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Global Existence of Solutions to Nonlinear Hyperbolic Systems of Conservation Laws*

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1. INTRODUCTION

We introduce a new method of constructing solutions to the Cauchy problem for nonlinear hyperbolic systems of conservation laws in one space dimension. We consider systems which are strictly hyperbolic and genuinely nonlinear in the sense of Lax [10]. We present the method here in the setting of systems of two conservation laws,

$$U_t + G(U)_x = 0, \qquad -\infty < x < \infty, \quad t > 0, \tag{1.1}$$

where $U = \{{u_1 \atop u_2}\}$ and $G = \{{g_1 \atop g_2}\}$ is smooth nonlinear mapping from R^2 to R^2.

It is well known that, in general, smooth solutions of the Cauchy problem do not exist for all time no matter how regular the initial data. As a consequence we seek a weak solution of the Cauchy problem, i.e., a bounded measurable vector-field $U = U(x, t)$ which takes on the prescribed data at $t = 0$ and satisfies (1.1) in the sense of distributions.

Since weak solutions are not uniquely determined by their initial data, one usually imposes an admissibility condition on solutions. Following Lax [10], we impose an "entropy" admissibility condition; i.e., we require solutions to satisfy the entropy inequality,

$$\eta(U)_t + q(U)_x \leqslant 0, \tag{1.2}$$

in the sense of distributions, where $\eta = \eta(U)$ is a convex entropy for (1.1) with associated entropy flux q.

Solutions of the Cauchy problem with arbitrary initial data having small total variation exist by a theorem of Glimm [6]. These solutions are constructed as the limit of a sequence of approximate solutions V_h which consist locally of solutions of Riemann problems, i.e., of Cauchy problems with data

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$U_0(x) = U_l$ for $x < 0$, $U_0(x) = U_r$ for $x > 0$ where U_l and U_r are constant states. The functions V_h are approximate solutions in the sense that the equations and the entropy condition are satisfied modulo certain error terms (that approach zero as the parameter h approaches zero).

The process of constructing V_h begins at $t = 0$ with a piecewise constant approximation $V_h(x, 0)$ of the prescribed initial data. The function V_h is defined in the strip $0 \leqslant t < t_h$ by solving the Riemann problems generated along the line $t = 0$ by the discontinuities of $V_h(x, 0)$. The time t_h is chosen so small that the shock and rarefaction waves generated by the discontinuities of $V_h(x, 0)$ do not interact in the strip $0 \leqslant t < t_h$ and, consequently, V_h is an exact solution in $0 \leqslant t < t_h$. The function V_h is redefined at time $t = t_h$ as a piecewise constant function $V_h(x, t_h)$ and then extended into the strip $t_h \leqslant t < 2t_h$ by solving the Riemann problems generated by $V_h(x, t_h)$. The continuation of this process yields a globally defined approximate solution V_h which is an exact solution in each strip $nt_h \leqslant t < (n + 1)\, t_h$.

The method of redefining V_h at the interfaces $t = nt_h$ has a probabilistic feature: the values of V_h along the line $t = nt_h$ depend upon a random choice of mesh points. Using uniform variational estimates, it follows, by a compactness argument, that there exists a subsequence V_{h_k} which converges to a solution of the Cauchy problem for almost all choices of mesh points.

We present here a method of constructing solutions of the Cauchy problem as the limit of a sequence of piecewise constant approximating solutions U_h. Each vector-field U_h is an exact weak solution but is an "approximating" solution in the sense that the entropy condition is only satisfied modulo an error term. We show that the error term is of order h and, thus, approaches zero as the parameter h approaches zero.

The global construction process for U_h begins at $t = 0$ with a piecewise constant approximation $U_h(x, 0)$ of the prescribed data. In the neighborhood of each point of discontinuity of $U_h(x, 0)$, we solve the generated Riemann problems within the class of piecewise constant functions. Under this constraint, we are forced to introduce into the solution lines of discontinuity which do not satisfy the entropy condition. However, by the method of construction, these discontinuities are very weak and contribute in toto only an error of order h to the entropy condition.

The waves generated by the discontinuities of $U_h(x, 0)$ evolve until the time t_1 at which the first set of wave interactions takes place. Along the line $t = t_1$, U_h is a piecewise constant function and, as such, generates a family of Riemann problems. The solution U_h is continued beyond t_1 by solving the latter Riemann problems within the class of piecewise constant functions and, then, allowing the generated waves to evolve until the time t_2 at which the next set of interactions takes place, etc.

We show that the process of regenerating the solution by solving local

Riemann problems yields an approximating solution U_h that is globally defined and that contains, moreover, only a finite number of discontinuities in any compact subset of the x–t plane. It follows, from a natural adaptation of the estimates and compactness argument of Glimm [6], that there exists a subsequence U_{h_k} which converges to a solution of the Cauchy problem with the prescribed data.

We are concerned with systems of two laws and data with small total variation. The method of Glimm has been used to prove existence of solutions to special classes of systems of two laws with data having large total variation [1, 4, 5, 8, 9, 12, 13] and to the general system of n laws data having small total variation [6]. In principle, our scheme is applicable to both of these cases. The scheme is also applicable to a single conservation law and reduces, in that case, to a method which is similar to the method of polygonal approximations introduced by Dafermos [2].

We note that our method is adaptable for numerical calculations but that more information is necessary to determine the efficiency of the method.

2. Preliminaries

We recall that system (1.1) is strictly hyperbolic if the 2×2 matrix $dG(U)$ has real and distinct eigenvalues $\lambda_1(U) < \lambda_2(U)$. Since we are concerned here with solutions having small variation, we may assume without loss of generality that

$$\lambda_1(U) < 0 < \lambda_2(U). \tag{2.1}$$

We also recall that (1.1) is genuinely nonlinear in the sense of Lax if the right eigenvectors $r_j(U)$, $j = 1, 2$, of dG satisfy $r_j \cdot \nabla\lambda_j \neq 0$. Following Lax, we normalize the direction of r_j by requiring

$$r_j \cdot \nabla\lambda_j > 0, \qquad j = 1, 2. \tag{2.2}$$

Since the Riemann problem plays a central role in the construction of solutions, we briefly recall, in the case of two laws, the structure of the solution constructed by Lax [10]. Let U_l, U_r be the Riemann data. The solution is a function of the ratio x/t and consists of three constant states U_l, U_m and U_r. If the middle state U_m is distinct from the left state U_l, then U_l is connected to U_m by either a 1-shock or a 1-rarefaction wave. If U_m is distinct from the right state U_r, then U_m is connected to U_r by either a 2-shock or a 2-rarefaction wave. By condition (2.1), 1-waves propagate to the left in the x–t plane and 2-waves to the right.

The elementary waves of (1.1) are themselves governed by a pair of approximate conservation laws. These laws are the key to the variational estimates on solutions which establish BV and L^1 as the natural function spaces for (1.1).

In connection with the quantitative measurement of the magnitude or "strength" of waves (and for the purpose of constructing solutions to the Riemann problem), special coordinate functions w_j, $j = 1, 2$, are introduced in a natural association with the jth mode of propagation.

DEFINITION. A coordinate transformation $(u_1, u_2) \to (w_1, w_2)$ is said to consist of Riemann invariants if

$$r_i \cdot \nabla w_j = 0, \qquad i \neq j, \tag{2.3}$$

and

$$r_j \cdot \nabla w_j > 0, \qquad j = 1, 2. \tag*{(2.3)$'$}$$

We note that (2.3) is the statement that w_j is an i-Riemann invariant. Thus, $(u_1, u_2) \to (w_1, w_2)$ maps smooth solutions of (1.1) onto smooth solutions of the characteristic system

$$\frac{\partial}{\partial t} w_1 + \lambda_1 \frac{\partial}{\partial x} w_1 = 0, \qquad \frac{\partial}{\partial t} w_2 + \lambda_2 \frac{\partial}{\partial x} w_2 = 0.$$

Let v_l and v_r denote the states on the left and right of a j-wave. Following Glimm [6], we call the quantity

$$\epsilon_j = w_j(v_r) - w_j(v_l) \tag{2.4}$$

the magnitude of the j-wave. Using the normalizations (2.2) and (2.3)$'$, it follows that w_j and λ_j increase (decrease) from left to right across a j-rarefaction wave (j-shock) and, therefore, that rarefaction waves have positive magnitude and shocks negative.

