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An Algebraic Approach to Simple Hyperbolic Splines on the Real Line

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INTRODUCTION

In this paper we discuss the question of existence and uniqueness of generalized splines on R^1 for the linear differential operator $L = D(D - \alpha)$. We view the problem from an algebraic standpoint, and, utilizing the theory of doubly infinite Toeplitz matrices, prove an existence and uniqueness theorem of broad application. The behavior of the splines is also considered for the limiting cases $\alpha \to 0$ and $\alpha \to \infty$.

Hyperbolic splines of this type have been considered by Schweikert [7] as a means of obtaining a more acceptable "faired" curve to a given set of interpolation data in cases where the cubic spline introduces what he calls extraneous inflection points—points at which the cubic spline interpolant changes the direction of concavity but at which the draftsman's faired curve would contain no such reversal. He shows that the assumptions leading to cubic spline interpolation as the mathematical model of what a draftsman actually does in fairing a curve do not accurately account for the process and that the spline in tension (hyperbolic spline) better represents the draftsman's techniques.

The theory of cubic and higher-order polynomial spline functions on the real line has been studied by Schoenberg [5], [6], Ahlberg, Nilson, and Walsh [3], and Ahlberg and Nilson [2] among others. For splines on finite intervals the theory is extensive and there are two basic approaches—the algebraic and what Ahlberg, Nilson, and Walsh [4] call the intrinsic. The intrinsic approach is used to obtain existence, uniqueness, and convergence results for generalized splines on finite intervals. It entails establishing certain integral relations and inequalities satisfied by generalized spline

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functions, such as the minimum norm property and the best approximation property [4]. It may be possible to extend this approach to apply to generalized splines on the real line; however, we utilize, instead, the extension of the algebraic approach to infinite intervals.

1. Existence and Uniqueness

Let us consider the linear differential operator [7]

$$L = D(D - \alpha), \qquad \alpha \neq 0$$

and without loss of generality, as will be seen momentarily, let us take $\alpha > 0$. Let $\Delta = \{x_j = jl : j = 0, \pm 1, \pm 2,...\}$; i.e., on R^1, Δ is the uniform mesh of size $l > 0$. We call $S_\Delta(x)$ a *simple hyperbolic spline on* R^1 for the uniform mesh Δ if (i) on each mesh interval $[x_j, x_{j+1}]$, $S_\Delta(x)$ satisfies $L^*LS_\Delta(x) = 0$, and (ii) $S_\Delta(x)$ is $C^2(R^1)$. Furthermore, $S_\Delta(x)$ is a simple hyperbolic spline of interpolation if, in addition to (i) and (ii), (iii) $S_\Delta(x_j) = y_j$, $j = 0, \pm 1, \pm 2,...$ for prescribed interpolation data $\{y_j\}$.

On $[x_j, x_{j+1}]$, $S_\Delta(x)$ is a linear combination of

$$1,\ x,\ \sinh \alpha x,\ \cosh \alpha x$$

since $L^*L = D^2(D^2 - \alpha^2)$. (Note that $L = D(D + \alpha)$, $\alpha > 0$, results in the same differential equation being satisfied on any mesh interval, for again $L^*L = D^2(D^2 - \alpha^2)$). Writing

$$S_\Delta(x) = c_1^j + c_2^j x + c_3^j \sinh \alpha x + c_4^j \cosh \alpha x$$

on $[x_j, x_{j+1}]$, we wish to find c_i^j, $i = 1,..., 4$, $j = 0, \pm 1, \pm 2,...$ so that (ii) and (iii) are satisfied. We proceed in the same manner as for splines on finite intervals. Letting $M_j = S_\Delta''(x_j)$ be the unknown second derivative of our spline of interpolation at the mesh points, we require for all j

$$\begin{aligned} c_1^j + c_2^j x_j + c_3^j \sinh \alpha x_j + c_4^j \cosh \alpha x_j &= y_j, \\ c_1^j + c_2^j x_{j+1} + c_3^j \sinh \alpha x_{j+1} + c_4^j \cosh \alpha x_{j+1} &= y_{j+1}, \\ \alpha^2 c_3^j \sinh \alpha x_j + \alpha^2 c_4^j \cosh \alpha x_j &= M_j, \\ \alpha^2 c_3^j \sinh \alpha x_{j+1} + \alpha^2 c_4^j \cosh \alpha x_{j+1} &= M_{j+1}, \end{aligned} \tag{1.1}$$

