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UNITARITY BOUNDS ON DIFFRACTION DISSOCIATION *

U.P. SUKHATME
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge, England

F.S. HENYEY
Department of Physics, University of Michigan, Ann Arbor, Michigan

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Using s-channel unitarity and the standard picture that diffraction dissociation and elastic scattering are the shadow of non-diffractive particle production, we derive rigorous upper bounds for the diffraction dissociation cross section. The bounds are valid at each impact parameter, and are derived for an arbitrary number N of diffractive channels. Our results are a generalization of previously derived bounds for the special simple case of $N = 2$ channels.

1. Formulation of the problem

It is well-known that the requirement of s-channel unitarity restricts the amount of allowed inelastic diffraction dissociation. What makes such bounds especially interesting and useful is the fact that diffraction is often substantial, and the bounds are close to being saturated (for example in pp scattering at CERN ISR energies). Therefore, the manner in which these bounds are attained is a valuable guide for building models of diffraction dissociation and multiparticle production.

The simplest upper bound on the diffraction dissociation cross section is

$$\sigma_{\rm diff}(b) \leqslant \tfrac{1}{2}\sigma_{\rm T}(b) - \sigma_{\rm el}(b)\,. \qquad (1)$$

This bound, which was first published by Pumplin [1], is valid at each impact parameter b. A simple derivation is given below. Reasonable additional assumptions (about the elastic scattering of diffractively excited channels) lead to considerably stronger bounds on $\sigma_{\rm diff}$. In this article we derive the *lowest* upper bounds on diffraction for an *arbitrary* number N of diffractive channels. This is a generalization of previously derived bounds for the simple case of $N = 2$ channels [2]. The results depend fairly

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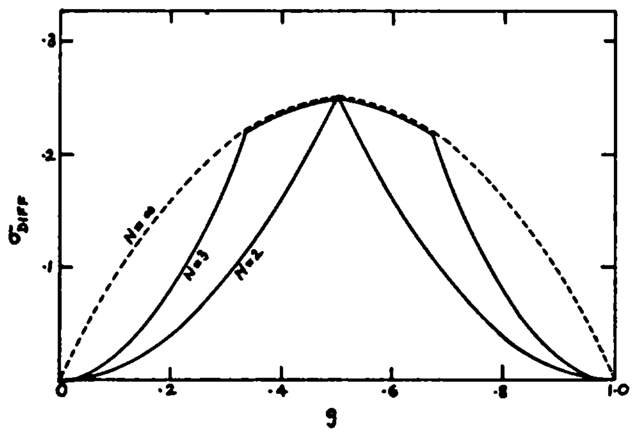

Fig. 1. General bounds on diffraction as a function of the number of diffractive channels N and the elastic amplitude g for the case of equal diagonal entries (sect. 2). The dashed inverted parabola $g(1-g)$ is the simple bound of eq. (1), and it corresponds to $N = \infty$. The bounds on diffraction are more stringent when the number of diffractive channels is small.

substantially on the value of N, especially for small N, as can readily be seen from figs. 1 and 2. Note that there is a substantial improvement over the simple bound of eq. (1), especially near the "edges", where the elastic scattering amplitude is close to zero or one. In general, we also obtain much improved bounds in other regions, as is discussed in detail in sect. 3.

The case of only two diffractive channels ($N = 2$) was recently treated in detail in ref. [2]. Although the mathematical bounds derived there are correct, the subsequent physical reasoning is somewhat misleading since all the physically present diffractive channels are combined into a single "effective" channel in order to make use of $N = 2$ results. This point will be further discussed later (sect. 4).

