

## Scattering Length and Perturbations of $-\Delta$ by Positive Potentials

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### INTRODUCTION

In [5], Kac and Luttinger gave an elegant connection between the scattering length of a positive potential and Brownian motion. The purpose of this paper is to develop this notion further into a tool for studying the effectiveness of such a potential, as a perturbation of  $-\Delta$ .

In the first section we define scattering length, prove Kac and Luttinger's formula, and state a few simple properties of scattering length. In the next two sections we look for conditions on when a sequence  $v_j$  of positive potentials has, in the limit, a negligible effect on  $-\Delta$  and when it "solidifies" to a compact set  $K$ , leading to a Dirichlet problem for  $\Delta$  on the complement of  $K$ . We also produce a two-sided bound for the lowest eigenvalue of  $-\Delta + v$  on a bounded region  $\Omega$ , with Neumann conditions on  $\partial\Omega$ , in terms of the scattering length of  $v$ .

The fourth section treats an analog of a problem dealt with by Kac and Luttinger involving randomly placed potentials. In a sense this is intermediate between two special cases handled in Sections 2 and 3 and more delicate. The last section introduces several notions of regularity of a compact set, generalizing that of Stroock, and looks at the application to potential theory.

For simplicity, we work on  $\mathbb{R}^n$  only for  $n \geq 3$ .

Several problems investigated in this paper are analogous to problems involving obstacles investigated in [9].

### 1. DEFINITION OF SCATTERING LENGTH

Let  $v \in L^\infty_{\text{comp}}(\mathbb{R}^n)$  be  $\geq 0$ ; we'll say  $v \in \mathcal{D}^+$ . Associated to  $v$  is a function  $U_v$ , which we will call the capacity potential of  $v$ , defined by

$$\begin{aligned} U_v(x) &= (v - \Delta)^{-1} v(x) \\ &= \lim_{\epsilon \downarrow 0} (\epsilon + v - \Delta)^{-1} v(x). \end{aligned}$$

The limit is taken in  $L^2_{loc}(\mathbb{R}^n)$ . We will show that

- (i)  $U_v$  exists,
- (ii)  $0 \leq U_v \leq 1$ ,
- (iii)  $v \leq w \Rightarrow U_v \leq U_w$ .

It follows that  $U_v$  solves the differential equation

$$\Delta U_v = -v(x) \quad (1 - U_v(x)).$$

Thus,  $-\Delta U_v = \mu_v$ , a positive measure on  $\mathbb{R}^n$ . In analogy with classical potential theory, we consider the total mass of  $\mu_v$ .

DEFINITION.  $\Gamma(v) = \int d\mu_v(x)$  is the scattering length of  $v$ .

The first proposition simultaneously establishes assertions (i)–(iii) above and gives an elegant formula for  $U_v(x)$ , due to Kac and Luttinger [5]. Here  $E_x$  is the expectation with respect to Wiener measure on the set of Brownian paths starting at  $x$ .

PROPOSITION 1.1.  $U_v(x) = E_x\{1 - \exp(-\int_0^\infty v(w(\tau)) d\tau)\}$ , in  $L^2_{loc}$ .

*Proof.*

$$\begin{aligned} \lim_{\epsilon \downarrow 0} (\epsilon + v - \Delta)^{-1} v &= \lim_{\epsilon \downarrow 0} \int_0^\infty e^{-\epsilon t} e^{t(\Delta+v)} v(x) dt \\ &= \lim_{\epsilon \downarrow 0} \int_0^\infty E_x \left\{ \exp \left( - \int_0^t \{v(w(\tau)) - \epsilon\} d\tau \right) v(w(t)) \right\} dt \\ &= \int_0^\infty E_x \left\{ \exp \left( - \int_0^t v(w(\tau)) d\tau \right) v(w(t)) \right\} dt \\ &= \int_0^\infty E_x \left\{ - \frac{d}{dt} \exp \left( - \int_0^t v(w(\tau)) d\tau \right) \right\} dt \\ &= E_x \left\{ 1 - \exp \left( - \int_0^\infty v(w(\tau)) d\tau \right) \right\}. \end{aligned}$$

The second identity uses the Feynman–Kac formula. ■

Recall that if  $K$  is a compact set satisfying a mild regularity condition, its capacity potential is given by

$$U_K(x) = E_x \left\{ 1 - \exp \left( - \int_0^\infty v_K(w(\tau)) d\tau \right) \right\},$$

where  $v_K(x) = +\infty$  if  $x \in K$ , 0 if  $x \notin K$ .

Thus Kac and Luttinger’s formula makes it natural that scattering length is analogous to capacity. One phenomenon you might expect is that if  $v_n \rightarrow +\infty$  on  $K$  and 0 off  $K$ , then  $\Gamma(v_n) \rightarrow \text{cap } K$ , as  $n \rightarrow \infty$ . As we’ll see in later sections, the situation is more complicated than this, and some interesting phenomena arise, especially if  $K$  is not Kac-regular.

We proceed to establish a few elementary properties of  $U_v$  and  $\Gamma(v)$ : monotonicity, subadditivity, limit properties, and the fact that  $U_v$  is small in the Sobolev space  $H^1_{\text{loc}}$  if  $\Gamma(v)$  is small.

PROPOSITION 1.2. *If  $v \leq w$ , then  $\Gamma(v) \leq \Gamma(w)$ .*

*Proof.* We know that  $U_v \leq U_w$ . Now let  $K$  be a compact neighborhood of  $\text{supp } v \cup \text{supp } w$ ,  $\nu_K$  its equilibrium measure, and  $U_K = -\Delta^{-1}\nu_K$  its capacity potential. Thus

$$\begin{aligned} \int_{\mathbb{R}^n} U_v(x) d\nu_K(x) &= - \int_{\mathbb{R}^n} U_K(x) \Delta U_v(x) dx \\ &= \int U_K(x) d\mu_v(x) = \Gamma(v). \end{aligned} \tag{1.1}$$

This formula makes the monotonicity of  $\Gamma(v)$  evident.

PROPOSITION 1.3.  *$\Gamma(u + v) \leq \Gamma(u) + \Gamma(v)$ .*

*Proof.* In view of (1.1), it is only necessary to show that  $U_{u+v} \leq U_u + U_v$ , but this is an easy consequence of Proposition 1.1. ■

PROPOSITION 1.4. *If  $v_n \uparrow v$ , then  $U_{v_n} \uparrow U_v$  and  $\Gamma(v_n) \uparrow \Gamma(v)$ .*

*Proof.* The convergence of  $U_{v_n}$  follows from Proposition 1.1 and the monotone convergence theorem. This established, the convergence of  $\Gamma(v_n)$  follows from (1.1) and the monotone convergence theorem. ■

PROPOSITION 1.5.  *$U_v \in C^{2-\epsilon}(\mathbb{R}^n)$ . Furthermore,  $\Gamma(v_n) \rightarrow 0 \Rightarrow U_{v_n} \rightarrow 0$  in  $H^1_{\text{loc}}(\mathbb{R}^n)$ .*

*Proof.*  $\Delta U_v = -v(1 - U_v) \in L^\infty \Rightarrow U_v \in C^{2-\epsilon}(\mathbb{R}^n)$ . To finish the proof, we estimate  $(\Delta U_v, U_v)$  and  $\int_S |U_v|$ , where  $S$  is a given compact subset of  $\mathbb{R}^n$ .

First of all,  $-(\Delta U_v, U_v) = \int U_v(x) d\mu_v(x) \leq \Gamma(v)$  because  $0 \leq U_v \leq 1$ . Next,

$$\begin{aligned} \int_S U_v(x) dx &= -C_n \int_S \int \frac{\Delta U_v(y)}{|x - y|^{n-2}} dy dx \\ &= C_n \int \left( \int_S \frac{dx}{|x - y|^{n-2}} \right) d\mu_v(y) \\ &\leq \alpha_S \Gamma(v) \end{aligned}$$

where

$$\alpha_S = C_n \sup_v \int_S \frac{dx}{|x - y|^{n-2}}. \blacksquare$$

Finally, we mention two crude estimates that will be useful.

PROPOSITION 1.6.  $\Gamma(v) \leq \text{cap}(\text{supp } v)$  and  $\Gamma(v) \leq \|v\|_{L^1}$ .

