^{2d}(\vec{r},\vec{r'})\,dA' \\
&\quad - \int_{A'} J_z^*\frac{dG^{2d}(\vec{r},\vec{r'})}{dy}dA'. \qquad (2.18)
\end{aligned}$$

It is now observed that

$$E_x^s - \frac{iZ_0}{k_0}\int_{A'}\left(-J_x\delta(\vec{r}-\vec{r'})\right)dA' \;=\; E_x^s + \frac{iZ_0}{k_0}J_x$$

$$
\begin{aligned}
&= E_x^s + \frac{iZ_0}{k_0}\left(-ik_0Y_0(\varepsilon_r - 1)(E_x^{inc} + E_x^s)\right) \\
&= E_x^s + (\varepsilon_r - 1)(E_x^{inc} + E_x^s) \\
&= \varepsilon_r(E_x^{inc} + E_x^s) - E_x^{inc} \\
&= \varepsilon_r E_x^{tot} - E_x^{inc}. \qquad (2.19)
\end{aligned}
$$

Furthermore, recalling that $J_x = -ik_0Y_0(\varepsilon_r - 1)E_x^{tot}$, we find

$$
\begin{aligned}
E_x^{tot} &= \frac{iZ_0}{(\varepsilon - 1)k_0} J_x \\
&= RJ_x, \qquad (2.20)
\end{aligned}
$$

where R is usually referred to as the resistivity. Substituting (2.19) and (2.20) into (2.16) we obtain

$$
E_x^{inc} = \varepsilon_r RJ_x + \frac{iZ_0}{k_0}\int_{A'}\left(J_x\frac{d^2}{dy^2} - J_y\frac{d^2}{dxdy}\right)G^{2d}(\vec{r},\vec{r}')\,dA' + \int_{A'} J_z^*\frac{dG^{2d}(\vec{r},\vec{r}')}{dy}dA'. \qquad (2.21)
$$

The E_y, H_x, and H_y equations are obtained in a similar manner. We note, however, that for the E_z and H_z components, we must also make the substitutions:

$$
E_z^s = E_z^{tot} - E_z^{inc} = RJ_z - E_z^{inc} \qquad (2.22)
$$

$$
H_z^s = H_z^{tot} - H_z^{inc} = R^*J_z^* - H_z^{inc}. \qquad (2.23)
$$

Summarizing, we have

$$
\begin{aligned}
E_x^{inc} &= \varepsilon_r RJ_x + \frac{iZ_0}{k_0}\int_{A'}\left[J_x\frac{d^2}{dy^2} - J_y\frac{d^2}{dxdy}\right]G^{2d}(\vec{r},\vec{r}')\,dA' + \int_{A'} J_z^*\frac{dG^{2d}(\vec{r},\vec{r}')}{dy}dA' \\
E_y^{inc} &= \varepsilon_r RJ_y + \frac{iZ_0}{k_0}\int_{A'}\left[-J_x\frac{d^2}{dydx} + J_y\frac{d^2}{dx^2}\right]G^{2d}(\vec{r},\vec{r}')\,dA' - \int_{A'} J_z^*\frac{dG^{2d}(\vec{r},\vec{r}')}{dx}dA' \\
E_z^{inc} &= RJ_z - ik_0Z_0\int_{A'} J_zG^{2d}(\vec{r},\vec{r}')\,dA' + \int_{A'}\left[J_y^*\frac{d}{dx} - J_x^*\frac{d}{dy}\right]G^{2d}(\vec{r},\vec{r}')\,dA'
\end{aligned}
$$

$$
\begin{aligned}
H_x^{inc} &= \mu_r R^* J_x^* + \frac{iY_0}{k_0}\int_{A'}\left[J_x^*\frac{d^2}{dy^2} - J_y^*\frac{d^2}{dxdy}\right]G^{2d}(\vec{r},\vec{r}')\,dA' - \int_{A'} J_z\frac{dG^{2d}(\vec{r},\vec{r}')}{dy}dA' \\
H_y^{inc} &= \mu_r R^* J_y^* + \frac{iY_0}{k_0}\int_{A'}\left[-J_x^*\frac{d^2}{dydx} + J_y^*\frac{d^2}{dx^2}\right]G^{2d}(\vec{r},\vec{r}')\,dA' + \int_{A'} J_z\frac{dG^{2d}(\vec{r},\vec{r}')}{dx}dA' \\
H_z^{inc} &= R^* J_z^* - ik_0Y_0\int_{A'} J_z^* G^{2d}(\vec{r},\vec{r}')\,dA' - \int_{A'}\left[J_y\frac{d}{dx} - J_x\frac{d}{dy}\right]G^{2d}(\vec{r},\vec{r}')\,dA'. \quad (2.24)
\end{aligned}
$$

The above form a set of integral equations applicable to dielectric/magnetic cylinders of arbitrary cross section and material composition. Their direct numerical implementation, however, can become cumbersome for scatterers associated with curvalinear surface material boundaries. Therefore, it is instructive to obtain a set of integral equations suitable for modeling curvalinear layers of material. Such a set must then utilize the local directions of the layer over the integration. We denote these directions as $\hat{s}'$ and $\hat{n}'$, where $\hat{s}'$ is tangent to the curvalinear layer at (x', y'), the integration point and $\hat{n}'$ is the corresponding normal to the layer so that $\hat{s}' \times \hat{n}' = \hat{z}'$. Furthermore, we may express the location of each integration point in the new coordinate frame $(\hat{s}', \hat{n}', \hat{z})$ by (s', n'). Similar transformations may be introduced for the observation point so that the directions $(\hat{x}, \hat{y}, \hat{z})$ are transformed to $((\hat{s}, \hat{n}, \hat{z})$ and the coordinates (x, y) to (s, n). We now proceed to implement the above variable substitutions in the integral equations (2.24). Again, attention will be confined to the E_x integral equation with the understanding that similar steps apply for the integral equations involving the other components.

Suppose we have some arbitrary cylindrical material configuration which has been partitioned into thin layer where the observation point is within the material and is defined by the coordinates (s, n). Since it is immaterial in which direction we orient the x and y axes, it is to our advantage to orient them so that $\hat{x} = \hat{s}$, and $\hat{y} = \hat{n}$. From (2.24)

$$E_s^{inc} = \varepsilon_r R J_s + \frac{iZ_0}{k_0} \int_{A'} \left[J_s \frac{d^2}{dn^2} - J_n \frac{d^2}{dsdn} \right] G^{2d}(\vec{r}, \vec{r'})\, ds'dn' + \int_{A'} J_z^* \frac{dG^{2d}(\vec{r}, \vec{r'})}{dn} ds'dn'. \tag{2.25}$$

The integrals in the above equation are over $\hat{s}'$ and $\hat{n}'$, it is necessary that the corresponding integrands be expressed solely in terms of these variables rather than s and n. The second integrand in (2.25) is already in this form and it is required to only consider the first integrand of (2.25). We have

$$\begin{aligned}
\left(J_s \frac{d^2}{dn^2} - J_n \frac{d^2}{dsdn} \right) G^{2d}(\vec{r}, \vec{r'}) &= \frac{i}{4} \left(J_{s'}(\hat{s} \cdot \hat{s'}) + J_{n'}(\hat{s} \cdot \hat{n'}) \right) \frac{d^2 H_0^{(1)}(k_0\rho)}{dn^2} \\
&\quad - \frac{i}{4} \left(J_{s'}(\hat{n} \cdot \hat{s'}) + J_{n'}(\hat{n} \cdot \hat{n'}) \right) \frac{d^2 H_0^{(1)}(k_0\rho)}{dsdn} \\
&= \frac{d}{dn} \left[\frac{-ik_0}{4} \left(J_{s'}(\hat{s} \cdot \hat{s'}) + J_{n'}(\hat{s} \cdot \hat{n'}) \right) (\hat{\rho} \cdot \hat{n}) H_1^{(1)}(k_0\rho) \right] \\
&\quad - \frac{d}{dn} \left[\frac{-ik_0}{4} \left(J_{s'}(\hat{n} \cdot \hat{s'}) + J_{n'}(\hat{n} \cdot \hat{n'}) \right) (\hat{\rho} \cdot \hat{s}) H_1^{(1)}(k_0\rho) \right] \\
&= \frac{d}{dn} \left[\frac{ik_0}{4} J_{s'} \left((\hat{n} \cdot \hat{s'})(\hat{\rho} \cdot \hat{s}) - (\hat{\rho} \cdot \hat{n})(\hat{s} \cdot \hat{s'}) \right) H_1^{(1)}(k_0\rho) \right] \\
&\quad + \frac{d}{dn} \left[\frac{ik_0}{4} J_{n'} \left((\hat{n} \cdot \hat{n'})(\hat{\rho} \cdot \hat{s}) - (\hat{\rho} \cdot \hat{n})(\hat{s} \cdot \hat{n'}) \right) H_1^{(1)}(k_0\rho) \right].
\end{aligned} \tag{2.26}$$

However,

$$\begin{aligned}
(\hat{n} \cdot \hat{s'})(\hat{\rho} \cdot \hat{s}) - (\hat{\rho} \cdot \hat{n})(\hat{s} \cdot \hat{s'}) &= (\hat{n} \times \hat{s}) \cdot (\hat{s'} \times \hat{\rho}) \\
&= -\hat{z} \cdot (\hat{s'} \times \hat{\rho}) \\
&= \hat{z} \cdot (\hat{\rho} \times \hat{s'}) \\
&= \hat{\rho} \cdot (\hat{s'} \times \hat{z}) \\
&= -\hat{\rho} \cdot \hat{n'}
\end{aligned} \tag{2.27}$$

and

$$
\begin{aligned}
(\hat{n}\cdot\hat{n'})(\hat{\rho}\cdot\hat{s})-(\hat{\rho}\cdot\hat{n})(\hat{s}\cdot\hat{n'}) &= (\hat{n}\times\hat{s})\cdot(\hat{n'}\times\hat{\rho}) \\
&= -\hat{z}\cdot(\hat{n'}\times\hat{\rho}) \\
&= \hat{z}\cdot(\hat{\rho}\times\hat{n'}) \\
&= \hat{\rho}\cdot(\hat{n'}\times\hat{z}) \\
&= \hat{\rho}\cdot\hat{s'} \qquad (2.28)
\end{aligned}
$$

implying that

$$
\begin{aligned}
\left(J_s\frac{d^2}{dn^2}-J_n\frac{d^2}{dsdn}\right)G^{2d}(\vec{r},\vec{r'}) &= \frac{d}{dn}\left[\frac{ik_0}{4}\left(J_{n'}(\hat{\rho}\cdot\hat{s'})-J_{s'}(\hat{\rho}\cdot\hat{n'})\right)H_1^{(1)}(k_0\rho)\right] \\
&= \frac{d}{dn}\left(J_{n'}\frac{dG^{2d}(\vec{r},\vec{r'})}{ds'}-J_{s'}\frac{dG^{2d}(\vec{r},\vec{r'})}{dn'}\right). \qquad (2.29)
\end{aligned}
$$

Thus, E_s^{inc} may be written as

$$
\begin{aligned}
E_s^{inc} &= \varepsilon_r R J_s+\frac{iZ_0}{k_0}\int_{A'}\frac{d}{dn}\left(J_{n'}\frac{dG^{2d}(\vec{r},\vec{r'})}{ds'}-J_{s'}\frac{dG^{2d}(\vec{r},\vec{r'})}{dn'}\right)ds'dn' \\
&\quad+\int_{A'}J_z^*\frac{dG^{2d}(\vec{r},\vec{r'})}{dn}ds'dn'. \qquad (2.30)
\end{aligned}
$$

As discussed earlier the integral equations for the other components follow in a similar manner. Introducing the definitions $\widetilde{J}=Z_0J$, $\widetilde{R}=R/Z_0$, and $\widetilde{R}^*=R^*/Y_0$, the complete set of integral equations in terms of localized variables is

$$
\begin{aligned}
\frac{E_s^{inc}}{\varepsilon_r\widetilde{R}} &= \widetilde{J}_s \quad -\frac{i}{k_0\varepsilon_r\widetilde{R}}\int_{A'}\frac{d}{dn}\left(\widetilde{J}_{s'}\frac{dG^{2d}(\vec{r},\vec{r'})}{dn'}-\widetilde{J}_{n'}\frac{dG^{2d}(\vec{r},\vec{r'})}{ds'}\right)ds'dn' \\
&\quad+\frac{1}{\varepsilon_r\widetilde{R}}\int_{A'}J_z^*\frac{dG^{2d}(\vec{r},\vec{r'})}{dn}ds'dn' \\
\frac{E_n^{inc}}{\varepsilon_r\widetilde{R}} &= \widetilde{J}_n \quad +\frac{i}{k_0\varepsilon_r\widetilde{R}}\int_{A'}\frac{d}{ds}\left(\widetilde{J}_{s'}\frac{dG^{2d}(\vec{r},\vec{r'})}{dn'}-\widetilde{J}_{n'}\frac{dG^{2d}(\vec{r},\vec{r'})}{ds'}\right)ds'dn'
\end{aligned}
$$

$$
\begin{aligned}
& && -\frac{1}{\varepsilon_r \tilde{R}} \int_{A'} J_z^* \frac{dG^{2d}(\vec{r},\vec{r}')}{ds} ds'dn' \\
\frac{E_z^{inc}}{\tilde{R}} &= \tilde{J}_z && -\frac{ik_0}{\tilde{R}} \int_{A'} \tilde{J}_z G^{2d}(\vec{r},\vec{r}')\, ds'dn' \\
& && +\frac{1}{\tilde{R}} \int_{A'} \left(J_{s'}^* \frac{dG^{2d}(\vec{r},\vec{r}')}{dn'} - J_{n'}^* \frac{dG^{2d}(\vec{r},\vec{r}')}{ds'} \right) ds'dn' \\
\frac{Z_0 H_s^{inc}}{\mu_r \tilde{R}^*} &= J_s^* && -\frac{i}{k_0 \mu_r \tilde{R}^*} \int_{A'} \frac{d}{dn} \left(J_{s'}^* \frac{dG^{2d}(\vec{r},\vec{r}')}{dn'} - J_{n'}^* \frac{dG^{2d}(\vec{r},\vec{r}')}{ds'} \right) ds'dn' \\
& && -\frac{1}{\mu_r \tilde{R}^*} \int_{A'} \tilde{J}_z \frac{dG^{2d}(\vec{r},\vec{r}')}{dn} ds'dn' \\
\frac{Z_0 H_n^{inc}}{\mu_r \tilde{R}^*} &= J_n * && +\frac{i}{k_0 \mu_r \tilde{R}^*} \int_{A'} \frac{d}{ds} \left(J_{s'}^* \frac{dG^{2d}(\vec{r},\vec{r}')}{dn'} - J_{n'}^* \frac{dG^{2d}(\vec{r},\vec{r}')}{ds'} \right) ds'dn' \\
& && +\frac{1}{\mu_r \tilde{R}^*} \int_{A'} \tilde{J}_z \frac{dG^{2d}(\vec{r},\vec{r}')}{ds} ds'dn' \\
\frac{Z_0 H_z^{inc}}{\tilde{R}^*} &= J_z^* && -\frac{ik_0}{\tilde{R}^*} \int_{A'} J_z^* G^{2d}(\vec{r},\vec{r}')\, ds'dn' \\
& && -\frac{1}{\tilde{R}^*} \int_{A'} \left(\tilde{J}_{s'} \frac{dG^{2d}(\vec{r},\vec{r}')}{dn'} - \tilde{J}_{n'} \frac{dG^{2d}(\vec{r},\vec{r}')}{ds'} \right) ds'dn'. \qquad (2.31)
\end{aligned}
$$

Chapter 3

Development of Compact Set of Integral Equations

The integral equations developed in the previous chapter involve a total of six unknown current components which reduce to three provided we assume an $E_z(H_z^i = 0)$ or $H_z(E_z^i = 0)$ incidence as is usually the case. In particular, for the H_z incidence case the three relevant integral equations are

$$\begin{aligned}
\frac{E_s^{inc}}{\varepsilon_r \tilde{R}} &= \tilde{J}_s - \frac{i}{k_0 \varepsilon_r \tilde{R}} \int_{A'} \frac{d}{dn} \left(\tilde{J}_{s'} \frac{dG^{2d}(\vec{r},\vec{r}')}{dn'} - \tilde{J}_{n'} \frac{dG^{2d}(\vec{r},\vec{r}')}{ds'} \right) ds'dn' \\
&\quad + \frac{1}{\varepsilon_r \tilde{R}} \int_{A'} J_z^* \frac{dG^{2d}(\vec{r},\vec{r}')}{dn} ds'dn' \\
\frac{E_n^{inc}}{\varepsilon_r \tilde{R}} &= \tilde{J}_n + \frac{i}{k_0 \varepsilon_r \tilde{R}} \int_{A'} \frac{d}{ds} \left(\tilde{J}_{s'} \frac{dG^{2d}(\vec{r},\vec{r}')}{dn'} - \tilde{J}_{n'} \frac{dG^{2d}(\vec{r},\vec{r}')}{ds'} \right) ds'dn' \\
&\quad - \frac{1}{\varepsilon_r \tilde{R}} \int_{A'} J_z^* \frac{dG^{2d}(\vec{r},\vec{r}')}{ds} ds'dn' \\
\frac{Z_0 H_z^{inc}}{\tilde{R}^*} &= J_z^* - \frac{ik_0}{\tilde{R}^*} \int_{A'} J_z^* G^{2d}(\vec{r},\vec{r}')\, ds'dn' \\
&\quad - \frac{1}{\tilde{R}^*} \int_{A'} \left(\tilde{J}_{s'} \frac{dG^{2d}(\vec{r},\vec{r}')}{dn'} - \tilde{J}_{n'} \frac{dG^{2d}(\vec{r},\vec{r}')}{ds'} \right) ds'dn'. \qquad (3.1)
\end{aligned}$$

As will be shown in this chapter, it is possible to reduce that number of unknowns in the above equation set by introducing a known equivalence between electric and magnetic currents [2]. Before proceeding with the application of this equivalence it is necessary to first introduce certain modifications in the integral equation (3.1). Noting the identies

$$\begin{aligned}
\frac{d}{dn}\left(\frac{dG^{2d}(\vec{r},\vec{r}')}{ds'}\right) &= \frac{d}{ds'}\left(\frac{dG^{2d}(\vec{r},\vec{r}')}{dn}\right) \\
\frac{d}{ds}\left(\frac{dG^{2d}(\vec{r},\vec{r}')}{ds'}\right) &= \frac{d}{ds'}\left(\frac{dG^{2d}(\vec{r},\vec{r}')}{ds}\right)
\end{aligned} \tag{3.2}$$

and employing integration by parts in the integrals of (3.1) involving $\widetilde{J}_n$ we find that

$$\begin{aligned}
\frac{i}{k_0\varepsilon_r\widetilde{R}}\int_{A'}\widetilde{J}_{n'}\frac{d}{dn}\left(\frac{dG^{2d}(\vec{r},\vec{r}')}{ds'}\right)ds'dn' &= \frac{i}{k_0\varepsilon_r\widetilde{R}}\int_{C'}\widetilde{J}_{n'}\frac{dG^{2d}(\vec{r},\vec{r}')}{dn}dn'\,|_{endpoints} \\
&- \frac{i}{k_0\varepsilon_r\widetilde{R}}\int_{A'}\frac{d\widetilde{J}_{n'}}{ds'}\frac{dG^{2d}(\vec{r},\vec{r}')}{dn}ds'dn' \\
\frac{-i}{k_0\varepsilon_r\widetilde{R}}\int_{A'}\widetilde{J}_{n'}\frac{d}{ds}\left(\frac{dG^{2d}(\vec{r},\vec{r}')}{ds'}\right)ds'dn' &= \frac{-i}{k_0\varepsilon_r\widetilde{R}}\int_{C'}\widetilde{J}_{n'}\frac{dG^{2d}(\vec{r},\vec{r}')}{ds}dn'\,|_{endpoints} \\
&+ \frac{i}{k_0\varepsilon_r\widetilde{R}}\int_{A'}\frac{d\widetilde{J}_{n'}}{ds'}\frac{dG^{2d}(\vec{r},\vec{r}')}{dn}ds'dn' \\
\frac{1}{\widetilde{R}^*}\int_{A'}\widetilde{J}_{n'}\frac{dG^{2d}(\vec{r},\vec{r}')}{ds'}ds'dn' &= \frac{1}{\widetilde{R}^*}\int_{C'}\widetilde{J}_{n'}G^{2d}(\vec{r},\vec{r}')\,dn'\,|_{endpoints} \\
&- \frac{1}{\widetilde{R}^*}\int_{A'}\frac{d\widetilde{J}_{n'}}{ds'}G^{2d}(\vec{r},\vec{r}')\,ds'dn'.
\end{aligned} \tag{3.3}$$

Substituting these expressions back into (3.1) and introducing the equivalence [2] $\widetilde{J}_z^* = J_z^* - \frac{i}{k_0}\frac{d\widetilde{J}_{n'}}{ds}$ we obtain

$$\begin{aligned}
\frac{E_s^{inc}}{\varepsilon_r\widetilde{R}} &= \widetilde{J}_s - \frac{i}{k_0\varepsilon_r\widetilde{R}}\int_{A'}\widetilde{J}_{s'}\frac{d}{dn'}\left(\frac{dG^{2d}(\vec{r},\vec{r}')}{dn}\right)ds'dn' \\
&+\frac{1}{\varepsilon_r\widetilde{R}}\int_{A'}\widetilde{J}_z^*\frac{dG^{2d}(\vec{r},\vec{r}')}{dn}ds'dn' \\
&+\frac{i}{k_0\varepsilon_r\widetilde{R}}\int_{C'}\widetilde{J}_{n'}\frac{dG^{2d}(\vec{r},\vec{r}')}{dn}dn'\,|_{endpoints}
\end{aligned}$$

$$
\begin{aligned}
\frac{E_n^{inc}}{\varepsilon_r \widetilde{R}} &= \widetilde{J}_n + \frac{i}{k_0 \varepsilon_r \widetilde{R}} \int_{A'} \widetilde{J}_{s'} \frac{d}{dn'} \left(\frac{dG^{2d}(\vec{r},\vec{r}')}{ds} \right) ds'dn' \\
& -\frac{1}{\varepsilon_r \widetilde{R}} \int_{A'} \widetilde{J}_z^* \frac{dG^{2d}(\vec{r},\vec{r}')}{ds} ds'dn' \\
& -\frac{i}{k_0 \varepsilon_r \widetilde{R}} \int_{C'} \widetilde{J}_{n'} \frac{dG^{2d}(\vec{r},\vec{r}')}{ds} dn' \mid_{endpoints} \\
\frac{Z_0 H_z^{inc}}{\widetilde{R}^*} &= J_z^* + \frac{-ik_0}{\widetilde{R}^*} \int_{A'} \widetilde{J}_z^* G^{2d}(\vec{r},\vec{r}') ds'dn' \\
& -\frac{1}{\widetilde{R}^*} \int_{A'} \widetilde{J}_{s'} \frac{dG^{2d}(\vec{r},\vec{r}')}{dn'} ds'dn' \\
& +\frac{1}{\widetilde{R}^*} \int_{C'} \widetilde{J}_{n'} G^{2d}(\vec{r},\vec{r}') dn' \mid_{endpoints} . \qquad (3.4)
\end{aligned}
$$

To remove the remaining component of J_n in the E_n integral equation we first diffentiate that equation with respect to s, multiply by $-i/k_0$ and then add it to the H_z integral equation. The resulting equations are

$$
\begin{aligned}
\frac{E_s^{inc}}{\varepsilon_r \widetilde{R}} &= \widetilde{J}_s - \frac{i}{k_0 \varepsilon_r \widetilde{R}} \int_{A'} \widetilde{J}_{s'} \frac{d}{dn'} \left(\frac{dG^{2d}(\vec{r},\vec{r}')}{dn} \right) ds'dn' \\
& +\frac{1}{\varepsilon_r \widetilde{R}} \int_{A'} \widetilde{J}_z^* \frac{dG^{2d}(\vec{r},\vec{r}')}{dn} ds'dn' \\
& +\frac{i}{k_0 \varepsilon_r \widetilde{R}} \int_{C'} \widetilde{J}_{n'} \frac{dG^{2d}(\vec{r},\vec{r}')}{dn} dn' \mid_{endpoints}, \qquad (3.5) \\
\frac{Z_0 H_z^{inc}}{\widetilde{R}^*} - \frac{i}{k_0 \varepsilon_r \widetilde{R}} \frac{dE_n^{inc}}{ds} &- \frac{iE_n^{inc}}{k_0} \frac{d}{ds} \left(\frac{1}{\varepsilon_r \widetilde{R}} \right) = \widetilde{J}_z^* \\
&- \int_{A'} \widetilde{J}_{s'} \frac{d}{dn'} \left[\frac{1}{\widetilde{R}^*} - \frac{1}{k_0^2} \frac{d}{ds} \left(\frac{1}{\varepsilon_r \widetilde{R}} \right) \frac{d}{ds} - \frac{1}{k_0^2 \varepsilon_r \widetilde{R}} \frac{d^2}{ds^2} \right] G^{2d}(\vec{r},\vec{r}') ds'dn' \\
&- \int_{A'} ik_0 \widetilde{J}_z^* \left[\frac{1}{\widetilde{R}^*} - \frac{1}{k_0^2} \frac{d}{ds} \left(\frac{1}{\varepsilon_r \widetilde{R}} \right) \frac{d}{ds} - \frac{1}{k_0^2 \varepsilon_r \widetilde{R}} \frac{d^2}{ds^2} \right] G^{2d}(\vec{r},\vec{r}') ds'dn' \\
&+ \int_{C'} \widetilde{J}_{n'} \left[\frac{1}{\widetilde{R}^*} - \frac{1}{k_0^2} \frac{d}{ds} \left(\frac{1}{\varepsilon_r \widetilde{R}} \right) \frac{d}{ds} - \frac{1}{k_0^2 \varepsilon_r \widetilde{R}} \frac{d^2}{ds^2} \right] G^{2d}(\vec{r},\vec{r}') dn' \mid_{endpoints} \\
& (3.6)
\end{aligned}
$$

Clearly these involve only two unknowns throughout the cross section of the inhomogeneous cylinder, namely $\widetilde{J}_s$ and the equivalent magnetic current $\widetilde{J}_z^*$. In addition, there remains a presence of the $\tilde{J}_n$ components only over the outer surface of the cylinder coincident with the $\hat{n}$ direction. When considering a layer simulation of the structure, these $\tilde{J}_n$ components are in fact the currents at the two ends of the inhomogeneous layer in the direction normal to the layer. It should be obvious that since the bulk of the unknowns is certainly within the cross section of the cylinder, the presence of the above $\tilde{J}_n$ do not add appreciably to the total system unknowns. Thus, with the derivation of (3.6) we have in essence reduced the system unknowns from 3N to 2N. Considering that the computer time required for the numerical solution of a system is proportional to the cubic power of the number of unknowns, equations (3.6) call for a tremendous increase in the efficiency of the intended numerical solution in comparison with that required for a solution of (3.1). However, it should be noted that we have achieved such an efficiency by increasing the complexity of the resulting integral equations. As will be seen in the next chapter, the numerical implementation of the higher order derivatives appearing in (3.6) must be evaluated with extreme care especially at the self cell.

In the next chapter we consider in some detail the numerical implementation of the compact set of equations (3.6).

Chapter 4

Solution of the Compact System of Integral Equations

4.1 Formal Solution

Although the compact set (3.6) may be solved as is, it proves more convenient to utilize the E_n integral equation in (3.4) as an additional auxilliary equation to be enforced at the strip edges. This, of course, does not bring about an increase in the number of unknowns.

Before proceeding with the details of the numerical implementation of (3.6) the following definitions are made in an effort to simplify the discussion on the subsequent operations. Namely, we define

$$\begin{aligned}
g_1(\vec{r}) &= \frac{E_s^{inc}(\vec{r})}{\varepsilon_r(\vec{r})\tilde{R}(\vec{r})} \\
g_2(\vec{r}) &= \frac{Z_0 H_z^{inc}(\vec{r})}{\tilde{R}^*(\vec{r})} - \frac{i}{k_0 \varepsilon_r(\vec{r})\tilde{R}(\vec{r})} \frac{dE_n^{inc}(\vec{r})}{ds} - \frac{iE_n^{inc}(\vec{r})}{k_0} \frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r})\tilde{R}(\vec{r})}\right) \\
g_3(\vec{r}) &= \frac{E_n^{inc}(\vec{r})}{\varepsilon_r(\vec{r})\tilde{R}(\vec{r})} \\
f_1(\vec{r}, \vec{r}') &= \frac{1}{\varepsilon_r(\vec{r})\tilde{R}(\vec{r})} \frac{dG^{2D}(\vec{r}, \vec{r}')}{dn}
\end{aligned}$$

$$f_2(\vec{r},\vec{r'}) = \left[\frac{1}{\tilde{R}^*(\vec{r})} - \frac{1}{k_0^2}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r})\tilde{R}(\vec{r})}\right)\frac{d}{ds} - \frac{1}{k_0^2\varepsilon_r(\vec{r})\tilde{R}(\vec{r})}\frac{d}{ds^2}\right] G^{2D}(\vec{r},\vec{r'})$$

$$f_3(\vec{r},\vec{r'}) = \frac{1}{\varepsilon_r(\vec{r})\tilde{R}(\vec{r})}\frac{dG^{2D}(\vec{r},\vec{r'})}{ds}. \tag{4.1}$$

Furthermore, introducing the subscript $q = 1/2$ to denote leading/trailing edges, the new system augmented with the E_n integral equation of (3.4) may be written as

$$g_1(\vec{r}) = \tilde{J}_s(\vec{r}) + \int_{A'} \tilde{J}_{s'}(\vec{r'}) \left[\frac{-i}{k_0}\frac{df_1(\vec{r},\vec{r'})}{dn'}\right] dA' + \int_{A'} \tilde{J}_z^*(\vec{r'}) f_1(\vec{r},\vec{r'}) dA' + \int_{C'} \tilde{J}_{n'}(\vec{r'}) \left[\frac{i(-1)^q}{k_0} f_1(\vec{r},\vec{r'})\right] dn' \tag{4.2}$$

$$g_2(\vec{r}) = \tilde{J}_z^*(\vec{r}) + \int_{A'} \tilde{J}_{s'}(\vec{r'}) \left[\frac{-df_2(\vec{r},\vec{r'})}{dn'}\right] dA' + \int_{A'} \tilde{J}_z^*(\vec{r'})(-ik_0) f_2(\vec{r},\vec{r'}) dA' + \int_{C'} \tilde{J}_{n'}(\vec{r'}) \left[(-1)^q f_2(\vec{r},\vec{r'})\right] dn \tag{4.3}$$

$$g_3(\vec{r}) = \tilde{J}_n(\vec{r}) + \int_{A'} \tilde{J}_{s'}(\vec{r'}) \left[\frac{i}{k_0}\frac{df_3(\vec{r},\vec{r'})}{dn'}\right] dA' + \int_{A'} \tilde{J}_z^*(\vec{r'})[-f_3(\vec{r},\vec{r'})] dA' + \int_{C'} \tilde{J}_{n'}(\vec{r'}) \left[\frac{-i(-1)^q}{k_0} f_3(\vec{r},\vec{r'})\right] dn' \tag{4.4}$$

where (4.4) will be utilized only for generating the additional equations needed for the determination of $\tilde{J}_n(\vec{r})$ at the terminations of the layers. Assuming that the material has been partioned into thin layers, let each layer be further subdivided into a discrete number of expansion cells (totalling N for the entire body), whose area is denoted by $A_i, i = 1, N$ (see fig4.1). The currents $\tilde{J}_s$ and $\tilde{J}_z^*$ may now be expressed by a series of expansion functions with constant coefficients, viz.,

$$\tilde{J}_s(\vec{r}) = \sum_{i=1}^{N} \left[K_{si} P_i^{2d}(\vec{r}) + \Delta\tilde{J}_{si}(\vec{r})\right]$$

$$\tilde{J}_z^*(\vec{r}) = \sum_{i=1}^{N} \left[K_{zi}^* P_i^{2d}(\vec{r}) + \Delta\tilde{J}_{zi}^*(\vec{r})\right]$$

$$P_i^{2d}(\vec{r}) = \begin{cases} h_i^{2d}(\vec{r}), & \vec{r} \in A_i \\ 0, & \vec{r} \notin A_i \end{cases} \tag{4.5}$$