Loosely stated, the approximate conservation laws govern the magnitudes of elementary waves "during interactions." In order to formulate a precise statement, we consider two solutions U and V of the Riemann problem with initial data U_l, U_m and U_m, V_r respectively. Let $\gamma = (\gamma_1, \gamma_2)$ and $\delta = (\delta_1, \delta_2)$ denote the magnitude of waves U and V. We define the interaction of γ and δ as the pair $\epsilon = (\epsilon_1, \epsilon_2)$ consisting of the magnitudes ϵ_j of the j-waves which occur in the solution of the Riemann problem with data U_l, V_r.

For convenience, we shall identify waves with their magnitudes by calling γ_j and δ_j the incoming j-waves of the interaction and ϵ_j the outgoing j-wave. Now, the approximate laws state that the magnitudes of waves in an interaction are conserved up to linear terms and that the deviation from linearity is governed by a term of higher order involving only those incoming waves which are approaching in the following sense of Glimm.

DEFINITION. Incoming waves γ_k and δ_j are said to approach if

$$k > j$$

or if

$$k = j \quad \text{and} \quad \delta_k < 0 \quad \text{or} \quad \gamma_j < 0.$$

Specializing [6, Theorem 2.1] to systems of two laws, we have

THEOREM 2.1. *In the interaction of incoming waves γ and δ, the outgoing waves satisfy*

$$\epsilon_i = \gamma_i + \delta_i + O(\tau)\, D(\gamma, \delta), \qquad i = 1, 2, \tag{2.5}$$

where $\tau = \max\{|\gamma_i|, |\delta_k|\}$ *and*

$$D(\gamma, \delta) = \sum \{|\gamma_i|\,|\delta_k| : \gamma_i \text{ and } \delta_k \text{ are approaching}\}.$$

We note that pairs of j-rarefaction waves are excluded from D by the fact that their member waves do not meet in the x–t plane, when separated by a constant state, as a consequence of the equality of the slopes of the j-characteristics bounding the constant state. However, pairs of j-shocks and pairs consisting of a j-shock and a j-rarefaction wave are included in D by the fact that their member waves do meet in the x–t plane, when separated by a constant state, as a consequence of the Lax shock conditions. Secondly we note that the quantity D provides a measure of the strength of all wave interactions in the interaction of two solutions of Riemann problems.

The approximate conservation laws reveal two basic mechanisms of decay of solutions: the interaction of j-shocks (negative magnitude) with j-rarefaction waves (positive magnitude) and the recession of the outgoing waves after interaction. An interrelationship between these two mechanisms is expressed by the Glimm functional F_1. We recall the definition of F_1:

$$F_1(J) = L(J) + \text{const}\, Q(J), \tag{2.6}$$

where

$$L(J) = \sum \{|\gamma| : \gamma \text{ crosses } J\},$$

$$Q(J) = \sum \{|\gamma|\,|\delta| : \gamma \text{ and } \delta \text{ cross } J \text{ and approach}\}.$$

With respect to the approximate solutions of Glimm, the functional F_1 is defined on an order family of spacelike polygonal arcs J. Each arc J lies in some strip $0 \leqslant t < T < \infty$ and extends from $x = -\infty$ to $x = +\infty$. With respect to our approximating solutions, F_1 is defined on a similar ordered family of polygonal arcs.

In both of the frameworks above, F_1 decreases monotonically as J moves toward larger time. Since F_1 dominates the total variation norm, the decay of F_1 implies uniform estimates on the total x-variation of solutions along any line $t = t_0$ in terms of the total variation of the initial data.

In general, L is not a decreasing functional. However, Q is a decreasing functional whose decay and global nature can be employed to measure the strength of all wave interactions in the x–t plane and, thereby, extend the local conservation laws to global laws. The extension given by Glimm and Lax [7] is expressed in terms of quantities $D(\Lambda)$ and $C(\Lambda)$ which measure, respectively, the strength of all wave interactions and all wave cancellations in a domain Λ of the x–t plane. The quantity $D(\Lambda)$ is defined by

$$D(\Lambda) = \sum_{I \in \Lambda} D_I, \tag{2.7}$$

where the sum is taken over all interactions I and Λ and where

$$D_I = \sum \{ | \gamma | \, | \delta | : \gamma \text{ and } \delta \text{ are the incoming and approaching waves of } I \}.$$

Preliminary to defining $C(\Lambda)$, the local laws (2.5) are rewritten in [7] as laws separately governing the magnitudes of shock and rarefaction waves of the ith kind during interaction:

$$\begin{aligned} L_i^+ &= E_i^+ - C_i + O(\tau)\, D(\gamma, \delta), \\ L_i^- &= E_i^- + C_i + O(\tau)\, D(\gamma, \delta). \end{aligned} \tag{2.8}$$

Here $E_i^\pm$ and $L_i^\pm$ denote the sums of all the magnitudes of the incoming (E) and outgoing (L) shock ($-$) and rarefaction ($+$) waves of the ith kind; the quantity

$$C_i \overset{\text{def}}{=} \tfrac{1}{2}(| \gamma_i | + | \delta_i | - | \gamma_i + \delta_i |)$$

denotes the amount of cancellation experienced by the ith mode and $\tau = \max_{i=1,2}\{| \gamma_i |, | \delta_i |\}$.

Summing the local laws (2.8) over all interactions in the domain Λ yields the global laws,

$$L_i^\pm(\Lambda) = E_i^\pm \mp C_i(\Lambda) + O(\tau)\, D(\Lambda), \tag{2.9}$$

which relate the total strength of all shock ($-$) and rarefaction ($+$) waves of the ith kind entering (E) and leaving (L) the domain Λ to the quantities $D(\Lambda)$ and

$$C_i(\Lambda) = \sum_{\Lambda} C_i,$$

the total amount of cancellation of waves in the ith mode of propagation. Moreover, it is observed in [7] that there exists a small constant with the property that

$$(\text{oscillation of } U_0)\ V(L) \leqslant \text{const}$$

implies

$$D(\Lambda) \leqslant 2V^2(L), \tag{2.10}$$

where $V(L)$ is the total variation of the initial data U_0 over an interval L whose domain of determinancy contains Λ. The estimate (2.10) is essential for this paper.

3. Existence

We discuss first the local construction of the approximating solutions. For this purpose, we recall the following facts. Along a line of discontinuity $x = x(t)$, a solution U satisfies the Rankine–Hugoniot relations,

$$s[u_1] = [g_1], \qquad s[u_2] = [g_2],$$

where square bracket denotes the value on the left side of $x(t)$ minus the value on the right and $s = dx/dt$ the speed of propagation of $x(t)$. A line of discontinuity propagating to the left (right) in the x–t plane is called a 1-shock (2-shock) if it satisfies the entropy condition $[w_1] < 0$ ($[w_2] < 0$). Under the normalization (2.3)$'$, these entropy conditions are equivalent to the Lax shock conditions. Alternatively, a line of discontinuity propagating to the left (right) is called a 1-rarefaction shock (2-rarefaction shock) if it satisfies $[w_1] > 0$ ($[w_2] > 0$).

By the same method as in [10], it is possible to construct a solution U of the Riemann problem which is a piecewise constant function of the ratio x/t and which consists of a 1-wave propagating to the left and/or 2-wave propagating to the right. Let U_l, U_r be the Riemann data. The configuration of waves in U is given as follows. The half-plane $t \geqslant 0$ is the union of the closures of consecutive sectors $S_k = \{(x, t)\colon a_k < x/t < a_{k+1}\}$, $k = 1, 2, 3$, which are constant states of U. The sector S_1 is the constant state U_l, and S_3 is the constant state U_r. If U takes on distinct values in S_j and S_{j+1}, the common boundary of S_j and S_{j+1} is a j-shock or a j-rarefaction shock. We call this class of piecewise constant solutions the class K. It is the basis for the local construction of the approximating solutions U_h.

We define the magnitude of a j-rarefaction shock by (2.4) and note that, under this definition, the approximate laws (2.5) hold for the interaction of two solutions in K as a consequence of the equality of the rarefaction wave curves and the rarefaction shock curves of (1.1) up to third order terms in ϵ.

Secondly, we note that solutions in K are uniquely determined by their initial data.

The problem of constructing U_h reduces locally to the general problem of constructing the outgoing solution to the interaction of a finite number of waves at a point. The setting of the latter problem is the following. A solution U exists in some small region $\{(x, t): (x - x_0)^2 + (t - t_0)^2 \leqslant \text{const.}, t \leqslant t_0\}$ as a piecewise constant function of the ratio $(x - x_0)/(t - t_0)$. By an outgoing solution to the interaction at (x_0, t_0), we mean a solution to the Riemann problem with data $U_l = U(x_0 - 0, t_0)$, $U_r = (x_0 + 0, t_0)$ that is a piecewise constant function of the ratio $(x - x_0)/(t - t_0)$ defined in $t \geqslant t_0$. In Lemma 3.6, we construct an outgoing solution that has the same structure as the solutions in [10] except that j-rarefaction waves are replaced by certain piecewise constant waves which we call j-fans.