and so we get the following representation for $S_\Delta(x)$ on $[x_j, x_{j+1}]$ in terms of the known interpolation data y_j, y_{j+1} and the unknown spline second derivatives M_j, M_{j+1}:

$$\begin{aligned} S_\Delta(x) = y_j\left(\frac{x_{j+1}-x}{l}\right) + y_{j+1}\left(\frac{x-x_j}{l}\right) - \frac{M_j}{\alpha^2\sinh\alpha l} \\ \cdot\left[\left(\frac{x_{j+1}-x}{l}\right)\sinh\alpha l - \sinh\alpha(x_{j+1}-x)\right] \\ - \frac{M_{j+1}}{\alpha^2\sinh\alpha l}\left[\left(\frac{x-x_j}{l}\right)\sinh\alpha l - \sinh\alpha(x-x_j)\right]. \end{aligned} \tag{1.2}$$

By the use of this representation it is clear that $S_\Delta(x_j) = y_j$ and S_Δ and S_Δ'' are continuous at x_j for all j. The requirement that S_Δ' be continuous at x_j leads to a condition on M_j; namely,

$$\begin{aligned} M_{j-1} + 2\left(\frac{\alpha l\cosh\alpha l - \sinh\alpha l}{\sinh\alpha l - \alpha l}\right) M_j + M_{j+1} \\ = \frac{(\alpha l)^2\sinh\alpha l}{\sinh\alpha l - \alpha l}\left(\frac{y_{j+1} - 2y_j + y_{j-1}}{l^2}\right) \end{aligned} \tag{1.3}$$

for all j. In matrix form this is

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdots & 0 & 1 & 2\eta & 1 & 0 & \cdots & & & \\ & \cdots & 0 & 1 & 2\eta & 1 & 0 & \cdots & & \\ & & \cdots & 0 & 1 & 2\eta & 1 & 0 & \cdots & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \vdots \\ M_{-1} \\ M_0 \\ M_1 \\ \vdots \end{bmatrix} = \frac{(\alpha l)^2\sinh\alpha l}{\sinh\alpha l - \alpha l} \begin{bmatrix} \vdots \\ d_{-1} \\ d_0 \\ d_1 \\ \vdots \end{bmatrix} \tag{1.4}$$

where

$$\eta = \eta(\alpha l) = \frac{\alpha l\cosh\alpha l - \sinh\alpha l}{\sinh\alpha l - \alpha l} \quad \text{and} \quad d_j = \frac{y_{j+1} - 2y_j + y_{j-1}}{l^2}.$$

The matrix in (1.4) is a doubly infinite Toeplitz matrix. To solve (1.4) we must ensure that the infinite sums on the left-hand side are convergent (in our case this is obvious since all terms but three are zero); we must find a doubly infinite matrix, the inverse of the one in (1.4), with which to multiply both sides in order to obtain on the left a doubly infinite identity matrix multiplying the vector of M_j's; and then we must ensure that the inverse matrix, when applied to the right-hand side, yields convergent infinite sums. To this end we quote the following basic result of the theory of Toeplitz matrices.

Consider the matrix

$$T_\phi = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdots & c_0 & c_{-1} & c_{-2} & & \cdots \\ \cdots & c_1 & c_0 & c_{-1} & & \cdots \\ \cdots & c_2 & c_1 & c_0 & & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

and define $\phi(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$. If the $\{c_n\}$ are such that ϕ is a bounded function, then we have

THEOREM 1.1. *T_ϕ is invertible if $1/\phi$ is essentially bounded. If T_ϕ^{-1} exists, it satisfies $T_\phi^{-1} = T_{1/\phi}$ where $T_{1/\phi}$ is the Toeplitz matrix defined by the sequence of Fourier coefficients of* $1/\phi$ [8].