The proof is immediate.

2. FADING POTENTIALS AND EIGENVALUE ESTIMATES FOR  $\Delta - \tau$

Let  $\Omega$  be an open subset of  $\mathbb{R}^n$ ,  $v_j$  supported in  $\Omega$ . If  $\Gamma(v_j) \rightarrow 0$  as  $j \rightarrow \infty$ , then functions of  $\Delta - v_j$  tend to functions of  $\Delta$  as  $j \rightarrow \infty$ , i.e., the effect of the perturbation  $v_j$  disappears. (In [9], the authors showed that if  $K_j \subset \Omega$  are compact,  $\text{cap } K_j \rightarrow 0$ , then the Laplace operators  $\Delta_j$  on  $\Omega \setminus K_j$  with Dirichlet boundary conditions on  $\partial K_j$  behave like the unperturbed Laplace operator  $\Delta$  on  $\Omega$  as  $j \rightarrow \infty$ .) Thus  $\Gamma(v_j) \rightarrow 0 \Rightarrow$  the lowest eigenvalue  $\lambda_1(v_j)$  of  $v_j - \Delta$  on  $\Omega$  tends to 0 as  $j \rightarrow \infty$ , if  $\Omega$  is bounded; we prescribe Neumann boundary conditions on  $\partial\Omega$ . In fact  $\lambda_1 \leq C\Gamma(v_j)$  for  $\Gamma(v_j)$  small. (For compact sets  $K_j$  this is pointed out in [8].)

We also prove the more subtle converse inequality:

$$C^1\Gamma(v) \leq \lambda_1(v).$$

Thus  $\Gamma(v)$  provides a measure of the effectiveness of  $v$  as a perturber of  $\Delta$ .

THEOREM 2.1. *Let  $\Delta - v_j$  be defined on  $\Omega$ , with coercive self-adjoint boundary conditions on  $\partial\Omega$ . If  $\Gamma(v_j) \rightarrow 0$  as  $j \rightarrow \infty$ , then  $f(\Delta - v_j)u \rightarrow f(\Delta)u$  for all  $u \in L^2(\Omega)$ , where  $f$  is any bounded Borel function on  $(-\infty, 0]$  that is continuous on a neighborhood of  $\sigma(\Delta)$ .*

*Proof.* As observed in [9], it suffices to prove strong convergence of  $e^{t(\Delta - v_j)}$  to  $e^{t\Delta}$ , for some  $t > 0$ . For the sake of simplicity we suppose  $\Omega = \mathbb{R}^n$ . (See the remark at the end of Section 3.)  $e^{t(\Delta - v_j)}$  has a kernel  $p_j(x, y, t)$  given by

$$p_j(x, y, t) = E_x \left\{ \exp \left( - \int_0^t v_j(w(\tau)) d\tau \right) : w(t) = y \right\} p_0(x, y, t),$$

where  $p_0(x, y, t) = (4\pi t)^{-n/2} \exp(-|x - y|^2/4t)$  is the kernel of  $e^{t\Delta}$ . It follows that

$$\begin{aligned} \int |p_0(x, y, t) - p_j(x, y, t)| dy &= E_x \left\{ 1 - \exp \left( - \int_0^t v_j(w(\tau)) d\tau \right) \right\} \\ &\leq E_x \left\{ 1 - \exp \left( - \int_0^\infty v_j(w(\tau)) d\tau \right) \right\} \\ &= U_v(x). \end{aligned}$$

Thus if  $S$  is any compact subset of  $\mathbb{R}^n$ ,

$$\int_S \int_{\mathbb{R}^n} |p_0(x, y, t) - p_j(x, y, t)| dy dx \leq \alpha_S \Gamma(v_j),$$

where  $\alpha_S$  is the quantity defined in the proof of Proposition 1.5. Thus  $\Gamma(v_j) \rightarrow 0 \Rightarrow p_j(x, y, t)$  tends to  $p_0(x, y, t)$  in measure. Since  $p_j \leq p_0$ , strong operator convergence is a simple consequence of the Lebesgue dominated convergence theorem. ■

PROPOSITION 2.2. *Let  $-\Delta + v$  be defined by the Neumann boundary condition on  $\partial\Omega$ , with  $\lambda_1(v)$  its lowest eigenvalue. Then*

$$\lambda_1(v) \leq C\Gamma(v) \quad \text{if} \quad \Gamma(v) \text{ is small.}$$

We suppose  $\Omega$  is bounded.

*Proof.* We use the variational characterization of  $\lambda_1(v)$ :

$$\lambda_1(v) = \inf_{\phi \in H^1(\Omega)} \frac{\int_\Omega (|\nabla\phi|^2 + v\phi^2)}{\int_\Omega \phi^2}.$$

To prove the proposition it is only necessary to make a clever choice of  $\phi$ ; we take  $\phi = U_v - 1$ . Thus

$$\begin{aligned} \int_\Omega (|\nabla\phi|^2 + v\phi^2) &\leq \int_{\mathbb{R}^n} (-U_v\Delta U_v + vU_v(U_v - 1) - v(U_v - 1)) \\ &= \int d\mu_v(x) = \Gamma(v). \end{aligned}$$

On the other hand,  $\int_\Omega \phi^2 \approx \text{vol}(\Omega)$  if  $\Gamma(v)$  is small, by Proposition 1.5. This completes the proof.

PROPOSITION 2.3. *Let  $-\Delta + v$  be defined by the Neumann boundary condition on the bounded set  $\Omega$ , as above. Then*

$$C^1\Gamma(v) \leq \lambda_1(v). \tag{2.1}$$

*Proof.* We look for an estimate  $\|e^{t(\Delta-v)}\| \leq C e^{-\alpha t}$ . Let  $p(x, y, t)$  be the kernel of  $e^{t(\Delta-v)}$  on  $\Omega$ . Claim

$$\int_{\Omega} p(x, y, t) dy \leq \int_{\mathbb{R}^n} \tilde{p}(x, y, t) dy \leq M(t), \tag{2.2}$$

say, where  $\tilde{p}(x, y, t)$  is the fundamental kernel of  $e^{t(\Delta-v)}$  on  $\mathbb{R}^n$ . We postpone briefly the proof of (2.2).

As is well known, (2.2) implies that  $\|e^{t(\Delta-v)}\|_{L^2(\Omega)} \leq M(t)$ , so it remains to find a good estimate for  $M(t)$ . Let

$$U_v^t(x) = 1 - E_x \left\{ \exp \left( - \int_0^t v(w(\tau)) d\tau \right) \right\}.$$

Note that  $\lim_{t \rightarrow \infty} U_v^t(x) = U_v(x)$ . Also you can suppose

$$M(t) = \sup_x (1 - U_v^t(x)).$$

Note that

$$\begin{aligned} U_v(x) &= C_n \int \frac{d\mu_v(y)}{|x-y|^{n-2}} \Rightarrow U_v(x) \\ &\geq C_n \Gamma(v) (\text{diam } \Omega)^{2-n} \quad \text{for } x \in \Omega. \end{aligned} \tag{2.3}$$

Now we look at the difference  $U_v(x) - U_v^t(x)$ .

$$\begin{aligned} \int \tilde{p}(x, y, t) U_v^\tau(y) dy &= \int \tilde{p}(x, y, t) \left( 1 - \int \tilde{p}(y, z, \tau) dz \right) dy \\ &= \int \tilde{p}(x, y, t) dy - \int \tilde{p}(x, z, t + \tau) dz \\ &= U_v^{t+\tau}(x) - U_v^t(x). \\ \therefore U_v(x) - U_v^t(x) &= \int \tilde{p}(x, y, t) U_v(y) dy \text{ (letting } \tau \rightarrow \infty) \\ &\leq \int p_0(x, y, t) U_v(y) dy \\ &= C_n \iint \frac{d\mu_v(z)}{|z-y|^{n-2}} p_0(x, y, t) dy \\ &= C_n \int \left( \int \frac{p_0(x, y, t)}{|z-y|^{n-2}} dy \right) d\mu_v(z) \\ &\leq \Gamma(v) C_n \sup_{z \in \Omega} \int \frac{p_0(x, y, t)}{|y-z|^{n-2}} dy \\ &\leq \Gamma(v) (\alpha + \beta(x, t)) t^{-1/2}, \end{aligned} \tag{2.4}$$

where

$$\alpha = C_n \int_{\mathbb{R}^n} \frac{1}{|w|^{n-2}} e^{-|w|^2/4} dw$$

and  $\beta(x, t) \rightarrow 0$  as  $t \rightarrow \infty$ , locally uniformly in  $x$ .