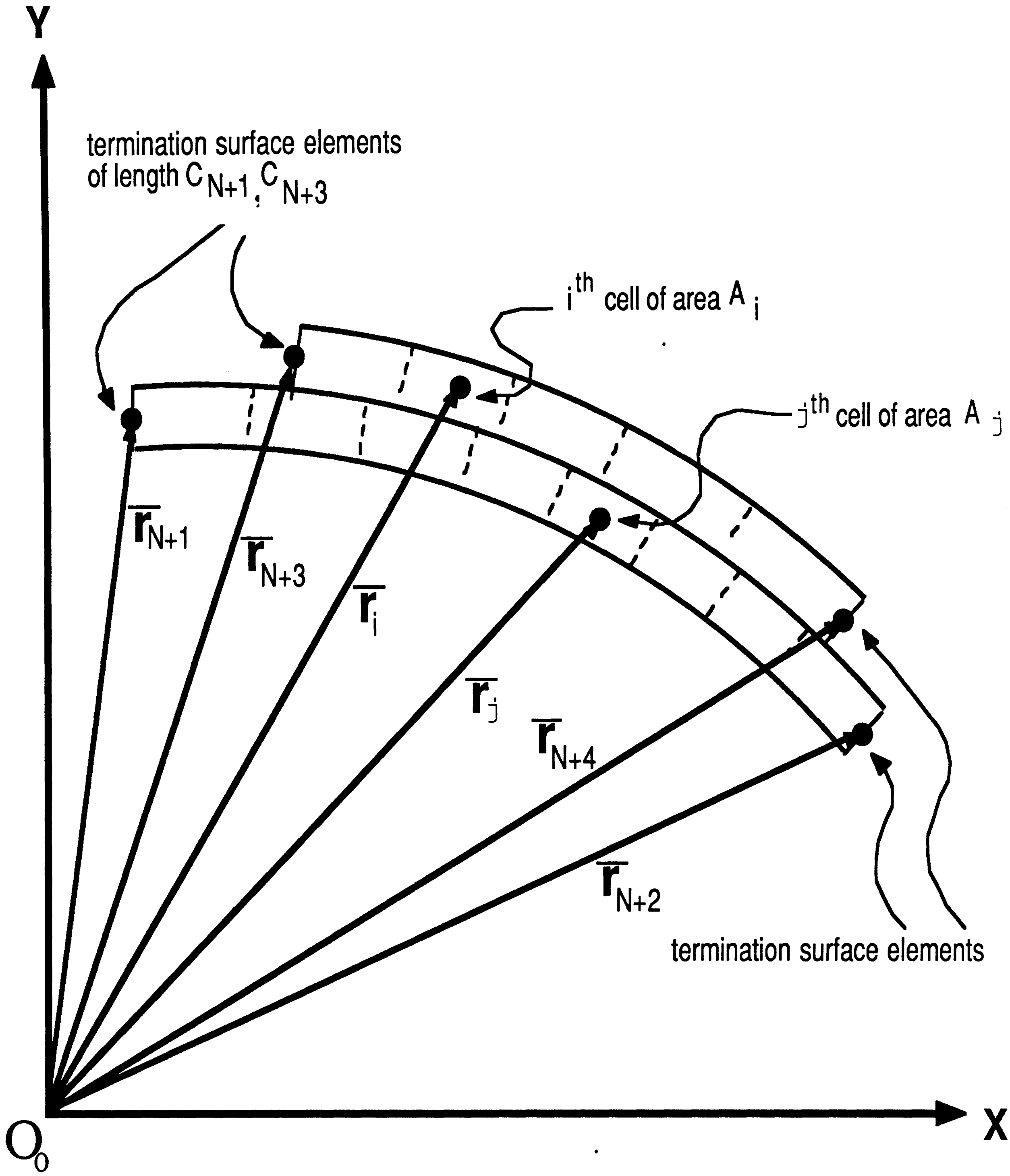

Figure 4.1: Discretization of the layers comprising the scatterer.

where $h_i^{2d}(\vec{r})$ is the basis function corresponding to the i^{th} cell and $\Delta\tilde{J}_{si}(\vec{r})$ with $\Delta\tilde{J}^*_{zi}(\vec{r})$ have been introduced only to make the equation exact. Let now N^{edge} be the number of expansion cell edges which lie along layer edges, denoted by $C_i, i = 1, N^{edge}$. Note also that since in the solution of (4.2) to (4.4), we require knowlege of $\tilde{J}_n(\vec{r})$ only along layer edges, we may replace $\tilde{J}_n(\vec{r})$ by $\tilde{J}_n^{edge}(\vec{r})$. The current component $\tilde{J}_n^{edge}(\vec{r})$ may subsequently be expressed as a series of expansion functions with constant coefficients (over $C_i, i = 1, N^{edge}$), augmented by a difference term, viz.,

$$\begin{aligned} \tilde{J}_n^{edge}(\vec{r}) &= \sum_{i=1}^{N^{edge}} \left[K_{ni}P_i^{1d}(\vec{r}) + \Delta\tilde{J}_{ni}^{edge}(\vec{r})\right] \\ P_i^{1d}(\vec{r}) &= \begin{cases} h_i^{1d}(\vec{r}), & \vec{r} \in C_i \\ 0, & \vec{r} \notin C_i \end{cases} \end{aligned} \tag{4.6}$$

It is clear that expressions (4.5) and (4.6) contain $2N + N^{edge}$ unknowns. These can now be substituted into (4.2) - (4.4) and the resulting equations may be rearranged so that the terms involving ΔJ are collected on the left hand side of each equation while the remainder of terms are gathered on the right hand side. In the application of the method of moments for their solution, it is then customary to multiply both sides of the equation by some weighting function. The integral equations can be subsequently integrated over the width of the weighting function which usually spans the extent of a single cell. Appropriate weighting functions are

$$W_i^{2d}(\vec{r}) = \begin{cases} w_i^{2d}(\vec{r}) & \vec{r} \in A_i \\ 0 & \vec{r} \notin A_i \end{cases} \tag{4.7}$$

for the case of (4.2) - (4.3) and

$$W_i^{1d}(\vec{r}) = \begin{cases} w_i^{1d}(\vec{r}) & \vec{r} \in C_i \\ 0 & \vec{r} \notin C_i \end{cases} \tag{4.8}$$

for the case of (4.4), where w_i^{2d}, (w_i^{1d}) is an arbitrary function corresponding to the i^{th} cell(edge). When these are now employed in the procedure duscussed above and the weighted integrals involving ΔJ are set to zero, we obtain

$$G_i^1 = u_i^{2d}K_{si} + \textstyle\sum_{j=1}^{N} F_{ij}^1 K_{sj} + \sum_{j=1}^{N} F_{ij}^2 K_{zj}^* + \sum_{j=1}^{N^{edge}} F_{ij}^3 K_{nj}^{edge} \quad ; i = 1, 2, \ldots N$$
$$G_i^2 = u_i^{2d}K_{zi}^* + \textstyle\sum_{j=1}^{N} F_{ij}^4 K_{sj} + \sum_{j=1}^{N} F_{ij}^5 K_{zj}^* + \sum_{j=1}^{N^{edge}} F_{ij}^6 K_{nj}^{edge} \quad ; i = 1, 2, \ldots N$$
$$G_i^3 = u_i^{1d}K_{ni}^{edge} + \textstyle\sum_{j=1}^{N} F_{ij}^7 K_{sj} + \sum_{j=1}^{N} F_{ij}^8 K_{zj}^* + \sum_{j=1}^{N^{edge}} F_{ij}^9 K_{nj}^{edge} \quad ; i = 1, 2, \ldots N^{edge}$$
$$(4.9)$$

where, upon definition of the operators L_i^{2d}, L_i^{1d}, M_j^{2d}, and M_j^{1d} as

$$\begin{aligned} L_i^{2d}(f) &= \int_{A_i} w_i^{2d}(\vec{r}) f(\vec{r}) dA \\ L_i^{1d}(f) &= \int_{C_i} w_i^{1d}(\vec{r}) f(\vec{r}) dn \\ M_j^{2d}(f) &= \int_{A_j} h_j^{2d}(\vec{r}') f(\vec{r}') dA' \\ M_j^{1d}(f) &= \int_{C_j} h_j^{1d}(\vec{r}') f(\vec{r}') dn', \end{aligned} \qquad (4.10)$$

we have

$$\begin{aligned} G_i^1 &= L_i^{2d}(g_1) \\ G_i^2 &= L_i^{2d}(g_2) \\ G_i^3 &= L_i^{1d}(g_3) \\ u_i^{2d} &= L_i^{2d}(h_i^{2d}) \\ u_i^{1d} &= L_i^{1d}(h_i^{1d}) \\ F_{ij}^1 &= L_i^{2d} M_j^{2d} \left(\frac{-i}{k_0} \frac{df_1(\vec{r}, \vec{r}')}{dn'} \right) \\ F_{ij}^2 &= L_i^{2d} M_j^{2d} \left(f_1(\vec{r}, \vec{r}') \right) \end{aligned}$$

$$
\begin{aligned}
F_{ij}^{3} &= L_i^{2d} M_j^{1d} \left(\frac{i(-1)^q}{k_0} f_1(\vec{r}, \vec{r}') \right) \\
F_{ij}^{4} &= L_i^{2d} M_j^{2d} \left((-1) \frac{df_2(\vec{r}, \vec{r}')}{dn'} \right) \\
F_{ij}^{5} &= L_i^{2d} M_j^{2d} (-ik_0 f_2(\vec{r}, \vec{r}')) \\
F_{ij}^{6} &= L_i^{2d} M_j^{1d} ((-1)^q f_2(\vec{r}, \vec{r}')) \\
F_{ij}^{7} &= L_i^{1d} M_j^{2d} \left(\frac{i}{k_0} \frac{df_3(\vec{r}, \vec{r}')}{dn'} \right) \\
F_{ij}^{8} &= L_i^{1d} M_j^{2d} ((-1) f_3(\vec{r}, \vec{r}')) \\
F_{ij}^{9} &= L_i^{1d} M_j^{1d} \left(\frac{-i(-1)^q}{k_0} f_3(\vec{r}, \vec{r}') \right) .
\end{aligned}
\tag{4.11}
$$

Equation (4.9) may be also written in matrix form as

$$
\begin{pmatrix} G^1 \\ G^2 \\ G^3 \end{pmatrix} = \begin{pmatrix} [\tilde{F}^1] & [F^2] & [F^3] \\ [F^4] & [\tilde{F}^5] & [F^6] \\ [F^7] & [F^8] & [\tilde{F}^9] \end{pmatrix} \begin{pmatrix} K_s \\ K_z^* \\ K_n^{edge} \end{pmatrix} \tag{4.12}
$$

where F_{ij}^p are the impedance matrix elements given in (4.11) and $\tilde{F}_{ij}^p$ are given by

$$
\begin{aligned}
\tilde{F}_{ij}^{1} &= F_{ij}^{1} + \gamma_{ij} u_i^{2d} \\
\tilde{F}_{ij}^{5} &= F_{ij}^{5} + \gamma_{ij} u_i^{2d} \\
\tilde{F}_{ij}^{9} &= F_{ij}^{9} + \gamma_{ij} u_i^{1d}
\end{aligned}
\tag{4.13}
$$

with γ_{ij} denoting the kronecker delta function. A solution for the current expansion coefficients K_s, K_z^* and K_n^{edge} may now be found via standard matrix inversion techniques. However, before this can be accomplished it is necessary that all integrals in (4.11) be first evaluated numerically or analytically. Specifically an analytic evaluation will be necessary for the self cells and a numerical one for most of the other cells.

4.2 Selection of Basis and Weighting Functions

To simplify the evaluation of the matrix elements we will employ a rather simple form of basis and weighting functions. Specifically we will employ pulse basis for the expansion of the current, implying

$$h_i^{1d}(\vec{r}) = 1, \quad h_i^{2d}(\vec{r}) = 1 \tag{4.14}$$

and the weighting functions will be set to

$$w_i^{1d}(\vec{r}) = \delta(\vec{r} - \vec{r}_i), \quad w_i^{2d}(\vec{r}) = \delta(\vec{r} - \vec{r}_i), \tag{4.15}$$

where, as shown in Fig. 4.2, $\vec{r}_i$ denotes the centroid of the i^{th} cell or the center of the i^{th} edge element. Using the above weighting and expansion functions, the elements of the excitation vector become

$$\begin{aligned}
G_i^1 = g_1(\vec{r}_i) &= \frac{E_s^{inc}(\vec{r}_i)}{\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)} \\
G_i^2 = g_2(\vec{r}_i) &= \frac{Z_0 H_z^{inc}(\vec{r}_i)}{\tilde{R}^*(\vec{r}_i)} - \frac{i}{k_0 \varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)} \frac{dE_n^{inc}(\vec{r}_i)}{ds} - \frac{iE_n^{inc}(\vec{r}_i)}{k_0} \frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)}\right) \\
G_i^3 = g_3(\vec{r}_i) &= \frac{E_n^{inc}(\vec{r}_i)}{\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)}.
\end{aligned} \tag{4.16}$$

Also the impedance elements may be summarized as

$$\begin{aligned}
\tilde{F}_{ij}^1 &= \gamma_{ij} + \int_{A_j} \left(\frac{-i}{k_0} \frac{df_1(\vec{r}, \vec{r}')}{dn'}\right)_{\vec{r}=\vec{r}_i} \\
F_{ij}^2 &= \int_{A_j} f_1(\vec{r}_i, \vec{r}') \\
F_{ij}^3 &= \int_{C_j} \frac{i(-1)^q}{k_0} f_1(\vec{r}_i, \vec{r}') \\
F_{ij}^4 &= \int_{A_j} \left(\frac{-df_2(\vec{r}, \vec{r}')}{dn'}\right)_{\vec{r}=\vec{r}_i}
\end{aligned}$$

$$
\begin{aligned}
\widetilde{F}_{ij}^{5} &= \gamma_{ij} + \int_{A_j} (-ik_0 f_2(\vec{r}_i, \vec{r}')) \\
F_{ij}^{6} &= \int_{C_j} (-1)^q f_2(\vec{r}_i, \vec{r}') \\
F_{ij}^{7} &= \int_{A_j} \left(\frac{i}{k_0} \frac{df_3(\vec{r}, \vec{r}')}{dn'} \right)_{\vec{r}=\vec{r}_i} \\
F_{ij}^{8} &= \int_{A_j} (-f_3(\vec{r}_i, \vec{r}')) \\
\widetilde{F}_{ij}^{9} &= \gamma_{ij} + \int_{C_j} \frac{-i(-1)^q}{k_0} f_3(\vec{r}_i, \vec{r}')
\end{aligned}
\qquad (4.17)
$$

Two different integration schemes will be considered for the evaluation of (4.17), depending upon whether a rectangular or uniformly curved (circular) cell is assumed. In the former case, the cell evaluations may be done with high degree of accuracy for all values of ρ, where, as usual, ρ is the distance from the point of observation to the point of integration. However for the latter case (uniform curvature) the cell evaluations will be in certain cases more approximate depending upon the magnitude of ρ and the size of the arc subtended by the cell of integration. Cells of non-uniform curvature (e.g. elliptical or parabolic) are not considered in this report. The type of integration employed will vary according to ρ as summarized in Table 4.1.

4.3 Definition of Coordinates

In the evaluation of the impedance elements requiring integration over rectangular cells, it is convenient to introduce the geometry shown in Fig. 4.2. The coordinates/components of the labeled points/vectors are the ones which are required in the subsequent analysis. It is further necessary that some of these be defined with respect to two coordinate frames in the case when a singular evalua-

Regime	Straight Edge(Rect. Cell)	Circular Cell
$k_0\rho > a_1$	(2-dim) 3pt. Simpson's rule	2-dim. 3pt. Simpson's rule
$a_1 > k_0\rho > a_2$	(2-dim) 5pt. Simpson's rule	2-dim. 5pt. Simpson's rule
$a_2 > k_0\rho$	analytical evaluation	2-dim. 5pt. Simpson's rule/analytical
self cell	analytical evaluation	approx. analytical evaluation

Table 4.1: Criteria Used for Integration of Impedance Cells

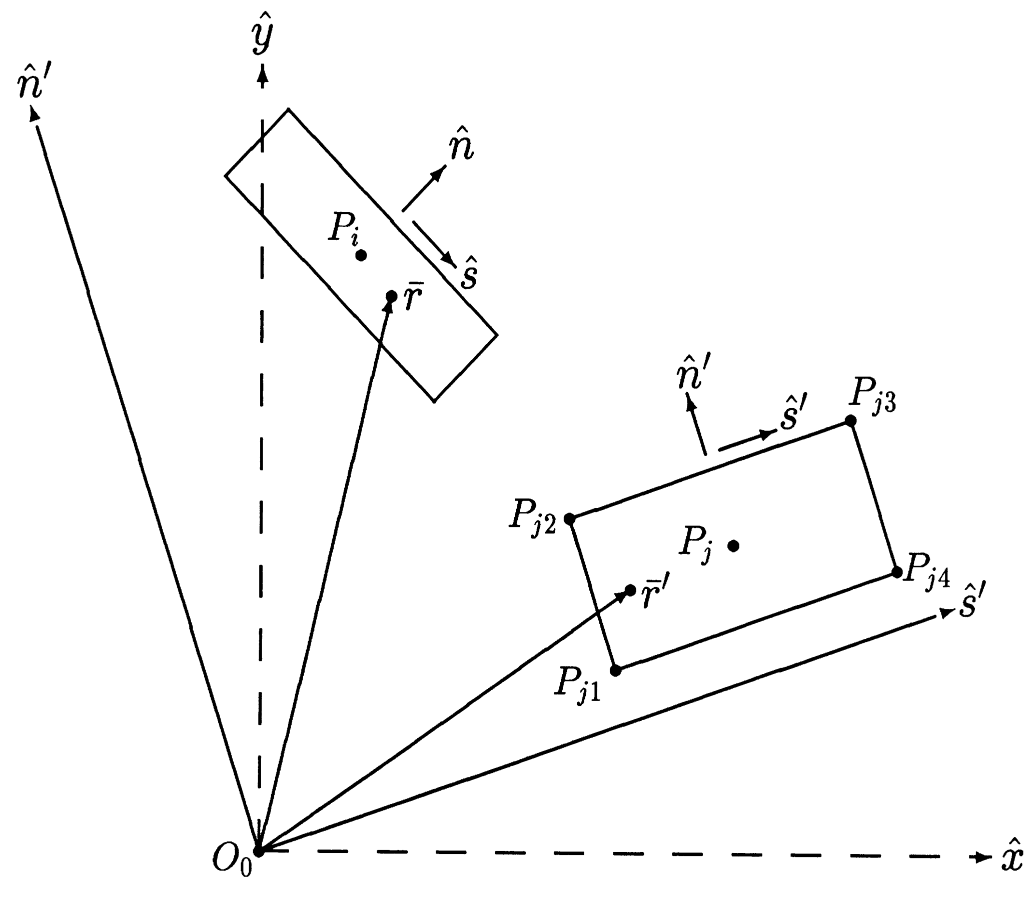

Figure 4.2: Coordinate frame used in the evaluation of impedance elements which require integration over a rectangular cell.

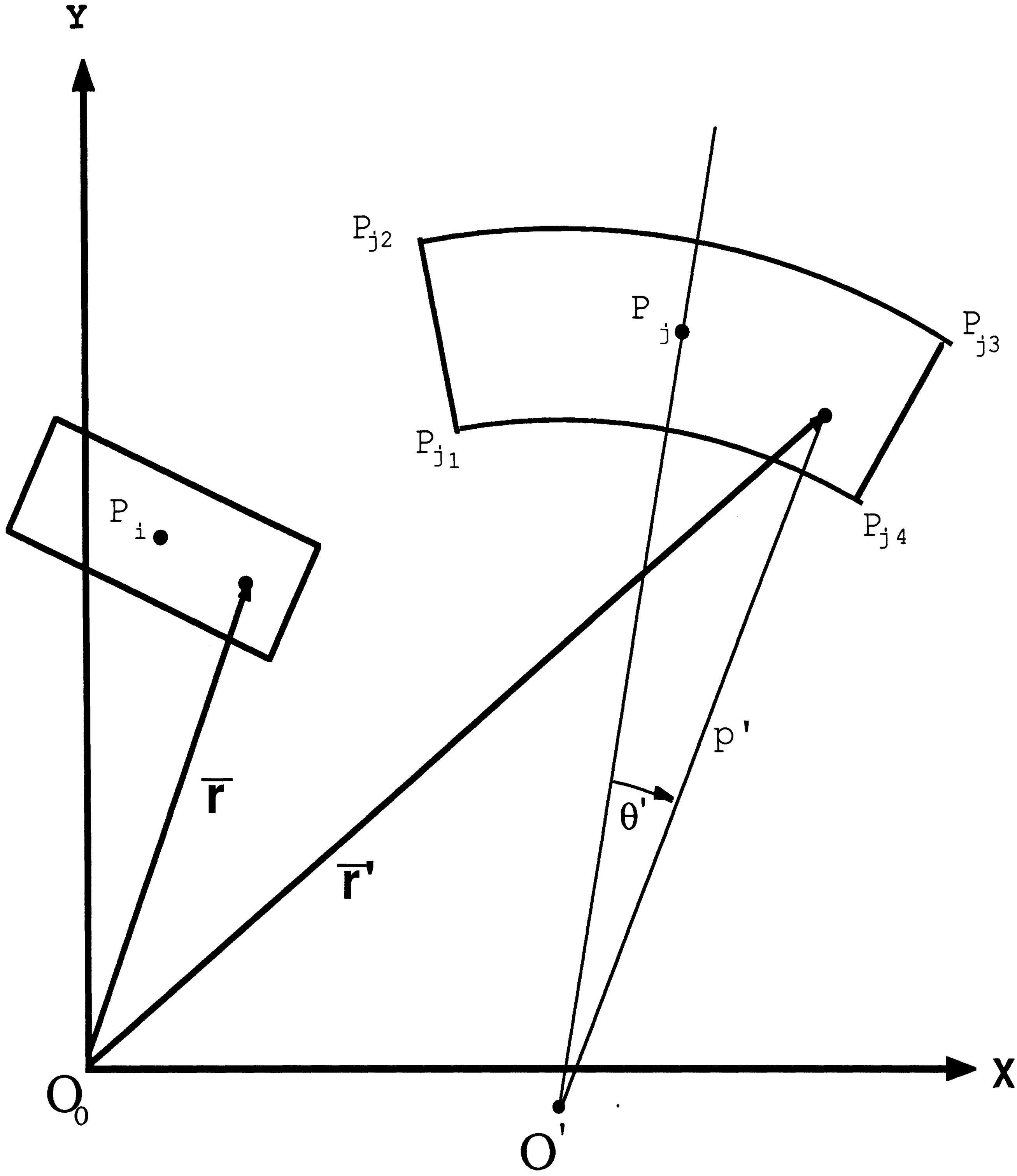

Figure 4.3: Coordinate frame used in the evaluation of impedance elements which require integration over a circular cell.

tion of the element is required. One of these coordinate frames is associated with directions $(\hat{s}', \hat{n}', \hat{z})$ and has its origin at O_0. With this coordinate frame as our reference, the following definitions will apply (the $\hat{z}$ directed component is suppressed because of the assumed two dimensional geometry):

(s_{obs}, n_{obs}) : $\hat{s}'$ and $\hat{n}'$ components of the vector $\bar{r}$

(s', n') : $\hat{s}'$ and $\hat{n}'$ components of the vector $\bar{r}'$

(s_i, n_i) : $\hat{s}'$ and $\hat{n}'$ coordinates of P_i

(s_j, n_j) : $\hat{s}'$ and $\hat{n}'$ coordinates of P_j

$(s_j - \delta_j/2, n_j - \tau_j/2)$: $\hat{s}'$ and $\hat{n}'$ coordinates of P_{j1}

$(s_j - \delta_j/2, n_j + \tau_j/2)$: $\hat{s}'$ and $\hat{n}'$ coordinates of P_{j2}

$(s_j + \delta_j/2, n_j + \tau_j/2)$: $\hat{s}'$ and $\hat{n}'$ coordinates of P_{j3}

$(s_j + \delta_j/2, n_j - \tau_j/2)$: $\hat{s}'$ and $\hat{n}'$ coordinates of P_{j4}.

The other coordinate frame required in the analysis is that associated with same directions $(\hat{s}', \hat{n}', \hat{z})$ but with origin shifted to P_i. With this coordinate frame as our reference, we introduce the following additional definitions:

$(s_j - \delta_j/2 - s_i, n_j - \tau_j/2 - n_i) = (x_1, y_1)$: $\hat{s}'$ and $\hat{n}'$coordinates of P_{j1}

$(s_j - \delta_j/2 - s_i, n_j + \tau_j/2 - n_i) = (x_2, y_2)$: $\hat{s}'$ and $\hat{n}'$coordinates of P_{j2}

$(s_j + \delta_j/2 - s_i, n_j + \tau_j/2 - n_i) = (x_3, y_3)$: $\hat{s}'$ and $\hat{n}'$coordinates of P_{j3}

$(s_j + \delta_j/2 - s_i, n_j - \tau_j/2 - n_i) = (x_4, y_4)$: $\hat{s}'$ and $\hat{n}'$coordinates of P_{j4}

$\rho_1 = \sqrt{(x_1)^2 + (y_1)^2}$

$\rho_2 = \sqrt{(x_2)^2 + (y_2)^2}$

$$\rho_3 = \sqrt{(x_3)^2 + (y_3)^2}$$
$$\rho_4 = \sqrt{(x_4)^2 + (y_4)^2}.$$

In the evaluation of the impedance elements requiring integration over circular cells, we introduce the geometry shown in Fig. 4.3. In this situation, it proves convenient to define a cylindrical coordinate system $(\hat{p}', \hat{\theta}', \hat{z})$ with origin O', the local center of curvature of the j^{th} cell, and axis $\theta' = 0$, which contains the point P_j. The following definitions apply with this coordinate frame as our reference:

(p_{obs}, θ_{obs})	: $\hat{p}'$ and $\hat{\theta}'$ components of the vector $\bar{r}$
(p', θ')	: $\hat{p}'$ and $\hat{\theta}'$ components of the vector $\bar{r}'$
(p_i, θ_i)	: $\hat{p}'$ and $\hat{\theta}'$ coordinates of P_i
$(p_j, 0)$	: $\hat{p}'$ and $\hat{\theta}'$ coordinates of P_j
$(p_j - \Delta p_j/2, -\Delta\theta_j/2)$	: $\hat{p}'$ and $\hat{\theta}'$ coordinates of P_{j1}
$(p_j + \Delta p_j/2, -\Delta\theta_j/2)$	: $\hat{p}'$ and $\hat{\theta}'$ coordinates of P_{j2}
$(p_j + \Delta p_j/2, \Delta\theta_j/2)$	: $\hat{p}'$ and $\hat{\theta}'$ coordinates of P_{j3}
$(p_j - \Delta p_j/2, \Delta\theta_j/2)$	: $\hat{p}'$ and $\hat{\theta}'$ coordinates of P_{j4}.

4.4 Explicit Forms of Impedance Elements for Numerical Integration

To perform the numerical integrations, a 5x5 grid of sample points is generated for each cell. The parameter ρ given in Table 4.1 above is taken to be the distance from the point of observation to the closest point on the sample grid. Below we derive the final expression for the elements F_{ij} as employed in their numerical implementation:

4.4.1 Element $\widetilde{F}^1_{ij}$, Rectangular Cell:

$$\begin{aligned}
\widetilde{F}^1_{ij} &= \gamma_{ij} + \int_{A_j} \frac{-i}{k_0}\frac{df_1(\vec{r}_i,\vec{r}')}{dn'}dA' \\
&= \gamma_{ij} + \int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{-i}{k_0}\frac{df_1(\vec{r}_i,\vec{r}')}{dn'}ds'dn' \\
&= \gamma_{ij} + \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \frac{-i}{k_0} f_1(\vec{r}_i,\vec{r}')\Big|_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} ds' \\
&= \gamma_{ij} + \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \frac{-i}{k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[\frac{dG(\vec{r},\vec{r}')}{dn}\Big|_{\vec{r}=\vec{r}_i}\right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} ds' \\
&= \gamma_{ij} + \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \frac{-i}{k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[\left(\frac{-ik_0}{4}(\hat{\rho}\cdot\hat{n})H_1^{(1)}(k_0\rho)\right)_{\vec{r}=\vec{r}_i}\right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} ds' \\
&= \gamma_{ij} - \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \frac{1}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[\left((\hat{\rho}\cdot\hat{n})H_1^{(1)}(k_0\rho)\right)_{\vec{r}=\vec{r}_i}\right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} ds' \qquad (4.18)
\end{aligned}$$

4.4.2 Element $\widetilde{F}^1_{ij}$, Circular Cell:

$$\begin{aligned}
\widetilde{F}^1_{ij} = \gamma_{ij} &+ \int_{p_j-\Delta p_j/2}^{p_j+\Delta p_j/2}\int_{-\Delta\theta_j/2}^{\Delta\theta_j/2} \frac{-i}{k_0}\frac{df_1(\vec{r}_i,\vec{r}')}{dp'}(p'd\theta'dp') \\
= \gamma_{ij} &+ \int_{-\Delta\theta_j/2}^{\Delta\theta_j/2} \frac{-i}{k_0}\left[p'f_1(\vec{r}_i,\vec{r}')\right]_{p'=p_j-\Delta p_j/2}^{p'=p_j+\Delta p_j/2} d\theta' + \int_{p_j-\Delta p_j/2}^{p_j+\Delta p_j/2}\int_{-\Delta\theta_j/2}^{\Delta\theta_j/2} \frac{i}{k_0} f_1(\vec{r}_i,\vec{r}')d\theta'dp' \\
= \gamma_{ij} &+ \int_{-\Delta\theta_j/2}^{\Delta\theta_j/2} \frac{-i}{k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[p'\frac{dG(\vec{r},\vec{r}')}{dn}\Big|_{\vec{r}=\vec{r}_i}\right]_{p'=p_j-\Delta p_j/2}^{p'=p_j+\Delta p_j/2} d\theta' \\
&+ \int_{p_j-\Delta p_j/2}^{p_j+\Delta p_j/2}\int_{-\Delta\theta_j/2}^{\Delta\theta_j/2} \frac{i}{k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left(\frac{dG(\vec{r},\vec{r}')}{dn}\right)_{\vec{r}=\vec{r}_i} d\theta'dp' \\
= \gamma_{ij} &+ \int_{-\Delta\theta_j/2}^{\Delta\theta_j/2} \frac{-i}{k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[\left(p'\frac{-ik_0}{4}(\hat{\rho}\cdot\hat{n})H_1^{(1)}(k_0\rho)\right)_{\vec{r}=\vec{r}_i}\right]_{p'=p_j-\Delta p_j/2}^{p'=p_j+\Delta p_j/2} d\theta' \\
&+ \int_{p_j-\Delta p_j/2}^{p_j+\Delta p_j/2}\int_{-\Delta\theta_j/2}^{\Delta\theta_j/2} \frac{i}{k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left(\frac{-ik_0}{4}(\hat{\rho}\cdot\hat{n})H_1^{(1)}(k_0\rho)\right)_{\vec{r}=\vec{r}_i} d\theta'dp' \\
= \gamma_{ij} &- \int_{-\Delta\theta_j/2}^{\Delta\theta_j/2} \frac{1}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[\left(p'(\hat{\rho}\cdot\hat{n})H_1^{(1)}(k_0\rho)\right)_{\vec{r}=\vec{r}_i}\right]_{p'=p_j-\Delta p_j/2}^{p'=p_j+\Delta p_j/2} d\theta' \\
&+ \int_{p_j-\Delta p_j/2}^{p_j+\Delta p_j/2}\int_{-\Delta\theta_j/2}^{\Delta\theta_j/2} \frac{1}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left((\hat{\rho}\cdot\hat{n})H_1^{(1)}(k_0\rho)\right)_{\vec{r}=\vec{r}_i} d\theta'dp' \qquad (4.19)
\end{aligned}$$

4.4.3 Element F_{ij}^2, Rectangular Cell:

$$
\begin{aligned}
F_{ij}^2 &= \int_{A_j} f_1(\vec{r}_i, \vec{r}')dA' \\
&= \int_{s_j-\delta_j/2}^{s_j+\tau_j/2} \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} f_1(\vec{r}_i, \vec{r}')ds'dn' \\
&= \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \left(\frac{dG(\vec{r}, \vec{r}')}{dn}\right)_{\vec{r}=\vec{r}_i} ds'dn' \\
&= \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \left[\frac{-ik_0}{4}(\hat{\rho}\cdot\hat{n})H_1^{(1)}(k_0\rho)\right]_{\vec{r}=\vec{r}_i} ds'dn' \\
&= \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{-ik_0}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \left[(\hat{\rho}\cdot\hat{n})H_1^{(1)}(k_0\rho)\right]_{\vec{r}=\vec{r}_i} ds'dn' \qquad (4.20)
\end{aligned}
$$