DEFINITION. A j-fan is a solution which is defined in some sector $\{(x, t): a < (x - x_0)/(t - t_0) < b,\ t \geqslant t_0\}$ as a piecewise constant function of the ratio $(x - x_0)/(t - t_0)$ such that all discontinuities are j-rarefaction shocks.

Furthermore, the outgoing solution is constructed in such a way that the incoming and outgoing waves obey a natural extension of the approximate laws in which we define the magnitude of a j-fan ϵ_j by

$$\epsilon_j = \sum \epsilon_j^{(i)},$$

where the summation is taken over all member waves $\epsilon_j^{(i)}$.

Next, we outline the method of constructing U_h. At $t = 0$, U_h is defined as a piecewise constant approximation of the initial data having compact support. In the neighborhood of each point of discontinuity of $U_h(x, 0)$, the Riemann problem is solved within the class of piecewise constant functions and under the constraint that all rarefaction shocks have magnitude less than h. The generated waves evolve until the time t_1 at which the first set of wave interactions takes place. Using the outgoing solution constructed in Lemma 3.6, these interactions are resolved and the outgoing waves evolve until the time t_2 at which the second set of interactions takes place, etc.

This method of construction yields a globally defined solution U_h with the following properties. Along every line $t = t_0$, U_h is a piecewise constant function with less than N discontinuities where N depends on h but is independent of t. All wave interactions in U_h take place at one of a discrete set of consecutive times t_n where $\lim_{n\to\infty} t_n = \infty$. All rarefaction shocks have magnitude less than $3h$.

The key fact which permits the continuation of the local construction process for all time is the following. The outgoing solution of Lemma 3.6 is constructed in such a way that the number of outgoing waves exceeds the number of incoming waves only if the strength of all wave interactions at the point in question exceeds h. Since the total strength of wave interactions in the x–t plane is bounded (recall (2.10)), the number of waves in U_h at time t is bounded in terms of the number of waves in U_h at time $t = 0+$.

The method of estimating U_h is based on three fundamentals of conservation laws: the local laws of Glimm, the global laws of Glimm and Lax, and the Glimm functional F_1. From the structure of U_h in the x–t plane, it follows that F_1 is adaptable to the family U_h, $h > 0$. Moreover, from the decay of F_1, it follows that the x-variation of U_h is bounded uniformly in h and t and, consequently, that there exists a subsequent U_{h_k} which converges in L^1_{loc} on every line $t = t_0$ to a solution of the Cauchy problem. The global laws are used to establish the upper bound $3h$ on the magnitudes of all rarefaction shocks in U_h.

We now proceed with the construction of the outgoing solution to the general interaction at a point. To this end, we classify the forms of a piecewise constant solution to a Riemann problem by an ordered pair (W_1, W_2) whose components W_j designate the type of j-wave; W_j equals S for shock, RS for rarefaction shock, F for fan, E for a wave which is either a shock or a rarefaction shock and ϕ for missing wave.

Consider a solution of the form (W_1, W_2) with data U_l, U_m and solution of the form (W_1', W_2') with data U_m, U_r. By the outgoing solution of the interaction of (W_1, W_2) with (W_1', W_2'), we mean a solution of the Riemann problem with data U_l, U_r that is a piecewise constant function of the ratio x/t. For the purposes of this section, we call interactions of the above form binary interactions and we identify all the translates $U(x - x_0, t - t_0)$ of an outgoing solution $U(x, t)$.

The outgoing solution to an interaction at a point is constructed by first resolving the interaction into a composition of binary interactions and then employing at each stage of composition an outgoing solution to the appropriate binary interaction. It follows from the method of decomposition that no generality is lost if we consider only binary interactions of the following type: a solution of the form (W_1, ϕ) having data U_l, U_m with a solution of the form (E, W_2) with data U_m, U_r.

The outgoing solutions of binary interactions of the above form are constructed in Lemmas 3.1–3.4. There are eight cases to be considered. However, the constructions for the cases given in Lemmas 3.1 and 3.4 essentially determine the constructions in the remaining cases; Lemma 3.1 serves as the inductive step in Lemma 3.2 and Lemma 3.2 as the inductive step in Lemma 3.4. The proofs of the corollaries are virtually identical to those of the corresponding lemmas and are omitted.

The construction proceeds as follows. Let N_i denote the number of incoming waves, N_0 the number of outgoing waves, $\gamma_j^{(i)}$ the magnitude of the ith incoming j-wave (labeled so that $\gamma_j^{(i)}$ lies to the left of $\gamma_j^{(i+1)}$), $\gamma_j^{(i)+} = \max\{\gamma_j^{(i)}, 0\}$, $\gamma_j^{(i)-} = \min\{\gamma_j^{(i)}, 0\}$, $D = \sum \{| \gamma_j^{(i)} | \, | \gamma_l^{(k)} | : \gamma_j^{(i)} \text{ and } \gamma_l^{(k)} \text{ approach}\}$, τ the oscillation and ϵ_j the magnitude of the outgoing j-wave. We prove two kinds of estimates, approximate conservation laws and bounds on the outgoing rarefaction shocks of the form $\epsilon_j^+ \leqslant \max\{h, \gamma_j^{(i)+} \text{ all } i\}$. The former are needed for proving the existence of solutions and the latter for proving the entropy condition.

LEMMA 3.1. *If τ is sufficiently small, there exists an outgoing solution to the interaction of (RS, ϕ) with (E, E) which satisfies $N_0 \leqslant N_i$ and*

$$\epsilon_1 = \gamma_1^{(1)} + \gamma_1^{(2)} + O(\tau)D, \qquad \epsilon_2 = \gamma_2^{(1)} + O(\tau)D \tag{3.1}$$

$$\epsilon_1 \leqslant \gamma_1^{(1)+} + \gamma_1^{(2)+}. \tag{3.2}$$

Moreover, if $\gamma_1^{(1)} \leqslant 4h$, the outgoing solution has the following properties.
If $\gamma_2^{(1)} > 0$, either $\epsilon_2 \leqslant \gamma_2^{(1)}$ or ϵ_2 is a 2-fan such that $\epsilon_2^{(1)} = h$ and $\epsilon_2^{(2)} = \gamma_2^{(1)}$.
If $\gamma_2^{(1)} \leqslant 0$, then $\epsilon_2 \leqslant h$

Proof. Consider Fig. 1. If $\gamma_1^{(2)} \geqslant 0$, the lemma is immediate; we define ϵ_1 to be the 1-fan with $\epsilon_1^{(1)} = \gamma_1^{(1)}$ and $\epsilon_1^{(2)} = \gamma_1^{(2)}$, and we define ϵ_2 to be the single wave with magnitude $\gamma_2^{(1)}$. Therefore, we assume $\gamma_1^{(2)} < 0$ and consider first the case where $\gamma_2^{(1)} \geqslant 0$. In this case, we define the outgoing solution to be the solution in K with data U_l, U_r. Hence, (3.1) is satisfied.

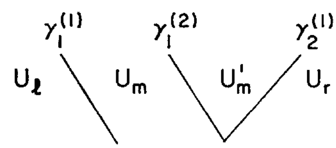

FIGURE 1

For small τ, it follows from (3.1) that

$$\epsilon_1 = \gamma_1^{(1)} + (\gamma_1^{(2)} + O(\tau) \mid \gamma_1^{(1)} \mid \mid \gamma_1^{(2)} \mid) \leqslant \gamma_1^{(1)} = \gamma_1^{(1)+} + \gamma_1^{(2)+},$$

and, if $\gamma^{(1)} \leqslant 4h$, that

$$\epsilon_2 = \gamma_2^{(1)} + O(\tau) \mid \gamma_1^{(1)} \mid \mid \gamma_1^{(2)} \mid \leqslant O(\tau)\, 4h \mid \gamma_1^{(2)} \mid \leqslant h.$$

Next, we consider the case where $\gamma_2^{(1)} > 0$. Let U be a solution in K with data U_l, U_m'. Let $\tilde{\epsilon}_j$ be the magnitude of the j-wave in U. If $\tilde{\epsilon}_2 < 0$, we define the outgoing solution to be the solution in K with data U_l, U_r. If $\tilde{\epsilon}_2 \geqslant 0$, we define ϵ_1 to be a single wave with magnitude $\tilde{\epsilon}_1$ and ϵ_2 to be a

2-fan with $\epsilon_2^{(1)} = \tilde{\epsilon}_2$ and $\epsilon_2^{(2)} = \gamma_2^{(1)}$. In both subcases, (3.1) and (3.2) clearly hold. Moreover, if τ is small, we have in the former case,

$$\epsilon_2 = \gamma_2^{(1)} + (\tilde{\epsilon}_2 + O(\tau) \mid \tilde{\epsilon}_2 \mid \mid \gamma_2 \mid) \leqslant \gamma_2^{(1)},$$

and in the latter case,

$$\begin{aligned} \epsilon_2^{(1)} &= \tilde{\epsilon}_2 = O(\tau) \mid \gamma_1^{(1)} \mid \mid \gamma_1^{(2)} \mid \leqslant O(\tau)\, 4h \mid \gamma_1^{(2)} \mid \leqslant h, \\ \epsilon_2^{(2)} &= \gamma_2^{(1)}. \end{aligned}$$

The proof is complete.