Pick  $T$  so large that  $(\alpha + \beta(x, T)) T^{-1/2} < (C_n/2) (\text{diam } \Omega)^{2-n}$ . From (2.3) and (2.4) it follows that

$$M(T) \leq 1 - (C_n/2) (\text{diam } \Omega)^{2-n} \Gamma(v) = 1 - C\Gamma(v) \leq e^{-C\Gamma(v)}.$$

Hence  $e^{-T\lambda_1(v)} = \|e^{T(\Delta-v)}\| \leq e^{-C\Gamma(v)}$ , so  $\lambda_1(v) \geq (C^1/T) \Gamma(v)$ , as asserted.

It remains to establish (2.2). Since in this proposition we are not interested in the dependence of  $C^1$  on  $\Omega$ , we will assume  $\Omega$  is a cube in (2.2), without loss of generality as far as (2.1) is concerned. In such a case, let  $v \in L^2(\Omega)$  define  $\tilde{v} \in L^2_{loc}(\mathbb{R}^n)$  by the method of images, and let  $\tilde{p}(x, y, t)$  be the kernel of  $e^{t(\Delta-\tilde{v})}$  on  $\mathbb{R}^n$ . Then

$$\int_{\Omega} p(x, y, t) dy = \int_{\mathbb{R}^n} \tilde{p}(x, y, t) dy \quad \text{if } x \in \Omega,$$

and

$$\tilde{p}(x, y, t) \leq \tilde{p}(x, y, t).$$

From these, (2.2) is evident. ■

**PROPOSITION 2.4.** *If  $\Omega \subset \mathbb{R}^n$  is a given bounded open set,  $\lambda_1(K)$  is the lowest eigenvalue for  $-\Delta$  on  $\Omega - K$  with Dirichlet boundary conditions on  $\partial K$ , Neumann on  $\partial\Omega$ ,  $K \subset \Omega$  compact, then*

$$C^1 \text{cap}(K) \leq \lambda_1(K) \leq C \text{cap}(K),$$

*the latter inequality holding for  $\text{cap}(K)$  small.*

*Proof.* Same as for Propositions 2.2 and 2.3.

### 3. SOLIDIFYING POTENTIALS

In this section we examine some conditions under which a function  $f(\Delta - v_j)$  tends to  $f(\Delta_K)$ , where  $\Delta_K$  is defined on  $\Omega - K$ , with Dirichlet boundary conditions on  $\partial K$ . In such a case, one would say  $\{v_j\}$  *solidifies* to  $K$ .

We say that a compact set  $K$  is Kac-regular if almost every Brownian path that touches  $K$  spends a positive amount of time in  $K$ . If  $K$  is Kac-regular, its capacity potential is given by

$$U_K(x) = E_x \left\{ 1 - \exp \left( - \int_0^\infty V_K(w(\tau)) d\tau \right) \right\},$$

where  $V_K(x) = +\infty$  for  $x \in K$ , 0 for  $x \notin K$ .

**THEOREM 3.1.** *Let  $K \subset \mathbb{R}^n$  be Kac-regular, and suppose  $v \in \mathcal{P}^+$  is supported on  $K$ ,  $v(x) > 0$  quasi-everywhere on  $K$ . Then  $f(\Delta - \alpha v) u \rightarrow f(\Delta_K) u$  in  $L^2$  for all  $u \in L^2$ , as  $\alpha \uparrow + \infty$ , for any bounded continuous  $f$  on  $(-\infty, 0]$ .*

*Proof.* Convergence of the kernels  $p_\alpha(x, y, t)$  of  $e^{t(\Delta - \alpha v)}$  to the kernel  $p(x, y, t)$  of  $e^{t\Delta_K}$  is immediate from the formula

$$p_\alpha(x, y, t) = E_x \left\{ \exp \left( - \int_0^t \alpha v(w(\tau)) d\tau \right) : w(t) = y \right\} p_0(x, y, t).$$

From this operator convergence follows as in the proof of Theorem 2.1. ■

Such results as the preceding are known; see [7]. We now look at some more delicate solidifying phenomena. In the first case we consider,  $v$  is the sum of many potentials of small support and large amplitude.

To be more precise, let  $\Omega$  be a given region,  $K \subset \Omega$  compact. Suppose  $K$  has smooth boundary. Let  $j$  points  $\xi_1, \dots, \xi_j$ , be picked, roughly evenly spread over  $\text{int } K$ , the interior of  $K$ .

Let  $v_j(x) = V_j$  if  $|x - \xi_j| \leq r_j$ , some  $v_j(x) = 0$  otherwise. We suppose  $V_j \uparrow \infty$  and  $r_j \downarrow 0$  as  $j \rightarrow \infty$ , and ask the question: When does  $\Delta - v_j$  tend to  $\Delta_K$ ? On  $\partial\Omega$ , any coercive self-adjoint boundary condition can be placed.

This can be handled in a manner similar to that of Theorem 4.4 of [9] and the example following it. Thus if  $f \in L^2(\Omega)$  is given and if  $u_j = (1 - \Delta + v_j)^{-1} f$ , then  $\{u_j\}$  is bounded in  $H^1(\Omega)$ , so has a weak limit point  $u \in H^1(\Omega)$ , and clearly  $\Delta u = f$  on  $\Omega \setminus K$ . It is also easy to see that  $u$  satisfies the right boundary conditions on  $\partial\Omega$ . The only problem is to prove that  $u$  vanishes on  $K$ . To do this we estimate the lowest eigenvalue  $\lambda_j$  of  $-\Delta + v_j$  on  $K$ , with Neumann boundary conditions on  $\partial K$ . Since

$$\int_K |u_j|^2 \leq (\lambda_j + 1)^{-1} \int_K u_j f \leq C(\lambda_j + 1)^{-1},$$

it follows that

$$\int_K |u|^2 = 0 \quad \text{provided} \quad \lambda_j \rightarrow \infty. \tag{3.1}$$

To say that  $\xi_j$  are evenly spaced in  $\text{int } K$ , we mean  $K$  can be covered with balls  $B_{j,v}$  of radius  $R_j$  such that no  $x \in K$  belongs to more than some fixed number  $N$  of the  $B_{j,v}$ , for each  $j$ . Thus  $R_j^n \approx (c/j)$ . It is clear that  $\lambda_j$  can be estimated from above and below by  $\tilde{\lambda}_j$ , the lowest eigenvalue of  $-\Delta + \tilde{v}_j$  on  $B_{R_j} = \{x : |x| < R_j\}$ , with Neumann boundary conditions on  $\partial B_{R_j}$ , where  $\tilde{v}_j(x) = V_j$  for  $|x| < r_j$ , 0 otherwise. Since the eigenfunction corresponding to  $\tilde{\lambda}_j$  is radially symmetric, this estimate can be reduced to a one dimensional problem.



For later purposes, we treat a general one dimensional problem. Given  $v \in L^2(a, b)$ ,  $\phi \in C(a, b)$ , both  $\geq 0$ , we want to estimate

$$\inf_{u \in C^\infty[a,b]} \frac{\int_a^b (u'^2 + u^2 v) \phi \, dt}{\int_a^b u^2 \phi \, dt} = \lambda. \tag{3.2}$$

A similar problem was treated in [9, Lemma 4.5], but (3.2) is tougher to estimate since no boundary condition is imposed on  $u$ .

Suppose  $\int_a^b u^2 \phi \, dt = 1$ . Then  $u(t)^2 \geq \frac{1}{2}(1/\int \phi)$  somewhere, and in fact (picking some  $T_1 \in (a, b)$ ) either

(I)  $u(t_1) \geq 2^{-1/2} (\int \phi)^{-1/2}$  for some  $t_1 \geq T_1$ , or

(II)  $\int_a^{T_1} u^2 \phi \, dt \geq 1 - \int_{T_1}^b \frac{1}{2} \left(\int \phi\right)^{-1} \phi \, dt \geq \frac{1}{2}$ .