4.4.4 Element F_{ij}^2, Circular Cell:

$$
\begin{aligned}
F_{ij}^2 &= \int_{p_j-\Delta p_j/2}^{p_j+\Delta p_j/2} \int_{-\Delta\theta_j/2}^{\Delta\theta_j/2} f_1(\vec{r}_i, \vec{r}')(p'd\theta'dp') \\
&= \int_{p_j-\Delta p_j/2}^{p_j+\Delta p_j/2} \int_{-\Delta\theta_j/2}^{\Delta\theta_j/2} \frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \left(\frac{dG(\vec{r}, \vec{r}')}{dn}\right)_{\vec{r}=\vec{r}_i} p'd\theta'dp' \\
&= \int_{p_j-\Delta p_j/2}^{p_j+\Delta p_j/2} \int_{-\Delta\theta_j/2}^{\Delta\theta_j/2} \frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \left[\frac{-ik_0}{4}(\hat{\rho}\cdot\hat{n})H_1^{(1)}(k_0\rho)\right]_{\vec{r}=\vec{r}_i} p'd\theta'dp' \\
&= \int_{p_j-\Delta p_j/2}^{p_j+\Delta p_j/2} \int_{-\Delta\theta_j/2}^{\Delta\theta_j/2} \frac{-ik_0}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \left[(\hat{\rho}\cdot\hat{n})H_1^{(1)}(k_0\rho)\right]_{\vec{r}=\vec{r}_i} p'd\theta'dp' \qquad (4.21)
\end{aligned}
$$

4.4.5 Element F_{ij}^3:

$$
\begin{aligned}
F_{ij}^3 &= \int_{C_j} \frac{i(-1)^q}{k_0} f_1(\vec{r}_i, \vec{r}')dn' \\
&= \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{i(-1)^q}{k_0} f_1(\vec{r}_i, \vec{r}')dn' \\
&= \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{i(-1)^q}{k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \left(\frac{dG(\vec{r}, \vec{r}')}{dn}\right)_{\vec{r}=\vec{r}_i} dn' \\
&= \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{i(-1)^q}{k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \left[\frac{-ik_0}{4}(\hat{\rho}\cdot\hat{n})H_1^{(1)}(k_0\rho)\right]_{\vec{r}=\vec{r}_i} dn' \\
&= \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{(-1)^q}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \left[(\hat{\rho}\cdot\hat{n})H_1^{(1)}(k_0\rho)\right]_{\vec{r}=\vec{r}_i} dn' \qquad (4.22)
\end{aligned}
$$

4.4.6 Element F_{ij}^4, Rectangular Cell:

$$
\begin{aligned}
F_{ij}^4 &= \int_{A_j} \frac{-df_2(\vec{r}_i,\vec{r}')}{dn'} dA' \\
&= \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{-df_2(\vec{r}_i,\vec{r}')}{dn'} ds'dn' \\
&= \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} (-1) f_2(\vec{r}_i,\vec{r}')|_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} ds' \\
&= \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} (-1) \left[\left(\frac{1}{\widetilde{R}^*(\vec{r}_i)} - \frac{1}{k_0^2 \varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \frac{d}{ds^2} \right) G(\vec{r},\vec{r}')_{\vec{r}=\vec{r}_i} \right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} ds' \\
&+ \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \left[\frac{1}{k_0^2} \frac{d}{ds} \left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \right) \frac{d}{ds} G(\vec{r},\vec{r}')_{\vec{r}=\vec{r}_i} \right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} ds' \\
&= \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \frac{1}{\widetilde{R}^*(\vec{r}_i)} \frac{-iH_0^{(1)}(k_0\rho)_{\vec{r}=\vec{r}_i}}{4} ds'|_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} \\
&+ \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \frac{1}{k_0^2} \frac{d}{ds} \left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \right) \frac{-ik_0}{4} \left((\hat{\rho}\cdot\hat{s}) H_1^{(1)}(k_0\rho) \right)_{\vec{r}=\vec{r}_i} ds'|_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} \\
&+ \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \frac{1}{k_0^2 \varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \frac{i}{4} \left(k_0^2 \left((\hat{\rho}\cdot\hat{s})^2 - (\hat{\rho}\cdot\hat{n})^2 \right) \frac{H_1^{(1)}(k_0\rho)}{k_0\rho} \right)_{\vec{r}=\vec{r}_i} ds'|_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} \\
&+ \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \frac{1}{k_0^2 \varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \frac{i}{4} \left(-k_0^2 (\hat{\rho}\cdot\hat{s})^2 H_0^{(1)}(k_0\rho) \right)_{\vec{r}=\vec{r}_i} ds'|_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} \\
&= \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \frac{-i}{4} \left(\frac{1}{\widetilde{R}^*(\vec{r}_i)} + \frac{(\hat{\rho}\cdot\hat{s})^2_{\vec{r}=\vec{r}_i}}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \right) H_0^{(1)}(k_0\rho)_{\vec{r}=\vec{r}_i} ds'|_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} \\
&+ \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \frac{-i}{4k_0} \frac{d}{ds} \left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \right) \left((\hat{\rho}\cdot\hat{s}) H_1^{(1)}(k_0\rho) \right)_{\vec{r}=\vec{r}_i} ds'|_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} \\
&+ \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \frac{i}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \left(\left((\hat{\rho}\cdot\hat{s})^2 - (\hat{\rho}\cdot\hat{n})^2 \right) \frac{H_1^{(1)}(k_0\rho)}{k_0\rho} \right)_{\vec{r}=\vec{r}_i} ds'|_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2}
\end{aligned}
\tag{4.23}
$$

4.4.7 Element F_{ij}^4, Circular Cell:

$$
\begin{aligned}
F_{ij}^4 &= \int_{p_j-\Delta p_j/2}^{p_j+\Delta p_j/2} \int_{-\Delta\theta_j/2}^{\Delta\theta_j/2} \frac{-df_2(\vec{r}_i,\vec{r}')}{dp'} (p'd\theta'dp') \\
&= \int_{-\Delta\theta_j/2}^{\Delta\theta_j/2} [-p' f_2(\vec{r}_i,\vec{r}')]_{p'=p_j-\Delta p_j/2}^{p'=p_j+\Delta p_j/2} d\theta' + \int_{p_j-\Delta p_j/2}^{p_j+\Delta p_j/2} \int_{-\Delta\theta_j/2}^{\Delta\theta_j/2} f_2(\vec{r}_i,\vec{r}') d\theta' dp'
\end{aligned}
$$

$$
\begin{aligned}
= & \int_{-\Delta\theta_j/2}^{+\Delta\theta_j/2} \frac{-i}{4}\left(\frac{1}{\widetilde{R}^*(\vec{r}_i)} + \frac{(\hat{\rho}\cdot\hat{s})^2_{\vec{r}=\vec{r}_i}}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right) p' H_0^{(1)}(k_0\rho)_{\vec{r}=\vec{r}_i} d\theta' \Big|_{p'=p_j-\Delta p_j/2}^{p'=p_j+\Delta p_j/2} \\
& + \int_{-\Delta\theta_j/2}^{+\Delta\theta_j/2} \frac{-i}{4k_0}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)\left(p'(\hat{\rho}\cdot\hat{s})H_1^{(1)}(k_0\rho)\right)_{\vec{r}=\vec{r}_i} d\theta' \Big|_{p'=p_j-\Delta p_j/2}^{p'=p_j+\Delta p_j/2} \\
& + \int_{-\Delta\theta_j/2}^{+\Delta\theta_j/2} \frac{i}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left(p'\left((\hat{\rho}\cdot\hat{s})^2 - (\hat{\rho}\cdot\hat{n})^2\right)\frac{H_1^{(1)}(k_0\rho)}{k_0\rho}\right)_{\vec{r}=\vec{r}_i} d\theta' \Big|_{p'=p_j-\Delta p_j/2}^{p'=p_j+\Delta p_j/2} \\
& + \int_{-\Delta\theta_j/2}^{+\Delta\theta_j/2}\int_{p_j-\Delta p_j/2}^{p_j+\Delta p_j/2} \frac{i}{4}\left(\frac{1}{\widetilde{R}^*(\vec{r}_i)} + \frac{(\hat{\rho}\cdot\hat{s})^2_{\vec{r}=\vec{r}_i}}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right) H_0^{(1)}(k_0\rho)_{\vec{r}=\vec{r}_i} d\theta' dp' \\
& + \int_{-\Delta\theta_j/2}^{+\Delta\theta_j/2}\int_{p_j-\Delta p_j/2}^{p_j+\Delta p_j/2} \frac{i}{4k_0}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)\left((\hat{\rho}\cdot\hat{s})H_1^{(1)}(k_0\rho)\right)_{\vec{r}=\vec{r}_i} d\theta' dp' \\
& + \int_{-\Delta\theta_j/2}^{+\Delta\theta_j/2}\int_{p_j-\Delta p_j/2}^{p_j+\Delta p_j/2} \frac{-i}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left(\left((\hat{\rho}\cdot\hat{s})^2 - (\hat{\rho}\cdot\hat{n})^2\right)\frac{H_1^{(1)}(k_0\rho)}{k_0\rho}\right)_{\vec{r}=\vec{r}_i} d\theta' dp'
\end{aligned}
\tag{4.24}
$$

4.4.8 Element $\widetilde{F}^5_{ij}$, Rectangular Cell:

$$
\begin{aligned}
\widetilde{F}^5_{ij} = & \ \gamma_{ij} + \int_{A_j} (-ik_0) f_2(\vec{r}_i, \vec{r}')dA' \\
= & \ \gamma_{ij} + \int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\int_{n_j-\tau_j/2}^{n_j+\tau_j/2} (-ik_0) f_2(\vec{r}_i, \vec{r}')ds'dn' \\
= & \ \gamma_{ij} + \int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\int_{n_j-\tau_j/2}^{n_j+\tau_j/2} (-ik_0)\left[\left(\frac{1}{\widetilde{R}^*(\vec{r}_i)} - \frac{1}{k_0^2\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\frac{d}{ds^2}\right) G(\vec{r}, \vec{r}')_{\vec{r}=\vec{r}_i}\right] ds'dn' \\
& + \int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\int_{n_j-\tau_j/2}^{n_j+\tau_j/2} (ik_0)\left[\left(\frac{1}{k_0^2}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)\frac{d}{ds}\right) G(\vec{r}, \vec{r}')_{\vec{r}=\vec{r}_i}\right] ds'dn' \\
= & \ \gamma_{ij} + \int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{-ik_0}{\widetilde{R}^*(\vec{r}_i)}\frac{iH_0^{(1)}(k_0\rho)_{\vec{r}=\vec{r}_i}}{4} ds'dn' \\
& + \int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\int_{n_j-\tau_j/2}^{n_j+\tau_j/2} (ik_0)\frac{1}{k_0^2}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)\frac{-ik_0}{4}\left((\hat{\rho}\cdot\hat{s})H_1^{(1)}(k_0\rho)\right)_{\vec{r}=\vec{r}_i} ds'dn' \\
& + \int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\int_{n_j-\tau_j/2}^{n_j+\tau_j/2} (ik_0)\frac{1}{k_0^2\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\frac{i}{4}\left(k_0^2\left((\hat{\rho}\cdot\hat{s})^2 - (\hat{\rho}\cdot\hat{n})^2\right)\frac{H_1^{(1)}(k_0\rho)}{k_0\rho}\right)_{\vec{r}=\vec{r}_i} ds'dn' \\
& + \int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\int_{n_j-\tau_j/2}^{n_j+\tau_j/2} (ik_0)\frac{1}{k_0^2\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\frac{i}{4}\left(-k_0^2(\hat{\rho}\cdot\hat{s})^2 H_0^{(1)}(k_0\rho)\right)_{\vec{r}=\vec{r}_i} ds'dn' \\
= & \ \gamma_{ij} + \int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{k_0}{4}\left(\frac{1}{\widetilde{R}^*(\vec{r}_i)} + \frac{(\hat{\rho}\cdot\hat{s})^2_{\vec{r}=\vec{r}_i}}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right) H_0^{(1)}(k_0\rho)_{\vec{r}=\vec{r}_i} ds'dn'
\end{aligned}
$$

$$+\int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\int_{n_j-\tau_j/2}^{n_j+\tau_j/2}\frac{1}{4}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)\left((\hat{\rho}\cdot\hat{s})H_1^{(1)}(k_0\rho)\right)_{\vec{r}=\vec{r}_i}ds'dn'$$

$$+\int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\int_{n_j-\tau_j/2}^{n_j+\tau_j/2}\frac{-k_0}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left(\left((\hat{\rho}\cdot\hat{s})^2-(\hat{\rho}\cdot\hat{n})^2\right)\frac{H_1^{(1)}(k_0\rho)}{k_0\rho}\right)_{\vec{r}=\vec{r}_i}ds'dn' \quad (4.25)$$

4.4.9 Element $\widetilde{F}_{ij}^5$, Circular Cell:

$$\begin{aligned}
\widetilde{F}_{ij}^5 =& \gamma_{ij} + \int_{p_j-\Delta p_j/2}^{p_j+\Delta p_j/2}\int_{-\Delta\theta_j/2}^{\Delta\theta_j/2}(-ik_0)f_2(\vec{r}_i,\vec{r}')(p'd\theta'dp') \\
=& \gamma_{ij} + \int_{-\Delta\theta_j/2}^{+\Delta\theta_j/2}\int_{p_j-\Delta p_j/2}^{p_j+\Delta p_j/2}\frac{k_0}{4}\left(\frac{1}{\widetilde{R}^*(\vec{r}_i)}+\frac{(\hat{\rho}\cdot\hat{s})^2_{\vec{r}=\vec{r}_i}}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)H_0^{(1)}(k_0\rho)_{\vec{r}=\vec{r}_i}p'd\theta'dp' \\
&+\int_{-\Delta\theta_j/2}^{+\Delta\theta_j/2}\int_{p_j-\Delta p_j/2}^{p_j+\Delta p_j/2}\frac{1}{4}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)\left((\hat{\rho}\cdot\hat{s})H_1^{(1)}(k_0\rho)\right)_{\vec{r}=\vec{r}_i}p'd\theta'dp' \\
&+\int_{-\Delta\theta_j/2}^{+\Delta\theta_j/2}\int_{p_j-\Delta p_j/2}^{p_j+\Delta p_j/2}\frac{-k_0}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left(\left((\hat{\rho}\cdot\hat{s})^2-(\hat{\rho}\cdot\hat{n})^2\right)\frac{H_1^{(1)}(k_0\rho)}{k_0\rho}\right)_{\vec{r}=\vec{r}_i}p'd\theta'dp'
\end{aligned}$$

(4.26)

4.4.10 Element F_{ij}^6:

$$\begin{aligned}
F_{ij}^6 =& \int_{C_j}(-1)^q f_2(\vec{r}_i,\vec{r}')dn' \\
=& \int_{n_j-\tau_j/2}^{n_j+\tau_j/2}(-1)^q f_2(\vec{r}_i,\vec{r}')dn' \\
=& \int_{n_j-\tau_j/2}^{n_j+\tau_j/2}(-1)^q\left[\left(\frac{1}{\widetilde{R}^*(\vec{r}_i)}-\frac{1}{k_0^2}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)\frac{d}{ds}-\frac{1}{k_0^2\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\frac{d}{ds^2}\right)G(\vec{r},\vec{r}')_{\vec{r}=\vec{r}_i}\right]dn' \\
=& \int_{n_j-\tau_j/2}^{n_j+\tau_j/2}\frac{(-1)^q}{\widetilde{R}^*(\vec{r}_i)}\frac{iH_0^{(1)}(k_0\rho)_{\vec{r}=\vec{r}_i}}{4}dn' \\
&+\int_{n_j-\tau_j/2}^{n_j+\tau_j/2}(-1)^{q+1}\frac{1}{k_0^2}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)\frac{-ik_0}{4}\left((\hat{\rho}\cdot\hat{s})H_1^{(1)}(k_0\rho)\right)_{\vec{r}=\vec{r}_i}dn' \\
&+\int_{n_j-\tau_j/2}^{n_j+\tau_j/2}(-1)^{q+1}\frac{1}{k_0^2\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\frac{i}{4}\left(k_0^2\left((\hat{\rho}\cdot\hat{s})^2-(\hat{\rho}\cdot\hat{n})^2\right)\frac{H_1^{(1)}(k_0\rho)}{k_0\rho}\right)_{\vec{r}=\vec{r}_i}dn' \\
&+\int_{n_j-\tau_j/2}^{n_j+\tau_j/2}(-1)^{q+1}\frac{1}{k_0^2\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\frac{i}{4}\left(-k_0^2(\hat{\rho}\cdot\hat{s})^2H_0^{(1)}(k_0\rho)\right)_{\vec{r}=\vec{r}_i}dn' \\
=& \int_{n_j-\tau_j/2}^{n_j+\tau_j/2}\frac{i(-1)^q}{4}\left(\frac{1}{\widetilde{R}^*(\vec{r}_i)}+\frac{(\hat{\rho}\cdot\hat{s})^2_{\vec{r}=\vec{r}_i}}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)H_0^{(1)}(k_0\rho)_{\vec{r}=\vec{r}_i}dn'
\end{aligned}$$

$$+\int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{i(-1)^q}{4k_0}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)\left((\hat{\rho}\cdot\hat{s})H_1^{(1)}(k_0\rho)\right)_{\vec{r}=\vec{r}_i} dn'$$

$$+\int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{-i(-1)^q}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left(\left((\hat{\rho}\cdot\hat{s})^2-(\hat{\rho}\cdot\hat{n})^2\right)\frac{H_1^{(1)}(k_0\rho)}{k_0\rho}\right)_{\vec{r}=\vec{r}_i} dn' \quad (4.27)$$

4.4.11 Element F^7_{ij}, Rectangular Cell:

$$\begin{aligned}
F^7_{ij} &= \int_{A_j} \frac{i}{k_0}\frac{df_3(\vec{r}_i,\vec{r}')}{dn'}dA' \\
&= \int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{i}{k_0}\frac{df_3(\vec{r}_i,\vec{r}')}{dn'}ds'dn' \\
&= \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \frac{i}{k_0} f_3(\vec{r}_i,\vec{r}')\Big|_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} ds' \\
&= \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \frac{i}{k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[\frac{dG(\vec{r},\vec{r}')}{ds}\bigg|_{\vec{r}=\vec{r}_i}\right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} ds' \\
&= \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \frac{i}{k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[\left(\frac{-ik_0}{4}(\hat{\rho}\cdot\hat{s})H_1^{(1)}(k_0\rho)\right)_{\vec{r}=\vec{r}_i}\right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} ds' \\
&= \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \frac{1}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[\left((\hat{\rho}\cdot\hat{s})H_1^{(1)}(k_0\rho)\right)_{\vec{r}=\vec{r}_i}\right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} ds' \qquad (4.28)
\end{aligned}$$

4.4.12 Element F^7_{ij}, Circular Cell:

$$\begin{aligned}
F^7_{ij} &= \int_{p_j-\Delta p_j/2}^{p_j+\Delta p_j/2}\int_{-\Delta\theta_j/2}^{\Delta\theta_j/2} \frac{i}{k_0}\frac{df_3(\vec{r}_i,\vec{r}')}{dp'}(p'd\theta'dp') \\
&= \int_{-\Delta\theta_j/2}^{\Delta\theta_j/2} \frac{i}{k_0}\left[p' f_3(\vec{r}_i,\vec{r}')\right]_{p'=p_j-\Delta p_j/2}^{p'=p_j+\Delta p_j/2} d\theta' + \int_{p_j-\Delta p_j/2}^{p_j+\Delta p_j/2}\int_{-\Delta\theta_j/2}^{\Delta\theta_j/2} \frac{-i}{k_0} f_3(\vec{r}_i,\vec{r}')d\theta'dp' \\
&= \int_{-\Delta\theta_j/2}^{\Delta\theta_j/2} \frac{i}{k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[p'\frac{dG(\vec{r},\vec{r}')}{ds}\bigg|_{\vec{r}=\vec{r}_i}\right]_{p'=p_j-\Delta p_j/2}^{p'=p_j+\Delta p_j/2} d\theta' \\
&\quad + \int_{p_j-\Delta p_j/2}^{p_j+\Delta p_j/2}\int_{-\Delta\theta_j/2}^{\Delta\theta_j/2} \frac{-i}{k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left(\frac{dG(\vec{r},\vec{r}')}{ds}\right)_{\vec{r}=\vec{r}_i} d\theta'dp' \\
&= \int_{-\Delta\theta_j/2}^{\Delta\theta_j/2} \frac{i}{k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[\left(p'\frac{-ik_0}{4}(\hat{\rho}\cdot\hat{s})H_1^{(1)}(k_0\rho)\right)_{\vec{r}=\vec{r}_i}\right]_{p'=p_j-\Delta p_j/2}^{p'=p_j+\Delta p_j/2} d\theta' \\
&\quad + \int_{p_j-\Delta p_j/2}^{p_j+\Delta p_j/2}\int_{-\Delta\theta_j/2}^{\Delta\theta_j/2} \frac{-i}{k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left(\frac{-ik_0}{4}(\hat{\rho}\cdot\hat{s})H_1^{(1)}(k_0\rho)\right)_{\vec{r}=\vec{r}_i} d\theta'dp'
\end{aligned}$$

$$= \int_{-\Delta\theta_j/2}^{\Delta\theta_j/2} \frac{1}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \left[\left(p'(\hat{\rho}\cdot\hat{s})H_1^{(1)}(k_0\rho)\right)_{\vec{r}=\vec{r}_i}\right]_{p'=p_j-\Delta p_j/2}^{p'=p_j+\Delta p_j/2} d\theta'$$

$$+ \int_{p_j-\Delta p_j/2}^{p_j+\Delta p_j/2} \int_{-\Delta\theta_j/2}^{\Delta\theta_j/2} \frac{-1}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \left((\hat{\rho}\cdot\hat{s})H_1^{(1)}(k_0\rho)\right)_{\vec{r}=\vec{r}_i} d\theta' dp' \quad (4.29)$$

4.4.13 Element F_{ij}^8, Rectangular Cell:

$$\begin{aligned}
F_{ij}^8 &= \int_{A_j} (-1) f_3(\vec{r}_i, \vec{r}') dA' \\
&= \int_{s_j-\delta_j/2}^{s_j+\tau_j/2} \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} (-1) f_3(\vec{r}_i, \vec{r}') ds' dn' \\
&= \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \left(\frac{-dG(\vec{r}, \vec{r}')}{ds}\right)_{\vec{r}=\vec{r}_i} ds' dn' \\
&= \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \left[\frac{ik_0}{4}(\hat{\rho}\cdot\hat{s})H_1^{(1)}(k_0\rho)\right]_{\vec{r}=\vec{r}_i} ds' dn' \\
&= \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{i}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \left[(\hat{\rho}\cdot\hat{s})H_1^{(1)}(k_0\rho)\right]_{\vec{r}=\vec{r}_i} ds' dn' \quad (4.30)
\end{aligned}$$

4.4.14 Element F_{ij}^8, Circular Cell:

$$\begin{aligned}
F_{ij}^8 &= \int_{p_j-\Delta p_j/2}^{p_j+\Delta p_j/2} \int_{-\Delta\theta_j/2}^{\Delta\theta_j/2} (-1) f_3(\vec{r}_i, \vec{r}')(p' d\theta' dp') \\
&= \int_{p_j-\Delta p_j/2}^{p_j+\Delta p_j/2} \int_{-\Delta\theta_j/2}^{\Delta\theta_j/2} \frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \left(\frac{-dG(\vec{r}, \vec{r}')}{ds}\right)_{\vec{r}=\vec{r}_i} p' d\theta' dp' \\
&= \int_{p_j-\Delta p_j/2}^{p_j+\Delta p_j/2} \int_{-\Delta\theta_j/2}^{\Delta\theta_j/2} \frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \left[\frac{ik_0}{4}(\hat{\rho}\cdot\hat{s})H_1^{(1)}(k_0\rho)\right]_{\vec{r}=\vec{r}_i} p' d\theta' dp' \\
&= \int_{p_j-\Delta p_j/2}^{p_j+\Delta p_j/2} \int_{-\Delta\theta_j/2}^{\Delta\theta_j/2} \frac{ik_0}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \left[(\hat{\rho}\cdot\hat{s})H_1^{(1)}(k_0\rho)\right]_{\vec{r}=\vec{r}_i} p' d\theta' dp' \quad (4.31)
\end{aligned}$$

4.4.15 Element $\widetilde{F}_{ij}^9$:

$$\begin{aligned}
\widetilde{F}_{ij}^9 &= \gamma_{ij} + \int_{C_j} \frac{-i(-1)^q}{k_0} f_3(\vec{r}_i, \vec{r}') dn' \\
&= \gamma_{ij} + \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{-i(-1)^q}{k_0} f_3(\vec{r}_i, \vec{r}') dn' \\
&= \gamma_{ij} + \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{-i(-1)^q}{k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \left(\frac{dG(\vec{r}, \vec{r}')}{ds}\right)_{\vec{r}=\vec{r}_i} dn'
\end{aligned}$$

$$
\begin{aligned}
&= \gamma_{ij} + \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{-i(-1)^q}{k_0\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)} \left[\frac{-ik_0}{4}(\hat{\rho}\cdot\hat{s})H_1^{(1)}(k_0\rho)\right]_{\vec{r}=\vec{r}_i} dn' \\
&= \gamma_{ij} + \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{-(-1)^q}{4\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)} \left[(\hat{\rho}\cdot\hat{s})H_1^{(1)}(k_0\rho)\right]_{\vec{r}=\vec{r}_i} dn' \qquad (4.32)
\end{aligned}
$$

4.5 Evaluation of the Singular Impedance Elements

In the subsequent derivations, each of the impedance elements will be expressed in terms of four generic integrals. These are defined as

$$
\begin{aligned}
J_0(\alpha,\beta) &= \lim_{\epsilon\to 0}\int_\epsilon^\alpha\int_\epsilon^\beta H_0^{(1)}(k_0\sqrt{t^2+u^2})dtdu \\
J_1(\alpha,\beta) &= \lim_{\epsilon\to 0}\int_\epsilon^\alpha H_0^{(1)}(k_0\sqrt{t^2+\beta^2})dt \\
J_2(\alpha,\beta) &= \lim_{\epsilon\to 0}\frac{d}{d\beta}\int_\epsilon^\alpha H_0^{(1)}(k_0\sqrt{t^2+\beta^2})dt \\
J_3(\alpha,\beta) &= \lim_{\epsilon\to 0}\frac{d^2}{d\beta^2}\int_\epsilon^\alpha H_0^{(1)}(k_0\sqrt{t^2+\beta^2})dt
\end{aligned}
\qquad (4.33)
$$

and their analytical evaluation is given in the appendix.

In the computations that follow, use will be made of the substitutions

$$\tilde{s} = s' - s_{obs} \qquad (4.34)$$

$$\tilde{n} = n' - n_{obs}. \qquad (4.35)$$

These imply the identities

$$
\begin{aligned}
d\tilde{s} &= ds' \text{ (for integration over primed coordinates)} \\
d\tilde{n} &= dn' \text{ (for integration over primed coordinates)} \\
\frac{dH_0^{(1)}}{ds_{obs}} &= \frac{-dH_0^{(1)}}{d\tilde{s}} \\
\frac{dH_0^{(1)}}{dn_{obs}} &= \frac{-dH_0^{(1)}}{d\tilde{n}}.
\end{aligned}
\qquad (4.36)
$$

4.5.1 Element $\tilde{F}^1_{ij}$:

$$\begin{aligned}\tilde{F}^1_{ij} &= \gamma_{ij} - \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \frac{1}{4\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)} \left[\left\{(\hat{\rho}\cdot\hat{n})H_1^{(1)}(k_0\rho)\right\}_{\vec{r}=\vec{r}_i}\right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} ds' \\ &= \gamma_{ij} + \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \frac{1}{4k_0\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)} \left[\frac{dH_0^{(1)}(k_0\rho)}{dn}\bigg|_{\vec{r}=\vec{r}_i}\right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} ds'.\end{aligned}$$

Utilizing the expansion

$$\begin{aligned}\frac{dH_0^{(1)}}{dn} &= (\hat{n}\cdot\hat{s}')\frac{dH_0^{(1)}}{ds_{obs}} + (\hat{n}\cdot\hat{n}')\frac{dH_0^{(1)}}{dn_{obs}} \\ &= -(\hat{n}\cdot\hat{s}')\frac{dH_0^{(1)}}{ds'} + (\hat{n}\cdot\hat{n}')\frac{dH_0^{(1)}}{dn_{obs}}\end{aligned}$$

the above equation may be rewritten as

$$\begin{aligned}\tilde{F}^1_{ij} &= \gamma_{ij} + \frac{(\hat{n}\cdot\hat{n}')}{4k_0\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)} \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \left[\left\{\frac{d}{dn_{obs}}H_0^{(1)}(k_0\sqrt{(s'-s_{obs})^2+(n'-n_{obs})^2})\right\}_{\vec{r}=\vec{r}_i}\right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} d. \\ &\quad - \frac{(\hat{n}\cdot\hat{s}')}{4k_0\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)} \left[\left[H_0^{(1)}(k_0\sqrt{(s'-s_i)^2+(n'-n_i)^2})\right]_{s'=s_j-\delta_j/2}^{s'=s_j+\delta_j/2}\right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2}.\end{aligned}$$

Introducing the substitutions (4.34) and (4.35) we have

$$\begin{aligned}\tilde{F}^1_{ij} &= \gamma_{ij} - \frac{(\hat{n}\cdot\hat{n}')}{4k_0\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)} \left\{\int_{s_j-\delta_j/2-s_{obs}}^{s_j+\delta_j/2-s_{obs}} \left[\frac{d}{d\tilde{n}}H_0^{(1)}(k_0\sqrt{\tilde{s}^2+\tilde{n}^2})\right]_{\tilde{n}=n_j-\tau_j/2-n_{obs}}^{\tilde{n}=n_j+\tau_j/2-n_{obs}} d\tilde{s}\right\}_{\vec{r}=\vec{r}_i} \\ &\quad - \frac{(\hat{n}\cdot\hat{s}')}{4k_0\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)} \left[H_0^{(1)}(k_0\rho_3) - H_0^{(1)}(k_0\rho_2) - H_0^{(1)}(k_0\rho_4) + H_0^{(1)}(k_0\rho_1)\right]\end{aligned}$$

which may be rewritten in terms of the pre-defined generic integrals as

$$\begin{aligned}\tilde{F}^1_{ij} &= \gamma_{ij} - \frac{(\hat{n}\cdot\hat{n}')}{4k_0\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)} [J_2(x_3,y_3) - J_2(x_2,y_2) - J_2(x_4,y_4) + J_2(x_1,y_1)] \\ &\quad - \frac{(\hat{n}\cdot\hat{s}')}{4k_0\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)} \left[H_0^{(1)}(k_0\rho_3) - H_0^{(1)}(k_0\rho_2) - H_0^{(1)}(k_0\rho_4) + H_0^{(1)}(k_0\rho_1)\right].\end{aligned} \tag{4.37}$$

4.5.2 Element F_{ij}^2:

$$
\begin{aligned}
F_{ij}^2 &= \int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{-ik_0}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \left\{(\hat{\rho}\cdot\hat{n})H_1^{(1)}(k_0\rho)\right\}_{\vec{r}=\vec{r}_i} ds'dn' \\
&= \int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{i}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \left\{\frac{d}{dn}H_0^{(1)}(k_0\rho)\right\}_{\vec{r}=\vec{r}_i} ds'dn' \\
&= \int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{i}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \left\{\frac{d}{dn}H_0^{(1)}(k_0\sqrt{(s'-s_{obs})^2+(n'-n_{obs})^2})\right\}_{\vec{r}=\vec{r}_i} ds'dn'.
\end{aligned}
$$

Again, employing the expansion

$$
\begin{aligned}
\frac{dH_0^{(1)}}{dn} &= (\hat{n}\cdot\hat{s}')\frac{dH_0^{(1)}}{ds_{obs}} + (\hat{n}\cdot\hat{n}')\frac{dH_0^{(1)}}{dn_{obs}} \\
&= -(\hat{n}\cdot\hat{s}')\frac{dH_0^{(1)}}{ds'} - (\hat{n}\cdot\hat{n}')\frac{dH_0^{(1)}}{dn'}
\end{aligned}
$$

F_{ij}^2 may be expressed as

$$
\begin{aligned}
F_{ij}^2 = &-\frac{i(\hat{n}\cdot\hat{n}')}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \left[H_0^{(1)}(k_0\sqrt{(s'-s_i)^2+(n'-n_i)^2})\right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} ds' \\
&-\frac{i(\hat{n}\cdot\hat{s}')}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \left[H_0^{(1)}(k_0\sqrt{(s'-s_i)^2+(n'-n_i)^2})\right]_{s'=s_j-\delta_j/2}^{s'=s_j+\delta_j/2} dn'.
\end{aligned}
$$