Let us now establish our notation and formulate the problem clearly. The S-matrix is $S = I + iT$. The rows and columns label the various initial and final states e.g. pp, pp*, p*p, p*p*, *etc.* Let us assume that there are N diffractive states, and that the scattering sub-matrix between these states is pure imaginary and denoted by iG, where G is a real, symmetric $N \times N$ matrix. The eigenvalues of G are real numbers f_i ($i = 1, ..., N$). s-channel unitarity and the standard requirement that the scattering of diffractive eigenstates be purely absorptive (shadow of non-diffractive particle production) constrains the eigenvalues f_i to lie in the closed interval 0 to 1 [1,2]. $f_i = 0$ corresponds to a transparent eigenchannel, whereas $f_i = 1$ is a completely opaque eigenchannel. Let Q denote the orthogonal matrix which diagonalizes G,

$$F = Q^{T}GQ\ , \qquad F_{ij} = f_i\delta_{ij}\ , \qquad 0 \leqslant f_i \leqslant 1\ , \qquad Q^{T}Q = QQ^{T} = I\ . \qquad (2)$$

These relations are true at each impact parameter. We want to investigate bounds on diffraction dissociation cross sections imposed by unitarity. (For concreteness, we can think about pp scattering and let the row label 1 correspond to the initial pp

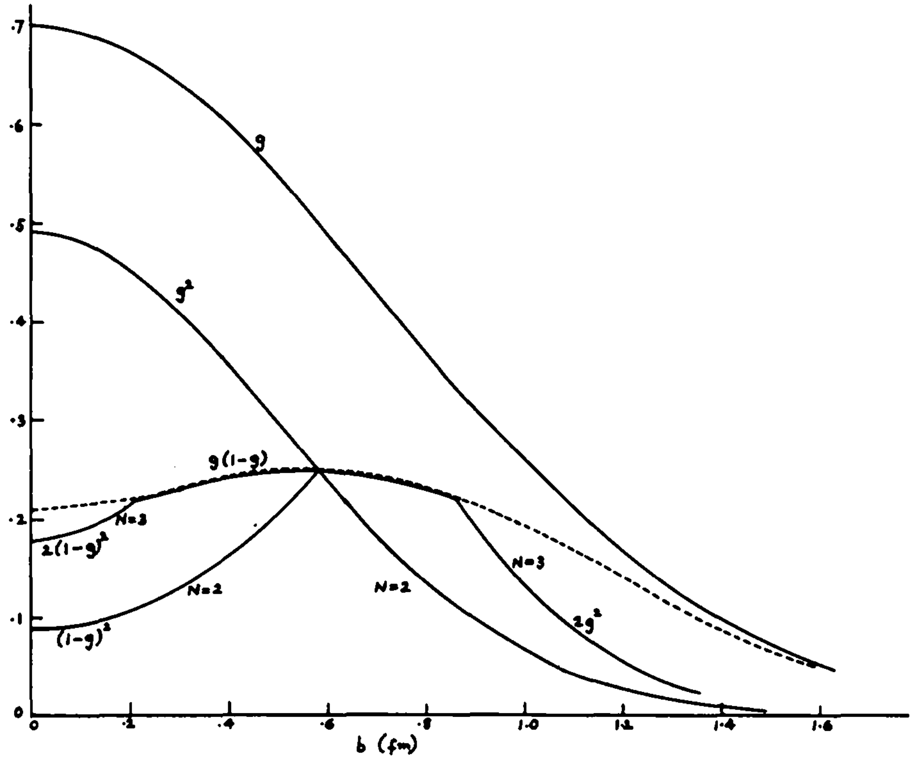

Fig. 2. Bounds on diffraction dissociation for pp scattering at ISR energies. The curve labelled g is the pp elastic amplitude in impact parameter space. We have chosen a convenient Gaussian representation of the data $g \equiv g_{11}(b) = 0.7\mathrm{e}^{-b^2}$, where b is in fm. The dashed curve marked $g(1-g)$ is the simplest bound on the diffractive cross section $\sigma_{\mathrm{diff}} \leqslant g(1-g)$ (eq. (1)), and is seen to be mildly peripheral. The solid line (marked $N = 2$) is a much stronger and more peripheral bound on diffraction – it corresponds to the case of two diffractive channels with equal elastic scattering discussed in sect. 2 of the text. The bound corresponding to three diffractive channels is labelled by $N = 3$. Note that as N increases, the improvement of the diffractive bound over the simple dashed curve gradually decreases, and occurs only near the edges.