Corollary 3.1. *If τ is sufficiently small, there exists an outgoing solution to the interaction of (S, ϕ) with (E, E) which satisfies all of the conclusions of Lemma* 3.1.

Lemma 3.2. *If τ is sufficiently small, there exists an outgoing solution to the interaction of (RS, ϕ) with (E, F) which satisfies $N_0 \leqslant N_i$ and*

$$\epsilon_1 = \gamma_1^{(1)} + \gamma_1^{(2)} + O(\tau)D, \qquad \epsilon_2 = \gamma_2 + O(\tau)D, \tag{3.3}$$

$$\epsilon_1 \leqslant \gamma_1^{(1)+} + \gamma_1^{(2)+}. \tag{3.4}$$

Moreover, if $\gamma_1^{(1)} \leqslant 4h$, the outgoing solution has the following property. Either ϵ_2 is a shock or ϵ_2 is a fan such that

$$\epsilon_2^{(i)} \leqslant \gamma_2^{(i+p)}, \qquad i \geqslant 1, \textit{for some } p, \tag{3.5}$$

or

$$\epsilon_2^{(1)} \leqslant h \qquad \textit{and} \qquad \epsilon_2^{(i)} \leqslant \gamma_2^{(i+q)}, \qquad i \geqslant 2, \textit{for some } q. \tag{3.6}$$

Proof. Consider Fig. 2. If $\gamma_1^{(2)} \geqslant 0$, the lemma is immediate; we define ϵ_1 to be a 1-fan with $\epsilon_1^{(1)} = \gamma_1^{(1)}$ and $\epsilon_1^{(2)} = \gamma_1^{(2)}$ and we define ϵ_2 to be the 2-fan γ_2, i.e., $\epsilon_2^{(i)} = \gamma_2^{(i)}$. Next, we assume $\gamma_1^{(2)} > 0$. In this case, we construct

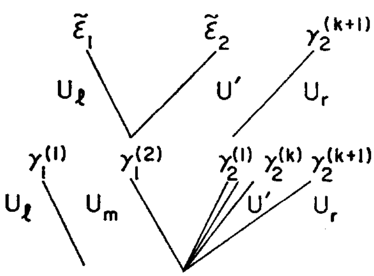

Figure 2

an outgoing solution whose left wave ϵ_1 is a single wave. The construction proceeds by induction on the number n of waves in the 2-fan γ_2. The case $n = 1$ follows from Lemma 3.1. We assume the case $n = k$. Let U be the outgoing solution with waves $\tilde{\epsilon}_j$ given by the induction hypothesis. If $\tilde{\epsilon}_2$ is a fan, $\tilde{\epsilon}_2 = \sum_{i=1}^{r} \epsilon_2^{(i)}$, we define ϵ_1 to be a single wave with magnitude $\tilde{\epsilon}_1$ and we define ϵ_2 to be a 2-fan with $\epsilon_2^{(i)} = \tilde{\epsilon}_2^{(i)}$, $1 \leqslant i \leqslant r$ and $\epsilon_2^{(r+1)} = \gamma_2^{(k+1)}$. With this definition, the lemma clearly holds if $\tilde{\epsilon}_2$ is a fan.

Next, we consider the case where $\tilde{\epsilon}_2$ is a shock. In this case, we define the outgoing solution to be the solution in K with data U_l, U_r. We prove (3.3) first. We have

$$\epsilon_1 = \tilde{\epsilon}_1 + O(\tau) \mid \gamma_2^{(k+1)} \mid \mid \tilde{\epsilon}_2 \mid, \tag{3.7}$$

$$\tilde{\epsilon}_2 = \sum_{i=1}^{k} \gamma_2^{(i)} + O(\tau) \mid \gamma_1^{(1)} \mid \mid \gamma_1^{(2)} \mid. \tag{3.8}$$

Since $\tilde{\epsilon}_2 < 0$ and $\gamma_2^{(i)} > 0$,

$$\mid \tilde{\epsilon}_2 \mid \leqslant O(\tau) \mid \gamma_1^{(1)} \mid \mid \gamma_1^{(2)} \mid. \tag{3.9}$$

By (3.9) and the induction hypothesis on $\tilde{\epsilon}_1$, it follows from (3.1) that

$$\begin{aligned} \epsilon_1 &= (\gamma_1^{(1)} + \gamma_1^{(2)} + O(\tau) \mid \gamma_1^{(1)} \mid \mid \gamma_1^{(2)} \mid) + O(\tau) \mid \gamma_2^{(k+1)} \mid O(\tau) \mid \gamma_1^{(1)} \mid \mid \gamma_1^{(2)} \mid \\ &= \gamma_1^{(1)} + \gamma_1^{(2)} + O(\tau)(1 + \mid \gamma_2^{(k+1)} \mid) \mid \gamma_1^{(1)} \mid \mid \gamma_1^{(2)} \mid \\ &= \gamma_1^{(1)} + \gamma_1^{(2)} + O(\tau) \mid \gamma_1^{(1)} \mid \mid \gamma_1^{(2)} \mid = \gamma_1^{(1)} + \gamma_1^{(2)} + O(\tau)D. \end{aligned}$$

We note that the uniformity of $O(\tau)$ follows from the fact that $\sum \gamma_2^{(i)} \leqslant 1$ for small τ. Similarly, we have

$$\begin{aligned} \epsilon_2 &= \tilde{\epsilon}_2 + \gamma_2^{(k+1)} + O(\tau) \mid \gamma_2^{(k+1)} \mid \mid \tilde{\epsilon}_2 \mid \\ &= \left(\sum_{i=1}^{k} \gamma_2^{(i)} + O(\tau) \mid \gamma_1^{(1)} \mid \mid \gamma_1^{(2)} \mid \right) + \gamma_2^{(k+1)} + O(\tau) \mid \gamma_2^{(k+1)} \mid \mid \tilde{\epsilon}_2 \mid \\ &= \sum_{i=1}^{k+1} \gamma_2^{(i)} + O(\tau) \mid \gamma_1^{(1)} \mid \mid \gamma_1^{(2)} \mid \\ &= \gamma_2 + O(\tau)D. \end{aligned} \tag{3.10}$$

Lastly, in the case where $\tilde{\epsilon}_2$ is a shock, it follows from (3.10) and $\tilde{\epsilon}_2 < 0$ that $\epsilon_2 \leqslant \gamma_2^{(k+1)}$ for small τ. Thus, (3.5) holds in this case without appeal to the inequality $\gamma_1^{(1)} \leqslant 4h$.

The remaining inequality (3.4) holds by (3.3) and by the method of constructing the outgoing solution.

COROLLARY 3.2. *If τ is sufficiently small, there exists an outgoing solution to the interaction of (S, ϕ) with (E, F) which satisfies all of the conclusions of Lemma* 3.2.

Next, we prove

LEMMA 3.3. *If τ is sufficiently small, there exists an outgoing solution to the interaction of (F, ϕ) with (E, F) satisfying $N_0 \leqslant N_i$ and*

$$\epsilon_1 = \sum_i \gamma_1^{(i)} + O(\tau)D, \qquad \epsilon_2 = \sum_i \gamma_2^{(i)} + O(\tau)D.$$

Moreover, if $\gamma_1^{(i)} \leqslant 4h$, the outgoing solution has the following properties.

Either ϵ_1 is a shock or ϵ_1 is a fan with $\epsilon_1^{(i)} \leqslant \gamma_1^{(i)}$ for all i.

Either ϵ_2 is a shock or ϵ_2 is a fan satisfying $\epsilon_2^{(i)} \leqslant \gamma_2^{(i+p)}$, for some p or satisfying $\epsilon_2^{(i)} \leqslant h$, $1 \leqslant i \leqslant q$, $\epsilon_2^{(i)} \leqslant \gamma_2^{(i+r)}$, $i \geqslant q + 1$, for some q, r.