Case I. Suppose  $\int_a^b u^2 v \phi = \alpha$ . Then  $\exists t_0$  such that  $u(t_0)^2 \leq 2\alpha (\int_a^b v \phi)^{-1}$ , and in fact the  $v \phi \, dt$  measure of the set of such  $t_0$  is  $\geq \frac{1}{2} \int_a^b v \phi$ . Thus there exists a  $T_0$  and a  $t_0 \geq T_0$  such that

$$u(t_0) \leq (2\alpha)^{1/2} \left(\int v \phi\right)^{-1/2} \quad \text{and} \quad \int_a^{T_0} v \phi = \frac{1}{2} \int_a^b v \phi.$$

Let  $T = \min(T_0, T_1)$ . Now

$$\int_T^b |u'| \geq 2^{-1/2} \left(\int \phi\right)^{-1/2} - 2\alpha^{1/2} \left(\int v \phi\right)^{-1/2}. \tag{3.3}$$

Since

$$\int_T^b |u'| \leq \left(\int_T^b |u'|^2 \phi\right)^{1/2} \left(\int_T^b \frac{1}{\phi}\right)^{1/2},$$

it follows that

$$\begin{aligned} \int |u'|^2 \phi &\geq \left(\int_T^b |u'| \right)^2 \left(\int_T^b \frac{1}{\phi}\right)^{-1} \\ &\geq \left(\int_T^b \frac{1}{\phi}\right)^{-1} \left[ 2^{-1/2} \left(\int \phi\right)^{-1/2} - (2\alpha)^{1/2} \left(\int v \phi\right)^{-1/2} \right]^2, \end{aligned}$$

provided the right-hand side of (3.3) is positive. Let

$$A = \left(\int_T^b \frac{1}{\phi}\right)^{-1}, \quad B = \left(2 \int \phi\right)^{-1/2}, \quad C = \left(\frac{1}{2} \int v \phi\right)^{-1/2}$$

$$\begin{aligned} \therefore \int (|u'|^2 + vu^2) \phi &\geq \alpha + A[B - (\alpha)^{1/2} C]^2 \quad (\text{if } (\alpha)^{1/2} C \leq B) \\ &\geq \alpha \text{ in general.} \end{aligned} \tag{3.4}$$

If you pick  $\alpha$  to minimize the right-hand side of (3.4), this will provide a lower bound for  $\lambda$ . By elementary calculus the minimum is seen to be  $AB^2/(1 + AC^2)$ , so

$$\lambda \geq \frac{\frac{1}{2} (\int \phi)^{-1} (\int_T^b (1/\phi))^{-1}}{1 + 2 (\int v\phi)^{-1} (\int_T^b (1/\phi))^{-1}}, \tag{3.5}$$

in Case I.

Case II. If  $\int_a^{T_1} u^2 \phi dt \leq \frac{1}{2}$ , then

$$\begin{aligned} \int_a^b (|u'|^2 + u^2 v\phi) &\geq \int_a^{T_1} v u^2 \phi \\ &\geq \frac{1}{2} \inf_{[a, T_1]} v(t). \\ \therefore \lambda &\geq \frac{1}{2} \inf_{a \leq t \leq T_1} v(t), \end{aligned}$$

in Case II.

The division into Cases I and II would not have been necessary if we had taken  $T_1 = a$ . However, for some particularly interesting  $\phi$ ,  $\int_a^b (1/\phi) = \infty$ . This would lead to a poor estimate in (3.5). The kind of function  $v$  we have in mind to get a good estimate has its maximum at  $a$ , where it does not peak too fast. This will include the cases we encounter below, but it may be desirable to generalize the above one-dimensional eigenvalue estimate.

We return to the estimate for  $\tilde{\lambda}_j$ . This is of the form (3.2), with  $a = 0$ ,  $b = R_j$ ,  $\phi(t) = t^{n-1}$ , and  $v = V_j$  on  $[0, r_j]$ ,  $0$  on  $[r_j, R_j]$ . You can take  $T = T_1 = 2^{-1/n} r_j$ , and expect Case I to apply. Plugging these quantities into (3.5), evaluating, and simplifying yields ( $n \geq 3$ )

$$\begin{aligned} \tilde{\lambda}_j &\geq C' \frac{r_j^{n-2}}{R_j^n} \frac{V_j r_j^2}{1 + V_j r_j^2} \\ &\geq C_j r_j^{n-2} \frac{V_j r_j^2}{1 + V_j r_j^2}. \end{aligned}$$

We have proved the following.

PROPOSITION 3.2. *The  $v_j$  defined above solidify to  $K$ , provided*

$$j r_j^{n-2} \frac{V_j r_j^2}{1 + V_j r_j^2} \rightarrow \infty \quad \text{as} \quad j \rightarrow \infty. \tag{3.6}$$

For (3.6) to hold, it is necessary that  $j r_j^{n-2} \rightarrow \infty$ , the condition for solidification of obstacles consisting of many tiny balls found in [9]. (3.6) is perhaps an unlikely looking condition, but as we shall see, it is rather sharp.

Let us derive an estimate for  $\Gamma(v_j)$ . Since  $\Gamma$  is subadditive,

$$\Gamma(v_j) \leq j\Gamma(v_{r,j}, v_j),$$

where

$$\begin{aligned} v_{r,v}(x) &= V && \text{for } |x| < r \\ &= 0 && \text{otherwise.} \end{aligned}$$

Thus we want a good estimate for  $\Gamma(v_{r,v})$ . We claim

$$\Gamma(V_{r,v}) \approx r^{n-2} \frac{Vr^2}{1 + Vr^2} \tag{3.7}$$

in the sense that the quotient of these two quantities lies in a compact subset of  $(0, \infty)$ ,  $0 < r < \infty$ ,  $0 < V < \infty$ .

To take care of the dependence on  $r$ , we scale. Thus if  $w_r(x) = (1/r^2) w(x/r)$ , it's easy to see that  $U_{w_r}(x) = U_w(x/r)$ , so

$$\begin{aligned} \Gamma(w_r) &= \frac{1}{r^2} \int w\left(\frac{x}{r}\right) \left(1 - U_w\left(\frac{x}{r}\right)\right) dx \\ &= r^{n-2} \Gamma(w). \end{aligned}$$

Hence  $\Gamma(v_{r,v}) = r^{n-2} \Gamma(v_{1,Vr^2})$ , so to verify (3.7) we can assume  $r = 1$ ; claim  $\Gamma(v_{1,v}) \approx V/(1 + V)$ . Since  $\Gamma(v_{1,v}) \leq \text{cap } B_1$ , there is no problem for  $V \geq 1$ . Since  $\Gamma(v_{1,v}) \leq \|v_{1,v}\|_{L^1} \leq CV$ , we have

$$\Gamma(v_{1,v}) \leq C \frac{V}{1 + V} \quad \text{for } V \leq 1.$$

It remains only to show that  $\Gamma(v_{1,v})$  does not approach 0 faster than  $V$ . But

$$\Gamma(v) = V \int_{|x| \leq 1} E_x \left\{ \exp \left( - \int_0^\infty v(w(\tau)) d\tau \right) \right\} dx,$$

and for  $V$  small, clearly  $E_x \{ \exp(- \int_0^\infty v_{1,v}(w(\tau)) d\tau) \} \approx 1$ , so we have established (3.7).

It follows that, if the left-hand side of (3.6) approaches 0 as  $j \rightarrow \infty$ , then  $\Gamma(v_j) \rightarrow 0$ , so  $\{v_j\}$  fades, by Theorem 2.1.

We now give a more general criterion for solidifying. As in Proposition 3.2, let  $K = \overline{\text{int } K}$  be a compact subset of  $\Omega$ , with smooth boundary. For each  $j$ , partition  $K$  into  $n$ -cubes  $K_{j,k}$  of edge  $1/j$  (intersected with  $K$ ), sides parallel to the coordinate axes. Let  $\chi_S$  be the characteristic function of a set  $S$ .