With the substitutions

$$\widetilde{s} = s' - s_i$$

$$\widetilde{n} = n' - n_i$$

we may write

$$
\begin{aligned}
F_{ij}^2 = &-\frac{i(\hat{n}\cdot\hat{n}')}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \int_{s_j-\delta_j/2-s_i}^{s_j+\delta_j/2-s_i} \left[H_0^{(1)}(k_0\sqrt{\widetilde{s}^2+\widetilde{n}^2})\right]_{\widetilde{n}=n_j-\tau_j/2-n_i}^{\widetilde{n}=n_j+\tau_j/2-n_i} d\widetilde{s} \\
&-\frac{i(\hat{n}\cdot\hat{s}')}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \int_{n_j-\tau_j/2-n_i}^{n_j+\tau_j/2-n_i} \left[H_0^{(1)}(k_0\sqrt{\widetilde{s}^2+\widetilde{n}^2})\right]_{\widetilde{s}=s_j-\delta_j/2-s_i}^{\widetilde{s}=s_j+\delta_j/2-s_i} d\widetilde{n} \\
= &-\frac{i(\hat{n}\cdot\hat{n}')}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[J_1(x_3,y_3) - J_1(x_2,y_2) - J_1(x_4,y_4) + J_1(x_1,y_1)\right]
\end{aligned}
$$

$$-\frac{i(\hat{n}\cdot\hat{s}')}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[J_1(y_3,x_3)-J_1(y_2,x_2)-J_1(y_4,x_4)+J_1(y_1,x_1)\right] \quad (4.38)$$

4.5.3 Element F^3_{ij}:

$$\begin{aligned}
F^3_{ij} &= \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{(-1)^q}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left\{(\hat{\rho}\cdot\hat{n})H_1^{(1)}(k_0\rho)\right\}_{\vec{r}=\vec{r}_i} dn' \\
&= \frac{(-1)^{q+1}}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{dH_0^{(1)}(k_0\rho)}{dn}\bigg|_{\vec{r}=\vec{r}_i} dn' \\
&= \frac{(-1)^{q+1}}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \left\{\left((\hat{n}\cdot\hat{s}')\frac{d}{ds_{obs}}-(\hat{n}\cdot\hat{n}')\frac{d}{dn'}\right)H_0^{(1)}(k_0\rho)\right\}_{\vec{r}=\vec{r}_i} dn' \\
&= \frac{(-1)^{q+1}(\hat{n}\cdot\hat{s}')}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{dH_0^{(1)}(k_0\rho)}{ds_{obs}}\bigg|_{\vec{r}=\vec{r}_i} dn' \\
&\quad +\frac{(-1)^{q}(\hat{n}\cdot\hat{n}')}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[H_0^{(1)}(k_0\rho)_{\vec{r}=\vec{r}_i}\right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} \\
&= \frac{(-1)^{q+1}(\hat{n}\cdot\hat{s}')}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \left\{\frac{d}{ds_{obs}}H_0^{(1)}(k_0\sqrt{(s_j+(-1)^q\delta_j/2-s_{obs})^2+(n'-n_{obs})^2})\right\}_{\vec{r}=\vec{r}_i} dn' \\
&\quad +\frac{(-1)^{q}(\hat{n}\cdot\hat{n}')}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[H_0^{(1)}(k_0\sqrt{(s_j+(-1)^q\delta_j/2-s_i)^2+(n'-n_i)^2})\right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2}.
\end{aligned}$$

For this case, the appropriate substitutions are

$$\tilde{s} = s_j + (-1)^q\delta/2 - s_{obs}$$

$$\tilde{n} = n' - n_{obs}.$$

These allow F^3_{ij} to be written as

$$\begin{aligned}
F^3_{ij} &= \frac{(-1)^{q}(\hat{n}\cdot\hat{s}')}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left\{\int_{n_j-\tau_j/2-n_{obs}}^{n_j+\tau_j/2-n_{obs}} \frac{d}{d\tilde{s}}H_0^{(1)}(k_0\sqrt{\tilde{s}^2+\tilde{n}^2})d\tilde{n}\right\}_{\vec{r}=\vec{r}_i} \\
&\quad +\frac{(-1)^{q}(\hat{n}\cdot\hat{n}')}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[H_0^{(1)}(k_0\sqrt{(s_j+(-1)^p\delta/2-s_i)^2+(n'-n_i)^2})\right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2}.
\end{aligned}$$

Thus for leading edges we may write

$$
\begin{aligned}
F_{ij}^3 &= \frac{-(\hat{n}\cdot\hat{s}')}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[J_2(y_2,x_2)-J_2(y_1,x_1)\right] \\
&\quad +\frac{-(\hat{n}\cdot\hat{n}')}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[H_0^{(1)}(k_0\rho_2)-H_0^{(1)}(k_0\rho_1)\right],
\end{aligned}
$$

while for trailing edges the appropriate expression is

$$
\begin{aligned}
F_{ij}^3 &= \frac{(\hat{n}\cdot\hat{s}')}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[J_2(y_3,x_3)-J_2(y_4,x_4)\right] \\
&\quad +\frac{(\hat{n}\cdot\hat{n}')}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[H_0^{(1)}(k_0\rho_3)-H_0^{(1)}(k_0\rho_4)\right]. \qquad (4.39)
\end{aligned}
$$

4.5.4 Element F_{ij}^4:

$$
\begin{aligned}
F_{ij}^4 &= \int_{s_j-\delta_j/2}^{s_j+\delta_j/2}(-1)\left[\left\{\left(\frac{1}{\widetilde{R}^*(\vec{r}_i)}-\frac{1}{k_0^2\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\frac{d}{ds^2}\right)G(\vec{r},\vec{r}')\right\}_{\vec{r}=\vec{r}_i}\right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2}ds' \\
&\quad +\int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\left[\frac{1}{k_0^2}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)\frac{dG(\vec{r},\vec{r}')}{ds}_{\vec{r}=\vec{r}_i}\right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2}ds' \\
&= \int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\left[\frac{-i}{4\widetilde{R}^*(\vec{r}_i)}H_0^{(1)}(k_0\rho)_{\vec{r}=\vec{r}_i}\right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2}ds' \\
&\quad +\int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\left[\frac{i}{4k_0^2\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\frac{d^2H_0^{(1)}(k_0\rho)}{ds^2}_{\vec{r}=\vec{r}_i}\right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2}ds' \\
&\quad +\int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\left[\frac{i}{4k_0^2}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)\frac{dH_0^{(1)}(k_0\rho)}{ds}_{\vec{r}=\vec{r}_i}\right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2}ds' \\
&= Int_1+Int_2+Int_3.
\end{aligned}
$$

By introducing the substitutions (4.34) and (4.35) we obtain

$$
\begin{aligned}
Int_1 &= \int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\left[\frac{-i}{4\widetilde{R}^*(\vec{r}_i)}H_0^{(1)}(k_0\sqrt{(s'-s_{obs})^2+(n'-n_{obs})^2})_{\vec{r}=\vec{r}_i}\right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2}ds' \\
&= \left\{\int_{s_j-\delta_j/2-s_{obs}}^{s_j+\delta_j/2-s_{obs}}\left[\frac{-i}{4\widetilde{R}^*(\vec{r}_i)}H_0^{(1)}(k_0\sqrt{\tilde{s}^2+\tilde{n}^2})\right]_{\tilde{n}=n_j-\tau_j/2-n_{obs}}^{\tilde{n}=n_j+\tau_j/2-n_{obs}}d\tilde{s}\right\}_{\vec{r}=\vec{r}_i}
\end{aligned}
$$

$$= \int_{s_j-\delta_j/2-s_i}^{s_j+\delta_j/2-s_i} \left[\frac{-i}{4\widetilde{R}^*(\vec{r}_i)} H_0^{(1)}(k_0\sqrt{\widetilde{s}^2+\widetilde{n}^2}) \right]_{\widetilde{n}=n_j-\tau_j/2-n_i}^{\widetilde{n}=n_j+\tau_j/2-n_i} d\widetilde{s}$$

$$= \frac{-i}{4\widetilde{R}^*(\vec{r}_i)} \left[J_1(x_3,y_3) - J_1(x_2,y_2) - J_1(x_4,y_4) + J_1(x_1,y_1) \right].$$

In order to simplify Int_2, we first rewrite $d^2H_0^{(1)}/ds^2$ as

$$\begin{aligned}
\frac{d^2H_0^{(1)}}{ds^2} &= \left[(\hat{s}\cdot\hat{s}')\frac{d}{ds_{obs}} + (\hat{s}\cdot\hat{n}')\frac{d}{dn_{obs}} \right] \left[(\hat{s}\cdot\hat{s}')\frac{d}{ds_{obs}} + (\hat{s}\cdot\hat{n}')\frac{d}{dn_{obs}} \right] H_0^{(1)} \\
&= (\hat{s}\cdot\hat{s}')^2 \frac{d^2H_0^{(1)}}{ds_{obs}^2} + 2(\hat{s}\cdot\hat{n}')(\hat{s}\cdot\hat{s}')\frac{d^2H_0^{(1)}}{ds_{obs}dn_{obs}} + (\hat{s}\cdot\hat{n}')^2 \frac{{}^2dH_0^{(1)}}{dn_{obs}^2} \\
&= -(\hat{s}\cdot\hat{s}')^2 \frac{d}{ds'}\frac{dH_0^{(1)}}{ds_{obs}} - 2(\hat{s}\cdot\hat{n}')(\hat{s}\cdot\hat{s}')\frac{d}{ds'}\frac{dH_0^{(1)}}{dn_{obs}} + (\hat{s}\cdot\hat{n}')^2 \frac{d^2H_0^{(1)}}{dn_{obs}^2}.
\end{aligned}$$

Employing this expression in Int_2, we obtain

$$\begin{aligned}
Int_2 &= \frac{i}{4k_0^2\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \left[\frac{d^2H_0^{(1)}(k_0\rho)}{ds^2} \Big|_{\vec{r}=\vec{r}_i} \right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} ds' \\
&= \frac{-i(\hat{s}\cdot\hat{s}')^2}{4k_0^2\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \left[\frac{d}{ds'}\frac{dH_0^{(1)}(k_0\rho)}{ds_{obs}} \Big|_{\vec{r}=\vec{r}_i} \right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} ds' \\
&+ \frac{-i(\hat{s}\cdot\hat{n}')(\hat{s}\cdot\hat{s}')}{2k_0^2\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \left[\frac{d}{ds'}\frac{dH_0^{(1)}(k_0\rho)}{dn_{obs}} \Big|_{\vec{r}=\vec{r}_i} \right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} ds' \\
&+ \frac{i(\hat{s}\cdot\hat{n}')^2}{4k_0^2\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \left[\frac{d^2H_0^{(1)}(k_0\rho)}{dn_{obs}^2} \Big|_{\vec{r}=\vec{r}_i} \right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} ds' \\
&= \frac{i(\hat{s}\cdot\hat{s}')^2}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \left[\left[\left\{ (\hat{\rho}\cdot\hat{s}')H_1^{(1)}(k_0\sqrt{(s'-s_{obs})^2+(n'-n_{obs})^2}) \right\}_{\vec{r}=\vec{r}_i} \right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} \right]_{s'=_j-\delta_j/2}^{s'=s_j+\delta_j/2} \\
&+ \frac{i(\hat{s}\cdot\hat{n}')(\hat{s}\cdot\hat{s}')}{2k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \left[\left[\left\{ (\hat{\rho}\cdot\hat{n}')H_1^{(1)}(k_0\sqrt{(s'-s_{obs})^2+(n'-n_{obs})^2}) \right\}_{\vec{r}=\vec{r}_i} \right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} \right]_{s'=s_j-\delta_j/2}^{s'=s_j+\delta_j/2} \\
&+ \frac{i(\hat{s}\cdot\hat{n}')^2}{4k_0^2\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \left[\left\{ \frac{d^2}{dn_{obs}^2} H_0^{(1)}(k_0\sqrt{(s'-s_{obs})^2+(n'-n_{obs})^2}) \right\}_{\vec{r}=\vec{r}_i} \right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} ds'.
\end{aligned}$$

In the last term of the above expression, we further make the substitutions (4.34) and (4.35) to yield

$$\begin{aligned}
Int_2 &= \frac{i(\hat{s}\cdot\hat{s}')^2}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[\frac{x_3}{\rho_3}H_1^{(1)}(k_0\rho_3)-\frac{x_2}{\rho_2}H_1^{(1)}(k_0\rho_2)-\frac{x_4}{\rho_4}H_1^{(1)}(k_0\rho_4)+\frac{x_1}{\rho_1}H_1^{(1)}(k_0\rho_1)\right]\\
&\quad+\frac{i(\hat{s}\cdot\hat{n}')(\hat{s}\cdot\hat{s}')}{2k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[\frac{y_3}{\rho_3}H_1^{(1)}(k_0\rho_3)-\frac{y_2}{\rho_2}H_1^{(1)}(k_0\rho_2)-\frac{y_4}{\rho_4}H_1^{(1)}(k_0\rho_4)+\frac{y_1}{\rho_1}H_1^{(1)}(k_0\rho_1)\right]\\
&\quad+\frac{i(\hat{s}\cdot\hat{n}')^2}{4k_0^2\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left\{\int_{s_j-\delta_j/2-s_{obs}}^{s_j+\delta_j/2-s_{obs}}\left[\frac{d^2}{d\widetilde{n}^2}H_0^{(1)}(k_0\sqrt{\widetilde{s}^2+\widetilde{n}^2})\right]_{\widetilde{n}=n_j-\tau_j/2-n_{obs}}^{\widetilde{n}=n_j+\tau/2-n_{obs}}d\widetilde{s}\right\}_{\vec{r}=\vec{r}_i}\\
&= \frac{i(\hat{s}\cdot\hat{s}')^2}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[\frac{x_3}{\rho_3}H_1^{(1)}(k_0\rho_3)-\frac{x_2}{\rho_2}H_1^{(1)}(k_0\rho_2)-\frac{x_4}{\rho_4}H_1^{(1)}(k_0\rho_4)+\frac{x_1}{\rho_1}H_1^{(1)}(k_0\rho_1)\right]\\
&\quad+\frac{i(\hat{s}\cdot\hat{n}')(\hat{s}\cdot\hat{s}')}{2k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[\frac{y_3}{\rho_3}H_1^{(1)}(k_0\rho_3)-\frac{y_2}{\rho_2}H_1^{(1)}(k_0\rho_2)-\frac{y_4}{\rho_4}H_1^{(1)}(k_0\rho_4)+\frac{y_1}{\rho_1}H_1^{(1)}(k_0\rho_1)\right]\\
&\quad+\frac{i(\hat{s}\cdot\hat{n}')^2}{4k_0^2\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[J_3(x_3,y_3)-J_3(x_2,y_2)-J_3(x_4,y_4)+J_3(x_1,y_1)\right].
\end{aligned}$$

Similarly, for Int_3 we have (with the usual definitions for $\widetilde{s}$ and $\widetilde{n}$)

$$\begin{aligned}
Int_3 &= \int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\left[\frac{i}{4k_0^2}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)\frac{d}{ds}H_0^{(1)}(k_0\rho)_{\vec{r}=\vec{r}_i}\right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2}ds'\\
&= \frac{-i(\hat{s}\cdot\hat{s}')}{4k_0^2}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)\left[\left[H_0^{(1)}(k_0\sqrt{(s'-s_{obs})^2+(n'-n_{obs})^2})_{\vec{r}=\vec{r}_i}\right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2}\right]_{s'=s_j-\delta_j/2}^{s'=s_j+\delta_j/2}\\
&\quad+\frac{i(\hat{s}\cdot\hat{n}')}{4k_0^2}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)\int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\left[\left\{\frac{d}{dn_{obs}}H_0^{(1)}(k_0\sqrt{(s'-s_{obs})^2+(n'-n_{obs})^2})\right\}_{\vec{r}=\vec{r}_i}ds'\right]_{n'=n_j-}^{n'=n_j+}\\
&= \frac{-i(\hat{s}\cdot\hat{s}')}{4k_0^2}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)\left[H_0^{(1)}(k_0\rho_3)-H_0^{(1)}(k_0\rho_2)-H_0^{(1)}(k_0\rho_4)+H_0^{(1)}(k_0\rho_1)\right]\\
&\quad+\frac{-i(\hat{s}\cdot\hat{n}')}{4k_0^2}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)\left\{\left[\int_{s_j-\delta_j/2-s_{obs}}^{s_j+\delta_j/2-s_{obs}}\frac{d}{d\widetilde{n}}H_0^{(1)}(k_0\sqrt{\widetilde{s}^2+\widetilde{n}^2})d\widetilde{s}\right]_{\widetilde{n}=n_j-\tau_j/2-n_{obs}}^{\widetilde{n}=n_j+\tau_j/2-n_{obs}}\right\}_{\vec{r}=\vec{r}_i}\\
&= \frac{-i(\hat{s}\cdot\hat{s}')}{4k_0^2}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)\left[H_0^{(1)}(k_0\rho_3)-H_0^{(1)}(k_0\rho_2)-H_0^{(1)}(k_0\rho_4)+H_0^{(1)}(k_0\rho_1)\right]
\end{aligned}$$

$$+\frac{-i(\hat{s}\cdot\hat{n}')}{4k_0^2}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)}\right)[J_2(x_3,y_3)-J_2(x_2,y_2)-J_2(x_4,y_4)+J_2(x_1,y_1)].$$

Hence, F_{ij}^4 can be expressed as

$$\begin{aligned}
F_{ij}^4 &= \frac{-i}{4\tilde{R}^*(\vec{r}_i)}[J_1(x_3,y_3)-J_1(x_2,y_2)-J_1(x_4,y_4)+J_1(x_1,y_1)] \\
&+\frac{i(\hat{s}\cdot\hat{n}')^2}{4k_0^2\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)}[J_3(x_3,y_3)-J_3(x_2,y_2)-J_3(x_4,y_4)+J_3(x_1,y_1)] \\
&+\frac{i(\hat{s}\cdot\hat{s}')^2}{4k_0\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)}\left[\frac{x_3}{\rho_3}H_1^{(1)}(k_0\rho_3)-\frac{x_2}{\rho_2}H_1^{(1)}(k_0\rho_2)-\frac{x_4}{\rho_4}H_1^{(1)}(k_0\rho_4)+\frac{x_1}{\rho_1}H_1^{(1)}(k_0\rho_1)\right] \\
&+\frac{i(\hat{s}\cdot\hat{n}')(\hat{s}\cdot\hat{s}')}{2k_0\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)}\left[\frac{y_3}{\rho_3}H_1^{(1)}(k_0\rho_3)-\frac{y_2}{\rho_2}H_1^{(1)}(k_0\rho_2)-\frac{y_4}{\rho_4}H_1^{(1)}(k_0\rho_4)+\frac{y_1}{\rho_1}H_1^{(1)}(k_0\rho_1)\right] \\
&+\frac{-i(\hat{s}\cdot\hat{s}')}{4k_0^2}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)}\right)\left[H_0^{(1)}(k_0\rho_3)-H_0^{(1)}(k_0\rho_2)-H_0^{(1)}(k_0\rho_4)+H_0^{(1)}(k_0\rho_1)\right] \\
&+\frac{-i(\hat{s}\cdot\hat{n}')}{4k_0^2}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)}\right)[J_2(x_3,y_3)-J_2(x_2,y_2)-J_2(x_4,y_4)+J_2(x_1,y_1)]
\end{aligned} \tag{4.40}$$

4.5.5 Element $\tilde{F}_{ij}^5$:

$$\begin{aligned}
\tilde{F}_{ij}^5 = \gamma_{ij} &+\int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\int_{n_j-\tau_j/2}^{n_j+\tau_j/2}(-ik_0)\left\{\left(\frac{1}{\tilde{R}^*(\vec{r}_i)}-\frac{1}{k_0^2\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)}\frac{d}{ds^2}\right)G(\vec{r},\vec{r}')\right\}_{\vec{r}=\vec{r}_i}ds'dn' \\
&+\int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\int_{n_j-\tau_j/2}^{n_j+\tau_j/2}\frac{i}{k_0}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)}\right)\frac{dG(\vec{r},\vec{r}')}{ds}_{\vec{r}=\vec{r}_i}ds'dn' \\
= \gamma_{ij} &+\int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\int_{n_j-\tau_j/2}^{n_j+\tau_j/2}\frac{k_0}{4\tilde{R}^*(\vec{r}_i)}H_0^{(1)}(k_0\rho)_{\vec{r}=\vec{r}_i}ds'dn' \\
&+\int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\int_{n_j-\tau_j/2}^{n_j+\tau_j/2}\frac{-1}{4k_0\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)}\frac{d^2H_0^{(1)}(k_0\rho)}{ds^2}_{\vec{r}=\vec{r}_i}ds'dn' \\
&+\int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\int_{n_j-\tau_j/2}^{n_j+\tau_j/2}\frac{-1}{4k_0}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)}\right)\frac{dH_0^{(1)}(k_0\rho)}{ds}_{\vec{r}=\vec{r}_i}ds'dn' \\
= \gamma_{ij} &+Int_1+Int_2+Int_3.
\end{aligned}$$

The substitutions (4.34) and (4.35) allow Int_1 to be written as

$$Int_1 = \frac{k_0}{4\tilde{R}^*(\vec{r}_i)}\int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\int_{n_j-\tau_j/2}^{n_j+\tau_j/2}H_0^{(1)}(k_0\sqrt{(s'-s_{obs})^2+(n'-n_{obs})^2})_{\vec{r}=\vec{r}_i}ds'dn'$$

$$
\begin{aligned}
&= \frac{k_0}{4\widetilde{R}^*(\vec{r}_i)} \left\{ \int_{s_j-\delta_j/2-s_{obs}}^{s_j+\delta_j/2-s_{obs}} \int_{n_j-\tau_j/2-n_{obs}}^{n_j+\tau_j/2-n_{obs}} H_0^{(1)}(k_0\sqrt{\tilde{s}^2+\tilde{n}^2}) d\tilde{s} d\tilde{n} \right\}_{\vec{r}=\vec{r}_i} \\
&= \frac{k_0}{4\widetilde{R}^*(\vec{r}_i)} \int_{s_j-\delta_j/2-s_i}^{s_j+\delta_j/2-s_i} \int_{n_j-\tau_j/2-n_i}^{n_j+\tau_j/2-n_i} H_0^{(1)}(k_0\sqrt{\tilde{s}^2+\tilde{n}^2}) d\tilde{s} d\tilde{n} \\
&= \frac{k_0}{4\widetilde{R}^*(\vec{r}_i)} \left[J_0(x_3,y_3) - J_0(x_2,y_2) - J_0(x_4,y_4) + J_0(x_1,y_1) \right].
\end{aligned}
$$

To simplify Int_2, we first expand $d^2H_0^{(1)}/ds^2$ as

$$
\begin{aligned}
\frac{d^2H_0^{(1)}}{ds^2} &= \left[(\hat{s}\cdot\hat{s}')\frac{d}{ds_{obs}} + (\hat{s}\cdot\hat{n}')\frac{d}{dn_{obs}} \right] \left[(\hat{s}\cdot\hat{s}')\frac{d}{ds_{obs}} + (\hat{s}\cdot\hat{n}')\frac{d}{dn_{obs}} \right] H_0^{(1)} \\
&= (\hat{s}\cdot\hat{s}')^2 \frac{d^2H_0^{(1)}}{ds_{obs}^2} + 2(\hat{s}\cdot\hat{n}')(\hat{s}\cdot\hat{s}')\frac{d^2H_0^{(1)}}{ds_{obs}dn_{obs}} + (\hat{s}\cdot\hat{n}')^2 \frac{d^2H_0^{(1)}}{dn_{obs}^2} \\
&= -(\hat{s}\cdot\hat{s}')^2 \frac{d}{ds'}\frac{dH_0^{(1)}}{ds_{obs}} + 2(\hat{s}\cdot\hat{n}')(\hat{s}\cdot\hat{s}')\left(\frac{d}{ds'}\right)\left(\frac{d}{dn'}\right) H_0^{(1)} - (\hat{s}\cdot\hat{n}')^2 \frac{d}{dn'}\frac{dH_0^{(1)}}{dn_{obs}}.
\end{aligned}
$$

Substituting the above expansion into Int_2 yields

$$
\begin{aligned}
Int_2 &= \frac{-1}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{d^2H_0^{(1)}(k_0\rho)}{ds^2}_{\vec{r}=\vec{r}_i} ds'dn' \\
&= \frac{(\hat{s}\cdot\hat{s}')^2}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{d}{ds'}\frac{dH_0^{(1)}(k_0\rho)}{ds_{obs}}_{\vec{r}=\vec{r}_i} ds'dn' \\
&+ \frac{-(\hat{s}\cdot\hat{n}')(\hat{s}\cdot\hat{s}')}{2k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{d}{ds'}\frac{d}{dn'} H_0^{(1)}(k_0\rho)_{\vec{r}=\vec{r}_i} ds'dn' \\
&+ \frac{(\hat{s}\cdot\hat{n}')^2}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{d}{dn'}\frac{dH_0^{(1)}(k_0\rho)}{dn_{obs}}_{\vec{r}=\vec{r}_i} ds'dn' \\
&= \frac{(\hat{s}\cdot\hat{s}')^2}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \left[\left\{ \frac{d}{ds_{obs}} H_0^{(1)}(k_0\sqrt{(s'-s_{obs})^2+(n'-n_{obs})^2}) \right\}_{\vec{r}=\vec{r}_i} \right]_{s'=s_j-\delta_j/2}^{s'=s_j+\delta_j/2} dn' \\
&+ \frac{-(\hat{s}\cdot\hat{n}')(\hat{s}\cdot\hat{s}')}{2k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \left[\left[H_0^{(1)}(k_0\sqrt{(s'-s_i)^2+(n'-n_i)^2}) \right]_{s'=s_j-\delta_j/2}^{s'=s_j+\delta_j/2} \right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} \\
&+ \frac{(\hat{s}\cdot\hat{n}')^2}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \int_{s_j-\delta_j/2}^{s_j+\delta_j/2} \left[\left\{ \frac{d}{dn_{obs}} H_0^{(1)}(k_0\sqrt{(s'-s_{obs})^2+(n'-n_{obs})^2}) \right\}_{\vec{r}=\vec{r}_i} \right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} ds'.
\end{aligned}
$$

For the first and second terms, we use (4.34) and (4.35) to express Int_2 as

$$\begin{aligned}
Int_2 &= \frac{-(\hat{s}\cdot\hat{s}')^2}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left\{\int_{n_j-\tau_j/2-n_{obs}}^{n_j+\tau_j/2-n_{obs}}\left[\frac{d}{d\widetilde{s}}H_0^{(1)}(k_0\sqrt{\widetilde{s}^2+\widetilde{n}^2})\right]_{\widetilde{s}=s_j-\delta_j/2-s_{obs}}^{\widetilde{s}=s_j+\delta_j/2-s_{obs}} d\widetilde{n}\right\}_{\vec{r}=\vec{r}_i}\\
&+\frac{-(\hat{s}\cdot\hat{n}')(\hat{s}\cdot\hat{s}')}{2k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[\left[H_0^{(1)}(k_0\sqrt{(s'-s_i)^2+(n'-n_i)^2})\right]_{s'=s_j-\delta_j/2}^{s'=s_j+\delta_j/2}\right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2}\\
&+\frac{-(\hat{s}\cdot\hat{n}')^2}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left\{\int_{s_j-\delta_j/2-s_{obs}}^{s_j+\delta_j/2-s_{obs}}\left[\frac{d}{d\widetilde{n}}H_0^{(1)}(k_0\sqrt{\widetilde{s}^2+\widetilde{n}^2})\right]_{\widetilde{n}=n_j-\tau_j/2-n_{obs}}^{\widetilde{n}=n_j+\tau_j/2-n_{obs}} d\widetilde{s}\right\}_{\vec{r}=\vec{r}_i}\\
&= \frac{-(\hat{s}\cdot\hat{s}')^2}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[J_2(y_3,x_3)-J_2(y_2,x_2)-J_2(y_4,x_4)+J_2(y_1,x_1)\right]\\
&+\frac{-(\hat{s}\cdot\hat{n}')(\hat{s}\cdot\hat{s}')}{2k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[H_0^{(1)}(k_0\rho_3)-H_0^{(1)}(k_0\rho_2)-H_0^{(1)}(k_0\rho_4)+H_0^{(1)}(k_0\rho_1)\right]\\
&+\frac{-(\hat{s}\cdot\hat{n}')^2}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[J_2(x_3,y_3)-J_2(x_2,y_2)-J_2(x_4,y_4)+J_2(x_1,y_1)\right].
\end{aligned}$$

Similarly, for Int_3, we have (with the usual definitions for $\widetilde{s}$ and $\widetilde{n}$)

$$\begin{aligned}
Int_3 &= \frac{-1}{4k_0}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)\int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\int_{n_j-\tau_j/2}^{n_j+\tau_j/2}\left.\frac{dH_0^{(1)}(k_0\rho)}{ds}\right|_{\vec{r}=\vec{r}_i} ds'dn'\\
&= \frac{-1}{4k_0}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)\int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\int_{n_j-\tau_j/2}^{n_j+\tau_j/2}\left\{\left(-(\hat{s}\cdot\hat{s}')\frac{d}{ds'}-(\hat{s}\cdot\hat{n}')\frac{d}{dn'}\right)H_0^{(1)}(k_0\rho)\right\}_{\vec{r}=\vec{r}_i} ds'dn'\\
&= \frac{(\hat{s}\cdot\hat{s}')}{4k_0}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)\int_{n_j-\tau_j/2}^{n_j+\tau_j/2}\left[H_0^{(1)}(k_0\sqrt{(s'-s_{obs})^2+(n'-n_{obs})^2})_{\vec{r}=\vec{r}_i}\right]_{s'=s_j-\delta_j/2}^{s'=s_j+\delta_j/2} dn'\\
&+\frac{(\hat{s}\cdot\hat{n}')}{4k_0}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)\int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\left[H_0^{(1)}(k_0\sqrt{(s'-s_{obs})^2+(n'-n_{obs})^2})_{\vec{r}=\vec{r}_i}\right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} ds'\\
&= \frac{(\hat{s}\cdot\hat{s}')}{4k_0}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)\left\{\int_{n_j-\tau_j/2-n_{obs}}^{n_j+\tau_j/2-n_{obs}}\left[H_0^{(1)}(k_0\sqrt{\widetilde{s}^2+\widetilde{n}^2})\right]_{\widetilde{s}=s_j-\delta_j/2-s_{obs}}^{\widetilde{s}=s_j+\delta_j/2-s_{obs}} d\widetilde{n}\right\}_{\vec{r}=\vec{r}_i}\\
&+\frac{(\hat{s}\cdot\hat{n}')}{4k_0}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)\left\{\int_{s_j-\delta_j/2-s_{obs}}^{s_j+\delta_j/2-s_{obs}}\left[H_0^{(1)}(k_0\sqrt{\widetilde{s}^2+\widetilde{n}^2})\right]_{n'=n_j-\tau_j/2-n_{obs}}^{n'=n_j+\tau_j/2-n_{obs}} d\widetilde{s}\right\}_{\vec{r}=\vec{r}_i}\\
&= \frac{(\hat{s}\cdot\hat{s}')}{4k_0}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)\left[J_1(y_3,x_3)-J_1(y_2,x_2)-J_1(y_4,x_4)+J_1(y_1,x_1)\right]\\
&+\frac{(\hat{s}\cdot\hat{n}')}{4k_0}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)\left[J_1(x_3,y_3)-J_1(x_2,y_2)-J_1(x_4,y_4)+J_1(x_1,y_1)\right].
\end{aligned}$$

Hence, the expression for $\widetilde{F}_{ij}^5$ becomes

$$\begin{aligned}\widetilde{F}_{ij}^5 = \gamma_{ij} &+ \frac{k_0}{4\widetilde{R}^*(\vec{r}_i)}\left[J_0(x_3,y_3) - J_0(x_2,y_2) - J_0(x_4,y_4) + J_0(x_1,y_1)\right] \\ &+ \frac{-(\hat{s}\cdot\hat{s}')^2}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[J_2(y_3,x_3) - J_2(y_2,x_2) - J_2(y_4,x_4) + J_2(y_1,x_1)\right] \\ &+ \frac{-(\hat{s}\cdot\hat{n}')^2}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[J_2(x_3,y_3) - J_2(x_2,y_2) - J_2(x_4,y_4) + J_2(x_1,y_1)\right] \\ &+ \frac{-(\hat{s}\cdot\hat{n}')(\hat{s}\cdot\hat{s}')}{2k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[H_0^{(1)}(k_0\rho_3) - H_0^{(1)}(k_0\rho_2) - H_0^{(1)}(k_0\rho_4) + H_0^{(1)}(k_0\rho_1)\right] \\ &+ \frac{(\hat{s}\cdot\hat{s}')}{4k_0}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)\left[J_1(y_3,x_3) - J_1(y_2,x_2) - J_1(y_4,x_4) + J_1(y_1,x_1)\right] \\ &+ \frac{(\hat{s}\cdot\hat{n}')}{4k_0}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)\left[J_1(x_3,y_3) - J_1(x_2,y_2) - J_1(x_4,y_4) + J_1(x_1,y_1)\right].\end{aligned} \tag{4.41}$$