Proof. The outgoing solution is constructed in a manner similar to that of Lemmas 3.1 and 3.2. The proof is by induction on the number n of waves in the fan γ_1. The case $n = 1$ follows from Lemma 3.2. Details are omitted.

COROLLARY 3.3. *If τ is sufficiently small, there exists an outgoing solution to the interaction of (F, ϕ) with (E, S) satisfying $N_0 \leqslant N_i$ and*

$$\epsilon_1 = \sum \gamma_1^{(i)} + O(\tau)D, \qquad \epsilon_2 = \gamma_2^{(1)} + O(\tau)D.$$

Moreover, if $\gamma_1^{(i)} \leqslant 4h$, the outgoing solution has the following properties.

Either ϵ_1 is a shock or ϵ_1 is a fan with $\epsilon_1^{(i)} \leqslant \gamma_1^{(i)}$ for all i.

Either ϵ_2 is a shock or ϵ_2 is a fan with $\epsilon_2^{(i)} \leqslant h$ for all i.

In the interactions of the preceeding lemmas and corollaries, the number of outgoing waves did not exceed the number of incoming waves. In the next lemma, we construct an outgoing solution of a class of interactions for which N_0 may exceed N_i.

LEMMA 3.4. *If τ is sufficiently small, there exists an outgoing solution to the interaction of (S, ϕ) with (S, E) which satisfies*

$$\begin{aligned} \epsilon_1 &= \gamma_1^{(1)} + \gamma_1^{(2)} + O(\tau)D, \\ \epsilon_2 &= \gamma_2^{(1)} + O(\tau)D. \end{aligned} \tag{3.11}$$

Moreover,

$$N_0 > N_i \qquad \textit{only if} \quad D \geqslant h.$$

If $\gamma_2^{(1)}$ is a shock, either ϵ_2 is a shock or ϵ_2 is a fan with $\epsilon_2^{(i)} \leqslant h$ for all i.

If $\gamma_2^{(1)}$ is a rarefaction shock, either ϵ_2 is a shock or ϵ_2 is a fan with p waves satisfying $\epsilon_2^{(i)} \leqslant h$, $1 \leqslant i \leqslant p-1$ and $\epsilon_2^{(p)} \leqslant \gamma_2^{(1)}$.

Proof. Consider Fig. 1. We discuss first the case where γ_2 is a shock. Let $\tilde{U}$ be the solution in K with data U_l, U_r and let $\tilde{\epsilon}_j$ be the magnitude of the j-wave in $\tilde{U}$. If $\tilde{\epsilon}_2 \leqslant h$, we define $\tilde{U}$ to be the outgoing solution. If $\tilde{\epsilon}_2 > h$, we define the outgoing solution to be a solution U to the Riemann problem with data U_l, U_r that has the following properties. The solution U is a small perturbation of $\tilde{U}$ in the sense that the magnitude ϵ_j of the j-wave in U satisfies

$$| \epsilon_j - \tilde{\epsilon}_j | = O(\tilde{\epsilon}_2{}^3), \qquad j = 1, 2. \tag{3.12}$$

Furthermore, ϵ_1 is a shock and ϵ_2 a fan with $\epsilon_2^{(i)} \leqslant h$ for all i. Existence of solutions U with the properties above follows from straightforward considerations of the shock curves and the rarefaction shock curves of (1.1) and the fact that $| \tilde{\epsilon}_2 | \leqslant | \tilde{\epsilon}_1 |$ for small τ. In order to prove (3.11) we note that

$$\begin{aligned} \tilde{\epsilon}_1 &= \gamma_1^{(1)} + \gamma_1^{(2)} + O(\tau)D, \\ \tilde{\epsilon}_2 &= \gamma_2^{(1)} + O(\tau)D. \end{aligned} \tag{3.13}$$

But $| \tilde{\epsilon}_2 | = O(\tau)D$ since $\tilde{\epsilon}_2 \geqslant 0$ and $\gamma_2^{(1)} \leqslant 0$. Thus, (3.11) follows from (3.12) and (3.13).

If $N_i > N_0$, it follows from the definition of the outgoing solution that $\tilde{\epsilon}_2 > h$, and hence, $D \geqslant h$ for small τ.

Next, we consider the case where γ_2 is a rarefaction shock. Let U' be the solution in K with data U_l, U_m' and let ϵ_j' denote the magnitude of the j-wave in U'. We discuss two subcases: $\epsilon_2' \geqslant 0$ and $\epsilon_2' < 0$. Suppose $\epsilon_2' \geqslant 0$. If $0 \leqslant \epsilon_2' \leqslant h$, we define the outgoing solution to be the solution of the Riemann problem with data U_l, U_r whose 1-wave is ϵ_1' and whose 2-wave is a fan consisting of ϵ_2' and $\gamma_2^{(1)}$. This yields a solution with $N_0 \leqslant N_i$. If, however, $\epsilon_2' \geqslant h$, we construct a perturbation $\tilde{U}'$ of U' as above and define the 1-wave of the outgoing solution as the 1-wave of $\tilde{U}'$ and the 2-wave of the outgoing solution as the fan consisting of the 2-fan of $\tilde{U}'$ adjoined to $\gamma_2^{(1)}$. This yields a solution with $N_0 > N_i$ but with $D \geqslant h$.

The subcase $\epsilon_2' < 0$ is handled similarly.

COROLLARY 3.4. *If τ is sufficiently small, there exists an outgoing solution to the interaction of (S, ϕ) with (S, F) that satisfies*

$$\begin{aligned} \epsilon_1 &= \gamma_1^{(1)} + \gamma_1^{(2)} + O(\tau)D, \\ \epsilon_2 &= \gamma_2 + O(\tau)D. \end{aligned}$$

Moreover, the solution has the following properties.

$$N_0 > N_i \qquad \textit{only if} \quad D \geqslant h.$$

Either ϵ_2 *is a shock or* ϵ_2 *is a fan with* $\epsilon_2^{(1)} \leqslant h$, $1 \leqslant i \leqslant p-1$, $\epsilon_2^{(i)} \leqslant \gamma_2^{(i+q)}$, $i \geqslant p$ *for some* p *and* q.

This completes the construction of the outgoing solution to the interaction of (W_1, ϕ) with (E, W_2). In order to proceed to the general interaction at a point, we need the following lemma.

LEMMA 3.5. *If* τ *is sufficiently small, there exists an outgoing solution to the interaction of* (ϕ, F) *with* (E, ϕ) *that has the form* (E, F). *Furthermore, the number of waves in the incoming 2-fan equals the number of waves in the outgoing 2-fan and the outgoing waves satisfy*

$$\begin{gathered} \epsilon_1 = \gamma_1^{(1)} + O(\tau)D, \qquad \epsilon_2 = \gamma_2 + O(\tau)D, \\ \epsilon_2^{(k)} = \gamma_2^{(k)}(1 + O(\tau)\,\gamma_1^{(1)}). \end{gathered} \tag{3.14}$$

Proof. The outgoing solution is constructed by an induction on the number of waves in the incoming 2-fan. The construction process is illustrated in Fig. 3. Let $\gamma_2^{(i)}$, $1 \leqslant i \leqslant k$, denote the magnitudes of the waves in the incoming fan. We assume that the lemma is satisfied for the waves $\tilde{\epsilon}_j$ in the outgoing solution with data U^*, U_r. Next, we solve the Riemann problem in K with data U_l, U_m, obtaining a solution with waves δ_1 and $\delta_2 > 0$. We define the outgoing solution to be the solution to the Riemann problem with data U_l, U_r whose 1-wave is δ_1 and whose 2-wave is a fan ϵ_2 with

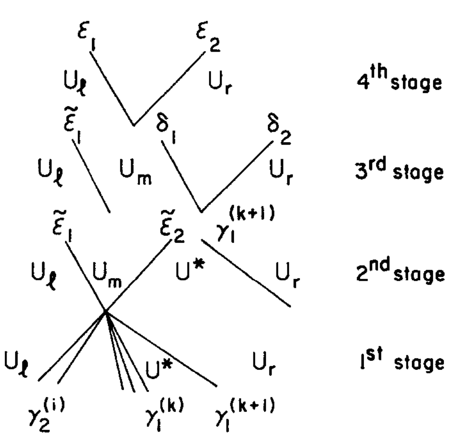

FIGURE 3

$\epsilon_2^{(1)} = \delta_2$ and $\epsilon_2^{(i)} = \epsilon_2^{(i)}$, $1 \geqslant 2$. The relations (3.14) follow from a straightforward induction.

We now construct the outgoing solution to the general interaction of a finite number of waves at a point.