**THEOREM 3.3.** *A sequence  $v_j \in \mathcal{P}^+$  supported by  $K$  solidifies to  $K$  as  $j \rightarrow \infty$ , provided there exists  $\alpha_j \uparrow \infty$  such that for each  $j$  and each cube  $K_{j,k}$*

$$\liminf_{l \rightarrow \infty} \Gamma(\chi_{K_{j,k}} v_l) \geq \alpha_j j^{-n}.$$

*Proof.* By (3.1) it suffices to show that  $\lambda_j$ , the lowest eigenvalue of  $-\Delta + v$  on  $K$ , with Neumann boundary condition on  $\partial K$ , tends to  $+\infty$ , and for this it suffices to show that  $\lambda_{j,k,l}$ , the lowest eigenvalue of  $-\Delta + \chi_{K_{j,k}} v_l$  on  $K_{j,k}$ , exceeds  $\beta_j \rightarrow \infty$ , as  $j \rightarrow \infty$ , for all large  $l$ , all  $k$ . In fact, scaling the  $K_{j,k}$  to unit cubes and applying Proposition 2.3 yields  $\lambda_{j,k,l} \geq C\alpha_j$ . ■

Theorem 3.3, together with the estimate (3.7), provides an alternate proof of Proposition 3.2, not using the estimate (3.5). However, the analysis via (3.5) is flexible enough to handle situations not covered by Theorem 3.3. Suppose for example that  $S \subset \mathbb{R}^n$  is a smooth closed hypersurface, possibly with smooth boundary. Let  $K_j = \{x: \text{dist}(x, S) \leq r_j\}$ , and set  $v_j(x) = V_j$  on  $K_j$ , 0 off  $K_j$ . Let  $r_j \downarrow 0$ ,  $V_j \uparrow \infty$ . We ask, when does  $\{v_j\}$  solidify to  $S$ ? This problem is treated in a manner similar to that of [9, example at end of Section 4].

Namely, if  $f \in L^2(\Omega)$  and  $u_j = (1 + v_j - \Delta)^{-1}f$ , then  $\{u_j\}$  is bounded in  $H^1(\Omega)$ . If  $u \in H^1(\Omega)$  is a weak limit point, then  $u$  satisfies the appropriate boundary conditions on  $\partial\Omega$  and  $(1 + v_j - \Delta)u = f$ , on  $\Omega \setminus S$ . It remains only to investigate when  $u|_S = 0$ .

We let  $U_H = \{x: \text{dist}(x, S) \leq H\}$  and claim that if  $V_j \uparrow \infty$  sufficiently fast, then  $\int_{U_H} |u|^2 \leq CH^2$ . Since the volume of  $U_H$  is roughly proportional to  $H$ , for  $H$  small, this implies that  $u|_S = 0$ . In order to derive such an estimate, we look for the lowest eigenvalue  $\lambda$  of  $-\Delta + v_j$  on  $U_H$ .

$$\lambda = \inf_{w \in C^\infty(U_H)} \frac{\int (|\nabla w|^2 + v_j |w|^2)}{\int |w|^2}. \tag{3.8}$$

Taking a unit vector field  $X$  normal to  $S$ , since  $\int |\nabla w|^2 \geq \int |Xw|^2$ , we can reduce the estimate of (3.8) to a one-dimensional problem of the form (3.2), with  $a = 0$ ,  $b = H$ ,  $v = V_j$  on  $[0, r_j]$ , 0 on  $(r_j, H]$ , and  $\phi = 1$ . The estimate we get, taking  $T = 0$ , is

$$\lambda \geq \frac{C}{H^2 + (1/V_j r_j)}.$$

Then  $\int_{U_H} |u_j|^2 \leq C(H^2 + (1/V_j r_j))$ , which implies

**PROPOSITION 3.4.** *The  $v_j$  defined above solidify to  $S$ , provided*

$$V_j r_j \rightarrow \infty \quad \text{as} \quad j \rightarrow \infty. \tag{3.9}$$

To see that (3.9) is sharp, let us estimate  $\Gamma(v)$ , where  $v = v_{r,v} = V$  for  $\text{dist}(x, S) \leq r$ , 0 otherwise. Since  $\Gamma(v) \leq \|v\|_{L^1}$ , clearly  $\Gamma(v) \leq Vr$ . By Theorem 2.1 it follows that the  $v_j$  fade if the left-hand side of (3.9) approaches 0 as  $j \rightarrow \infty$ .

We would like a more precise estimate of  $\Gamma(v)$ . We claim

$$\Gamma(v) \approx \frac{Vr}{1 + Vr} \tag{3.10}$$

( $0 < r < 1, 0 < V < \infty$ ,  $S$  being fixed).

Since  $\Gamma(v) \leq \text{cap}(\text{supp } v) \approx \text{cap } S$ , this is clear for  $Vr \geq 1$ , and since  $\Gamma(v) \leq \|v\|_{L^1}$ , we have  $\Gamma(v) \leq C(Vr/(1 + Vr))$  for  $Vr \leq 1$ . It remains only to show that  $\Gamma(v) \geq C(Vr/(1 + Vr))$  for  $Vr \leq 1$ . This time the scaling trick used to treat (3.7) will not work, and for this reason (3.10) seems much less straightforward. However, we have an eigenvalue estimate for  $-\Delta + v$  on  $U_H$ , and with Proposition 2.2 this yields the desired lower bound on  $\Gamma(v)$ .

For the problem treated in Proposition 3.4, there is an intermediate case, namely,  $V_j r_j = \alpha$ , a positive constant. In such a case, perturbation results of Kato [6] yield that the limiting behavior of  $f(\Delta - v_j)w$  as  $j \rightarrow \infty$  is given formally by  $f(\Delta - \alpha\mu)w$ , for  $w \in L^2(\Omega)$ , where  $\mu$  is surface measure on  $S$ . The operator  $\Delta - \alpha\mu$  is alternatively described as  $\Delta$  on  $\Omega \setminus S$ , on whose domain functions  $u$  satisfy a transmission condition

$$\begin{aligned} u_+(x) &= u_-(x), \\ \frac{\partial u_+}{\partial v_+} + \frac{\partial u_-}{\partial v_-} &= \alpha u_+(x), \quad x \in S, \end{aligned}$$

where  $u_{\pm}$  are the values of  $u$  on either side of  $S$  and  $\partial u_+/\partial v_+, \partial u_-/\partial v_-$  their normal derivatives.

It is convenient to note that solidifying or fading properties of potentials  $v_n$  can be localized in the following fashion. Suppose  $v_n \in \mathcal{P}^+$  are supported in a compact subset  $K$  of a domain  $\Omega$ , on which  $\Delta$  is defined with coercive boundary conditions, making  $\Delta$  negative self-adjoint. We'd like to say that for  $f \in L^2(\Omega)$ ,  $(1 + v_n - \Delta)^{-1}f$  converges to  $(1 - \Delta)^{-1}f$  in  $L^2(\Omega)$ , where  $\Delta$  is defined on  $\Omega$  in the case of fading  $v_n$  or on  $\Omega \setminus K$  with Dirichlet boundary condition on  $\partial K$  in the case of solidifying. Suppose the behavior of  $v_n$  is known with respect to the Laplacian on  $\mathbb{R}^m$ . As we have seen, the Feynman-Kac formula provides a nice tool for determining such behavior. It turns out that the  $v_n$  must enjoy the same sort of limiting behavior with respect to  $\Delta$  on  $\Omega$ . This is proved easily as follows.