state.) At any particular impact parameter b, the total, elastic and diffractive cross sections are given by

$$\sigma_{\mathrm{T}}(b) = 2g_{11}\,, \; \sigma_{\mathrm{el}}(b) = g_{11}^2\,, \qquad \sigma_{\mathrm{diff}}(b) = \sum_{k=2}^{N} g_{1k}^2\,. \tag{3}$$

Integration over b yields the three types of measurable cross sections

$$\sigma_J = 2\pi \int_0^\infty b\,\mathrm{d}b\,\sigma_J(b)\,, \qquad J = \mathrm{T, el, diff}\,.$$

Using eq. (2) $G = QFQ^{\mathrm{T}}$, we can re-write eq. (3) in the equivalent form

$$\sigma_{\mathrm{T}}(b) = 2\sum_i Q_{1i}^2 f_i\,, \qquad \sigma_{\mathrm{el}}(b) = (\sum_i Q_{1i}^2 f_i)^2\,,$$

Since $0 \leqslant f_i \leqslant 1$, we have $f_i^2 \leqslant f_i$. Therefore

$$\sigma_{\text{diff}}(b) + \sigma_{\text{el}}(b) = \sum_i Q_{1i}^2 f_i^2 \leqslant \sum_i Q_{1i}^2 f_i = \tfrac{1}{2}\sigma_{\text{T}}(b) ,$$

or

$$\sigma_{\text{diff}}(b) \leqslant \tfrac{1}{2}\sigma_{\text{T}}(b) - \sigma_{\text{el}}(b) = g_{11}(1 - g_{11}) ,$$

which establishes eq. (1). Since this simple bound is true at each impact parameter b, it is of course also valid when we integrate over b. The dashed curve in fig. 2 shows this bound for pp scattering at ISR energies. If only g_{11} were known, this would be the best we could do. However, one usually has a reasonable idea about the b-dependence of elastic scattering in other channels too, i.e. the diagonal elements of G. With this additional information, one should get improved bounds on σ_{diff}. This leads us to formulate the following problem:

"Assuming that all elastic scattering amplitudes are known (i.e. the diagonal elements of G are given functions of impact parameter), what bounds does unitarity impose on σ_{diff}?"

A more precise mathematical formulation is

"G is a real, symmetric $N \times N$ matrix whose diagonal entries g_{ii} ($i = 1, ..., N$) are given. The eigenvalues f_i are required to lie in the range $0 \leqslant f_i \leqslant 1$. What is the maximum value which the quantity $\sigma_{\text{diff}} \equiv \Sigma_{k=2}^{N} g_{1k}^2$ can have?"

For clarity, we first solve the problem in sect. 2 for the special (but interesting and physically relevant) case when all the diagonal entries are equal. The more general case of arbitrary diagonal elements in G is treated in sect. 3.

Before we begin, it is good to note the following useful theorem on diffractive bounds.

Theorem A: If $D_{\text{B}}(g_{ii})$ is an attainable upper bound for diffraction in a real, symmetric matrix G with given diagonal elements g_{ii} and eigenvalues $f_i(g_{ii})$ between 0 and 1, then $D_{\text{B}}(1 - g_{ii})$ is also an upper bound.

Proof: Consider the matrices G and $I - G$. They both have the same amount of diffraction σ_{diff}. The eigenvalues of $I - G$ are $1 - f_i(g_{ii})$, and they lie between 0 and 1, since $0 \leqslant f_i(g_{ii}) \leqslant 1$. Therefore, if $D_{\text{B}}(g_{ii})$ is a diffractive upper bound for G, it is also an upper bound for $I - G$, whose diagonal elements are $1 - g_{ii}$. Relabelling the matrix elements of $I - G$ by the transformation $1 - g_{ii} \rightarrow g_{ii}$, it follows that $D_{\text{B}}(1 - g_{ii})$ is an upper bound on diffraction for the transformed matrix $\widetilde{G}$ with diagonal elements g_{ii}. Note that the eigenvalues of $\widetilde{G}$ are $\widetilde{f}_i = 1 - f_i(1 - g_{ii})$.