LEMMA 3.6. *Let a finite number of waves $\gamma_j^{(i)}$ interact at some point P. If $\sum_{i,j} |\gamma_j^{(i)}|$ is sufficiently small, there exists an outgoing solution of the interaction at P which satisfies*

$$\epsilon_j = \sum_i \gamma_j^{(i)} + O(\tau)D \tag{3.15}$$

as the oscillation τ approaches zero. Furthermore, $O(\tau)$ is uniform in $\sum_i |\gamma_j^{(i)}|$ and $N_0 > N_i$ only if $D \geqslant h$.

Proof. The outgoing solution is constructed by induction on the number of incoming waves $\gamma_j^{(i)}$. In Fig. 4 we illustrate the construction process in the case where the induction is on an incoming 1-wave. The case for 2-waves is symmetrical.

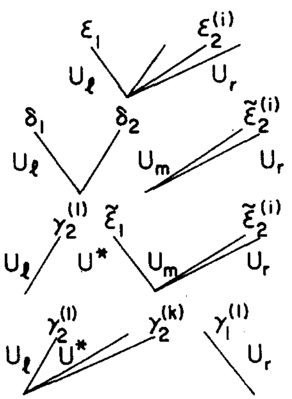

FIGURE 4

We assume that the outgoing solution to the interaction of $\gamma_2^{(i)}, \gamma_1^{(1)}, \gamma_1^{(2)}, \ldots, \gamma_1^{(k)}$ has been constructed and that its outgoing waves $\tilde{\epsilon}_j$ satisfy the lemma. Let U_m be the middle state of this solution. We define first a solution U to the Riemann problem with data U_m, U_r in the following way. If $\tilde{\epsilon}_2$ is a shock, let U be the solution in K; if $\tilde{\epsilon}_2$ is a fan, let U be the solution given by Lemma 3.4. We observe that, in both cases, the 1-wave δ_1 of U is a single wave, i.e., a shock or a rarefaction shock. Therefore, the interaction of $\tilde{\epsilon}_1$ with U is a binary interaction whose type is covered by the previous lemmas and corollaries. We define the outgoing solution of the interaction of $\gamma_j^{(i)}$ to be the outgoing solution to the binary interaction of $\tilde{\epsilon}_1$ with U that is constructed in Lemmas 3.1 to 3.4 and their corollaries.

The approximate conservation laws (3.15) follow by a straightforward induction on the number of incoming $\gamma_j^{(i)}$ using the approximate laws established in Lemmas 3.1 to 3.4 and their corollaries. We now prove that $N_0 > N_i$ only if $D \geqslant h$. Let

$$D_p = \sum \{| \gamma_j^{(m)} | \, | \gamma_l^{(n)} | : \gamma_j^{(m)}, \gamma_l^{(n)} \text{ approach}; \; m \leqslant p \text{ if } j = 1, n \leqslant p \text{ if } l = 1\}.$$

If $N_0 > N_i$ at the second stage, it follows from the induction hypothesis that $D \equiv D_{n+1} \geqslant D_n \geqslant h$. Hence, we may assume that $N_0 \leqslant N_i$ at the second stage. Since the interaction of $\tilde{\epsilon}_2$ and $\gamma_1^{(k+2)}$ does not increase the number of waves, we have $N_0 \leqslant N_i$ at the third stage. By construction, the number of waves increases from the third to fourth stage only if $\tilde{\epsilon}_1$ and δ_1 are both shocks and $| \tilde{\epsilon}_1 | \, | \delta_1 | \geqslant h$. Under these latter conditions, we shall show that

$$| \tilde{\epsilon}_1 | \, | \delta_1 | \leqslant D_{k+1} . \tag{3.16}$$

In estimating $| \tilde{\epsilon}_1 | \, | \delta_1 |$ we use

$$| \tilde{\epsilon}_1 | \leqslant \sum_{i=1}^{k} | \gamma_1^{(i)} | + O(\tau) \, D_k , \tag{3.17}$$

$$| \tilde{\epsilon}_2 | \leqslant \sum_{i} | \gamma_2^{(i)} | + O(\tau) \, D_k , \tag{3.18}$$

and

$$| \delta_1 | \leqslant | \gamma_1^{(k+1)} | + O(\tau) \, | \gamma_1^{(k+1)} | \, | \tilde{\epsilon}_2 |. \tag{3.19}$$

We substitute the right-hand side of (3.18) into (3.19) and multiply this quantity by the right-hand side of (3.17), obtaining, for small τ, the estimate

$$| \tilde{\epsilon}_1 | \, | \delta_1 | \leqslant | \gamma_1^{(k+1)} | \sum_{i=1}^{k} | \gamma_1^{(i)} | + \tfrac{1}{2} \sum | \gamma_2^{(i)} | \sum_{i=1}^{k} | \gamma_1^{(i)} | + \tfrac{1}{2} D_k . \tag{3.20}$$

We observe that, by construction, δ_1 is a shock if and only if $\gamma_1^{(k+1)}$ is a shock. Hence, $\gamma_1^{(k+1)}$ approaches $\gamma_1^{(i)}$ for $i \leqslant k$. Estimating the second term of (3.20) by D_k, we have

$$| \tilde{\epsilon}_1 | \, | \delta_1 | < | \gamma_1^{(k+1)} | \sum_{i=1}^{k} | \gamma_1^{(i)} | + D_k < D_{k+1} .$$

The proof of the lemma is complete.

We recall that a bounded measurable vector-field $U(x, t)$ is a weak solution of (1.1) with initial data $U_0(x)$ if

$$\iint_{t>0} U\phi_t + G(U)\,\phi_x\,dx\,dt + \int_{t=0} U_0(x)\,\phi(x, 0)\,dx = 0$$

for $\phi \in C^\infty$ with compact support in $t \geqslant 0$. We now prove the main existence theorem. Fix a constant state $\tilde{U}$ and let the initial data $U_0(x)$ be a function of finite total variation.

THEOREM 3.1. *If* $\| U_0 - \tilde{U} \|_\infty + TVU_0$ *is sufficiently small, there exists a weak solution* $U(x, t)$ *of the Cauchy problem of* (1.1) *with initial data* $U_0(x)$ *that satisfies*

$$TV_xU(x, t) \leqslant \text{const } TVU_0\,, \tag{3.21}$$

$$\int_{-\infty}^{\infty} | U(x, t_1) - U(x, t_2)|\,dx \leqslant \text{const } | t_1 - t_2 |, \tag{3.22}$$

where the constant depends only on Eq. (1.1).

Proof. First, we fix h and construct the approximating solution U_h. This is accomplished by inductively constructing a sequence of consecutive strips $S_k = \{(x, t): t_k \leqslant t < t_{k+1}\}$ with the following properties. For every n, U_h exists in $\bigcup_{k=1}^n S_k$ as a solution of (1.1). Each strip S_k is the union of a finite number of regions

$$R_k^{(j)} = \{(x, t) : a_k^{(j)} < x \leqslant a_k^{(j+1)},\ t_k \leqslant t < t_{k+1}\},$$

$j = 1, 2,..., m$, in which U_h exists as a piecewise constant function of the ratio $(x - x_j)/(t - t_k)$ for some x_j between $a_k^{(j)}$ and $a_k^{(j+1)}$. All points of interaction of U_h lie on some interface $t = t_k$. For each n, only a finite number of points of interaction lie in $\bigcup_{k=1}^n S_k$.

The construction of U_h proceeds as follows. Let $U_+ = \lim_{x\to+\infty} U_0(x)$ and $U_- = \lim_{x\to-\infty} U_0(x)$. Let $U_0^{(h)}(x)$, $h > 0$, be a family of piecewise constant functions with the properties that $U_0^{(h)}(x) \equiv U_-$ for small x, $U_0^{(h)}(x) \equiv U_+$ for large x and

$$\begin{aligned} \lim_{h\to 0} U_0^{(h)}(x) &= U_0(x) \qquad \text{a.e.,} \\ TVU_0^{(h)}(x) &\leqslant 2TVU_0(x). \end{aligned} \tag{3.23}$$

It is not difficult to show that a piecewise constant solution U_h of (1.1) exists in some strip $S_1 = \{(x, t): 0 \leqslant t \leqslant t_1\}$, takes on the initial data $U_0^{(h)}(x)$ and has the required structure with respect to S_1 as described above.

Furthermore, U_h can be constructed so that all rarefaction shocks of U_h in S_1 have magnitude less than h and so that

$$TV_x U_h(x, t) \leqslant \text{const } TVU_0, \qquad \text{for} \quad t \leqslant t_1,$$

where the constant depends only on the Eq. (1.1).