4.5.6 Element F_{ij}^6:

$$\begin{aligned}F_{ij}^6 &= \int_{n_j-\tau_j/2}^{n_j+\tau_j/2}(-1)^q\left\{\left(\frac{1}{\widetilde{R}^*(\vec{r}_i)} - \frac{1}{k_0^2\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\frac{d}{ds^2}\right)G(\vec{r},\vec{r}')\right\}_{\vec{r}=\vec{r}_i} dn' \\ &\quad + \int_{n_j-\tau_j/2}^{n_j+\tau_j/2}\frac{(-1)^{q+1}}{k_0^2}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)\frac{dG(\vec{r},\vec{r}')}{ds}_{\vec{r}=\vec{r}_i} dn' \\ &= \int_{n_j-\tau_j/2}^{n_j+\tau_j/2}\frac{i(-1)^q}{4\widetilde{R}^*(\vec{r}_i)}H_0^{(1)}(k_0\rho)_{\vec{r}=\vec{r}_i}dn' \\ &\quad + \int_{n_j-\tau_j/2}^{n_j+\tau_j/2}\frac{i(-1)^{q+1}}{4k_0^2\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\frac{d^2H_0^{(1)}(k_0\rho)}{ds^2}_{\vec{r}=\vec{r}_i} dn' \\ &\quad + \int_{n_j-\tau_j/2}^{n_j+\tau_j/2}\frac{i(-1)^{q+1}}{4k_0^2}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)\frac{dH_0^{(1)}(k_0\rho)}{ds}_{\vec{r}=\vec{r}_i} dn' \\ &= Int_1 + Int_2 + Int_3.\end{aligned}$$

Making the substitutions

$$\widetilde{s} = s_j + (-1)^q\delta_j/2 - s_{obs}$$

$$\widetilde{n} = n' - n_{obs}$$

for Int_1 yields

$$\begin{aligned}
Int_1 &= \frac{i(-1)^q}{4\widetilde{R}^*(\vec{r}_i)} \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} H_0^{(1)}(k_0\sqrt{(s_j+(-1)^q\delta_j/2-s_{obs})^2+(n'-n_{obs})^2})_{\vec{r}=\vec{r}_i} dn' \\
&= \frac{i(-1)^q}{4\widetilde{R}^*(\vec{r}_i)} \left\{ \int_{n_j-\tau_j/2-n_{obs}}^{n_j+\tau_j/2-n_{obs}} H_0^{(1)}(k_0\sqrt{\widetilde{s}^2+\widetilde{n}^2})d\widetilde{n} \right\}_{\vec{r}=\vec{r}_i} \\
&= \frac{i(-1)^q}{4\widetilde{R}^*(\vec{r}_i)} \int_{n_j-\tau_j/2-n_i}^{n_j+\tau_j/2-n_i} H_0^{(1)}(k_0\sqrt{\widetilde{s}^2+\widetilde{n}^2})d\widetilde{n} \\
&= \begin{cases} \frac{-i}{4\widetilde{R}^*(\vec{r}_i)} \left[J_1(y_2,x_2) - J_1(y_1,x_1) \right] & \text{; leading edge} \\ \frac{i}{4\widetilde{R}^*(\vec{r}_i)} \left[J_1(y_3,x_3) - J_1(y_4,x_4) \right] & \text{; trailing edge.} \end{cases}
\end{aligned}$$

To simplify Int_2, we first expand $d^2H_0^{(1)}/ds^2$ as

$$\begin{aligned}
\frac{d^2H_0^{(1)}}{ds^2} &= \left[(\hat{s}\cdot\hat{s}')\frac{d}{ds_{obs}} + (\hat{s}\cdot\hat{n}')\frac{d}{dn_{obs}} \right] \left[(\hat{s}\cdot\hat{s}')\frac{d}{ds_{obs}} + (\hat{s}\cdot\hat{n}')\frac{d}{dn_{obs}} \right] H_0^{(1)} \\
&= (\hat{s}\cdot\hat{s}')^2 \frac{d^2H_0^{(1)}}{ds_{obs}^2} + 2(\hat{s}\cdot\hat{n}')(\hat{s}\cdot\hat{s}')\frac{d^2H_0^{(1)}}{ds_{obs}dn_{obs}} + (\hat{s}\cdot\hat{n}')^2\frac{d^2H_0^{(1)}}{dn_{obs}^2} \\
&= (\hat{s}\cdot\hat{s}')^2 \frac{d^2H_0^{(1)}}{ds_{obs}^2} - 2(\hat{s}\cdot\hat{n}')(\hat{s}\cdot\hat{s}')\frac{d}{dn'}\frac{dH_0^{(1)}}{ds_{obs}} - (\hat{s}\cdot\hat{n}')^2\frac{d}{dn'}\frac{dH_0^{(1)}}{dn_{obs}}.
\end{aligned}$$

When this is substituted in Int_2 we have

$$\begin{aligned}
Int_2 =& \frac{i(-1)^{q+1}}{4k_0^2\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{d^2H_0^{(1)}(k_0\rho)}{ds^2}_{\vec{r}=\vec{r}_i} dn' \\
=& \frac{i(-1)^{q+1}(\hat{s}\cdot\hat{s}')^2}{4k_0^2\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{d^2H_0^{(1)}(k_0\rho)}{ds_{obs}^2}_{\vec{r}=\vec{r}_i} dn' \\
&+ \frac{i(-1)^q(\hat{s}\cdot\hat{n}')(\hat{s}\cdot\hat{s}')}{2k_0^2\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{d}{dn'}\frac{dH_0^{(1)}(k_0\rho)}{ds_{obs}}_{\vec{r}=\vec{r}_i} dn' \\
&+ \frac{i(-1)^q(\hat{s}\cdot\hat{n}')^2}{4k_0^2\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{d}{dn'}\frac{dH_0^{(1)}(k_0\rho)}{dn_{obs}}_{\vec{r}=\vec{r}_i} dn' \\
=& \frac{i(-1)^{q+1}(\hat{s}\cdot\hat{s}')^2}{4k_0^2\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \left\{ \frac{d^2}{ds_{obs}^2} H_0^{(1)}(k_0\sqrt{(s_j+(-1)^q\delta_j/2-s_{obs})^2+(n'-n_{obs})^2}) \right\}_{\vec{r}=\vec{r}_i} dn'
\end{aligned}$$

$$+\frac{i(-1)^q(\hat{s}\cdot\hat{n}')(\hat{s}\cdot\hat{s}')}{2k_0^2\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[\left\{\frac{d}{ds_{obs}}H_0^{(1)}(k_0\sqrt{(s_j+(-1)^q\delta_j/2-s_{obs})^2+(n'-n_{obs})^2})\right\}_{\vec{r}=\vec{r}_i}\right]_{n'=n_j-\tau_j}^{n'=n_j+\tau_j}$$

$$+\frac{i(-1)^q(\hat{s}\cdot\hat{n}')^2}{4k_0^2\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[\left\{\frac{d}{dn_{obs}}H_0^{(1)}(k_0\sqrt{(s_j+(-1)^q\delta_j/2-s_{obs})^2+(n'-n_{obs})^2})\right\}_{\vec{r}=\vec{r}_i}\right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2}$$

$$=\frac{i(-1)^{q+1}(\hat{s}\cdot\hat{s}')^2}{4k_0^2\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\int_{n_j-\tau_j/2}^{n_j+\tau_j/2}\left\{\frac{d^2}{ds_{obs}^2}H_0^{(1)}(k_0\sqrt{(s_j+(-1)^q\delta_j/2-s_{obs})^2+(n'-n_{obs})^2})\right\}_{\vec{r}=\vec{r}_i}dn'$$

$$-\frac{i(-1)^q(\hat{s}\cdot\hat{n}')(\hat{s}\cdot\hat{s}')}{2k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[\left\{(\hat{\rho}\cdot\hat{s}')H_0^{(1)}(k_0\sqrt{(s_j+(-1)^q\delta_j/2-s_{obs})^2+(n'-n_{obs})^2})\right\}_{\vec{r}=\vec{r}_i}\right]_{n'=n_j-\tau_j}^{n'=n_j+\tau_j}$$

$$-\frac{i(-1)^q(\hat{s}\cdot\hat{n}')^2}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[\left\{(\hat{\rho}\cdot\hat{n}')H_0^{(1)}(k_0\sqrt{(s_j+(-1)^q\delta_j/2-s_{obs})^2+(n'-n_{obs})^2})\right\}_{\vec{r}=\vec{r}_i}\right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2}.$$

We now introduce the substitutions

$$\widetilde{s}=s_j+(-1)^q\delta_j/2-s_{obs}$$

$$\widetilde{n}=n'-n_{obs}.$$

These allow Int_2 to be expressed as

$$\begin{aligned}
Int_2 &= \frac{i(-1)^{q+1}(\hat{s}\cdot\hat{s}')^2}{4k_0^2\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left\{\int_{n_j-\tau_j/2-n_{obs}}^{n_j+\tau_j/2-n_{obs}}\frac{d^2}{d\widetilde{s}^2}H_0^{(1)}(k_0\sqrt{\widetilde{s}^2+\widetilde{n}^2})d\widetilde{n}\right\}_{\vec{r}=\vec{r}_i}\\
&\quad+\frac{i(-1)^{q+1}(\hat{s}\cdot\hat{n}')(\hat{s}\cdot\hat{s}')}{2k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left\{\left[(\hat{\rho}\cdot\hat{s}')H_1^{(1)}(k_0\sqrt{\widetilde{s}^2+\widetilde{n}^2})\right]_{\widetilde{n}=n_j-\tau_j/2-n_{obs}}^{\widetilde{n}=n_j+\tau_j/2-n_{obs}}\right\}_{\vec{r}=\vec{r}_i}\\
&\quad+\frac{i(-1)^{q+1}(\hat{s}\cdot\hat{n}')^2}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left\{\left[(\hat{\rho}\cdot\hat{n}')H_1^{(1)}(k_0\sqrt{\widetilde{s}^2+\widetilde{n}^2})\right]_{\widetilde{n}=n_j-\tau_j/2-n_{obs}}^{\widetilde{n}=n_j+\tau_j/2-n_{obs}}\right\}_{\vec{r}=\vec{r}_i}\\
&= \frac{i(\hat{s}\cdot\hat{s}')^2}{4k_0^2\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[J_3(y_2,x_2)-J_3(y_1,x_1)\right]\\
&\quad+\frac{i(\hat{s}\cdot\hat{n}')(\hat{s}\cdot\hat{s}')}{2k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[\frac{x_2}{\rho_2}H_1^{(1)}(k_0\rho_2)-\frac{x_1}{\rho_1}H_1^{(1)}(k_0\rho_1)\right]\\
&\quad+\frac{i(\hat{s}\cdot\hat{n}')^2}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[\frac{y_2}{\rho_2}H_1^{(1)}(k_0\rho_2)-\frac{y_1}{\rho_1}H_1^{(1)}(k_0\rho_1)\right]\text{; leading edge}\\
&= \frac{-i(\hat{s}\cdot\hat{s}')^2}{4k_0^2\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[J_3(y_3,x_3)-J_3(y_4,x_4)\right]
\end{aligned}$$

$$+\frac{-i(\hat{s}\cdot\hat{n}')(\hat{s}\cdot\hat{s}')}{2k_0\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)}\left[\frac{x_3}{\rho_3}H_1^{(1)}(k_0\rho_3)-\frac{x_4}{\rho_4}H_1^{(1)}(k_0\rho_4)\right]$$
$$+\frac{-i(\hat{s}\cdot\hat{n}')^2}{4k_0\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)}\left[\frac{y_3}{\rho_3}H_1^{(1)}(k_0\rho_3)-\frac{y_4}{\rho_4}H_1^{(1)}(k_0\rho_4)\right];\text{ trailing edge.}$$

Similarly, for Int_3 we have (with the usual definitions for $\tilde{s}$ and $\tilde{n}$)

$$Int_3=\frac{i(-1)^{q+1}}{4k_0^2}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)}\right)\int_{n_j-\tau_j/2}^{n_j+\tau_j/2}\left((\hat{s}\cdot\hat{s}')\frac{d}{ds_{obs}}-(\hat{s}\cdot\hat{n}')\frac{d}{dn'}\right)H_0^{(1)}(k_0\rho)_{\vec{r}=\vec{r}_i}dn'$$
$$=\frac{i(-1)^{q+1}(\hat{s}\cdot\hat{s}')}{4k_0^2}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)}\right)\int_{n_j-\tau_j/2}^{n_j+\tau_j/2}\frac{d}{ds_{obs}}H_0^{(1)}(k_0\sqrt{(s_j+(-1)^q\delta_j/2-s_{obs})^2+(n'-n_{obs})^2})_{\vec{r}=\vec{r}}$$
$$+\frac{i(-1)^{q}(\hat{s}\cdot\hat{n}')}{4k_0^2}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)}\right)\left[H_0^{(1)}(k_0\sqrt{(s_j+(-1)^q\delta_j/2-s_i)^2+(n'-n_i)^2})\right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2}$$
$$=\frac{i(-1)^{q}(\hat{s}\cdot\hat{s}')}{4k_0^2}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)}\right)\left\{\int_{n_j-\tau_j/2-n_{obs}}^{n_j+\tau_j/2-n_{obs}}\frac{d}{d\tilde{s}}H_0^{(1)}(k_0\sqrt{\tilde{s}^2+\tilde{n}^2})d\tilde{n}\right\}_{\vec{r}=\vec{r}_i}$$
$$+\frac{i(-1)^{q}(\hat{s}\cdot\hat{n}')}{4k_0^2}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)}\right)\left[H_0^{(1)}(k_0\sqrt{(s_j+(-1)^q\delta_j/2-s_i)^2+(n'-n_i)^2})\right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2}$$
$$=\frac{-i(\hat{s}\cdot\hat{s}')}{4k_0^2}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)}\right)[J_2(y_2,x_2)-J_2(y_1,x_1)]$$
$$+\frac{-i(\hat{s}\cdot\hat{n}')}{4k_0^2}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)}\right)\left[H_0^{(1)}(k_0\rho_2)-H_0^{(1)}(k_0\rho_1)\right];\text{ leading edge}$$
$$=\frac{i(\hat{s}\cdot\hat{s}')}{4k_0^2}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)}\right)[J_2(y_3,x_3)-J_2(y_4,x_4)]$$
$$+\frac{i(\hat{s}\cdot\hat{n}')}{4k_0^2}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)}\right)\left[H_0^{(1)}(k_0\rho_3)-H_0^{(1)}(k_0\rho_4)\right];\text{ trailing edge.}$$

Hence, the expression for $\tilde{F}_{ij}^6$ becomes

$$F_{ij}^6 = \frac{-i}{4\tilde{R}^*(\vec{r}_i)}[J_1(y_2,x_2)-J_1(y_1,x_1)]$$
$$+\frac{i(\hat{s}\cdot\hat{s}')^2}{4k_0^2\varepsilon_r(\vec{r}_i)\tilde{R}(\vec{r}_i)}[J_3(y_2,x_2)-J_3(y_1,x_1)]$$

$$
\begin{aligned}
&+\frac{i(\hat{s}\cdot\hat{n}')(\hat{s}\cdot\hat{s}')}{2k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[\frac{x_2}{\rho_2}H_1^{(1)}(k_0\rho_2)-\frac{x_1}{\rho_1}H_1^{(1)}(k_0\rho_1)\right]\\
&+\frac{i(\hat{s}\cdot\hat{n}')^2}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[\frac{y_2}{\rho_2}H_1^{(1)}(k_0\rho_2)-\frac{y_1}{\rho_1}H_1^{(1)}(k_0\rho_1)\right]\\
&+\frac{-i(\hat{s}\cdot\hat{s}')}{4k_0^2}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)\left[J_2(y_2,x_2)-J_2(y_1,x_1)\right]\\
&+\frac{-i(\hat{s}\cdot\hat{n}')}{4k_0^2}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)\left[H_0^{(1)}(k_0\rho_2)-H_0^{(1)}(k_0\rho_1)\right]\text{ ; leading edge}\\
=\;&\frac{i}{4\widetilde{R}^*(\vec{r}_i)}\left[J_1(y_3,x_3)-J_1(y_4,x_4)\right]\\
&+\frac{-i(\hat{s}\cdot\hat{s}')^2}{4k_0^2\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[J_3(y_3,x_3)-J_3(y_4,x_4)\right]\\
&+\frac{-i(\hat{s}\cdot\hat{n}')(\hat{s}\cdot\hat{s}')}{2k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[\frac{x_3}{\rho_3}H_1^{(1)}(k_0\rho_3)-\frac{x_4}{\rho_4}H_1^{(1)}(k_0\rho_4)\right]\\
&+\frac{-i(\hat{s}\cdot\hat{n}')^2}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[\frac{y_3}{\rho_3}H_1^{(1)}(k_0\rho_3)-\frac{y_4}{\rho_4}H_1^{(1)}(k_0\rho_4)\right]\\
&+\frac{i(\hat{s}\cdot\hat{s}')}{4k_0^2}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)\left[J_2(y_3,x_3)-J_2(y_4,x_4)\right]\\
&+\frac{i(\hat{s}\cdot\hat{n}')}{4k_0^2}\frac{d}{ds}\left(\frac{1}{\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\right)\left[H_0^{(1)}(k_0\rho_3)-H_0^{(1)}(k_0\rho_4)\right]\text{ ; trailing edge.}
\end{aligned}
\tag{4.42}
$$

4.5.7 Element F_{ij}^7:

$$
\begin{aligned}
F_{ij}^7 &= \int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\frac{1}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[\left\{(\hat{\rho}\cdot\hat{s})H_1^{(1)}(k_0\rho)\right\}_{\vec{r}=\vec{r}_i}\right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2}ds'\\
&= \int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\frac{-1}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[\frac{dH_0^{(1)}(k_0\rho)}{ds}_{\vec{r}=\vec{r}_i}\right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2}ds'.
\end{aligned}
$$

Utilizing the expansion

$$
\frac{dH_0^{(1)}}{ds} = (\hat{s}\cdot\hat{s}')\frac{dH_0^{(1)}}{ds_{obs}}+(\hat{s}\cdot\hat{n}')\frac{dH_0^{(1)}}{dn_{obs}}
$$

$$= -(\hat{s}\cdot\hat{s}')\frac{dH_0^{(1)}}{ds'} + (\hat{s}\cdot\hat{n}')\frac{dH_0^{(1)}}{dn_{obs}},$$

we obtain

$$F_{ij}^7 = \frac{-(\hat{s}\cdot\hat{n}')}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\left[\left\{\frac{d}{dn_{obs}}H_0^{(1)}(k_0\sqrt{(s'-s_{obs})^2+(n'-n_{obs})^2})\right\}_{\vec{r}=\vec{r}_i}\right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} ds'$$
$$+\frac{(\hat{s}\cdot\hat{s}')}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[\left[H_0^{(1)}(k_0\sqrt{(s'-s_i)^2+(n'-n_i)^2})\right]_{s'=s_j-\delta_j/2}^{s'=s_j+\delta_j/2}\right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2}.$$

If we now introduce the substitutions (4.34) and (4.35) we obtain

$$F_{ij}^7 = \frac{(\hat{s}\cdot\hat{n}')}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left\{\int_{s_j-\delta_j/2-s_{obs}}^{s_j+\delta_j/2-s_{obs}}\left[\frac{d}{d\widetilde{n}}H_0^{(1)}(k_0\sqrt{\widetilde{s}^2+\widetilde{n}^2})\right]_{\widetilde{n}=n_j-\tau_j/2-n_{obs}}^{\widetilde{n}=n_j+\tau_j/2-n_{obs}} d\widetilde{s}\right\}_{\vec{r}=\vec{r}_i}$$
$$+\frac{(\hat{s}\cdot\hat{s}')}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[H_0^{(1)}(k_0\rho_3)-H_0^{(1)}(k_0\rho_2)-H_0^{(1)}(k_0\rho_4)+H_0^{(1)}(k_0\rho_1)\right].$$

Finally, introducing the pre-defined generic integrals, F_{ij}^7 becomes

$$F_{ij}^7 = \frac{(\hat{s}\cdot\hat{n}')}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[J_2(x_3,y_3)-J_2(x_2,y_2)-J_2(x_4,y_4)+J_2(x_1,y_1)\right]$$
$$+\frac{(\hat{s}\cdot\hat{s}')}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[H_0^{(1)}(k_0\rho_3)-H_0^{(1)}(k_0\rho_2)-H_0^{(1)}(k_0\rho_4)+H_0^{(1)}(k_0\rho_1)\right]. \tag{4.43}$$

4.5.8 Element F_{ij}^8:

$$F_{ij}^8 = \int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\int_{n_j-\tau_j/2}^{n_j+\tau_j/2}\frac{ik_0}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left\{(\hat{\rho}\cdot\hat{s})H_1^{(1)}(k_0\rho)\right\}_{\vec{r}=\vec{r}_i} ds'dn'$$
$$= \int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\int_{n_j-\tau_j/2}^{n_j+\tau_j/2}\frac{-i}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left\{\frac{d}{ds}H_0^{(1)}(k_0\rho)\right\}_{\vec{r}=\vec{r}_i} ds'dn'$$
$$= \int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\int_{n_j-\tau_j/2}^{n_j+\tau_j/2}\frac{-i}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left\{\frac{d}{ds}H_0^{(1)}(k_0\sqrt{(s'-s_{obs})^2+(n'-n_{obs})^2})\right\}_{\vec{r}=\vec{r}_i} ds'dn'.$$

Recalling that

$$\begin{aligned}\frac{dH_0^{(1)}}{ds} &= (\hat{s}\cdot\hat{s}')\frac{dH_0^{(1)}}{ds_{obs}} + (\hat{s}\cdot\hat{n}')\frac{dH_0^{(1)}}{dn_{obs}} \\ &= -(\hat{s}\cdot\hat{s}')\frac{dH_0^{(1)}}{ds'} - (\hat{s}\cdot\hat{n}')\frac{dH_0^{(1)}}{dn'},\end{aligned}$$

F_{ij}^8 may expressed as

$$\begin{aligned}F_{ij}^8 &= \frac{i(\hat{s}\cdot\hat{n}')}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\int_{s_j-\delta_j/2}^{s_j+\delta_j/2}\left[H_0^{(1)}(k_0\sqrt{(s'-s_i)^2+(n'-n_i)^2})\right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} ds' \\ &+\frac{i(\hat{s}\cdot\hat{s}')}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\int_{n_j-\tau_j/2}^{n_j+\tau_j/2}\left[H_0^{(1)}(k_0\sqrt{(s'-s_i)^2+(n'-n_i)^2})\right]_{s'=s_j-\delta_j/2}^{s'=s_j+\delta_j/2} dn'.\end{aligned}$$

Further, employing the substitutions

$$\widetilde{s} = s' - s_i$$

$$\widetilde{n} = n' - n_i$$

we obtain

$$\begin{aligned}F_{ij}^8 &= \frac{i(\hat{s}\cdot\hat{n}')}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\int_{s_j-\delta_j/2-s_i}^{s_j+\delta_j/2-s_i}\left[H_0^{(1)}(k_0\sqrt{\widetilde{s}^2+\widetilde{n}^2})\right]_{\widetilde{n}=n_j-\tau_j/2-n_i}^{\widetilde{n}=n_j+\tau_j/2-n_i} d\widetilde{s} \\ &+\frac{i(\hat{s}\cdot\hat{s}')}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\int_{n_j-\tau_j/2-n_i}^{n_j+\tau_j/2-n_i}\left[H_0^{(1)}(k_0\sqrt{\widetilde{s}^2+\widetilde{n}^2})\right]_{\widetilde{s}=s_j-\delta_j/2-s_i}^{\widetilde{s}=s_j+\delta_j/2-s_i} d\widetilde{n} \\ &= \frac{i(\hat{s}\cdot\hat{n}')}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[J_1(x_3,y_3)-J_1(x_2,y_2)-J_1(x_4,y_4)+J_1(x_1,y_1)\right] \\ &+\frac{i(\hat{s}\cdot\hat{s}')}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[J_1(y_3,x_3)-J_1(y_2,x_2)-J_1(y_4,x_4)+J_1(y_1,x_1)\right].\end{aligned} \tag{4.44}$$

4.5.9 Element $\widetilde{F}_{ij}^9$:

$$\widetilde{F}_{ij}^9 = \gamma_{ij} + \int_{n_j-\tau_j/2}^{n_j+\tau_j/2}\frac{(-1)^{q+1}}{4\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left\{(\hat{\rho}\cdot\hat{s})H_1^{(1)}(k_0\rho)\right\}_{\vec{r}=\vec{r}_i} dn'$$

$$
\begin{aligned}
&= \gamma_{ij} \quad + \frac{(-1)^q}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{dH_0^{(1)}(k_0\rho)}{ds}\bigg|_{\vec{r}=\vec{r}_i} dn' \\
&= \gamma_{ij} \quad + \frac{(-1)^q}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \left\{ \left((\hat{s}\cdot\hat{s}_{obs}) \frac{d}{ds_{obs}} + (\hat{s}\cdot\hat{n}_{obs}) \frac{d}{dn_{obs}} \right) H_0^{(1)}(k_0\rho) \right\}_{\vec{r}=\vec{r}_i} dn' \\
&= \gamma_{ij} \quad + \frac{(-1)^q}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \left\{ \left((\hat{s}\cdot\hat{s}') \frac{d}{ds_{obs}} - (\hat{s}\cdot\hat{n}') \frac{d}{dn'} \right) H_0^{(1)}(k_0\rho) \right\}_{\vec{r}=\vec{r}_i} dn' \\
&= \gamma_{ij} \quad + \frac{(-1)^q(\hat{s}\cdot\hat{s}')}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \frac{dH_0^{(1)}(k_0\rho)}{ds_{obs}}\bigg|_{\vec{r}=\vec{r}_i} dn' \\
&\qquad\quad + \frac{(-1)^{q+1}(\hat{s}\cdot\hat{n}')}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \left[H_0^{(1)}(k_0\rho)_{\vec{r}=\vec{r}_i} \right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2} \\
&= \gamma_{ij} \quad + \frac{(-1)^q(\hat{s}\cdot\hat{s}')}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \int_{n_j-\tau_j/2}^{n_j+\tau_j/2} \left\{ \frac{d}{ds_{obs}} H_0^{(1)}(k_0\sqrt{(s_j+(-1)^q\delta/2-s_{obs})^2+(n'-n_{obs})^2}) \right\}_{\vec{r}=\vec{r}_i} dn' \\
&\qquad\quad + \frac{(-1)^{q+1}(\hat{s}\cdot\hat{n}')}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \left[H_0^{(1)}(k_0\sqrt{(s_j+(-1)^q\delta/2-s_i)^2+(n'-n_i)^2}) \right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2}.
\end{aligned}
$$

Making the substitutions

$$
\widetilde{s} = s_j + (-1)^q\delta_j/2 - s_{obs}
$$

$$
\widetilde{n} = n' - n_{obs}
$$

we obtain

$$
\begin{aligned}
\widetilde{F}_{ij}^9 \quad &= \gamma_{ij} \quad + \frac{(-1)^{q+1}(\hat{s}\cdot\hat{s}')}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \left\{ \int_{n_j-\tau_j/2-n_{obs}}^{n_j+\tau_j/2-n_{obs}} \frac{d}{d\widetilde{s}} H_0^{(1)}(k_0\sqrt{\widetilde{s}^2+\widetilde{n}^2}) d\widetilde{n} \right\}_{\vec{r}=\vec{r}_i} \\
&\qquad\quad + \frac{(-1)^{q+1}(\hat{s}\cdot\hat{n}')}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \left[H_0^{(1)}(k_0\sqrt{(s_j+(-1)^p\delta_j/2-s_i)^2+(n'-n_i)^2}) \right]_{n'=n_j-\tau_j/2}^{n'=n_j+\tau_j/2}.
\end{aligned}
$$

Thus for leading edges the appropriate expression for $\widetilde{F}_{ij}^9$ is

$$
\widetilde{F}_{ij}^9 \quad = \gamma_{ij} \quad + \frac{(\hat{s}\cdot\hat{s}')}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)} \left[J_2(y_2, x_2) - J_2(y_1, x_1) \right]
$$

$$+\frac{(\hat{s}\cdot\hat{n}')}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[H_0^{(1)}(k_0\rho_2)-H_0^{(1)}(k_0\rho_1)\right],$$

while for trailing edges we have

$$\begin{aligned}\widetilde{F}_{ij}^{9} &= \gamma_{ij} \quad +\frac{-(\hat{s}\cdot\hat{s}')}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[J_2(y_3,x_3)-J_2(y_4,x_4)\right] \\ &\quad +\frac{-(\hat{s}\cdot\hat{n}')}{4k_0\varepsilon_r(\vec{r}_i)\widetilde{R}(\vec{r}_i)}\left[H_0^{(1)}(k_0\rho_3)-H_0^{(1)}(k_0\rho_4)\right]. \end{aligned} \tag{4.45}$$

Chapter 5

Far Field Computation

In this chapter we evaluate the scattered electric field produced by the equivalent currents computed in the previous chapter.