In our case $\phi(\theta) = 2\eta + e^{i\theta} + e^{-i\theta} = 2\eta + 2\cos\theta$, and so $1/\phi$ is essentially bounded since elementary computations reveal $\eta > 2$ for $\alpha > 0$. The Fourier coefficients of $1/\phi$ are given by

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-in\theta}\,d\theta}{2(\cos\theta + \eta)} \tag{1.6}$$

and the change of variables $\theta \to -\theta$ shows that $a_{-n} = a_n$. Since $1/\phi$ is real, the Fourier coefficients a_n are real and, therefore, we have

$$a_{-n} = a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\cos n\theta\,d\theta}{2(\cos\theta + \eta)} = \frac{1}{2(\eta^2 - 1)^{1/2}}((\eta^2 - 1)^{1/2} - \eta)^n. \tag{1.7}$$

Defining $\mu = (\eta^2 - 1)^{1/2} - \eta$, we have

$$\begin{bmatrix} \vdots \\ M_{-1} \\ M_0 \\ M_1 \\ \vdots \end{bmatrix} = \frac{(\alpha l)^2 \sinh \alpha l}{2(\eta^2 - 1)^{1/2}(\sinh \alpha l - \alpha l)} \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdots & \mu & 1 & \mu & \mu^2 & \mu^3 & \cdots \\ \cdots & \mu^2 & \mu & 1 & \mu & \mu^2 & \cdots \\ \cdots & \mu^3 & \mu^2 & \mu & 1 & \mu & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \vdots \\ d_{-1} \\ d_0 \\ d_1 \\ \vdots \end{bmatrix}. \tag{1.8}$$

The calculation of the M_j's from (1.8) uniquely determines the splines of interpolation; however, unlike the finite interval case, the invertibility of the Toeplitz matrix in (1.4) does not suffice to guarantee the existence of the M_j's. This is due to the fact that the inverse does not have a finite number of nonzero entries in any row. Denoting by Δy_j the second difference $y_{j+1} - 2y_j + y_{j-1}$, we have in general, from (1.8),

$$\begin{aligned} M_j &= \frac{\alpha^2 \sinh \alpha l}{2(\eta^2 - 1)^{1/2}(\sinh \alpha l - \alpha l)} [\Delta y_j + \mu(\Delta y_{j+1} + \Delta y_{j-1}) \\ &\quad + \mu^2(\Delta y_{j+2} + \Delta y_{j-2}) + \cdots + \mu^n(\Delta y_{j+n} + \Delta y_{j-n}) + \cdots] \\ &= \frac{\alpha^2 \sinh \alpha l}{2(\eta^2 - 1)^{1/2}(\sinh \alpha l - \alpha l)} [2y_j(\mu - 1) + (y_{j+1} + y_{j-1}) \\ &\quad \cdot (\mu - 1)^2 + (y_{j+2} + y_{j-2})(\mu - 1)^2 \mu + \cdots \\ &\quad + (y_{j+n} + y_{j-n})(\mu - 1)^2 \mu^{n-1} + \cdots] \\ &= \frac{\alpha^2 \sinh \alpha l \cdot (\mu - 1)}{2(\eta^2 - 1)^{1/2}(\sinh \alpha l - \alpha l)} \left(2y_j + (\mu - 1) \sum_{n=1}^{\infty} (y_{j+n} + y_{j-n}) \cdot \mu^{n-1}\right). \end{aligned} \tag{1.9}$$

The existence of the spline of interpolation is predicated on the existence of the M_j's. And so, we have

THEOREM 1.2. *A necessary and sufficient condition for the existence and uniqueness of the simple hyperbolic spline of interpolation on R^1 with interpolation data $\{y_j\}$ is that*

$$\sum_{n=1}^{\infty} (y_{j+n} + y_{j-n}) \mu^{n-1}$$

exists for all j; where, we recall, $\mu = (\eta^2 - 1)^{1/2} - \eta$ and $-1 < \mu < 0$.

In fact it can readily be shown that $3^{1/2} - 2 \leqslant \mu < 0$ since $2 \leqslant \eta < \infty$ for $\alpha \geqslant 0$. Certainly if

$$\lim_{n\to\infty} \frac{|y_{j+n+1} + y_{j-(n+1)}|}{|y_{j+n} + y_{j-n}|} < \frac{1}{|\mu|} \qquad \text{for all } j,$$

then, by the ratio test, the required sums will exist. For example, if there exists a K such that for $k > K$, $y_{\pm k} > 0$ and $y_{\pm(k+1)} < y_{\pm k}((1/|\mu|) - \epsilon)$ for a small positive number ϵ; i.e., sufficiently far out the interpolation data does not grow any faster than by a factor of $1/|\mu|$, then the required sums will exist. For,

$$\begin{aligned} &\frac{|y_{j+n+1} + y_{j-n-1}|}{|y_{j+n} + y_{j-n}|} \\ &= \frac{y_{j+n-1} + y_{j-n-1}}{y_{j+n} + y_{j+n}} < \frac{y_{j+n}\left(\frac{1}{|\mu|} - \epsilon\right)}{y_{j+n} + y_{j-n}} + \frac{y_{j-n}\left(\frac{1}{|\mu|} - \epsilon\right)}{y_{j+n} + y_{j-n}} \\ &= \frac{1}{|\mu|} - \epsilon \end{aligned}$$

and so

$$\lim_{n\to\infty} \frac{|y_{j+n+1} + y_{j-n-1}|}{|y_{j+n} + y_{j-n}|} \leqslant \frac{1}{|\mu|} - \epsilon < \frac{1}{|\mu|}.$$