Let  $u_n = (1 + v_n - \Delta)^{-1}f$  on  $L^2(\Omega)$ . We know that  $\{u_n\}$  is bounded in  $H^1(\Omega)$  and has a weak limit point  $u \in H^1(\Omega)$ . Passing to a subsequence,  $u_n \rightarrow u$  weakly in  $H^1(\Omega)$ . It is routine to see that  $(1 - \Delta)u = f$  on  $\Omega \setminus K$  and  $u$  satisfies the right boundary conditions on  $\partial\Omega$ . We need only see that  $u$  behaves properly in  $K$ . Pick  $\phi \in C_0^\infty(\Omega)$  such that  $\phi = 1$  on a neighborhood of  $K$ . Let

$$\begin{aligned} w_n &= \phi(1 + v_n - \Delta)^{-1}f = \phi u_n. \\ \therefore (1 + v_n - \Delta)w_n &= \phi f - (\Delta\phi)u_n - \nabla\phi \cdot \nabla u_n. \end{aligned}$$

Since  $\Delta\phi$  and  $\nabla\phi$  are supported in  $\Omega \setminus K$ , while  $u_n \rightarrow u$  weakly in  $H^1(\Omega)$ , it follows that

$$w_n \in \dot{H}^1(\mathbb{R}^m)$$

$$(1 + v_n - \Delta) w_n \rightarrow \phi f - (\Delta\phi) u - \nabla\phi \cdot \nabla u \quad \text{weakly in } L^2(\mathbb{R}^m).$$

Since the free-space behavior of  $\{v_n\}$  is presumed known, we conclude that  $w_n \rightarrow \phi u$  and

$$(1 - \Delta)(\phi u) = \phi f - (\Delta\phi) u - \nabla\phi \cdot \nabla u$$

on either  $\mathbb{R}^m$  or  $\mathbb{R}^m \setminus K$ , and  $\phi u$  satisfies the appropriate condition on  $\partial K$ . It follows that  $u$  has the correct behavior in  $K$ .

#### 4. A PROBLEM INVOLVING RANDOM POTENTIALS

The purpose of this section is to treat an analog of a problem solved in [5]. Namely, we consider  $\Delta - q_n^{(\xi)}(x)$  on  $\Omega$ , a domain in  $\mathbb{R}^3$ , where  $q_n^{(\xi)}(x)$  is a random potential on  $\Omega$  given as follows.  $\xi \in X = \Omega \times \Omega \times \dots$ , with probability measure the product of  $(1/\text{vol } \Omega)$  and the Lebesgue measure, in each factor;  $\xi = (\xi_1, \xi_2, \dots)$ . Then

$$q_n^{(\xi)}(x) = \sum_{j=1}^n v_n(x - \xi_j),$$

where  $v_n(x) = \lambda_n r_n^{-2}$  for  $|x| < r_n$ ,  $v_n(x) = 0$  otherwise. We assume that  $\lambda_n \uparrow \infty$  and  $r_n = \alpha/n$ , for some constant  $\alpha$ .

The problem is to determine the limiting behavior of  $\Delta - q_n^{(\xi)}$ , at least in probability on  $X$ . Dirichlet conditions are placed on  $\partial\Omega$ . Note that this case is intermediate between the case of solidifying treated in Proposition 3.2 and the case of fading treated in Theorem 2.1.

In [5], Kac and Luttinger treated a problem like this, with the  $q_n^{(\xi)}$  replaced by obstacles (one might say, with  $\lambda_n \equiv +\infty$ ). The analysis here is based on theirs, especially in regard to the use of the Wiener sausage. We also take a few tricks from [9]. One extra complication in the present case is that we must consider not only the probability that a Brownian path hits  $\text{supp } q_n^{(\xi)}$  but also the amount of time such a path stays in this set.

**THEOREM 4.1.** *Under the above assumptions, for any  $u \in L^2(\Omega)$ , any bounded continuous  $f$  on  $(-\infty, 0]$  we have*

$$f(\Delta - q_n^{(\xi)}) u \rightarrow f(\Delta - 2\pi\alpha (\text{vol } \Omega)^{-1}) u$$

*in  $L^2(\Omega)$  in probability on  $X$ .*

*Proof.* It suffices to show that  $\exp(t(\Delta - q_n^{(\xi)})) u \rightarrow e^{t(\Delta - \beta)} u, \beta = 2\pi\alpha/\text{vol } \Omega$ , and for this it suffices to show that their kernels converge.

The kernel of  $\exp(t(\Delta - q_n^{(\xi)}))$  is  $a_n(x, y, t, \xi) p_0(x, y, t)$ , where  $p_0(x, y, t)$  is the free space fundamental kernel of  $e^{t\Delta}$ , and

$$a_n(x, y, t, \xi) = E_x \left\{ \mathcal{E}_\Omega(w, t) \exp \left( - \int_0^t q_n^{(\xi)}(w(\tau)) d\tau \right) \mid w(t) = y \right\}.$$

Here  $\mathcal{E}_\Omega(w, t) = 1$  if  $w(\tau) \in \Omega$  for  $0 < \tau \leq t$ ,  $0$  otherwise.

Let  $\nu_{n,w}$  be the probability measure on  $[0, \infty)$  giving the distribution of  $\int_0^t \nu_n(w(\tau) - \xi_j) dt$ , as  $\xi$  varies over  $X$ . If  $\mu_{n,w}$  gives the probability distribution of  $\int_0^t q_n^{(\xi)}(w(\tau)) d\tau$ , then  $\mu_{n,w} = \nu_{n,w} * \dots * \nu_{n,w}$  ( $n$  times) and

$$\begin{aligned} \int_X \exp \left( - \int_0^t q_n^{(\xi)}(w(\tau)) d\tau \right) d\xi &= \int_0^\infty e^{-s} d\mu_{n,w}(s) \\ &= \left( \int_0^\infty e^{-s} d\nu_{n,w}(s) \right)^n. \end{aligned} \tag{4.1}$$

Our first goal will be to examine  $\nu_{n,w}$  carefully. Note that  $\nu_{n,w}(\{0\})$  is the probability that the random point  $\xi_j \in \Omega$  does not lie in the set

$$W_{r_n,w}(0, t) = \{x \in \Omega : |x - w(\tau)| \leq r_n \text{ for some } \tau \in [0, t]\}.$$

This set is known as the ‘‘Wiener sausage.’’ Thus

$$1 - \nu_{n,w}(\{0\}) = \text{vol } W_{r_n,w}(0, t) / \text{vol } \Omega.$$

Now there are precise estimates on the volume of the Wiener sausage (see [5, 11, 13]). If we replace  $r_n$  by a subsequence, we can suppose that for almost all paths  $w$ ,

$$\begin{aligned} \frac{\text{vol } W_{r_n,w}(0, t)}{r_n} &\rightarrow 2\pi t. \\ \therefore \nu_{n,w}(\{0\}) &= 1 - \frac{2\pi\alpha t}{\text{vol } \Omega} \frac{1}{n} + o\left(\frac{1}{n}\right). \end{aligned}$$

We would like to establish that

$$\int_{(0,\infty)} e^{-s} d\nu_{n,w}(s) = o\left(\frac{1}{n}\right) \tag{4.2}$$

in probability on path space.

Granted this, it follows immediately that (4.1) is equal to

$$\left(1 - \frac{2\pi\alpha t}{\text{vol } \Omega} + o\left(\frac{1}{n}\right)\right)^n, \tag{4.3}$$

in probability on path space.

To prove (4.2), it suffices to show that

$$r_n^{-1} \nu_{n,w}((0, a)) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

in probability on path space. But this quantity is approximately proportional to the probability that  $\int_0^t v_n(w(\tau) - \xi_j) d\tau \in (0, a)$ , given  $\xi_j \in W_{r_n, w}(0, t)$ . Call this function  $G_{n,a}(w)$ . Thus

$$G_{n,a}(w) = \text{prob. that } (\lambda_n/r_n^2) \times \text{time path } w \text{ spends within of } \xi_j, \text{ in time interval } (0, t) \in (0, a), \text{ given } \xi_j \in W_{r_n, w}(0, t),$$

the probability being with respect to  $\xi \in X$ . Now if

$$H_{n,b}(w) = \text{prob. that } ((1/r_n^2) \times \text{time path } w \text{ spends within } r_n \text{ of } \xi_j, \text{ in time interval } (0, t) \in (0, b), \text{ given } \xi_j \in W_{r_n, w}(0, t),$$

then scaling implies that the probability distribution of  $H_{n,b}$ , as a random variable on path space, is roughly independent of  $n$ . More precisely, if the above quantity were denoted  $H_{n,b}(w, t)$ , then  $H_{n,b}(\cdot, t)$  and  $H_{1,b}(\cdot, r_n^{-2}t)$  have the same probability distribution. Thus

$$P_x\{G_{n,a}(w) \geq \epsilon\} = P_x\{H_{n,a/\lambda_n}(w) \geq \epsilon\} \rightarrow 0$$

as  $n \rightarrow \infty$ , which establishes (4.2), and hence (4.3); passing to the limit in (4.3) yields

$$\int_X \exp\left(-\int_0^t q_n^{(\xi)}(w(\tau)) d\tau\right) d\xi \rightarrow \exp(-2\pi\alpha t/\text{vol } \Omega), \quad \text{as } n \rightarrow \infty$$

in probability on path space.

$$\therefore \int_X a_n(x, y, t, \xi) d\xi \rightarrow \exp(-2\pi\alpha t/\text{vol } \Omega) \mathcal{E}_\Omega(w),$$

so

$$\int_X \exp(t(\Delta - q_n^{(\xi)})) u d\xi \rightarrow e^{t(\Delta - \beta)} u, \quad \text{in } L^2(\Omega). \tag{4.4}$$

The limit formula (4.4) gives convergence of the *means* of the semigroups  $\exp(t(\Delta - q_n^{(\xi)}))$ ; from this, convergence in probability follows, using the semigroup property. This proof is given in [9, Lemma 6.5]. ■



Combining the above reasoning with the methods of [9], it can be proved that if the  $\xi_j$  are picked in  $\Omega$  according to a nonuniform continuous probability density  $\rho(x)$ , then  $\exp(t(\Delta - q_n^{(\xi)})) \rightarrow e^{t(\Delta - 2n\alpha)}$  strongly in probability on  $X$ .