Rewriting (2.18) in terms of (s, n, z) coordinates we have

$$\begin{aligned} E_s^s &= -\frac{i}{k_0}\int_{A'} \tilde{J}_s \delta(\vec{r}-\vec{r}')dA' + \frac{i}{k_0}\int_{A'} \left(-\tilde{J}_s\frac{d^2}{dn^2} + \tilde{J}_n\frac{d^2}{dsdn}\right) G^{2d}(\vec{r},\vec{r}')\, dA' \\ &\quad -\int_{A'} J_z^* \frac{dG^{2d}(\vec{r},\vec{r}')}{dn}dA'. \end{aligned} \tag{5.1}$$

If the point of observation lies in the far zone, the first term of (5.1) is identically zero. Furthermore, substituting (2.29) inton (5.1), we may write

$$\begin{aligned} E_s^s &= \frac{i}{k_0}\int_{A'} \frac{d}{dn}\left(\tilde{J}_{s'}\frac{dG^{2d}(\vec{r},\vec{r}')}{dn'} - \tilde{J}_{n'}\frac{dG^{2d}(\vec{r},\vec{r}')}{ds'}\right) dA' \\ &\quad -\int_{A'} J_z^* \frac{dG^{2d}(\vec{r},\vec{r}')}{dn}dA'. \end{aligned} \tag{5.2}$$

Using integration by parts to modify the $\tilde{J}_{n'}$ term, (5.2) may be written as

$$\begin{aligned} E_s^s &= \frac{i}{k_0}\int_{A'} \tilde{J}_{s'}\frac{d^2G^{2d}(\vec{r},\vec{r}')}{dndn'}dA' - \frac{i}{k_0}\int_{C'} \tilde{J}_{n'}\frac{dG^{2d}(\vec{r},\vec{r}')}{dn}dn'|_{endpoints} \\ &\quad +\frac{i}{k_0}\int_{A'} \frac{d\tilde{J}_{n'}}{ds'}\frac{dG^{2d}(\vec{r},\vec{r}')}{dn}dA' - \int_{A'} J_z^*\frac{dG^{2d}(\vec{r},\vec{r}')}{dn}dA' \\ &= \frac{i}{k_0}\int_{A'} \tilde{J}_{s'}\frac{d^2G^{2d}(\vec{r},\vec{r}')}{dndn'}dA' - \frac{i}{k_0}\int_{C'} \tilde{J}_{n'}(-1)^q\frac{dG^{2d}(\vec{r},\vec{r}')}{dn}dn' - \int_{A'} \tilde{J}_z^*\frac{dG^{2d}(\vec{r},\vec{r}')}{dn}dA' \end{aligned} \tag{5.3}$$

where $q = 1/2$ for leading/trailing edges and again $\tilde{J}_z^* = J_z^* - \frac{i}{k_0}\frac{d\tilde{J}_{n'}}{ds'}$. Expanding (5.3) we have

$$\begin{aligned} E_s^s &= \frac{i}{k_0}\int_{A'} \tilde{J}_{s'}\left(\frac{ik_0}{4}\right)\left[((\rho\cdot\hat{s})(\rho\cdot\hat{s}') - (\rho\cdot\hat{n})(\rho\cdot\hat{n}'))\frac{H_1^{(1)}(k_0\rho)}{\rho} + k_0(\rho\cdot\hat{n})(\rho\cdot\hat{n}')H_0^{(1)}(k_0\rho)\right] dA' \\ &-\frac{i}{k_0}\int_{C'} \tilde{J}_{n'}(-1)^q\frac{-ik_0}{4}(\rho\cdot\hat{n})H_1^{(1)}(k_0\rho)dn' - \int_{A'} \tilde{J}_z^*\frac{-ik_0}{4}(\rho\cdot\hat{n})H_1^{(1)}(k_0\rho)dA'. \end{aligned} \quad (5.4)$$

Since the point of observation is in the far zone, the term containing $H_1^{(1)}(k_0\rho)/\rho$ becomes negligible in comparison with the remaining ones and (5.4) may be simplified to

$$\begin{aligned} E_s^s &= \frac{-k_0}{4}\int_{A'} \tilde{J}_{s'}(\rho\cdot\hat{n})(\rho\cdot\hat{n}')H_0^{(1)}(k_0\rho)dA' \\ &-\frac{1}{4}\int_{C'} \tilde{J}_{n'}(-1)^q(\rho\cdot\hat{n})H_1^{(1)}(k_0\rho)dn' + \frac{ik_0}{4}\int_{A'} \tilde{J}_z^*(\rho\cdot\hat{n})H_1^{(1)}(k_0\rho)dA'. \end{aligned} \quad (5.5)$$

Furthermore, $(\hat{s},\hat{n})$ may be set to $\hat{s} = -\hat{\phi}$, $\hat{n} = \hat{\varrho}$, where (ϱ,ϕ) are the cylindrical coordinates of the point of observation. Because this point is in the far zone, $\hat{\rho} \approx \hat{\varrho}$ and we may thus write

$$-E_\phi^s = \frac{-k_0}{4}\int_{A'} \tilde{J}_{s'}(\hat{\varrho}\cdot\hat{n}')H_0^{(1)}(k_0\rho)dA' - \frac{1}{4}\int_{C'} \tilde{J}_{n'}(-1)^q H_1^{(1)}(k_0\rho)dn' + \frac{ik_0}{4}\int_{A'} \tilde{J}_z^* H_1^{(1)}(k_0\rho)dA'. \quad (5.6)$$

We now recall that the asymptotic expressions for the zeroth and first order Hankel functions are given by

$$H_0^{(1)}(k_0\rho) \sim \sqrt{\frac{2}{\pi k_0\rho}}e^{i(k_0\varrho-\pi/4)}e^{-ik_0(x'\cos\phi+y'\sin\phi)} \quad (5.7)$$

$$H_1^{(1)}(k_0\rho) \sim \sqrt{\frac{2}{\pi k_0\rho}}e^{i(k_0\varrho-\pi/4-\pi/2)}e^{-ik_0(x'\cos\phi+y'\sin\phi)}. \quad (5.8)$$

Substituting (5.8) (with ρ replaced by ϱ) and the current expansions (4.5) and (4.6) into (5.6) ,we find

$$\begin{aligned} E_\phi^s &= \sqrt{\frac{k_0}{8\pi\varrho}} e^{i(k_0\varrho-\pi/4)} \sum_{i=1}^{N} K_{si} \int_{A_i} (\hat{\varrho}\cdot\hat{n}') e^{-ik_0(x'\cos\phi+y'\sin\phi)} dA' \\ &- \sqrt{\frac{k_0}{8\pi\varrho}} e^{i(k_0\varrho-\pi/4)} \sum_{i=1}^{N} K_{zi}^* \int_{A_i} e^{-ik_0(x'\cos\phi+y'\sin\phi)} dA' \\ &- \sqrt{\frac{1}{8\pi k_0\varrho}} e^{i(k_0\varrho+\pi/4)} \sum_{j=1}^{N^{edge}} K_{nj}^{edge} \int_{C_j} (-1)^q e^{-ik_0(x'\cos\phi+y'\sin\phi)} dn'. \quad (5.9) \end{aligned}$$

Introducing the identity $\varrho = \cos\phi\hat{x} + \sin\phi\hat{y}$, we further obtain

$$\begin{aligned} E_\phi^s &= \sqrt{\frac{k_0}{8\pi\varrho}} e^{i(k_0\varrho-\pi/4)} \sum_{i=1}^{N} K_{si} \int_{A_i} [\cos\phi(\hat{x}\cdot\hat{n}') + \sin\phi(\hat{y}\cdot\hat{n}')] e^{-ik_0(x'\cos\phi+y'\sin\phi)} dA' \\ &- \sqrt{\frac{k_0}{8\pi\varrho}} e^{i(k_0\varrho-\pi/4)} \sum_{i=1}^{N} K_{zi}^* \int_{A_i} e^{-ik_0(x'\cos\phi+y'\sin\phi)} dA' \\ &- \sqrt{\frac{1}{8\pi k_0\varrho}} e^{i(k_0\varrho+\pi/4)} \sum_{j=1}^{N^{edge}} K_{nj}^{edge} \int_{C_j} (-1)^q e^{-ik_0(x'\cos\phi+y'\sin\phi)} dn'. \quad (5.10) \end{aligned}$$

If each cell of integration is now approximated by a rectangular cell, we may write $\hat{\varrho}\cdot\hat{n}' = \cos\phi(\hat{x}\cdot\hat{n}') + \sin\phi(\hat{y}\cdot\hat{n}') \approx \cos\phi(\hat{x}\cdot\hat{n}_i) + \sin\phi(\hat{y}\cdot\hat{n}_i)$ and (5.10) can be rewritten as

$$\begin{aligned} E_\phi^s &= \sqrt{\frac{k_0}{8\pi\varrho}} e^{i(k_0\varrho-\pi/4)} \sum_{i=1}^{N} K_{si} [\cos\phi(\hat{x}\cdot\hat{n}_i) + \sin\phi(\hat{y}\cdot\hat{n}_i)] \int_{A_i} e^{-ik_0(x'\cos\phi+y'\sin\phi)} dA' \\ &- \sqrt{\frac{k_0}{8\pi\varrho}} e^{i(k_0\varrho-\pi/4)} \sum_{i=1}^{N} K_{zi}^* \int_{A_i} e^{-ik_0(x'\cos\phi+y'\sin\phi)} dA' \\ &- \sqrt{\frac{1}{8\pi k_0\varrho}} e^{i(k_0\varrho+\pi/4)} \sum_{j=1}^{N^{edge}} K_{nj}^{edge} \int_{C_j} (-1)^q e^{-ik_0(x'\cos\phi+y'\sin\phi)} dn'. \quad (5.11) \end{aligned}$$

Employing the identities $x' = s'(\hat{s}'\cdot\hat{x}) + n'(\hat{n}'\cdot\hat{x})$ and $y' = s'(\hat{s}'\cdot\hat{y}) + n'(\hat{n}'\cdot\hat{y})$ along with the rectangular cell approximation, the remaining integrals in (5.11) may be evaluated as

$$\int_{s_i-\delta/2}^{s_i+\delta/2} \int_{n_i-\tau/2}^{n_i+\tau/2} e^{-ik_0(x'\cos\phi+y'\sin\phi)} ds' dn'$$

$$
\begin{aligned}
&= \int_{s_i-\delta/2}^{s_i+\delta/2}\int_{n_i-\tau/2}^{n_i+\tau/2} e^{-ik_0\{[s'(\hat{s}'\cdot\hat{x})+n'(\hat{n}'\cdot\hat{x})]\cos\phi+[s'(\hat{s}'\cdot\hat{y})+n'(\hat{n}'\cdot\hat{y})]\sin\phi\}}ds'dn' \\
&= \frac{e^{-ik_0 s'[(\hat{s}'\cdot\hat{x})\cos\phi+(\hat{s}'\cdot\hat{y})\sin\phi]}}{-ik_0\left[(\hat{s}'\cdot\hat{x})\cos\phi+(\hat{s}'\cdot\hat{y})\sin\phi\right]}\Big|_{s'=s_i-\delta_i/2}^{s'=s_i+\delta_i/2}\frac{e^{-ik_0 n'[(\hat{n}'\cdot\hat{x})\cos\phi+(\hat{n}'\cdot\hat{y})\sin\phi]}}{-ik_0\left[(\hat{n}'\cdot\hat{x})\cos\phi+(\hat{n}'\cdot\hat{y})\sin\phi\right]}\Big|_{n'=n_i-\tau_i/2}^{n'=n_i+\tau_i/2} \\
&= e^{-ik_0\{[s_i(\hat{s}_i\cdot\hat{x})+n_i(\hat{n}_i\cdot\hat{x})]\cos\phi+[s_i(\hat{s}_i\cdot\hat{y})+n_i(\hat{n}_i\cdot\hat{y})]\sin\phi\}}\times \\
&\quad \frac{e^{-ik_0\delta_i/2[(\hat{s}_i\cdot\hat{x})\cos\phi+(\hat{s}_i\cdot\hat{y})\sin\phi]}-e^{ik_0\delta_i/2[(\hat{s}_i\cdot\hat{x})\cos\phi+(\hat{s}_i\cdot\hat{y})\sin\phi]}}{-ik_0\left[(\hat{s}_i\cdot\hat{x})\cos\phi+(\hat{s}_i\cdot\hat{y})\sin\phi\right]}\times \\
&\quad \frac{e^{-ik_0\tau_i/2[(\hat{n}_i\cdot\hat{x})\cos\phi+(\hat{n}_i\cdot\hat{y})\sin\phi]}-e^{ik_0\tau_i/2[(\hat{n}_i\cdot\hat{x})\cos\phi+(\hat{n}_i\cdot\hat{y})\sin\phi]}}{-ik_0\left[(\hat{n}_i\cdot\hat{x})\cos\phi+(\hat{n}_i\cdot\hat{y})\sin\phi\right]} \\
&= \delta_i\tau_i\frac{\sin\xi_i}{\xi_i}\frac{\sin\nu_i}{\nu_i}e^{-ik_0(x_i\cos\phi+y_i\sin\phi)}, \qquad (5.12)
\end{aligned}
$$

where

$$
\begin{aligned}
\xi_i &= k_0\delta_i/2\left[(\hat{s}_i\cdot\hat{x})\cos\phi+(\hat{s}_i\cdot\hat{y})\sin\phi\right] \\
\nu_i &= k_0\tau_i/2\left[(\hat{n}_i\cdot\hat{x})\cos\phi+(\hat{n}_i\cdot\hat{y})\sin\phi\right]. \qquad (5.13)
\end{aligned}
$$

Substituting (5.12) into (5.11) we obtain the far zone scattered field

$$
\begin{aligned}
E_\phi^s &= \sqrt{\frac{k_0}{8\pi\varrho}}e^{i(k_0\varrho-\pi/4)}\sum_{i=1}^{N}\Big[K_{si}(\cos\phi(\hat{x}\cdot\hat{n}_i)+\sin\phi(\hat{y}\cdot\hat{n}_i))-K_{zi}^*\Big]\delta_i\tau_i\frac{\sin\xi_i}{\xi_i}\frac{\sin\nu_i}{\nu_i}e^{-ik_0(x_i\cos\phi+y_i\sin\phi)} \\
&\quad -\sqrt{\frac{1}{8\pi k_0\varrho}}e^{i(k_0\varrho+\pi/4)}\sum_{j=1}^{N^{edge}}K_{nj}^{edge}(-1)^q\tau_j\frac{\sin\nu_j}{\nu_j}e^{-ik_0(x_j\cos\phi+y_j\sin\phi)}. \qquad (5.14)
\end{aligned}
$$

The echo width is now given by

$$
\begin{aligned}
\sigma &= 2\pi\varrho\lim_{\varrho\longrightarrow\infty}\frac{|E^{scat}|^2}{|E^{inc}|^2} \\
&= \frac{k_0}{4}\Bigg\|\sum_{i=1}^{N}\left[K_{si}\left(\cos\phi(\hat{x}\cdot\hat{n}_i)+\sin\phi(\hat{y}\cdot\hat{n}_i)\right)-K_{zi}^*\right]\delta_i\tau_i\left(\frac{\sin\xi_i}{\xi_i}\right)\left(\frac{\sin\nu_i}{\nu_i}\right)e^{-ik_0(x_i\cos\phi+y_i\sin\phi)} \\
&- \frac{i}{k_0}\sum_{j=1}^{N^{edge}}K_{nj}^{edge}(-1)^q\tau_j\left(\frac{\sin\nu_j}{\nu_j}\right)e^{-ik_0(x_j\cos\phi+y_j\sin\phi)}\Bigg\|^2. \qquad (5.15)
\end{aligned}
$$

Chapter 6

Description of the Computer Code QRCOMB

A computer code was written to implement the numerical solution of equation (4.12) for the most general case of an inhomogeneous composite cylinder of arbitrary cross section. Either principal polarization of incidence can be specified with this code. The code, to be referred to as QRCOMB, provides the user with the equivalent current distribution and the echowidth of the modeled structure. The code is written in FORTRAN and is self-contained, requiring only the basic system supplied functions. The sequence of steps executed by the code are:

(1). Reading of input file

(2). Geometry generation/discretization

(3). Compution of R parameters for each cell

(4). Computation of matrix elements

(5). Computation of input vector for the given observation angle

(6). Equivalent current solution via matrix inversion

(7). Application of current tapering

(8). Computation of echo width for given observation angle

(9). Repetition of (5) through (8) if generation of the backscatter pattern was required
Repetition of (8) if bistatic pattern was requested

(10). Output echowidth data.

Currently, the code accepts inputs for rectangular and circular layers. The input process is accomplished in the subroutine GEOQRC where each layer is first subdivided into discrete cells of equal width. Subsequently a 5x5 sampling grid of points is generated for each cell with the exterior sampling points outlining the cell boundary.

GEOQRC also provides the geometrical and material specification of all layer edges in addition to sorting the edges common to more than one layer.

The constitutive parameters of each cell are assigned in accordance with a set of input data referred to as "tapering specifications". These input specifications allow an (ε_r, μ_r) profile to be imposed over a particular portion of the scattering body. This profile describes the manner in which ε_r and μ_r vary in the region between two points on the structure which have specified values of ε_r and μ_r. A directional derivative of the R parameter is also computed for each cell via a finite difference approach.

The matrix element evaluations are carried out in the subroutine MTXQRC via analytical or numerical means (see Table 4.1 for the pertinent regimes). To perform the analytical evaluations, the coordinates of the of the integration cell are first transformed to local coordinates as described in Chapter 4. The matrix elements are then expressed in terms of the generic integrals given in the Appendix.

The numerical integrations are carried out using the expressions of chapter 3. Each integral over an area is evaluated via a two dimensional 5pt. or 3pt. Simpson's rule, depending on the distance between the point of integration and the point of observation. In the same way, one dimensional integrals are evaluated utilizing a one dimensional 5pt. or 3pt. Simpson's rule, with the comments made above pertaining here as well.

The remaining tasks are performed directly within the main program. The current tapering is accomplished by multiplying each current element by a coefficient between zero and one. This feature is included to simulate two dimensional structures that are infinite in one direction. This code also incorporates the capability to model a structure placed upon an infinite ground plane by adding the contribution due to the image wave incident upon the structure at an angle of $360 - \theta_{inc}$. Finally, the code provides an input option whereby an offset in decibels to be added to each computed value of echo width to adust the output quantity to a three dimensional cross section for comparison with other data. This adjustment in the output data is particularly useful for comparisons with experimental data corresponding to an elongated three dimensional target.

Chapter 7

Code Validation

In this section we present a sequence of two-dimensional geometries modeled by the compact integral equations and, where possible, contrasted with results obtained via alternate formulations. The last include a traditional integral equation formulation, a finite element method [10] and several high frequency techniques [12], [11].

The modeled geometries include:

1. perfectly conducting half-planes (figs 7.1,7.2);
2. single strips of various constitutive parameters (figs 7.3,7.4,7.5);
3. Partially and fully coated perfectly conducting rectangular cylinders, (figs 7.6,7.7,7.8,7.9,7.10,7.11,7.12);
4. perfectly conducting triangular cylinders with and without material coating (figs 7.13,7.14
5. Partially coated perfectly conducting circular cylinders (figs 7.15,7.16);
6. perfectly conducting wedge-circular cylinders with and without material coating (figs 7.17,7.18).

The specific geometrical details associated with each scattering plot are included in the corresponding figure. As seen, the agreement between the compact integral equation results and those obtained via alternative methods is always excellent, thus demonstrating that the compact set of integral equations may be implemented in a robust manner to handle a wide variety of scatterer configurations.

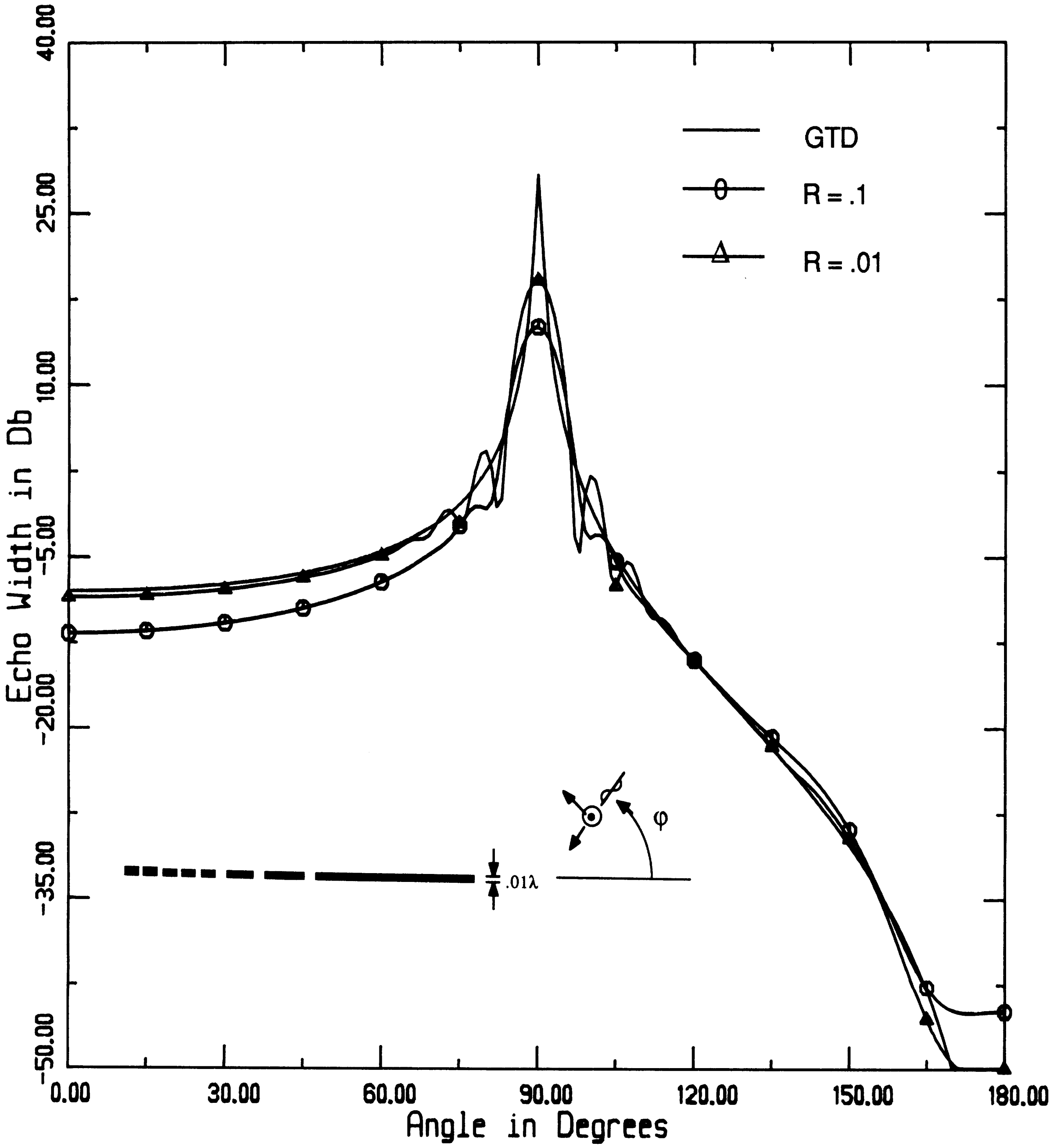

Figure 7.1: Ez backscatter echowidth of a perfectly conducting half-plane; comparison of results computed via the compact integral equations and a high frequency method for various values of resistivity, R.

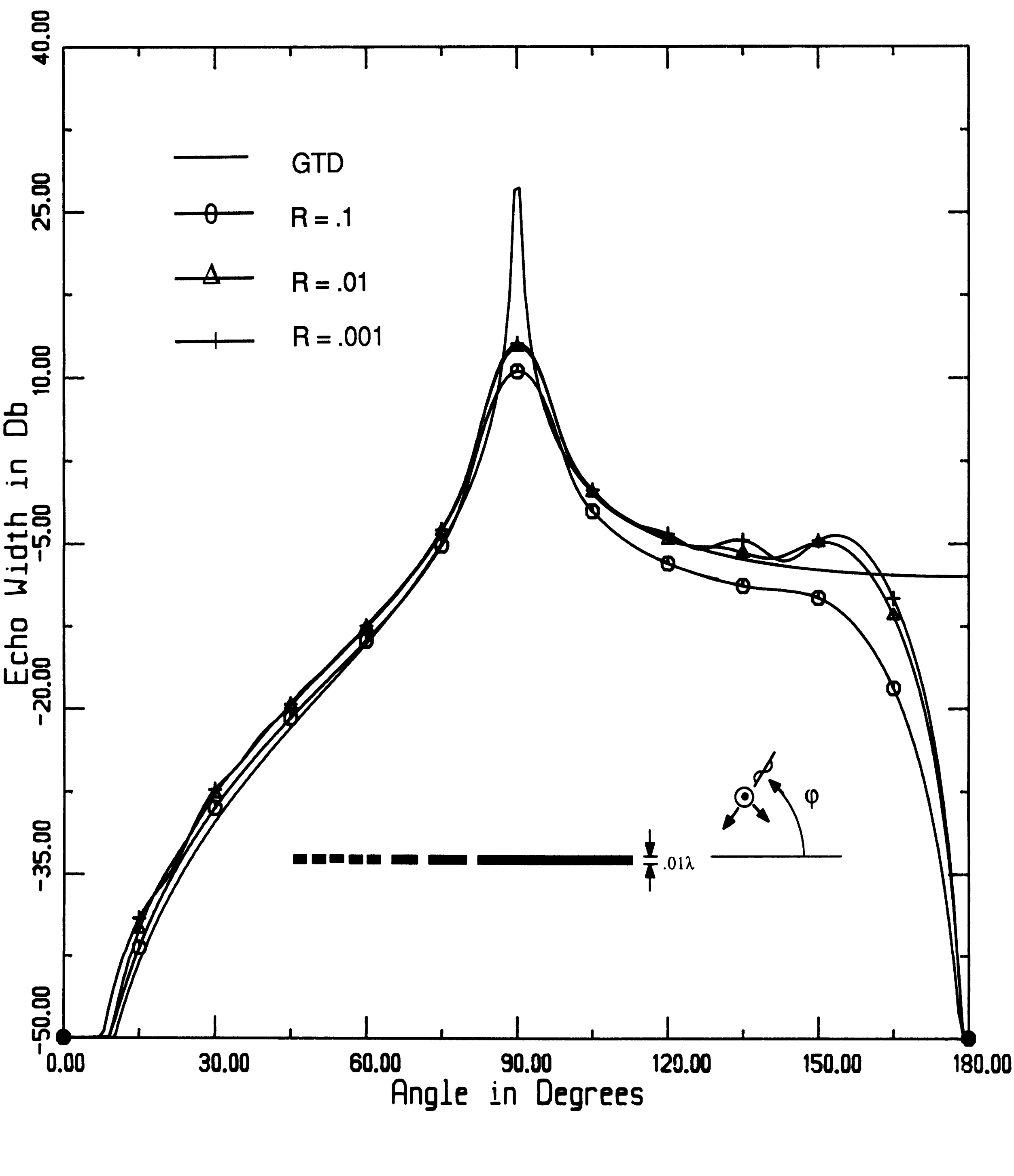

Figure 7.2: Hz backscatter echowidth of a perfectly conducting half-plane; comparison of results computed via the compact integral equations and a high frequency method for various values of resistivity, R.

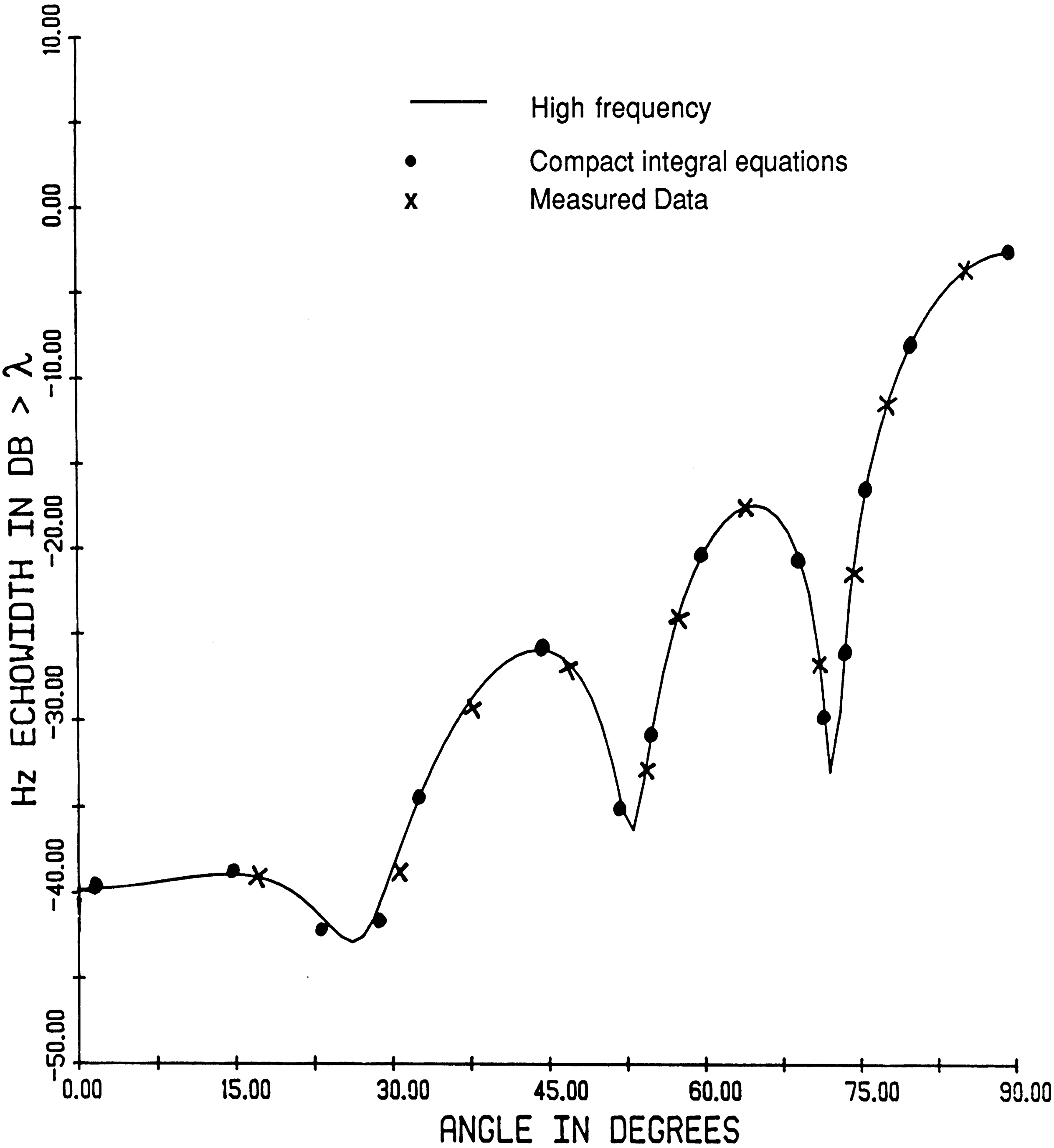

Figure 7.3: Hz backscatter echowidth by a 1.7 wavelengths wide and 0.01 wavelengths thick dielectric strip having $\varepsilon_r = 7.4 + i1.11$ and $\mu_r = 1.4 + i.672$; comparison of high frequency, moment method and measured results.

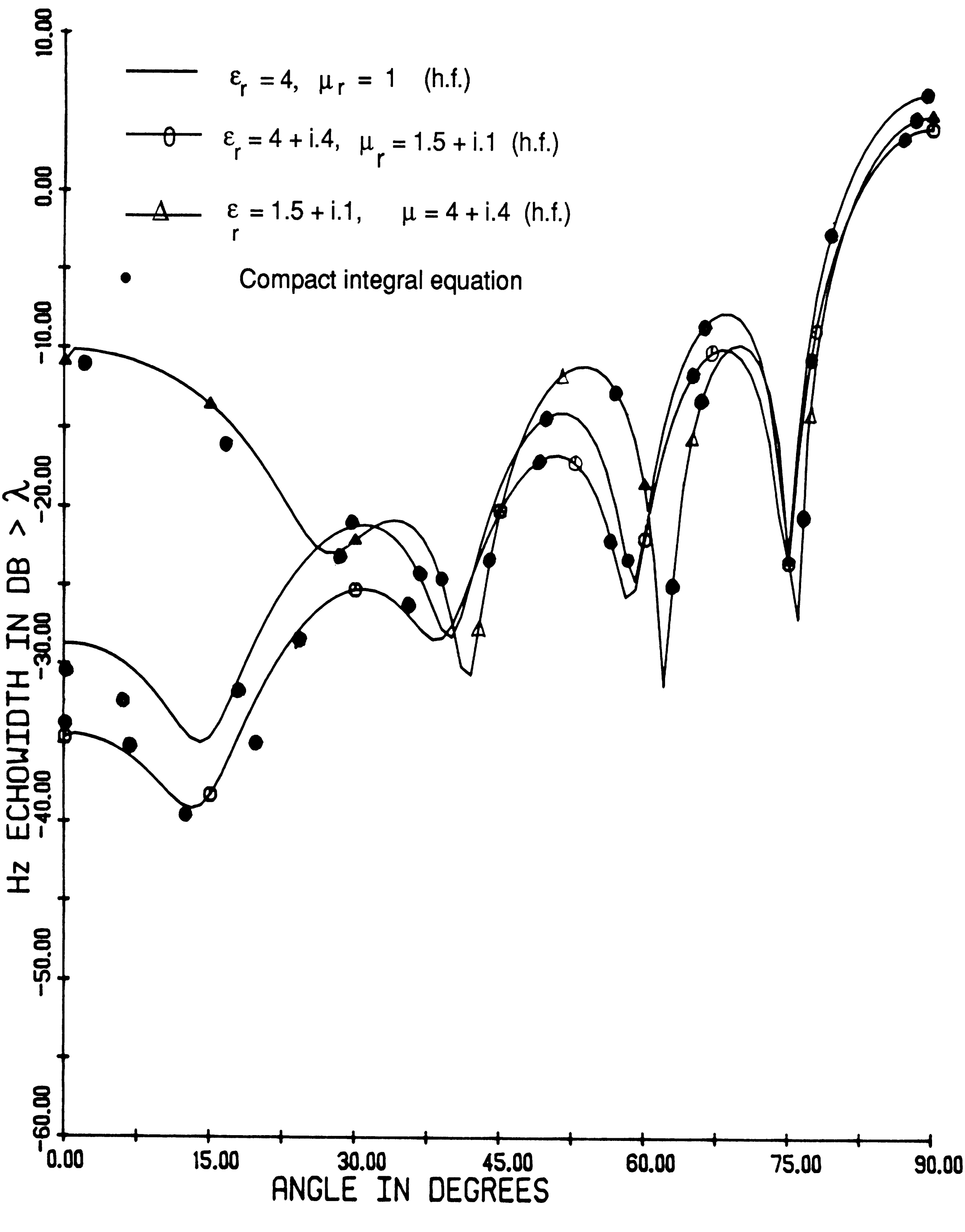

Figure 7.4: Hz backscatter echowidth of a 2.λ x .05λ material strip with ε_r and μ_r as indicated; comparison of results computed via the compact integral equations and a high frequency method.