In this case for $\mu = 3^{1/2} - 2$ ($\alpha = 0$), we would only require

$$y_{\pm(k+1)} < 3.7 y_{\pm k}$$

for $k > K$ and, for increasing α, $|\mu|$ decreases and the allowed growth factor $1/|\mu|$ which would still ensure convergence increases without limit.

It is interesting to note that with bounded interpolation data, $|y_j| < B$ for all j, the number of terms in the sums required for a given accuracy in the result decreases with increasing α; for

$$\left| \sum_{n=1}^{\infty} (y_{j+n} + y_{j-n}) \mu^{n-1} - \sum_{n=1}^{N} (y_{j+n} + y_{j-n}) \mu^{n-1} \right| < \frac{2B}{(1 - |\mu|)} |\mu|^N$$

and $|\mu| \to 0$ as $\alpha \to \infty$.

2. Asymptotic Behavior

A very useful spline is constructed if we take as interpolation data

$$y_0 = 1, \qquad y_j = 0, \quad j = \pm 1, \pm 2, \ldots . \tag{2.1}$$

We call such a spline a cardinal simple hyperbolic spline on R^1. Its importance stems from the fact that splines with arbitrary interpolation data can be expressed as linear combinations of cardinal splines, and that the cardinal spline centered at m (i.e., $y_m = 1$, $y_j = 0$, $j \neq m$) is just a translate of the cardinal spline centered at the origin (see Section 3).

From (1.8) with data (2.1), we have

$$M_0 = \frac{\alpha^2 \sinh \alpha l}{(\eta^2 - 1)^{1/2}(\sinh \alpha l - \alpha l)} (\mu - 1)$$

$$M_{\pm j} = \frac{\alpha^2 \sinh \alpha l}{2(\eta^2 - 1)^{1/2}(\sinh \alpha l - \alpha l)} (\mu - 1)^2 \mu^{j-1}, \qquad j = 1, 2, \ldots$$

and finally from (1.2), noting that $S_\Delta(-x) = S_\Delta(x)$, we get on $[0, l]$,

$$S_\Delta(x) = \frac{l - x}{l} + \frac{1}{(\eta^2 - 1)^{1/2}(\sinh \alpha l - \alpha l)} \left\{ (\eta + 1)\mu \left[\frac{x}{l} \sinh \alpha l - \sinh \alpha x \right] + (1 - \mu) \left[\frac{l - x}{l} \sinh \alpha l - \sinh \alpha (l - x) \right] \right\}$$

and on $[jl, (j+1)l]$,

$$S_\Delta(x) = \frac{(\eta+1)\mu^j}{(\eta^2-1)^{1/2}(\sinh \alpha l - \alpha l)} \left\{ \left[\frac{(j+1)\,l - x}{l} \sinh \alpha l \right.\right.$$

$$\left.\left. - \sinh \alpha((j+1)\,l - x) \right] + \mu \left[\frac{x - jl}{l} \sinh \alpha l - \sinh \alpha(x - jl) \right] \right\}. \tag{2.2}$$

Since $-1 < \mu < 0$, the factor μ^j gives S_Δ damped oscillatory behavior, as expected from the cubic case.

We now investigate the behavior of cardinal simple hyperbolic splines in the limiting cases $\alpha \to 0$ and $\alpha \to \infty$.