Given any bound, theorem A allows the immediate construction of a new bound by the interchange $g_{ii} \leftrightarrow 1 - g_{ii}$. Note that an application of theorem A to eq. (1) does not yield anything new, since the interchange $g_{11} \leftrightarrow 1 - g_{11}$ leaves the bound $g_{11}(1 - g_{11})$ unchanged.

2. Equal diagonal entries in G

Here we want to obtain the lowest upper bound on diffraction dissociation for the case of equal diagonal entries in the diffractive scattering submatrix G. Let $g_{11} = g_{22} = ... = g_{NN} = g$ (say). In this case, we will show that diffraction is bounded from above at any b by all of the following three bounds:

$$\sigma_{\text{diff}} \leqslant (N-1)g^2 \,, \tag{5a}$$

$$\sigma_{\text{diff}} \leqslant g(1-g) \,, \tag{5b}$$

$$\sigma_{\text{diff}} \leqslant (N-1)(1-g)^2 \,. \tag{5c}$$

Eq. (5b) is just the previously derived simple bound (eq. (1)). Eq. (5c) follows from (5a) by theorem A. So we need only establish (5a).

Proof:

$$\begin{aligned} Ng^2 &= g \operatorname{tr} G \\ &= g \operatorname{tr} F \\ &= (\sum_i Q_{1i}^2 f_i)(\sum_j f_j) \\ &= \sum_i Q_{1i}^2 f_i^2 + \sum_i Q_{1i}^2 f_i \sum_{j \neq i} f_j \geqslant \sum_i Q_{1i}^2 f_i^2 = \sigma_{\text{diff}} + \sigma_{\text{el}} \,. \end{aligned}$$

Table 1
Best upper bounds on diffraction for the case of N diffractive channels and equal diagonal entries g (examples of matrices G which saturate the bound are given in the last column)

Range of g	Lowest upper bound on σ_{diff}	Eigenvalues	Matrix G which attains bound
$0 \leqslant Ng \leqslant 1$	$\sigma_{\text{diff}} \leqslant (N-1)g^2$	$\underbrace{0, ..., 0}_{N-1}, Ng$	$g_{ij} = g$
$1 \leqslant Ng \leqslant N-1$	$\sigma_{\text{diff}} \leqslant g(1-g)$	$\underbrace{0, \frac{Ng-1}{N-2}, ..., 1}_{N-2}$	$g_{ii} = g$ $g_{1k} = g_{k1} = \sqrt{\frac{g(1-g)}{N-1}}\ (k \neq 1)$ $g_{jk} = \frac{(1-2g)}{N-2}\ (j \neq k; j, k \neq 1)$
$N-1 \leqslant Ng \leqslant N$	$\sigma_{\text{diff}} \leqslant (N-1)(1-g)^2$	$Ng-(N-1), \underbrace{1, ..., 1}_{N-1}$	$g_{ii} = g$ $g_{ij} = -(1-g)\ (i \neq j)$

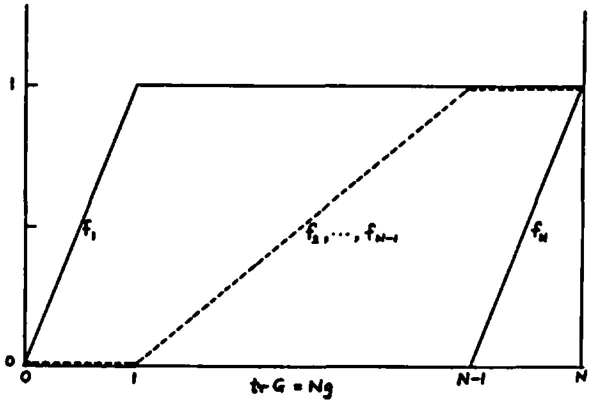

Fig. 3. The eigenvalues of G (as a function of tr G) which attain the diffractive bounds discussed in the equal diagonal entry case of sect. 2.

Therefore, $\sigma_{\rm diff} \leqslant (N-1)g^2$.