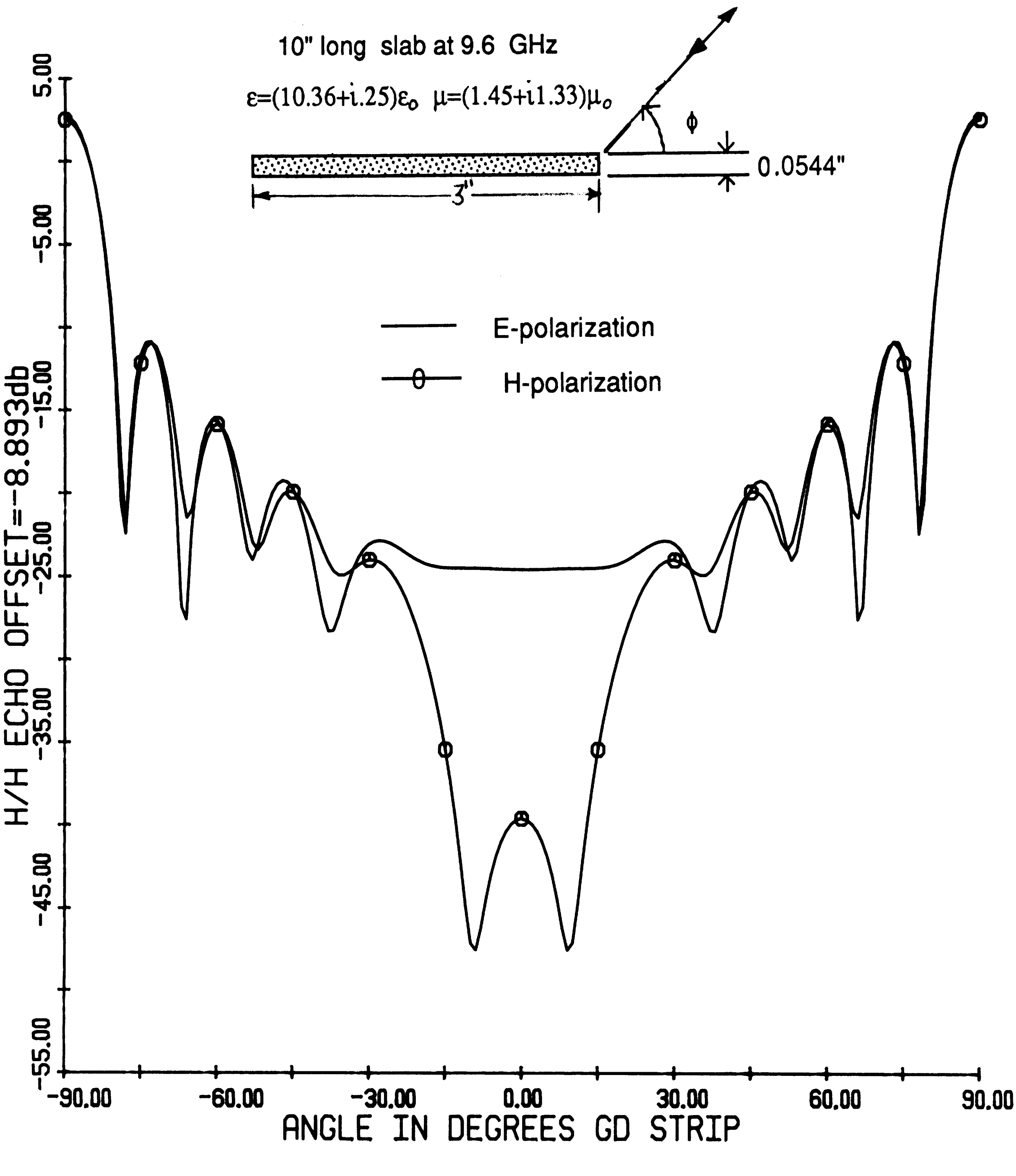

Figure 7.5: Backscatter echowidth of a 3"x.0544" material ($\varepsilon_r = 5. + i.5, \mu_r = 1.5 + i.5$) layer.

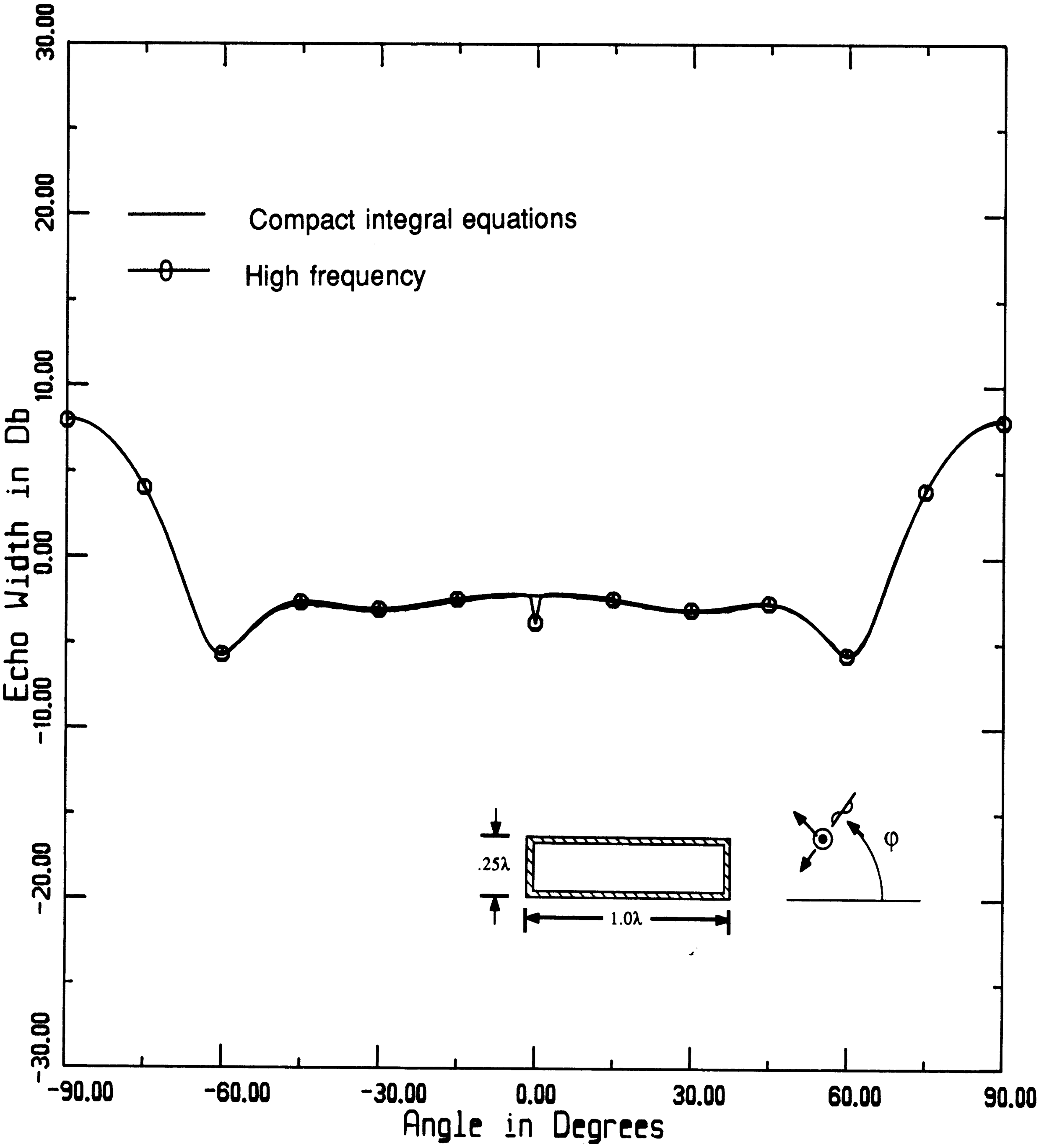

Figure 7.6: Ez backscatter echowidth of a 1.λ x 0.25λ perfectly conducting rectangular cylinder; comparison of high frequency method and compact integral equation solutions.

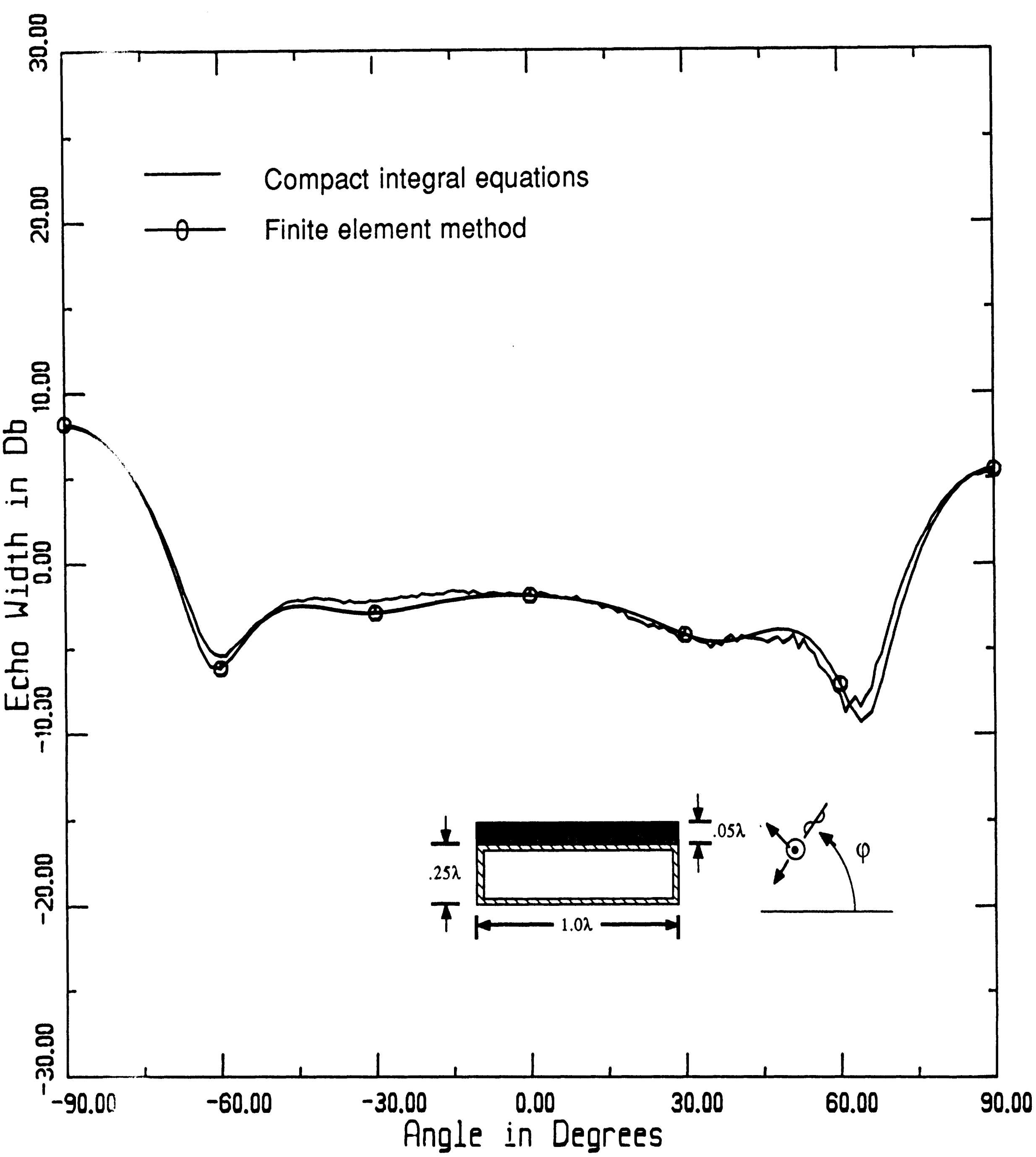

Figure 7.7: Ez backscatter echowidth of a .05λ thick material (ε_r = 5. + i.5, μ_r = 1.5 + i.5) layer upon a 1.λ x 0.25λ perfectly conducting rectangular cylinder; comparison of compact integral equation solution with finite element method.

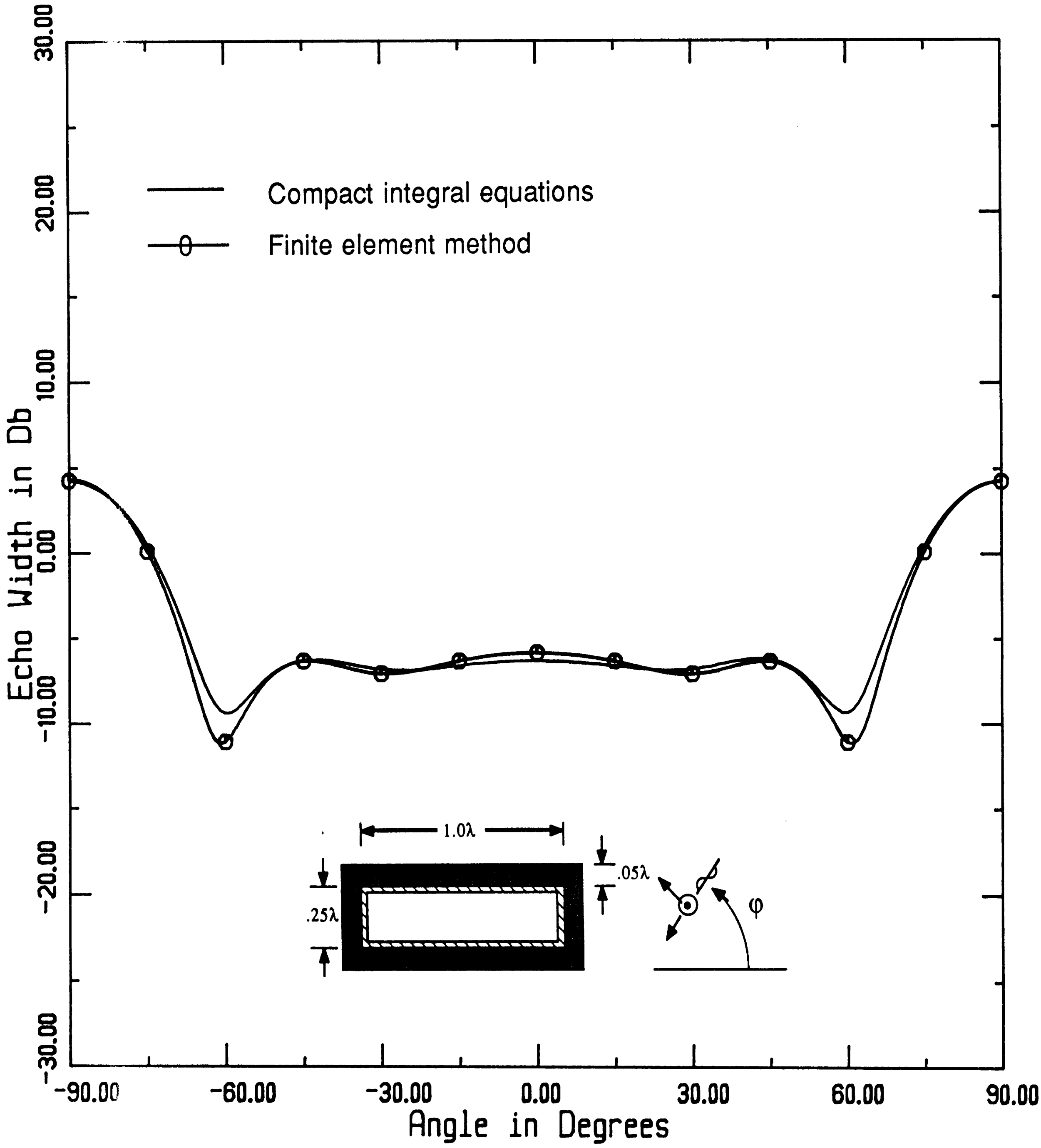

Figure 7.8: Ez backscatter echowidth of a 1.λ x 0.25λ perfectly conducting cylinder coated by a .05λ material ($\varepsilon_r = 5. + i.5, \mu_r = 1.5 + i0.5$) layer; comparison of solutions obtained via compact integral equations and the finite element method.

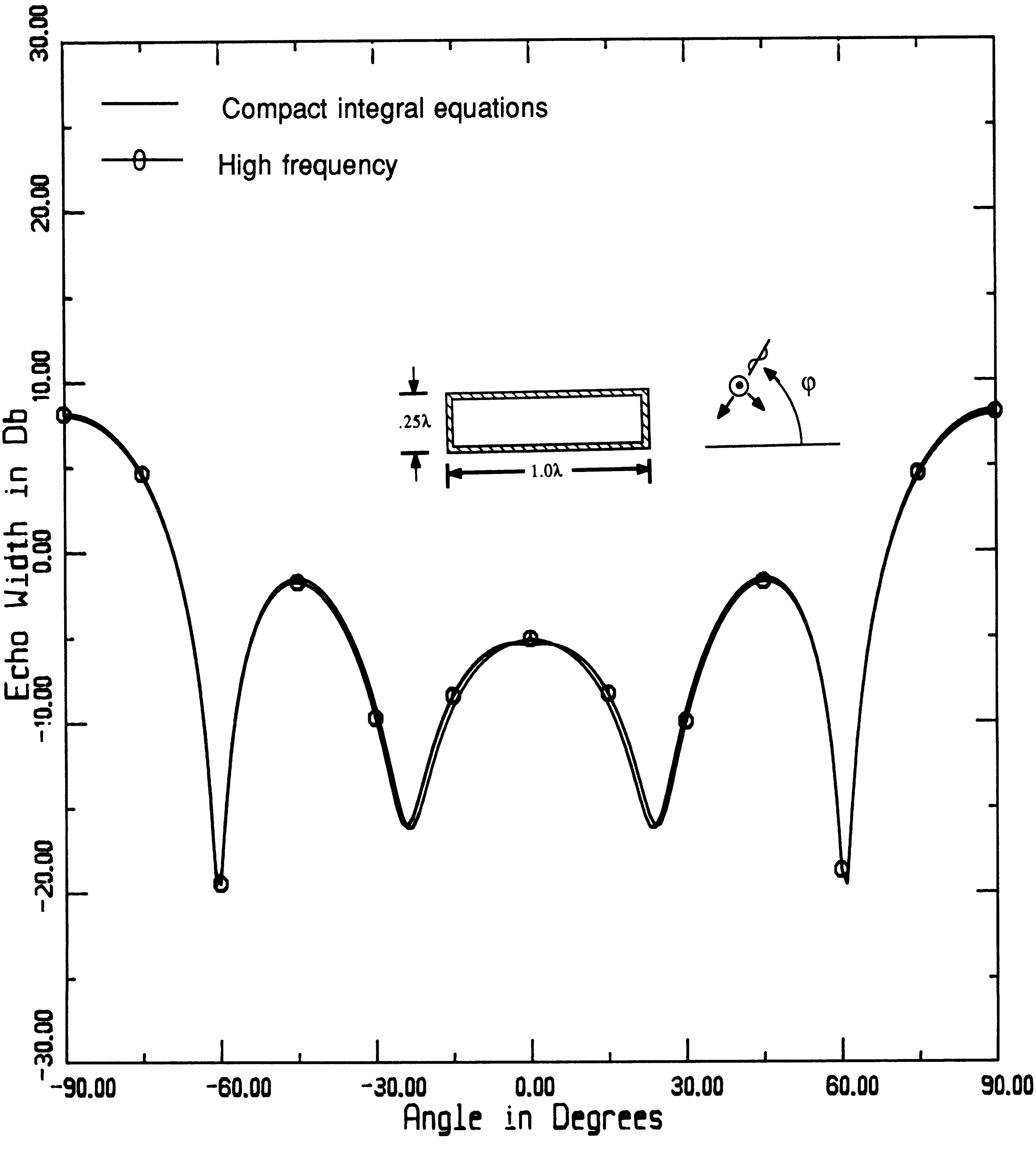

Figure 7.9: Hz backscatter echowidth of a $1.\lambda$ x 0.25λ perfectly conducting rectangular cylinder; comparison of high frequency method and compact integral equation solutions.

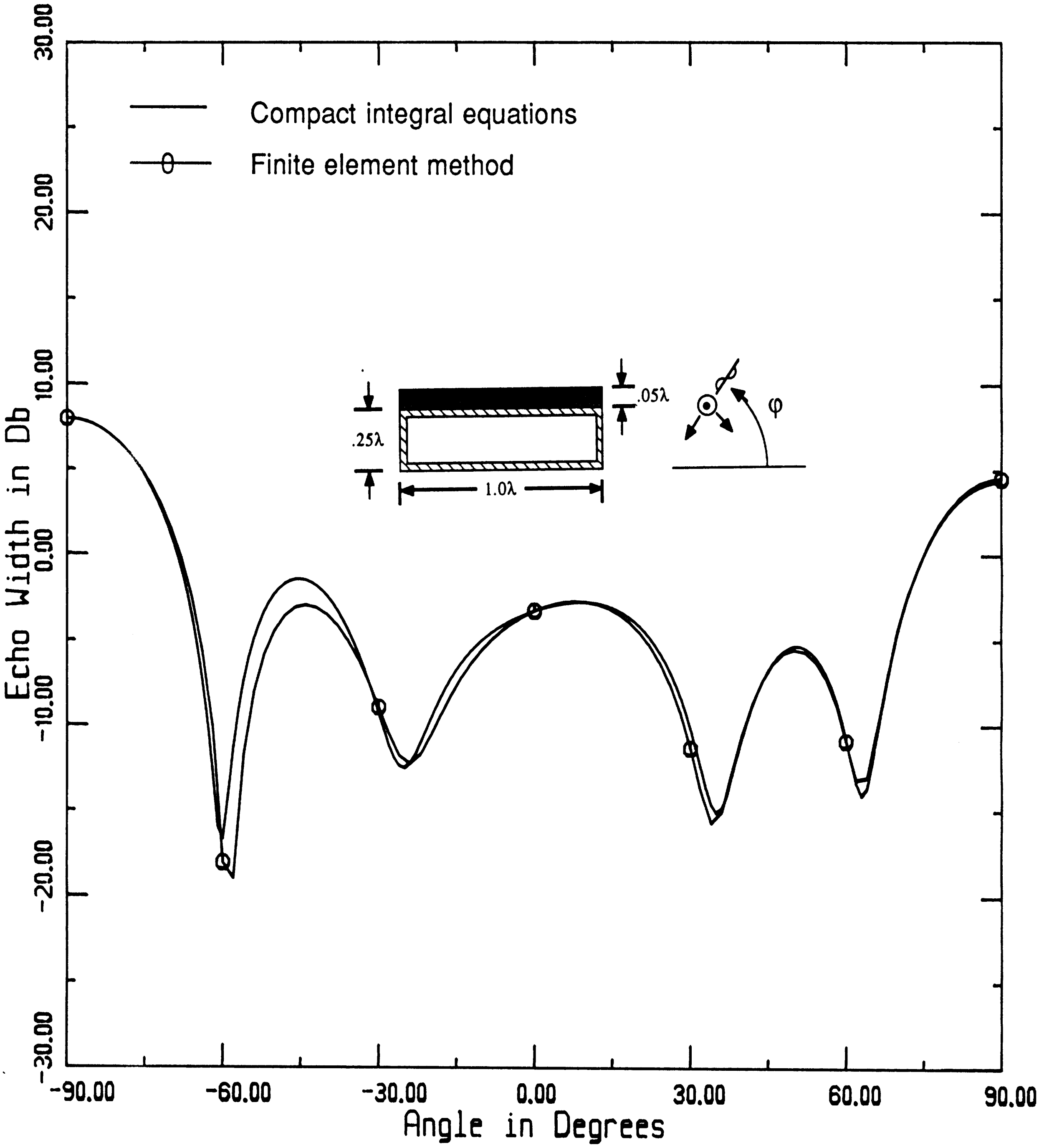

Figure 7.10: Hz backscatter echowidth of a .05λ thick material (ε_r = 5. + i.5, μ_r = 1.5 + i.5) layer upon a 1.λ x 0.25λ perfectly conducting rectangular cylinder; comparison of compact integral equation solution with finite element method.

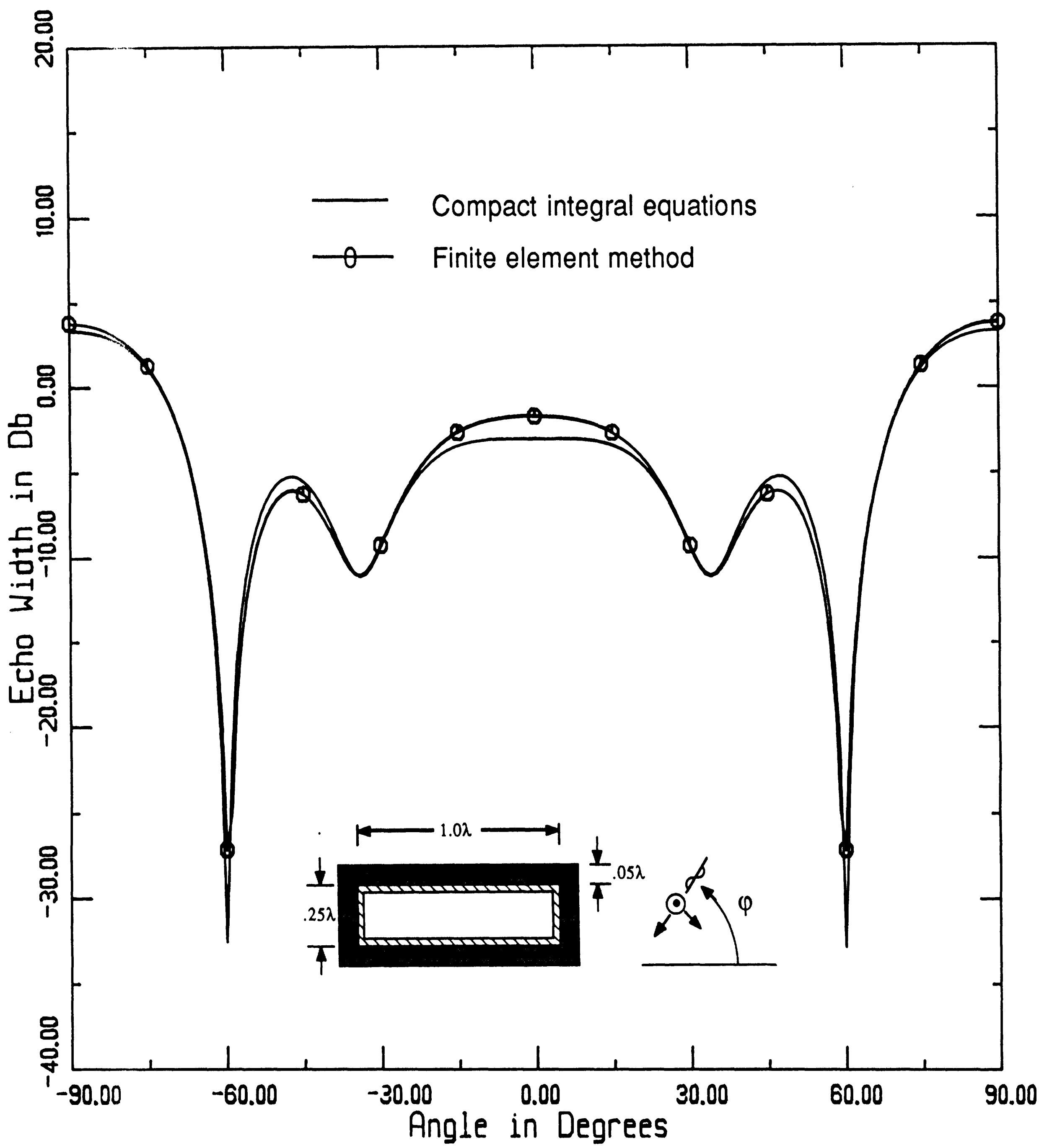

Figure 7.11: Hz backscatter echowidth of a $1.\lambda$ x 0.25λ perfectly conducting cylinder coated with a $.05\lambda$ thick material ($\varepsilon_r = 5. + i.5, \mu_r = 1.5 + i.5$) layer; comparison of solutions obtained via the compact integral equations and the finite element method.

$\varepsilon_r = 5 \quad \mu = 1$

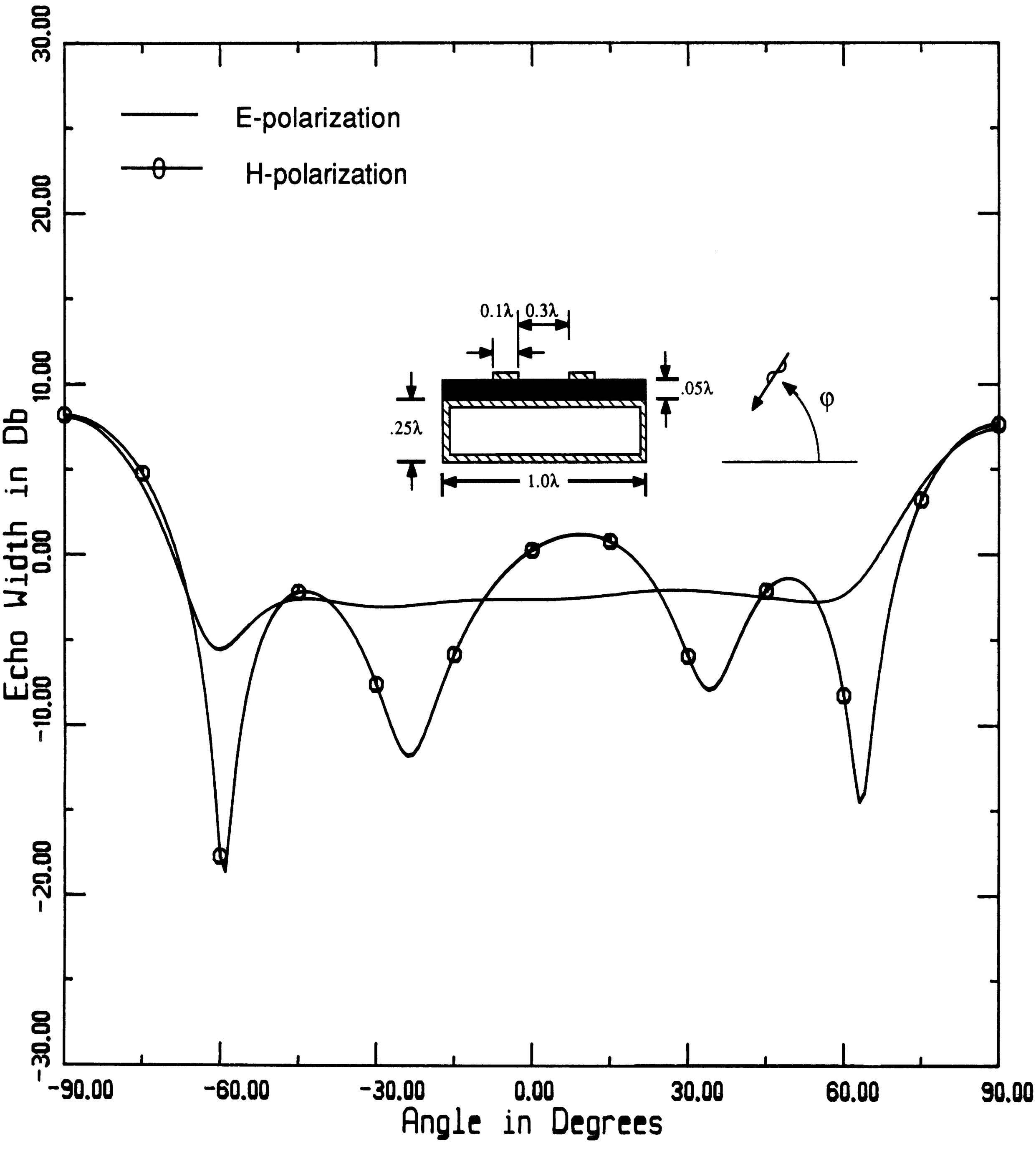

Figure 7.12: Backscatter echowidth of indicated microstrip geometry.

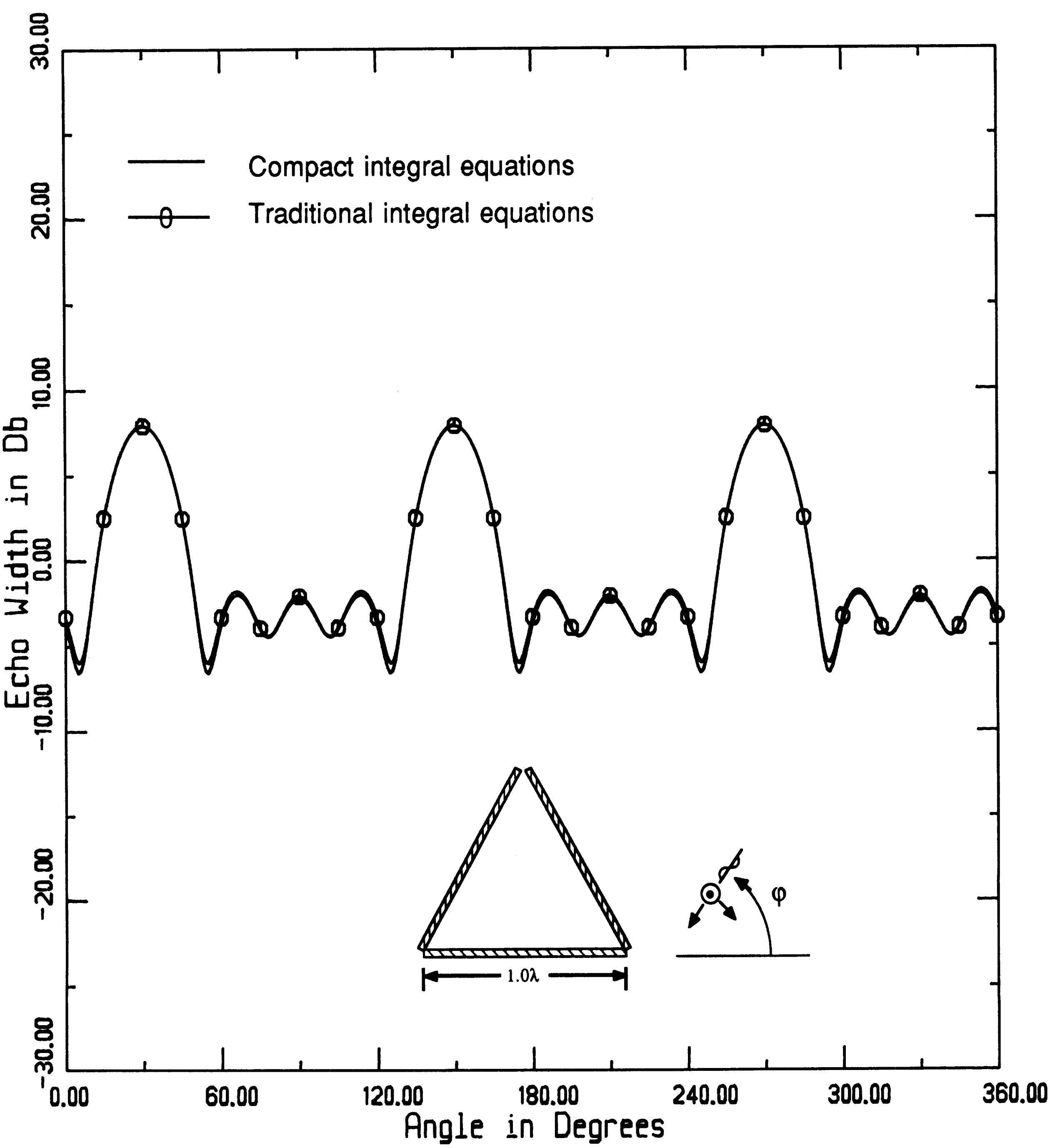

Figure 7.13: Hz backscatter echowidth of a 1.λ-per-side perfectly conducting equilateral triangular cylinder; comparison of solutions via the traditional and compact integral equations.