THEOREM 2.1. *As $\alpha \to 0$, the cardinal simple hyperbolic spline ($L = D(D - \alpha)$) converges to the cardinal cubic spline ($L = D^2$) on R^1.*

Proof. Fix l and let $\alpha \to 0$. On $[0, l]$ with $x = pl$, $0 \leqslant p \leqslant 1$, we have

$$\begin{aligned} S_\Delta(pl) = (1-p) + \frac{1}{(\eta^2-1)^{1/2}(\sinh \alpha l - \alpha l)} \{(\eta + 1 - (\eta^2-1)^{1/2}) \\ \cdot [(1-p)\sinh \alpha l - \sinh \alpha l(1-p)] - (\eta+1)(\eta - (\eta^2-1)^{1/2}) \\ \cdot [p \sinh \alpha l - \sinh \alpha lp]\}. \end{aligned} \tag{2.3}$$

As $\alpha \to 0$, $\eta(\alpha l) \to 2$ and so, by repeated application of L'Hôpital's rule to (2.3), we find that

$$\lim_{\alpha\to 0} S_\Delta(pl) = 1 + p^2(3 - 3\sqrt{3}) + p^3(-4 + 3\sqrt{3}). \tag{2.4.1}$$

Similarly, on $[jl, (j+1)l]$ with $x = jl + pl$, $0 \leqslant p \leqslant 1$, we find

$$\lim_{\alpha\to 0} S_\Delta((j+p)\,l) = 3(\sqrt{3}-2)^j[p - \sqrt{3}p^2 + (\sqrt{3}-1)\,p^3]. \tag{2.4.2}$$

Equations (2.4.1) and (2.4.2) are precisely the expressions for the cardinal cubic spline on R^1 [1].

In the other limiting case, letting $\alpha \to \infty$ we have the following expressions for the asymptotic behavior of various quantities in (2.2):

LEMMA 2.2.

$$\eta(\alpha l) = \alpha l - 1 + o\left(\frac{1}{e^{\epsilon \alpha l}}\right) \qquad \text{for} \quad \epsilon < 1$$

$$(\eta^2-1)^{1/2} - \eta = -\frac{1}{2(\alpha l - 1)} + o\left(\frac{1}{\alpha l}\right)$$

$$\sinh \alpha l - \alpha l = e^{\alpha l}/2 - \alpha l + o\left(\frac{1}{\alpha l}\right)$$

$$(1-p)\sinh \alpha l - \sinh \alpha l(1-p)$$
$$= \begin{cases} \dfrac{e^{\alpha l}}{2}(1 - p - e^{-\alpha l p}) + o\left(\dfrac{1}{\alpha l}\right) & \text{for } 0 < p < 1 \\ 0 & \text{for } p = 0, 1 \end{cases}$$

$$p \sinh \alpha l - \sinh \alpha l p$$
$$= \begin{cases} \dfrac{e^{\alpha l}}{2}(p - e^{-\alpha l(1-p)}) + o\left(\dfrac{1}{\alpha l}\right) & \text{for } 0 < p < 1 \\ 0 & \text{for } p = 0, 1 \end{cases}$$

as $\alpha \to \infty$.

Proof. These are all straightforward calculations. The second comes from the first and a Taylor expansion of the square root. The last three result from writing $\sinh x$ as $(e^x - e^{-x})/2$. We show the first.

$$| \eta(\alpha l) - \alpha l + 1 | \, e^{\epsilon \alpha l}$$
$$= \left| \frac{\alpha l \cosh \alpha l - \sinh \alpha l - \alpha l \sinh \alpha l + \alpha^2 l^2 + \sinh \alpha l - \alpha l}{\sinh \alpha l - \alpha l} \right| e^{\epsilon \alpha l}$$
$$= \frac{\alpha l(e^{-\alpha l} + \alpha l - 1)\, e^{\epsilon \alpha l}}{(e^{\alpha l} - e^{-\alpha l} - 2\alpha l)/2}$$
$$\to \frac{(\alpha l)^2 \, e^{\epsilon \alpha l}}{e^{\alpha l}/2} \to 2(\alpha l)^2 \, e^{-(1-\epsilon)\alpha l} \to 0$$

as $\alpha \to \infty$.

This gives us

THEOREM 2.3. *As* $\alpha \to \infty$, *the cardinal simple hyperbolic spline converges to the cardinal polygonal line approximation on* R^1.