Suppose that  $q_n(x)$  is supported on balls of radius  $r_n \rightarrow 0$  evenly spaced through  $\Omega$ , of total volume  $\beta \text{ vol } \Omega$ ,  $\beta$  a constant, and  $q_n(x) = \gamma$  on these balls. Then it is easy to see that

$$e^{t(\Delta - \sigma_n)} \rightarrow e^{t(\Delta - \beta\gamma)} \text{ strongly, as } n \rightarrow \infty. \tag{4.5}$$

There are other ways to let  $r_j \rightarrow 0$ ,  $V_j \rightarrow \infty$  so that the left side of (3.6) is constant. Surely there is a general result, containing (4.5) and Theorem 4.1 as special cases, but we do not know how to prove it.

### 5. $\alpha$ -REGULARITY OF COMPACT SETS

One major question we have in mind in this section is the following. If  $v_j(x) \rightarrow +\infty$  for  $x \in K$  and  $v_j(x) \rightarrow 0$  for  $x \notin K$ , when can one say that  $\Gamma(v_j) \rightarrow \text{cap } K$ ? If  $K$  is the closure of an open set with smooth boundary, the situation is simple, say, for  $v_j(x) = j\chi_K(x)$ . However, if  $K$  consists of a smooth hypersurface, then  $\Gamma(j\chi_K) = 0$ , while  $\text{cap}(K) > 0$ . This is related to the fact that while Brownian paths have a positive probability of hitting such  $K$ , they will not spend a positive amount of time in  $K$ . On the other hand, if  $v_j(x) = \lambda_j$  on  $\{x: \text{dist}(x, K) < 1/j\}$ , 0 off this set, results from Section 3 make it likely that  $\Gamma(v_j) \rightarrow \text{cap } K$  for a smooth surface  $K$ , provided  $\lambda_j/j \rightarrow \infty$ . As we see below, this is the case.

Stroock [12] has defined a notion of regularity that would treat functions  $v_j = j\chi_K$ . Namely, a compact set  $K$  is Kac-regular if almost every Brownian path that hits  $K$  spends a positive amount of time in  $K$ . To handle more general  $v_j$  as indicated above, we introduce here a couple of one-parameter families of notions of regularity, dealing with how long a path, hitting  $K$ , stays within a small neighborhood of  $K$ .

If  $K \subset \mathbb{R}^n$  is compact, let  $K_r = \{x: \text{dist}(x, K) \leq r\}$ . If  $w$  is a path, let  $\beta_{r,t}(w) =$  time spent in  $K_r$ , in time interval  $(0, t)$ , i.e.,

$$\beta_{r,t}(w) = \int_0^t \chi_{K_r}(w(\tau)) \, d\tau.$$

DEFINITION.  $x$  is  $\alpha$ -regular for  $K$  if any  $r_\nu \rightarrow 0$  has a subsequence  $r_j \rightarrow 0$  such that

$$P_x\{\liminf_{r_j \downarrow 0} r_j^{\alpha-2} \beta_{r_j,t}(w) > 0\} = 1, \quad \text{each } t > 0. \tag{5.1}$$

We suppose  $0 \leq \alpha \leq 2$ . Stroock calls  $x \in K$  “ $s$ -ext. regular” if almost all paths starting at  $x$  spend a positive amount of time in  $K$ , on any time interval  $(0, t)$ .

PROPOSITION 5.1.  $x \in K$  is 2-regular if and only if  $x$  is  $s$ -ext. regular.

*Proof.* The only nontrivial implication is that a 2-regular point  $x$  is  $s$ -ext. regular, so assume  $x$  is 2-regular. Let  $S_{r,t}(w) = \{s \in (0, t): w(s) \in K_r\}$ . Then there is a function  $\beta_t(w) > 0$  a.e. on path space such that

$$P_x\{\text{meas } S_{r_j,t}(w) \geq \beta_t(w), \text{ all } r_j > 0\} = 1.$$

If  $w$  is a path such that  $\text{meas } S_{r_j,t}(w) \geq \beta_t(w)$ , all  $j$ , then since  $S_{r_j,t}(w)$  are decreasing sets as  $r_j \downarrow 0$ , it follows that  $\text{meas } \bigcap_{r_j > 0} S_{r_j,t}(w) \geq \beta_t(w) > 0$ . But  $w(s) \in K$  if  $s \in \bigcap_j S_{r_j,t}(w)$ , so  $w$  spends a positive amount of time in  $K$ . ■

It will be desirable to weaken one notion of  $\alpha$ -regularity.

DEFINITION.  $x$  is weakly  $\alpha$ -regular for  $K(0 \leq \alpha \leq 2)$  if

$$E_x \left\{ \exp \left( -\lambda(r) r^{\alpha-2} \int_0^t \chi_{K_r}(w(\tau)) d\tau \right) \right\} \rightarrow 0, \tag{5.2}$$

as  $r \downarrow 0$ , for any  $\lambda$  such that  $\lambda(r) \uparrow \infty$  as  $r \downarrow 0$ .

Otherwise stated, the condition is that  $\beta_{r,t}(w)$  goes to zero almost as slowly as  $r^{2-\alpha}$ , in probability. Clearly any  $\alpha$ -regular  $x$  is weakly  $\alpha$ -regular.

PROPOSITION 5.2. Any  $x \in K$  is weakly 0-regular, for any compact  $K$ .

*Proof.* It suffices to consider  $K = \{x\}$ . Brownian scaling shows that

$$E_x\{\exp(-\lambda(r) r^{-2}\beta_{r,t}(w))\} = E_x\{\exp(-\lambda(r) \beta_{1,r^{-2}t}(w))\} \rightarrow 0 \quad \text{as } r \rightarrow 0. \quad \blacksquare$$

DEFINITION. A compact  $K$  is  $\alpha$ -regular if every  $x \in K$  is  $\alpha$ -regular, with the exception of a polar set.  $K$  is weakly  $\alpha$ -regular if given  $\lambda(r) \uparrow \infty$ , every  $r_v \rightarrow 0$  has a subsequence  $r_j \rightarrow 0$  such that  $E_x\{\exp(-\lambda(r_j) r_j^{\alpha-2}\beta_{r_j,t}(w))\} \rightarrow 0$  quasi-everywhere on  $K$ .

Thus every  $K$  is weakly 0-regular, while  $K$  is strongly 2-regular if and only if it is Kac-regular. Our next goal is to show what you can do with weakly  $\alpha$ -regular sets. Then we will give examples.

THEOREM 5.3. *Suppose  $K$  is weakly  $\alpha$ -regular. Let  $\lambda_n \uparrow \infty$  and*

$$\begin{aligned} v_n(x) &= \lambda_n r_n^{\alpha-2} && \text{on } K_{r_n} \\ &= 0 && \text{off} \end{aligned} \tag{5.3}$$

$r_n \downarrow 0$ . Then  $U_{v_n}(x) \rightarrow U_K(x)$  quasi-everywhere and  $\Gamma(v_n) \rightarrow \text{cap } K$ .