Next, we give the inductive step in the construction of U_h. Suppose U_h exists in the strip $\bigcup_{k=1}^{n-1} S_k$. From the structure of U_h in S_{n-1}, it follows that $U_h(x, t_n - 0)$ exists as a piecewise constant function having only a finite number of discontinuities at say x_j, $j = 1, 2,..., m$, $x_j < x_{j+1}$. In order to extend U_h, we first define solutions U_j of the Riemann problems generated at (x_j, t_n) by $U_h(x, t_n - 0)$: if (x_j, t_n) is a point of interaction, let U_j be the outgoing solution given by Lemma 3.6; otherwise, let U_j be the solution in K with data $U_h(x_j - 0, t_n)$, $U_h(x_j + 0, t_n)$.

Next, let t_j be the first time at which a 2-wave of U_j and a 1-wave of U_{j+1} interact and let $a_n^{(j+1)}$ be the x-coordinate of the point of interaction. Let $t_{n+1} = \min\{t_j\}$. We define U_h in $S_n = \{(x, t): t_n \leqslant t < t_{n+1}\}$ by

$$U_h(x, t) = U_j(x, t) \qquad \text{if} \quad (x, t) \in R_n^{(j)},$$

where

$$R_n^{(j)} \equiv \{(x, t) : a_n^{(j)} < x \leqslant a_n^{(j+1)};\ t_n \leqslant t < t_{n+1}\}.$$

In order to show that the construction process yields a globally defined solution U_h, i.e., that $\lim_{k\to\infty} t_k = \infty$, we show that the functional F_1 is adaptable to U_h. Since U_h satisfies the local approximate laws (3.15), the functional F_1 is monotonically decreasing. From this fact, it follows that U_h and consequently the speed of propagation of discontinuities in U_h are bounded in the sup norm uniformly in x, t and h and that the strength $D(\Lambda_n)$ of all wave interactions of U_h in $\Lambda_n = \bigcup_{k=1}^{n} S_k$ is bounded uniformly in n and h. By the method of construction, the latter implies that only a finite number of wave interactions increase the number of waves in U_h and, thus, that the number of waves in U_h along any line $t = t_0$ is a priori bounded in t_0 for fixed h. Using the uniform bound on propagation speed, it is not difficult to show that there exists at most a finite number of points of interaction in any compact set and that $\lim_{k\to\infty} t_k = \infty$.

Since the remainder of the proof is similar to [6], we shall only give a sketch. Fix n and consider the restriction of U_h to the strip $\Lambda_n = \bigcup_{k=1}^{n} S_k$. By construction, there exists only a finite number of points of interaction in Λ_n. Thus, U_h is bounded in Λ_n. Let

$$c = 2 \sup\{| \lambda_i(U_h(x, t)): i = 1, 2, (x, t) \in \Lambda_n\}.$$

As a consequence of the boundedness of U_h in x and t, we can construct a grid in the x–t plane that consists of a finite number of space-like lines $x + ct = \alpha_j$ and $x_j - ct = \beta_j$ and that has the following properties. All points of interaction lie in the interior of the diamond-shaped regions determined by the grid. Waves enter the diamonds through the WS and SE sides and leave through the WN and NE sides. Here N = north, S = south, etc.

Following [6], we define an I-curve J to be a continuous polygonal arc extending from $x = -\infty$ to $x = +\infty$, each of whose line segments coincides with some side NE, NW, SE, SW of some diamond and along which the coordinate x increases monotonically. The I curves are partially ordered by setting $J_2 \geqslant J_1$ if and only if J_2 lies towards larger time. The Glimm functional F_1 is defined on this family by (2.6).

If $\| U_0 - \tilde{U} \|_\infty + TVU_0$ is sufficiently small, it follows from [6] that F_1 is nonincreasing, i.e.,

$$F_1(J_2) \leqslant F_1(J_1) \qquad \text{if} \quad J_2 \geqslant J_1$$

and, hence, that

$$TV_x U_h(x, t) \leqslant \text{const } TVU_0^{(h)}(x), \tag{3.24}$$

$$\int_{-\infty}^{\infty} | U_h(x, t_1) - U_h(x, t_2)| \, dt \leqslant \text{const } | t_1 - t_2 |, \tag{3.25}$$

where the constants depend only on Eq. (1.1). Secondly, it follows from the decay of F_1 that the strength $D(\Lambda_n)$ of all wave interactions in Λ_n satisfies

$$D(\Lambda_n) \leqslant 2(TVU_0^{(h)})^2, \tag{3.26}$$

where D is defined by (2.7). Therefore, uniform estimates on the total variation and on strength of wave interactions follow from (3.23), (3.24), and (3.26).

The estimates (3.23), (3.24) and (3.25) imply that there exists a sequence of solutions U_{h_k} converging in L^1_{loc} on every line $t = t_0$ to a solution $U(x, t)$ of the Cauchy problem with initial data $U_0(x)$. (See [6].) The estimates (3.21) and (3.22) follow from (3.24) and (3.25) in the limit. The proof is complete.

By adapting the Glimm functional F_2 to the approximating solutions U_h, existence of solutions to (1.1) can be established under the weaker restriction that

$$\| \tilde{U} - U_0 \|_\infty (1 + TVU_0)$$

be small. Furthermore, the decay of F_2 implies, by [6], that the solution as constructed satisfies

$$\| U - \tilde{U} \|_\infty \leqslant \text{const} \| U_0 - \tilde{U} \|_\infty$$

where the constant depends only on the equations.

4. The Entropy Condition

In this section we prove that the solution U constructed in Theorem 3.1 satisfies the entropy inequality (1.2). (In connection with admissibility conditions, see [3].) In the setting of test functions, inequality (1.2) is the statement that

$$\iint_{t>0} \{\eta(U)\,\phi_t + q(U)\,\phi_x\}\,dx\,dt \geqslant 0 \tag{4.1}$$

for all positive $\phi \in C^\infty$ with compact support in $t > 0$.

The proof of (4.1) is based on the fact, which we establish below, that the magnitude of each rarefaction shock in U_h is less than $3h$ if $\| \tilde{U} - U_0 \|_\infty + TVU_0$ is sufficiently small. Granting this fact, (4.1) is proved as follows. Fix ϕ and choose T so large that $\Gamma = \{(x, t)\colon 0 < t < T\}$ contains the support of ϕ. Let $\eta_h = \eta(U_h)$ and $q_h = q(U_h)$. We observe that

$$\iint_\Gamma \{\eta_h \phi_t + q_h \phi_x\}\,dx\,dt = -\sum_{W\in\Gamma} \int_W \{s[\eta_h] - [q_h]\}\phi\,dt,$$

where the summation is taken over all waves W in Γ. It is shown in [11] that $s[\eta_h] - [q_h]$ is nonpositive for shocks and nonnegative for rarefaction shocks and, moreover, that

$$s[\eta_h] - [q_h] = O(\epsilon^3)$$

where ϵ is the magnitude of the wave. Therefore,

$$\begin{aligned}
&\iint_\Gamma \{\eta_h \phi_t + q_h \phi_x\}\,dx\,dt \\
&\qquad \geqslant -\sum_{RS\in\Gamma} \int_{RS} |\,O(\epsilon^3)|\,\phi\,dt \\
&\qquad \geqslant -\text{const}\,\|\phi\|_\infty\, Th^2 \sup_{0\leqslant t_0\leqslant T} \left\{\sum |\,\epsilon\,| : \epsilon \text{ is a } RS \text{ crossing } t = t_0\right\} \\
&\qquad \geqslant -\text{const}\,\|\phi\|_\infty\, Th^2 TVU_0\,.
\end{aligned}$$

The inequality (4.1) follows in the limit as h approaches zero.

In order to establish the upper bound $3h$ on rarefaction shocks, we prove first that the magnitude of a j-rarefaction shock is increased during an interaction by an amount which is dominated by terms involving the strengths of the incoming k-waves, $k \neq j$, and the strength D of the wave interactions at the point in question.

LEMMA 4.1. *There exists a constant α with the following property. If the incoming waves $\gamma^{(i)}$ of an interaction at a point satisfy*

$$\gamma_j^{(i)+} \leqslant 3h \qquad \text{and} \qquad \tau\Big(1 + D + \sum_{i,j} |\gamma_j^{(i)}|\Big) \leqslant \alpha, \tag{4.2}$$

then

$$\epsilon_j^{(i)} \leqslant \Big(1 + O(\tau)D + O(\tau)\sum_i |\gamma_k^{(i)}|\Big)\max\{h, \gamma_j^{(i)+} \text{ all } i\} \tag{4.3}$$

where $k \neq j$.

Proof. The proof is by induction on the number of incoming waves. We employ the decomposition given in Lemma 3.6 and consider the case in which the inductive step is on a 1-wave (see Fig. 4).