The best bound is the *least* of inequalities (5), and this depends on the value of g. These minimum values are in fact the *best* (i.e. the lowest) upper bounds since we will explicitly demonstrate that they are *attainable* by exhibiting an example which saturates them. Our example for the explicit matrices G, their eigenvalues, and the best bounds on diffraction corresponding to various values of g are shown in table 1. It is not hard to see that our example for the matrices G is not unique, but of course that does not affect any of our results. Note how the eigenvalues change as g is gradually increased (see fig. 3). At first only one eigenvalue is non-zero and it increases from 0 to 1. Since it is constrained to remain below unity, further increase in g is distributed among $N-2$ eigenvalues. Finally, when $N-1$ eigenvalues reach unity and g is increased still further the last eigenvalue increases from 0 to 1. Improvement over the simple bound of eq. (1) only occurs near the edges i.e. in the ranges $0 \leqslant g \leqslant 1/N$ and $1-1/N \leqslant g \leqslant 1$. Diffractive bounds for various values of N are shown in fig. 1, and the case of pp scattering in fig. 2. Note that when $N = 2$, there is a substantial improvement over the simple (dashed line) result of eq. (1). However, for $N \geqslant 3$ (as is very likely the case in practice) the improvement over the simple bound is not as marked, and as we have seen, occurs only at the edges. For an infinite number of channels and equal diagonal entries there is no improvement, and one recovers the Pumplin bound (eq. (1)).

3. Arbitrary diagonal entries in G

We will now establish the following bounds on diffraction for *arbitrary* diagonal entries $g_1, g_2, ..., g_N$ in the matrix G,

$$\sigma_{\rm diff} \leqslant g_1(g_2 + g_3 + ... + g_N)\,, \tag{6a}$$

$$\sigma_{\text{diff}} \leqslant g_1(1-g_1)\,, \tag{6b}$$

$$\sigma_{\text{diff}} \leqslant (1-g_1)(N-1-g_2-g_3-\ldots-g_N)\,, \tag{6c}$$

$$\sigma_{\text{diff}} \leqslant \tfrac{1}{2}[P+(\sum_i g_i - P)^2 - \sum_i g_i^2]\,, \tag{6d}$$

where P is any integer. The bound (6d) is best when P is the greatest integer in the trace $\Sigma_i g_i$. Note that (6b) is eq. (1), and (6c) follows from (6a) using theorem A. An application of theorem A to (6d) does not yield new results since the expression is invariant under $g_i \leftrightarrow 1-g_i$. So we only need to establish (6a) and (6d). The proof of (6a) is analogous to (5a),

$$g_1(\text{tr}\,G) \geqslant \sum_i Q_{1i}^2 f_i^2 = \sigma_{\text{diff}} + \sigma_{\text{el}}\,.$$

Hence, $\sigma_{\text{diff}} \leqslant g_1(g_2+g_3+\ldots+g_N)$, and this is a straightforward generalization of (5a).

The proof of (6d) follows from the three equations

$$\text{tr}\,G^2 = \sum_i g_{ii}^2 + 2\sum_{i<j} g_{ij}^2\,, \tag{7a}$$

$$\text{tr}\,G^2 = \text{tr}\,F^2 = \sum_i f_i^2\,, \tag{7b}$$

$$\sum_i f_i^2 \leqslant P + (P - \sum_i f_i)^2\,. \tag{7c}$$

The last inequality (7c) comes from maximizing $\Sigma_i f_i^2$ subject to the constraints that the trace $\Sigma_i f_i = \Sigma_i g_{ii}$ is given and $0 \leqslant f_i \leqslant 1$. The result follows from using