*Proof.* Let  $H$  be the set of paths that hit  $K$ .

$$\begin{aligned} U_{v_n}(x) &= E_x \left\{ 1 - \exp \left( - \int_0^\infty v_n(w(\tau)) d\tau \right) \right\} \\ &= E_x \left\{ \left( 1 - \exp \left( - \int_0^\infty v_n(w(\tau)) d\tau \right) \right) \chi_H(w) \right\} \\ &\quad + E_x \left\{ \left( 1 - \exp \left( - \int_0^\infty v_n(w(\tau)) d\tau \right) \right) \chi_{cH}(w) \right\} \\ &= A_n + B_n. \end{aligned}$$

Thus  $A_n$  and  $B_n$  are the Wiener integrals over the sets of paths that hit or miss  $K$ . Clearly  $B_n \rightarrow 0$  as  $n \rightarrow \infty$ , so we must show that  $A_n \rightarrow U_K(x)$ . Now

$$\begin{aligned} A_n &= U_K(x) - E_x \left\{ \exp \left( - \int_0^\infty v_n(w(\tau)) d\tau \right) \chi_H(w) \right\} \\ &= U_K(x) - A_n', \end{aligned}$$

so we need  $A_n' \rightarrow 0$  as  $n \rightarrow \infty$ . Passing to a subsequence  $j \rightarrow \infty$ , almost every path that hits  $K$  has its first hitting point at a  $y \in K$  such that  $E_y \{ \exp(- \int_0^\infty v_j(w(\tau)) d\tau) \} \rightarrow 0$ . By the strong Markov property this implies that  $A_j' \rightarrow 0$ .

Now that  $U_{v_j}(x) \rightarrow U_K(x)$  is established, the convergence of  $\Gamma(v_n)$  to  $\text{cap } K$  is a simple argument using (1.1). ■

Above we have discussed 0-regular and 2-regular sets. Now we give some examples of 1-regular sets.

PROPOSITION 5.4. *Let  $S$  be a piece of an  $n - 1$  dimensional linear subspace of  $\mathbb{R}^n$ , with smooth boundary. Then  $S$  is 1-regular.*

*Proof.* We may as well suppose  $S = \{x \in \mathbb{R}^n: x_1 = 0 \text{ and } (x_2, \dots, x_n) \in \Omega\}$ . We use the Brownian local time for one dimensional Brownian motion to treat this problem. Local time is defined as follows (see [4]),

$$\begin{aligned} l(w, t) &= \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_0^t \chi_{(-\epsilon, \epsilon)}(w(\tau)) d\tau \\ &= \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \quad (\text{time path spends in } (-\epsilon, \epsilon), \text{ in time interval } (0, t)). \end{aligned}$$

For almost all paths  $w$  with  $w(0) = 0$ , this limit exists and is  $> 0$ .

Now if a path  $w$  on  $\mathbb{R}^n$  starts out at  $(0, p_0)$ ,  $p_0 \in \text{int } \Omega$ , say,  $w = (w_1, w')$ , then let  $\tau$  be the first exit time of  $w'$  from  $\Omega$ . Since  $l(w_1, t \wedge \tau) > 0$  for almost all  $w_1$ , it follows that

$$P_{(0, p_0)} \left\{ \lim_{r \downarrow 0} \frac{1}{r} \beta_{r, t}(w) > 0 \right\} = 1,$$

so  $S$  is 1-regular. ■

We would conjecture that any smooth hypersurface  $S$  of  $\mathbb{R}^n$  is 1-regular. If  $S$  is a sphere, this can be proved in a manner as above, using the diffusion local time associated with the Bessel process, but this argument won't go through in the general case without some extra effort. One technical obstruction is that if  $y$  is a normal coordinate to  $S$ ,  $y(w)$  need not be a diffusion process. As we shall see below, a smooth hypersurface  $S$  is weakly 1-regular. First, a very general result.

**PROPOSITION 5.5.** *Suppose  $v_n$  solidifies to  $K$  in the sense of Section 3. Then passing to subsequences  $U_{v_n}(x) \rightarrow U_K(x)$  quasi-everywhere, and  $\Gamma(v_n) \rightarrow \text{cap } K$ . If any  $v_n$  of the form (5.3) solidifies to  $K$ , it follows that  $K$  is weakly  $\alpha$ -regular.*

*Proof.* Letting  $\Delta_K$  denote the Laplacian on  $\mathbb{R}^n \setminus K$ , with Dirichlet boundary conditions on  $K$ , we know that  $(1 - \Delta + v_n)^{-1} u \rightarrow (1 - \Delta_K)^{-1} u$  weakly in  $H^1$  for every  $u \in L^2$ . In fact, the convergence is in the strong topology of  $H^1(\mathbb{R}^n)$ . Indeed, letting  $f_n = (1 - \Delta + v_n)^{-1} u$ , we have

$$\|f_n\|_{H^1}^2 = (u, f_n) - \int v_n |f_n|^2, \quad \text{and} \quad (u, f_n) \rightarrow \|(1 - \Delta_K)^{-1} u\|_{H^1}^2,$$

so

$$\overline{\lim}_{n \rightarrow \infty} \|f_n\|_{H^1}^2 \leq \|(1 - \Delta_K)^{-1} u\|_{H^1}^2,$$

from which norm convergence is a consequence of weak convergence, as is well known. From this it is a simple consequence that

$$e^{t(\Delta - v_n)} u \rightarrow e^{t\Delta_K} u \quad \text{in } H^1(\mathbb{R}^n), \quad \forall u \in L^2(\mathbb{R}^n). \quad (5.4)$$

Now an element of  $H^1$  is defined pointwise except for a set of capacity zero (see [2, p. 307]), and passing to a subsequence in (5.4), we have pointwise convergence quasi-everywhere. Taking  $u = u_j$  to be the characteristic function of  $\{x \in \mathbb{R}^n: |x| \leq j\}$ , it follows that

$$E_x \left\{ \exp \left( - \int_0^t v_n(w(\tau)) d\tau \right) \right\} \rightarrow P_x(w(\tau) \notin K, 0 \leq \tau \leq t)$$

for all except for a polar set  $S$ , which establishes the first two assertions. If  $x \in K \setminus S$  and  $x$  is not an irregular point of  $K$ , you have

$$E_x \left\{ \exp \left( - \int_0^t v_n(w(\tau)) d\tau \right) \right\} \rightarrow 0,$$

which establishes the last assertion. ■

**COROLLARY 5.6.** *If  $S$  is a smooth hypersurface of  $\mathbb{R}^n$ ,  $S$  is weakly 1-regular.*

*Proof.* This is an immediate consequence of Propositions 5.5 and 3.4. ■

It seems that the degree of regularity of a set is connected to its dimension (or codimension). As another instance of this, we have the following.

**PROPOSITION 5.7.** *If  $\text{vol } K_r = o(r^{2-\alpha})$  as  $r \downarrow 0$ , then no  $x \in K$ , except perhaps for a polar set, is weakly  $\alpha$ -regular, and  $K$  is not weakly  $\alpha$ -regular unless  $\text{cap } K = 0$ .*

*Proof.* Let  $v_r(x) = \lambda(r) r^{\alpha-2}$ ,  $x \in K_r$ , 0 otherwise, where  $\lambda(r) \uparrow \infty$  but  $\lambda(r) r^{\alpha-2} \text{vol } K_r \rightarrow 0$ . Then  $\Gamma(v_r) \leq \|v_r\|_{L^1} \rightarrow 0$  as  $r \rightarrow 0$ , so by Proposition 1.5  $U_{v_r} \rightarrow 0$  in  $H_{\text{loc}}^1$ . Thus passing to a subsequence  $r_j \rightarrow 0$ ,  $U_{v_{r_j}}(0) \rightarrow 0$  for all  $x$  not in some polar set  $S$ .

$$\therefore E_x \left\{ \exp \left( - \int_0^\infty v_r(w(\tau)) d\tau \right) \right\} \rightarrow 1$$

as  $r = r_n \downarrow 0$ , for all  $x \notin S$ , which contradicts weak  $\alpha$ -regularity. ■

As a converse to this last proposition, one might expect something like the following: The set of points in  $K$  that are not weakly  $\alpha$ -regular have Hausdorff  $n - (2 - \alpha)$  dimensional measure zero. The truth may be a bit more complicated.

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