We assume inductively that the waves $\tilde{\epsilon}_j$, $j = 1, 2$, satisfy (4.3). First, we estimate δ_1 and δ_2. By Lemma 3.5,

$$\delta_1 = \gamma_1^{(k+1)} + O(\tau)\, |\gamma_1^{(k+1)}|\, |\epsilon_2| \tag{4.4}$$

and by the induction hypothesis,

$$|\tilde{\epsilon}_2| \leqslant \sum_i |\gamma_2^{(i)}| + O(\tau)\, D_k\,. \tag{4.5}$$

Hence,

$$\delta_1 \leqslant \gamma_1^{(k+1)+}(1 + O(\tau)\, |\tilde{\epsilon}_2|) \tag{4.6}$$

and, by substitution of (4.5) into (4.6),

$$\begin{aligned}\delta_1 &\leqslant \gamma_1^{(k+1)+}\Big(1 + O^2(\tau)\, D_k + O(\tau)\sum |\gamma_2^{(i)}|\Big)\\ &\leqslant \gamma_1^{(k+1)+}\Big(1 + O(\tau)\, D_{k+1} + O(\tau)\sum |\gamma_2^{(i)}|\Big)\end{aligned} \tag{4.7}$$

for small τ.

We estimate δ_2 next. From the construction in Lemma 3.6 it follows that $\tilde{\epsilon}_2$ and δ_2 are either both simultaneously fans or both simultaneously shocks. Suppose the former. Then, by Lemma 3.5,

$$\delta_2^{(i)} \leqslant \tilde{\epsilon}_2^{(i)}(1 + O(\tau)\, |\gamma_1^{(k+1)}|).$$

Therefore, using the induction hypothesis on $\tilde{\epsilon}_2^{(i)}$, we have

$$\delta_2^{(i)} \leqslant \left(1 + O(\tau)\, D_k + O(\tau) \sum_{i=1}^{k} |\, \gamma_1^{(i)} \,|\right) (1 + O(\tau) \,|\, \gamma_1^{(k+1)} \,|)$$

$$\max\{h, \gamma_2^{(i)}, \text{all } i\}. \tag{4.8}$$

Next, we show that the product P of the first two terms in (4.8) satisfies

$$P \leqslant \left(1 + O(\tau)\, D_{k+1} + O(\tau) \sum_{i=1}^{k+1} |\, \gamma_1^{(i)} \,|\right). \tag{4.9}$$

Expanding P, we have

$$P = 1 + O(\tau)(D_k + O(\tau) \,|\, \gamma_1^{(k+1)} \,|\, D_k) + O(\tau) \sum_{i=1}^{k} |\, \gamma_1^{(i)} \,|$$

$$+ O(\tau) \left(1 + O(\tau) \sum_{i=1}^{k} |\, \gamma_1^{(i)} \,|\right) |\, \gamma_1^{(k+1)} \,|. \tag{4.10}$$

Now, for sufficiently small α, we have

$$\begin{aligned} &D_k + O(\tau) \,|\, \gamma_1^{(k+1)} \,|\, D_k \leqslant D_{k+2}\,, \\ &O(\tau) \left(1 + O(\tau) \sum_{i=1}^{k} |\, \gamma_i^{(i)} \,|\right) \leqslant 1. \end{aligned} \tag{4.11}$$

Thus, inequality (4.9) follows from (4.10) and (4.11). We conclude that if δ_2 is a fan then

$$\delta_2^{(i)} \leqslant \left(1 + O(\tau)\, D_{k+1} + O(\tau) \sum_{i=1}^{k+1} |\, \gamma_1^{(i)} \,|\right) \max\{h, \gamma_2^{(i)+} \text{ all } i\}. \tag{4.12}$$

In order to estimate the magnitude of the outgoing rarefaction shocks at the fourth stage, we employ at the third stage the estimates (4.7), (4.12) and the induction hypothesis

$$\epsilon_1^{(i)} \leqslant \left(1 + O(\tau)D + O(\tau) \sum_{i} |\, \gamma_2^{(i)} \,|\right) \max\{h, \gamma_2^{(i)+} \text{ all } i\}. \tag{4.13}$$

We note that the solution of the general binary interaction has the following property. If the magnitude $\alpha_j^{(i)}$ of each rarefaction shock satisfies $\alpha_j^{(i)} \leqslant 4h$ then the magnitude $\epsilon_j^{(i)}$ of each outgoing rarefaction shock satisfies

$$\epsilon_j^{(i)} \leqslant \max\{h, \alpha_j^{(i)} \text{ all } i\}. \tag{4.14}$$

Therefore, if α is so small that

$$\left(1 + O(\tau)D + O(\tau) \sum |\gamma_j^{(i)}|\right) \leqslant 4/3$$

then all rarefaction shocks at the third stage have magnitude $\leqslant 4h$ by (4.7), (4.12) and (4.13). Thus, (4.3) holds at the fourth stage for $\epsilon_j^{(i)}$ by (4.7), (4.12), (4.13) and (4.14). The proof of the lemma is complete.

We can now prove the main lemma.

LEMMA 4.2. *There exists a constant β depending only on Eq.* (1.1) *with the property that all rarefaction shocks in U_h have magnitude $\leqslant 3h$ if $\| \tilde{U} - U_0 \|_\infty + TVU_0 \leqslant \beta$.*

Proof. Fix h and T and consider the restriction of U_h to $\{(x, t): 0 \leqslant t < T\}$. Let I be the set of all points of interaction in H. Let RS be a 1-rarefaction shock in H and let $P_0 \notin I$ be a point on RS.

We construct a polygonal arc BC, a backward continuation of RS, and estimate the strength of RS at its initial point P_0 in terms of the strength of BC at its end point P_n. The arc BC is the union of line segments RS_k, $k = 1, 2,..., n$, whose end points P_{k-1} and P_k lie in I for $k \geqslant 1$. Between P_k and P_{k+1}, RS_k coincides with some 2-rarefaction shock and does not intersect I. Relative to the point of interaction P_k, RS_k is an incoming 2-wave and RS_{k-1} an outgoing 2-wave. The segments RS_k are defined inductively as follows. The segment RS_1 equals RS between P_0 and the nearest point of interaction P_1 that lies on RS below P_0. Given RS_k, RS_{k+1} is determined by first considering those incoming 2-rarefaction shocks (if any) at P_k whose magnitude attains the maximum value of the magnitudes of all incoming 2-rarefaction shocks at P_k and then choosing the one RS_{k+1} with (say) slowest speed of propagation. The point P_{k+1} is defined to be the point in $I \cap RS_{k+1}$ nearest to P_k.

Next, we construct a region Λ to which the global laws (2.9) shall be applied. Let BC' be a polygonal arc lying slightly to the left of BC with the property that a 1-wave crosses BC' if and only if it is an incoming wave at some point P_k of interaction on BC. Let t_0 and t_n be the t-coordinates of P_0 and P_n. Let Λ be the region bounded by $t = t_0$, $t = t_n$, BC' and $x = M$ where M is so large that Λ contains the support of U_h in Λ. Applying the global laws, we have

$$|L_1^{\pm}(\Lambda)| \leqslant |E_1^{\pm}(\Lambda)| + O(\tau)\, D(\Lambda).$$

Since the total strength of waves entering Λ, i.e., crossing $t = t_n$, is bounded by a constant times TVU_0, we have, using (2.10),

$$|L_1^{\pm}(\Lambda)| \leqslant \text{const } TVU_0 . \tag{4.15}$$

We now estimate the magnitude of RS in terms of the magnitude of RS_n. If $t_n = 0$, it follows from the construction of U_h that the magnitude of RS_n is less than h. If $t_n > 0$, condition (4.3) implies that the magnitude of RS_n is less than $2h$ if $\| \tilde{U} - U_0 \|_\infty + TVU_0$ is sufficiently small. Therefore, by the construction of BC and by (4.3), we have

$$\text{magnitude}(RS) \leqslant 2h \prod_{k=1}^{n} (1 + O(\tau)\, \sigma_k), \tag{4.16}$$

where

$$\sigma_k = \sum \{| \gamma_1^{(i)} | : \gamma_1^{(i)} \text{ is an incoming 1-wave at } P_k\}$$
$$+ \sum \{| \gamma_j^{(i)} | \, | \gamma_m^{(l)} | : \gamma_j^{(i)}, \gamma_m^{(l)} \text{ are approaching at } P_k\}.$$

Thus, if $\| \tilde{U} - U_0 \|_\infty + TVU_0$ is sufficiently small, it follows from (4.15), (2.10) and (4.16) that the magnitude of RS at P_0 is less than $3h$. The proof is complete.

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