Proof. From (2.2) and the previous lemma, we have on $[0, l]$,

$$S_\Delta(pl) \to (1-p) + \frac{1}{\alpha l(e^{\alpha l}/2)} \left\{ \frac{e^{\alpha l}}{2}(1-p) - \alpha l \left(\frac{1}{2\alpha l}\right)\left(\frac{e^{\alpha l} p}{2}\right) \right\}$$
$$\to (1-p) + \frac{1}{\alpha l}\left(1 - p - \frac{p}{2}\right) \to (1-p) \quad \text{as} \quad \alpha \to \infty,$$

and on $[jl, (j+1)l]$,

$$S_\Delta((j+p)\, l) \to \left(-\frac{1}{2\alpha l}\right)^j (\alpha l) \left(\frac{(e^{\alpha l}/2)(1-p) - (e^{\alpha l} p/4\alpha l)}{\alpha l(e^{\alpha l}/2)}\right)$$
$$\to \left(-\frac{1}{2\alpha l}\right)^j \left(1 - p - \frac{p}{2\alpha l}\right) \to 0 \quad \text{as} \quad \alpha \to \infty.$$

The cardinal polygonal line approximation referred to in the theorem above results from connecting the interpolation data (2.1) by straight line segments.

Schweikert notes Theorem 2.3 in his thesis [7] without proof in the case of finite intervals. Intuitively, this result is not surprising, because for large values of α the dominant term in $L = D(D - \alpha)$ is $-\alpha D$ and the splines arising from $L^*L = \alpha D(-\alpha D) = -\alpha^2 D^2$ are continuous piecewise straight lines with discontinuous first derivatives at the mesh points. In some cases, as Schweikert has shown, the ability to choose α over such a wide range is helpful in making the approximating spline curve a more visually acceptable interpolant.

3. Application to Finite Intervals

Two basic approaches are available for the calculation of hyperbolic splines on finite intervals—the direct application of equations (1.2), (1.3) for $j = 1,\ldots, N - 1$ with appropriate boundary conditions at $j = 0$, N, [4], [7], or the utilization of the cardinal splines determined by (2.1) as we now consider. This approach is motivated by [1].

On the interval $[0, Nl]$, assume interpolation data $\{y_j\}_{j=0}^{N}$ to be given. Denote by $C_j(x)$ the cardinal simple hyperbolic spline centered at j (a translate of the spline defined by (2.2)). Let

$$S_\Delta(x) = \sum_{j=0}^{N} y_j C_j(x) + aC_{-1}(x) + bC_{N+1}(x). \tag{3.1}$$

Then, surely, $S_\Delta(x_j) = y_j$, $j = 0,\ldots, N$ and S_Δ is a spline of interpolation to $\{y_j\}$ on $[0, Nl]$.

Additional end conditions are required to specify the spline uniquely. Let us take them to be the values of the slopes at the endpoints, $S_\Delta'(0)$ and $S_\Delta'(Nl)$. (Other choices are possible.) It was to allow for these two parameters that the additional cardinal splines, $C_{-1}(x)$ and $C_{N+1}(x)$, were included in the representation (3.1).

Differentiation of (3.1) and evaluation of the result at 0, Nl yields

$$\begin{aligned} aC'_{-1}(0) + bC'_{N+1}(0) &= S_\Delta'(0) - \sum_{j=0}^{N} y_j C_j'(0) \\ aC'_{-1}(Nl) + bC'_{N+1}(Nl) &= S_\Delta'(Nl) - \sum_{j=0}^{N} y_j C_j'(Nl). \end{aligned} \tag{3.2}$$

So $S_\Delta(x)$, interpolating to $\{y_j\}_{j=0}^{N}$ and having prescribed slopes at the endpoints, is given by (3.1) with a, b to be found from (3.2). In order to solve (3.2) for a, b it is required only that

$$\begin{bmatrix} C'_{-1}(0) & C'_{N+1}(0) \\ C'_{-1}(Nl) & C'_{N+1}(Nl) \end{bmatrix} \tag{3.3}$$

have an inverse.

LEMMA 3.1. *The matrix* (3.3) *is invertible.*

Proof. Differentiation of (2.2) gives

$$\begin{aligned} C'_{-1}(0) &= (\eta + 1)\,\mu \\ C'_{N+1}(0) &= (\eta + 1)\,\mu^{N+1} \\ C'_{-1}(Nl) &= -(\eta + 1)\,\mu^{N+1} \\ C'_{N+1}(Nl) &= -(\eta + 1)\,\mu. \end{aligned}$$

So (3.3) becomes

$$\mu(\eta + 1)\begin{bmatrix} 1 & \mu^N \\ -\mu^N & -1 \end{bmatrix}$$

which is clearly invertible. In fact, for reasonably large N, (3.3) approximates

$$\mu(\eta + 1)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

since $|\mu| \leqslant 2 - 3^{1/2} < .3$, eliminating the need to actually invert (3.3) to solve for a, b.

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