Table 2
Bounds on diffraction for the case of $N = 2$ channels

Range of g_1, g_2	Lowest upper bound on σ_{diff}	Eigenvalues	Matrix G which attains bound
$0 \leqslant \text{tr}\,G \leqslant 1$	$\sigma_{\text{diff}} \leqslant g_1 g_2$	$0, g_1+g_2$	$\begin{bmatrix} g_1 & \sqrt{g_1 g_2} \\ \sqrt{g_1 g_2} & g_2 \end{bmatrix}$
$1 \leqslant \text{tr}\,G \leqslant 2$	$\sigma_{\text{diff}} \leqslant (1-g_1)(1-g_2)$	$g_1+g_2-1, 1$	$\begin{bmatrix} g_1 & \sqrt{(1-g_1)(1-g_2)} \\ \sqrt{(1-g_1)(1-g_2)} & g_2 \end{bmatrix}$

As mentioned in the text, our example for the matrices G is not unique. A change of sign of the off-diagonal elements g_{12} is another example which saturates the diffractive bounds.

the technique of Lagrange multipliers with inequality constraints. Therefore,

$$\sigma_{\rm diff}(b) = \sum_{k=2}^{N} g_{1k}^2 \leqslant \tfrac{1}{2}({\rm tr}\, G^2 - \sum_i g_{ii}^2) \leqslant \tfrac{1}{2}[P + (P - \sum_i g_{ii})^2 - \sum_i g_{ii}^2] \ ,$$

which proves (6d).

To understand the meaning of these bounds, let us study the special cases $N = 2$ and $N = 3$. For these cases, we will show that these are the lowest upper bounds by explicitly exhibiting matrices G which attain them. It is very possible that our results are in fact the lowest upper bounds for any N, but we will not demonstrate this. The results for $N = 2$ are shown in table 2, and those for $N = 3$ in table 3.

For the $N = 2$ case, note that the bounds are *always superior* to the simple bound of eq. (1), and are *considerably stronger.*

The bounds for $N = 3$ show four regions. There is the standard improvement near the edges, i.e. when $0 \leqslant tr\, G \leqslant 1$ and $2 \leqslant {\rm tr}\, G \leqslant 3$. In the central region, $\sigma_{\rm diff}$ is bounded by eq. (1). However, it is important to note the existence of large special regions ($g_1 + g_2 \geqslant 1, g_1 + g_3 \geqslant 1, 1 \leqslant {\rm tr}\, G \leqslant 2$) where further improvement occurs. These regions correspond to the last line in table 3, and it comes from (6d) with $P = 1$. Therefore, one has a *multiple-humped surface which bounds diffraction.* Unfortunately, this surface is not easy to draw graphically, but can be seen by examining table 3 carefully. These special central regions, where the least upper bound is lower than eq. (1), are a consequence of considering the general case of arbitrary diagonal entries in the matrix G.

4. Discussion

Let us consider our results in the context of pp scattering at ISR energies. Putting in approximate numbers in eq. (1) (integrated over b) gives

$$\sigma_{\rm diff} \leqslant \tfrac{1}{2}\sigma_{\rm T} - \sigma_{\rm el} \simeq 21.5 - 8.5 = 13 \text{ mb} \ .$$

This is not too far above the observed diffraction cross section of approximately 8 mb. If in addition we make the physically reasonable assumption of equal diagonal entries in the diffractive sub-matrix G, as is very likely to be the approximate situation in nature, then the bound on diffraction is more stringent, especially for a small number of diffractive channels. In fact, for $N = 2$, one finds $\sigma_{\rm diff} \lesssim \sigma_{\rm el} \simeq 8$ mb, which indicates a saturation of the two-channel bound and consequently a sharply peripheral diffractive cross section in impact parameter space (see the $N = 2$ curve in fig. 2). This point was recently emphasized at length in ref. [2], but is unfortunately not wholly correct. As we have seen, bounds on diffraction depend fairly substantially on the number N of available diffractive channels. It is therefore improper to combine these into one "effective" channel, especially since the combination procedure is im-