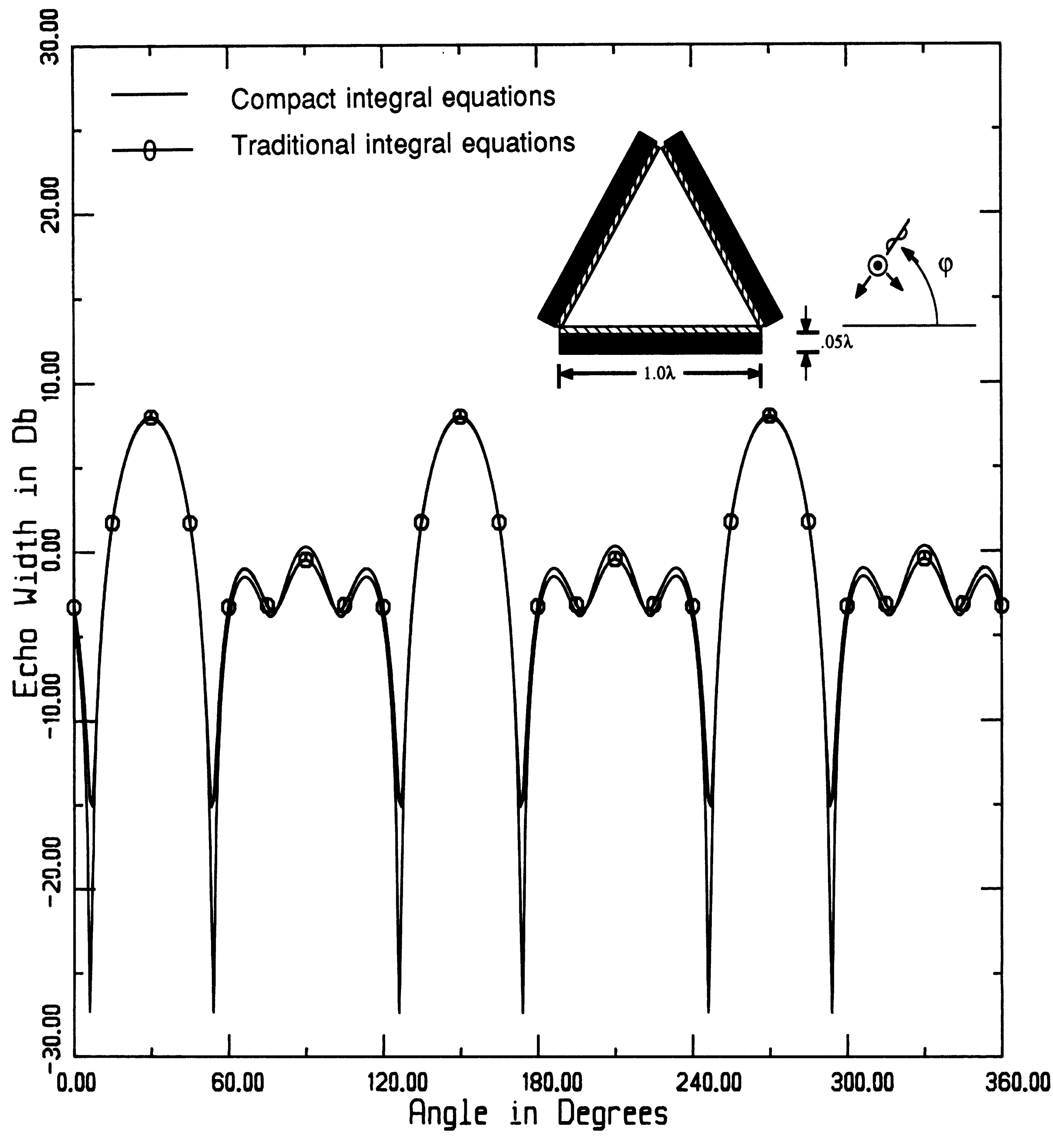

Figure 7.14: Hz backscatter echowidth of a 1.λ-per-side perfectly conducting equilateral triangular cylinder coated with a .05λ thick material ($\varepsilon_r = 4$) layer; comparison of solutions via the traditional and compact integral equations.

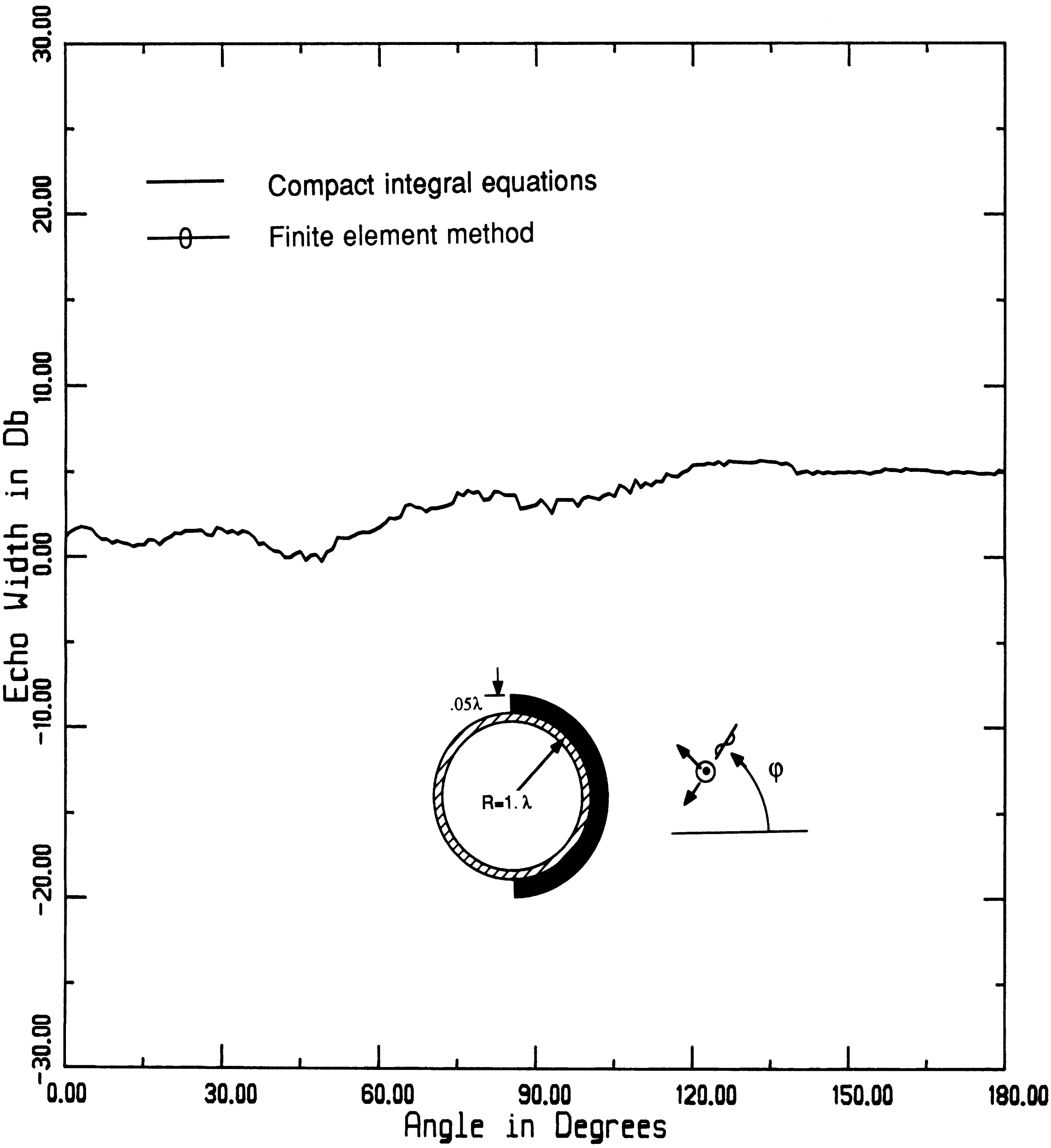

Figure 7.15: Ez backscatter echowidth of a partially coated circular perfectly conducting circular cylinder of radius $R = 1.0\lambda$. The coating is over half of the cylinder's surface, as shown, and its dielectric constants are $\varepsilon_r = 5. + i.5\&\ \mu_r = 1.5 + i.5$.

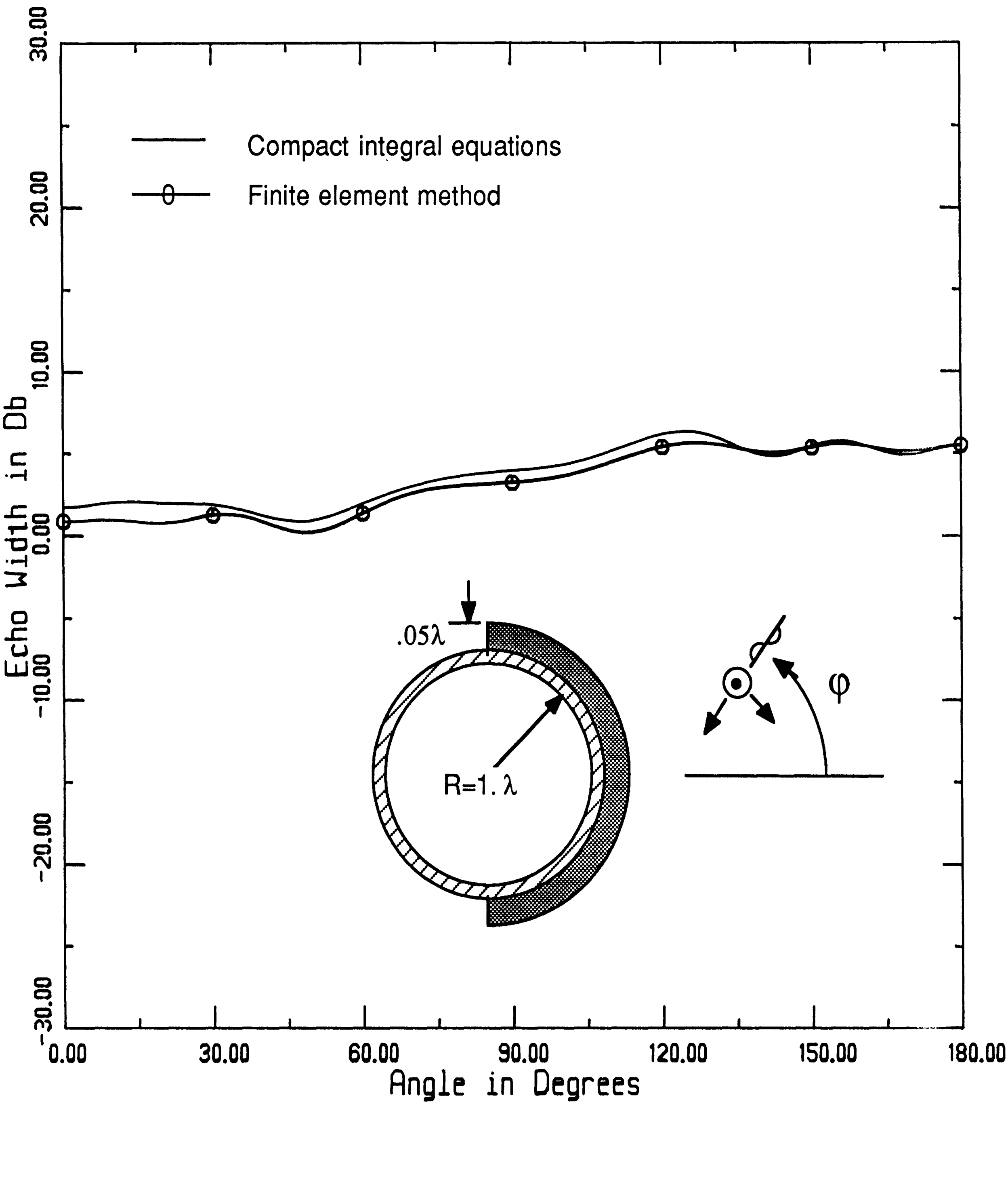

Figure 7.16: Hz backscatter echowidth of a partially coated circular perfectly conducting cylinder of radius $R = 1.0\lambda$. The coating is over half of the cylinder's surface, as shown, and its dielectric constants are $\varepsilon_r = 5. + i.5$& $\mu_r = 1.5 + i.5$.

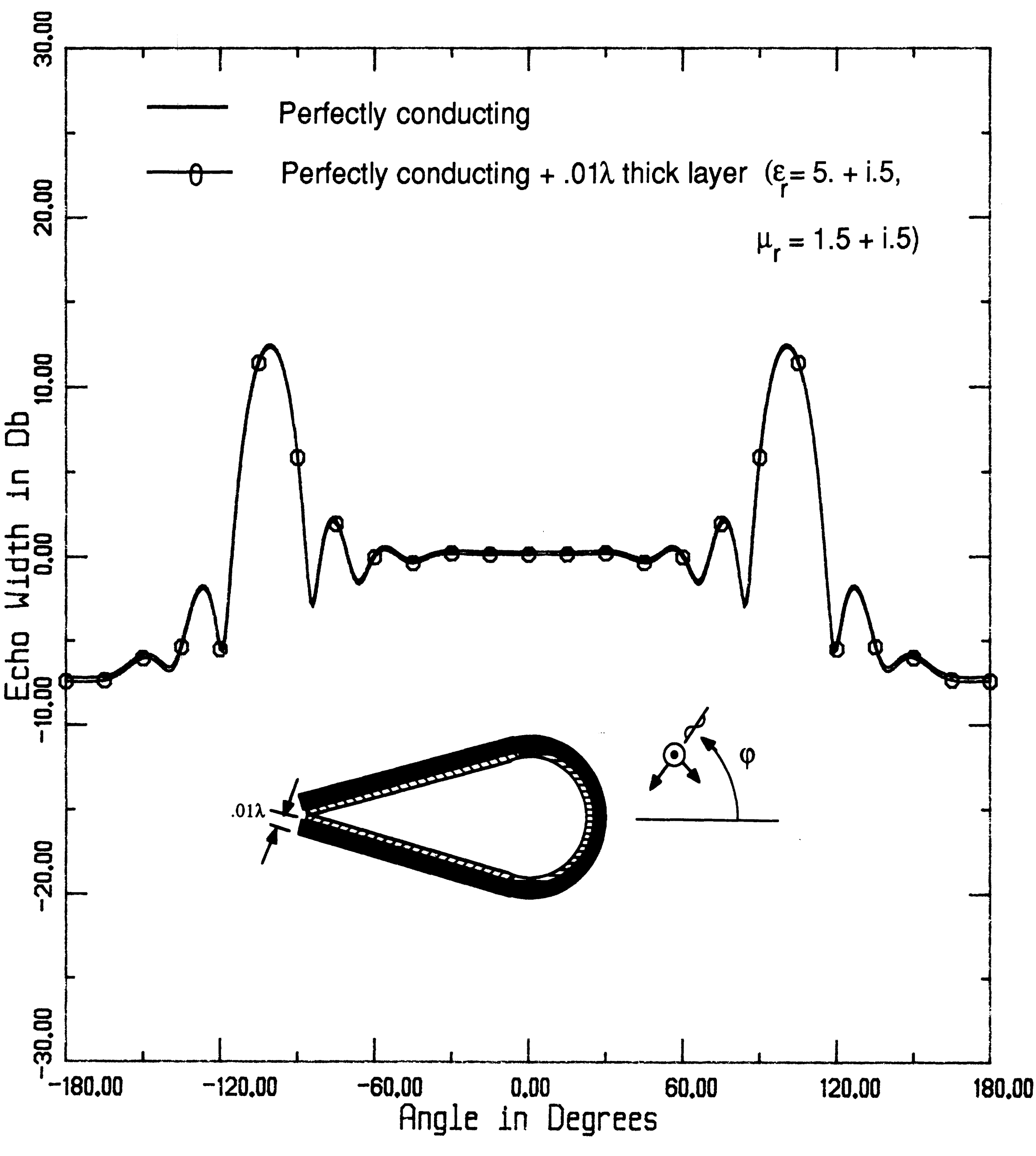

Figure 7.17: Ez backscatter echowidth of the shown coated and uncoated wedge-circular geometry.

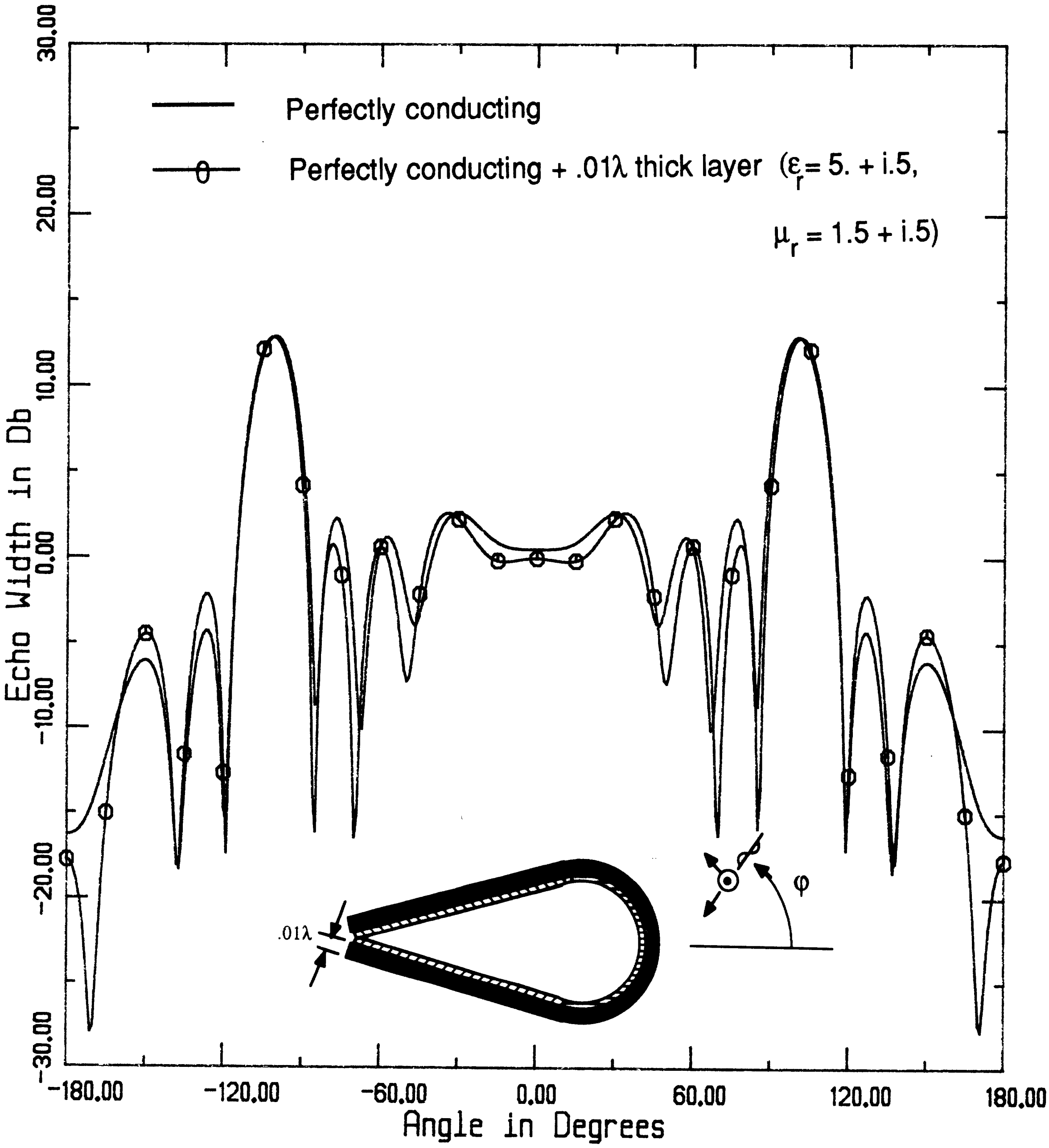

Figure 7.18: Hz backscatter echowidth of the shown coated and uncoated wedge-circular geometry.

Appendix A

The four singular expressions

$$\begin{aligned}
J_0(\alpha,\beta) &= \lim_{\epsilon\to 0}\int_\epsilon^\alpha\int_\epsilon^\beta H_0^{(1)}(k_0\sqrt{t^2+u^2})dtdu \\
J_1(\alpha,\beta) &= \lim_{\epsilon\to 0}\int_\epsilon^\alpha H_0^{(1)}(k_0\sqrt{t^2+\beta^2})dt \\
J_2(\alpha,\beta) &= \lim_{\epsilon\to 0}\frac{d}{d\beta}\int_\epsilon^\alpha H_0^{(1)}(k_0\sqrt{t^2+\beta^2})dt \\
J_3(\alpha,\beta) &= \lim_{\epsilon\to 0}\frac{d^2}{d\beta^2}\int_\epsilon^\alpha H_0^{(1)}(k_0\sqrt{t^2+\beta^2})dt
\end{aligned} \tag{A.1}$$

may be evaluated analytically when the argument of the Hankel function is small. Utilizing the small term expansion of H_0^1 to $O(4)$ in the above expressions for J_1,J_2 and J_3, we have

$$\begin{aligned}
J_1(\alpha,\beta) &= \lim_{\epsilon\to 0}\int_\epsilon^\alpha H_0^{(1)}(k_0\sqrt{t^2+\beta^2})dt = I_0+I_1+I_2+I_3+I_4 \quad ; \\
I_0 &= \frac{i\alpha}{\pi}\left[-i\pi+2\gamma-\ln 4\right], \\
I_1 &= \frac{i\alpha}{\pi}\left[2\ln k_0\sqrt{\alpha^2+\beta^2}-2+\frac{2\beta}{\alpha}\arctan\frac{\alpha}{\beta}\right], \\
I_2 &= \frac{i\alpha}{\pi}\left[\frac{i\pi}{2}+1-\gamma+\ln 2\right]\left[\frac{(k_0\alpha)^2}{6}+\frac{(k_0\beta)^2}{2}\right], \\
I_3 &= \frac{i\alpha}{\pi}\left[\frac{(k_0\beta)^2}{3}+\frac{(k_0\alpha)^2}{18}-\left(\frac{(k_0\beta)^2}{2}+\frac{(k_0\alpha)^2}{6}\right)\ln k_0\sqrt{\alpha^2+\beta^2}\right], \\
&\quad -\frac{i\alpha}{\pi}\left[\frac{(k_0\beta)^2}{3}\left(\frac{\beta}{\alpha}\right)\arctan\frac{\alpha}{\beta}\right],
\end{aligned}$$

$$
\begin{aligned}
I_4 &= \frac{i\alpha}{32\pi}\left[\frac{(k_0\alpha)^4}{5}+\frac{2(k_0\alpha)^2(k_0\beta)^2}{3}+(k_0\beta)^4\right]\left[\frac{-i\pi}{2}-\frac{17}{10}+\gamma-\ln 2+\ln k_0\sqrt{\alpha^2+\beta^2}\right], \\
&+\frac{i\alpha}{32\pi}\left[\frac{16}{15}\frac{\beta}{\alpha}(k_0\beta)^4\arctan\frac{\alpha}{\beta}-\frac{1}{15}\left(\frac{(k_0\beta)^2(k_0\alpha)^2}{3}+5(k_0\beta)^4\right)\right]. \qquad \text{(A.2)}
\end{aligned}
$$

$$
\begin{aligned}
J_2(\alpha,\beta) &= \lim_{\epsilon\to 0}\frac{d}{d\beta}\int_\epsilon^\alpha H_0^{(1)}(k_0\sqrt{t^2+\beta^2})dt = \frac{dI_1}{d\beta}+\frac{dI_2}{d\beta}+\frac{dI_3}{d\beta} \quad ; \\
\frac{dI_1}{d\beta} &= \frac{ik_0\alpha}{\pi}\left[\frac{2}{k_0\alpha}\arctan\frac{\alpha}{\beta}\right], \\
\frac{dI_2}{d\beta} &= \frac{ik_0\alpha}{\pi}\left[\frac{i\pi}{2}+1-\gamma+\ln 2\right]k_0\beta, \\
\frac{dI_3}{d\beta} &= \frac{ik_0\alpha}{\pi}\left[\frac{k_0\beta}{2}-k_0\beta\ln k_0\sqrt{\alpha^2+\beta^2}-k_0\beta\left(\frac{\beta}{\alpha}\right)\arctan\frac{\alpha}{\beta}\right], \\
\frac{dI_4}{d\beta} &= \frac{ik_0\alpha}{8\pi}\left[\frac{(k_0\alpha)^2}{3}+(k_0\beta)^2\right]\left[\frac{-i\pi}{2}-\frac{23}{12}+\gamma-\ln 2+\ln k_0\sqrt{\alpha^2+\beta^2}\right]k_0\beta, \\
&+\frac{ik_0\alpha}{8\pi}\left[\frac{2}{3}\frac{\beta}{\alpha}(k_0\beta)^2\arctan\frac{\alpha}{\beta}+\frac{(k_0\alpha)^2}{9}\right]k_0\beta. \qquad \text{(A.3)}
\end{aligned}
$$

$$
\begin{aligned}
J_3(\alpha,\beta) &= \lim_{\epsilon\to 0}\frac{d^2}{d\beta^2}\int_\epsilon^\alpha H_0^{(1)}(k_0\sqrt{t^2+\beta^2})dt = \frac{d^2I_1}{d\beta^2}+\frac{d^2I_2}{d\beta^2}+\frac{d^2I_3}{d\beta^2} \quad ; \\
\frac{d^2I_1}{d\beta^2} &= \frac{ik_0^2\alpha}{\pi}\frac{-2}{\beta^2+\alpha^2}, \\
\frac{d^2I_2}{d\beta^2} &= \frac{ik_0^2\alpha}{\pi}\left[\frac{i\pi}{2}+1-\gamma+\ln 2\right], \\
\frac{d^2I_3}{d\beta^2} &= \frac{ik_0^2\alpha}{\pi}\left[\frac{1}{2}-\ln k_0\sqrt{\alpha^2+\beta^2}-\left(\frac{2\beta}{\alpha}\right)\arctan\frac{\alpha}{\beta}\right], \\
\frac{d^2I_4}{d\beta^2} &= \frac{ik_0^2\alpha}{8\pi}\left[\frac{(k_0\alpha)^2}{3}+3(k_0\beta)^2\right]\left[\frac{-i\pi}{2}-\frac{65}{36}+\gamma-\ln 2+\ln k_0\sqrt{\alpha^2+\beta^2}\right], \\
&+\frac{ik_0^2\alpha}{8\pi}\left[\frac{8}{3}\frac{\beta}{\alpha}(k_0\beta)^2\arctan\frac{\alpha}{\beta}+\frac{2(k_0\alpha)^2}{27}\right]. \qquad \text{(A.4)}
\end{aligned}
$$

Combining these results, we obtain the following analytic expressions for $J_1(\alpha, \beta)$, $J_2(\alpha, \beta)$ and $J_3(\alpha, \beta)$:

$$
\begin{aligned}
J_1(\alpha,\beta) &= \lim_{\epsilon\to 0}\int_\epsilon^\alpha H_0^{(1)}(k_0\sqrt{t^2+\beta^2})dt \\
&= \frac{i\alpha}{\pi}\left[\gamma-\frac{i\pi}{2}-\frac{5}{3}+\ln\frac{k_0}{2}\sqrt{\alpha^2+\beta^2}\right]\left[2-\frac{(k_0\alpha)^2}{6}-\frac{(k_0\beta)^2}{2}+\frac{(k_0\alpha)^4}{160}+\frac{(k_0\alpha)^2(k_0\beta)^2}{48}+\frac{(k_0\beta}{32}\right. \\
&\quad +\frac{i\alpha}{3\pi}\left[4-\frac{(k_0\alpha)^2}{6}\right]-\frac{i\alpha}{480\pi}\left[\frac{(k_0\alpha)^4}{10}+\frac{2(k_0\alpha)^2(k_0\beta)^2}{3}+\frac{11(k_0\beta)^4}{2}\right] \\
&\quad +\frac{i\beta}{\pi}\left[2-\frac{(k_0\beta)^2}{3}+\frac{(k_0\beta)^4}{30}\right]\arctan\frac{\alpha}{\beta}, \qquad \text{(A}
\end{aligned}
$$

$$
\begin{aligned}
J_2(\alpha,\beta) &= \lim_{\epsilon\to 0}\frac{d}{d\beta}\int_\epsilon^\alpha H_0^{(1)}(k_0\sqrt{t^2+\beta^2})dt \\
&= \frac{ik_0^2\alpha\beta}{8\pi}\left[\gamma-\frac{i\pi}{2}-\ln 2-\frac{3}{2}+\ln k_0\sqrt{\alpha^2+\beta^2}\right]\left[\frac{(k_0\alpha)^2}{3}+(k_0\beta)^2-8\right] \\
&\quad -\frac{ik_0^2\alpha\beta}{96\pi}\left[\frac{(k_0\alpha)^2}{3}+5(k_0\beta)^2\right]+\frac{i}{\pi}\left[2-(k_0\beta)^2+\frac{(k_0\beta)^4}{12}\right]\arctan\frac{\alpha}{\beta}, \qquad \text{(A}
\end{aligned}
$$

$$
\begin{aligned}
J_3(\alpha,\beta) &= \lim_{\epsilon\to 0}\frac{d^2}{d\beta^2}\int_\epsilon^\alpha H_0^{(1)}(k_0\sqrt{t^2+\beta^2})dt \\
&= \frac{ik_0^2\alpha}{\pi}\left[\frac{i\pi}{2}+\frac{3}{2}-\gamma+\ln 2-\ln k_0\sqrt{\alpha^2+\beta^2}\right]\left[1-\frac{(k_0\alpha)^2}{24}-\frac{3(k_0\beta)^2}{8}\right] \\
&\quad -\frac{ik_0^2\alpha}{96\pi}\left[\frac{(k_0\alpha)^2}{3}+11(k_0\beta)^2\right]-\frac{ik_0^2\beta}{\pi}\left[2-\frac{(k_0\beta)^2}{3}\right]\arctan\frac{\alpha}{\beta} \\
&\quad -\frac{ik_0^2\alpha}{\pi}\left[\frac{2}{(k_0\alpha)^2+(k_0\beta)^2}\right] \qquad \text{(A}
\end{aligned}
$$

Finally, the expression for J_0 may be obtained by substituting the small term expansion of H_0^1 to $O(2)$ and utilizing cylindrical coordinates to perform the integration. This gives

$$
\begin{aligned}
J_0(\alpha,\beta) &= \lim_{\epsilon\to 0}\int_\epsilon^\alpha\int_\epsilon^\beta H_0^{(1)}(k_0\sqrt{t^2+u^2})dtdu \\
&= \frac{i\alpha\beta}{\pi}\left[\gamma-\frac{i\pi}{2}-\ln 2-\frac{3}{2}+\ln k_0\sqrt{\alpha^2+\beta^2}\right]\left[2-\frac{(k_0\alpha)^2+(k_0\beta)^2}{6}\right]
\end{aligned}
$$

$$+\frac{i\beta^2}{\pi}\left[1-\frac{(k_0\beta)^2}{12}\right]\arctan\frac{\alpha}{\beta}+\frac{i\alpha^2}{\pi}\left[1-\frac{(k_0\alpha)^2}{12}\right]\arctan\frac{\beta}{\alpha}$$
$$+\frac{i\alpha\beta}{\pi}\left[\frac{(k_0\alpha)^2+(k_0\beta)^2}{18}\right]. \tag{A.8}$$

Hence, although higher order terms for H_0^1 were retained in the evaluation of J_3, J_0 displays the same degree of approximation as J_3, with terms to $O(2)$ in α and β.

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