Table 3
Bounds on diffraction for $N = 3$ channels

Range of g_1, g_2, g_3	Lowest upper bound on σ_{diff}	Eigenvalues	Matrix G which attains bound (off-diagonal elements only)
$0 \leqslant \text{tr}\, G \leqslant 1$	$\sigma_{\text{diff}} \leqslant g_1(g_2+g_3)$	$0, 0, \text{tr}\, G$	$g_{12} = \sqrt{g_1 g_2}; g_{13} = \sqrt{g_1 g_3}; g_{23} = \sqrt{g_2 g_3}$
$2 \leqslant \text{tr}\, G \leqslant 3$	$\sigma_{\text{diff}} \leqslant (1-g_1)(2-g_2-g_3)$	$1, 1, \text{tr}\, G - 2$	$g_{12} = \sqrt{(1-g_1)(1-g_2)}; g_{13} = \sqrt{(1-g_1)(1-g_3)}$ $g_{23} = \sqrt{(1-g_2)(1-g_3)}$
$g_1+g_2 \leqslant 1$ $g_1+g_3 \leqslant 1$ $1 \leqslant \text{tr}\, G \leqslant 2$	$\sigma_{\text{diff}} \leqslant g_1(1-g_1)$	$0, 1, \text{tr}\, G - 1$	$g_{12} = \sqrt{\dfrac{g_1(1-g_1)(1-g_1-g_3)}{2-2g_1-g_2-g_3}}$ $g_{13} = \sqrt{\dfrac{g_1(1-g_1)(1-g_1-g_2)}{2-2g_1-g_2-g_3}}$ $g_{23} = \sqrt{1 - \text{tr}\, G + g_1 g_2 + g_2 g_3 + g_3 g_1 - g_1(1-g_1)}$
$g_1+g_2 \geqslant 1$ $g_1+g_3 \geqslant 1$ $1 \leqslant \text{tr}\, G \leqslant 2$	$\sigma_{\text{diff}} \leqslant g_1 g_2 g_3$ $+ (1-g_1)(1-g_2)(1-g_3)$	$0, 1, \text{tr}\, G - 1$	$g_{12} = \sqrt{\dfrac{g_2(1-g_2)(g_1+g_3-1)}{g_3-g_2}}$ $g_{13} = \sqrt{\dfrac{g_3(1-g_3)(g_1+g_2-1)}{g_2-g_3}}$ $g_{23} = 0$

The diagonal elements of G are the known quantities g_1, g_2, g_3 and the matrix G is symmetric.

pact parameter dependent. Thus, one cannot conclude from the two-channel bound of ref. [2] that inelastic diffraction must be peripheral, although that may very well be the case for other reasons connected with unitarity [3]. A similar conclusion has also been reached by considering explicit simple models for pp diffractive scattering [4].

This article has concentrated mainly on deriving mathematical bounds on diffraction. There are several reasons why these bounds should prove useful in physcial applications. Whenever diffraction is appreciable, the bounds will be saturated, at least in some regions of impact parameter. The bounds should also be kept in mind when studying low- and high-mass diffractive production, which is an interesting open problem. One needs a specific prescription or model to talk about the relevant number of available channels N at any energy. At very high energies, one expects a large number of open channels, and one has the simple Pumplin bound (eq. (1)). From the derivation, it is clear that the bound is saturated if for all i, $Q_{1i} = 0$, *or* $f_i^2 = f_i$ (i.e. eigenchannels are either transparent or black) [5]. Define the ratio R by

$$R(s) = \frac{\sigma_{\mathrm{el}} + \sigma_{\mathrm{diff}}}{\sigma_{\mathrm{T}}} = \frac{\sigma_{\mathrm{el}}}{\sigma_{\mathrm{T}}} \left(1 + \frac{\sigma_{\mathrm{diff}}}{\sigma_{\mathrm{el}}}\right).$$

For pp scattering at ISR energies, the value of R is approximately 0.4, not too far from the saturation value of 0.5. Also, over the ISR energy range $\sigma_{\mathrm{el}}/\sigma_{\mathrm{T}}$ is essentially constant. However, there are experimental hints and some data fits which suggest that the ratio $\sigma_{\mathrm{diff}}/\sigma_{\mathrm{el}}$ is at least a constant and may well be rising as a function of energy [6]. "Asymptotic saturation" of eq. (1) [$R(s) \to 0.5$ as $s \to \infty$] is an interesting possibility.

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