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Decaying States of Perturbed Wave Equations*

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We study the solutions of perturbed wave equations that represent free wave motion outside some ball. When there are no trapped rays, it is shown that every solution whose total energy decays to zero must be smooth. This extends results of Rauch to the even-dimensional case and to systems having more than one sound speed. In these results, obstacles are not considered. We show that, even allowing obstacles, waves with compact spatial support cannot decay, assuming a unique continuation hypothesis. An example with obstacle is given where nonsmooth, compactly supported, decaying waves exist.

1. SMOOTHNESS OF DECAYING STATES

Suppose u satisfies a symmetric hyperbolic equation

$$0 = u_t - Gu \equiv u_t + \sum_{j=1}^{n} A_j(t, x) \frac{\partial}{\partial x_j} u + B(t, x) u, \tag{1}$$

where the $A_j(t, x)$ are smooth hermitian matrix valued functions that are equal to constant matrices A_j, for $|x| \geqslant r$ and $B \in C_0^\infty(|x| \leqslant r)$. We make the following assumptions,

(A) *Strict hyperbolicity*: the matrix $\sum A_j(t, x)\xi_j$ has n real distinct eigenvalues for all $\xi \in \mathbb{R}^n \backslash 0$.

(B) *No trapped rays*: None of the rays of geometrical optics for $(\partial/\partial t) - G$ lie over a compact subset of x space (see [5]).

Under the additional hypothesis that G is elliptic, independent of time, and disspative ($G + G^* \leqslant 0$) a scattering theory for such systems has been developed by Lax and Phillips [4]. Wave operators are defined but they need not be isometries, and in particular, the operator W_+ of [4] annihilates all decaying states $u_0 \in L^2(\mathbb{R}^n)$. Recall that u_0 is called a decaying state if the

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solution u of (1) with $u|_{t=0} = u_0$ satisfies $\| u(t)\|_{L^2(\mathbb{R}^n)} \to 0$ as $t \to \infty$. It is of interest to study such objects and we prove the following.

THEOREM 1. *Under hypotheses* (A) and (B), *if u_0 is a decaying state, then* $u_0 \in C^\infty(\mathbb{R}^n)$.

As in [5], a big part in the proof is played by the spaces $D_\pm^\rho$ of incoming and outgoing states. To define these spaces; we use the Radon transform R, as developed by Lax and Phillips in [3, Sect. 1 and 2], modified for the symmetric hyperbolic case as in [2]. An important property of the Radon transform is that it gives a translation representation for solutions of the unperturbed hyperbolic system. That is, if $U_0(t)$ is the solution operator for the hyperbolic system $u_t - \sum_{j=1}^n A_j(\partial/\partial x_j)u = 0$, then

$$RU_0(t)f(x, \omega) = Rf(s + t, \omega).$$

R is an isometry of $L^2(\mathbb{R}^n)$ onto $W_{(n-1)/2}[\mathbb{R}, L^2(S^{n-1})]$, a Sobolev space of functions $g(s)$ with values in $L^2(S^{n-1})$. The norm in $W_{(n-1)/2}[\mathbb{R}, L^2(S^{n-1})]$ is given by

$$|||g|||^2 = \int_{-\infty}^{\infty} \| \hat{g}(\sigma)\|^2_{L^2(S^{n-1})} \, | \sigma |^{n-1} \, d\sigma,$$

where $\hat{g}$ is the Fourier transform of g. We write $Rf = Rf(s, \omega)$, with $s \in \mathbb{R}$, $\omega \in S^{n-1}$.

The outgoing spaces D_+^ρ are defined by

$$D_+^\rho = \{f \in L^2(\mathbb{R}^n)\colon K\partial_s^{n-1}Rf(s, \omega) = 0, \text{ for } s < \rho\};$$

similarly D_-^ρ consists of f with $K\partial_s^{n-1}Rf(x, \omega) = 0$ for $s > -\rho$. Here, K is the Hilbert transform acting on functions on $\mathbb{R}$ if n even, $K = I$ if n odd.

We are now ready to proceed with the

Proof of Theorem 1. As in [5], we show that if u_0 is not smooth, then (moving the origin if necessary as described below)

$$\pi_\rho^+ u(T) \neq 0, \qquad \text{for some} \quad T > 0, \tag{2}$$

where $\rho > r/C_{\min}$, and $C_{\min}$ is the smallest sound speed for $(\partial/\partial t) - G_0$, which is positive by assumption (B). As a consequence of (2) we have $\| u(t + T)\|_{L^2} \geqslant \| u(t + T)\|_{D_+^{\rho+t}} = \| u(T)\|_{D_+^\rho}$ for $t \geqslant 0$, which shows that u_0 is not a decaying state. (This argument eliminates to restriction to systems with only one sound speed).

To prove (2), pick an element of the wave front set $WF(u)$ as in [5] and follow it out along a ray until the ray becomes a straight line in the free space region $\{(x, t)\colon | x | > r\}$ passing through a point (x_0, T). Relabel coordinates in $\mathbb{R}^n$ so this line intersects the time axis $\{(x, t): t < T, x = 0\}$. Alter r so the perturbation of G_0 is still in the region $|x| < r$.

If T is taken large enough, we may assume the ray through (x_0, T) intersects the time axis at a time $T_0 < T - \rho$. Now if $v(t)$ is the solution to the free space equation $(\partial/\partial t)v - G_0 v = 0$, with $v(T) = u(T)$, it follows that any neighborhood of $(0, T_0)$ in $\mathbb{R}^{n+1}$ has nonempty intersection with sing supp v. For this propagation of singularities result see [1]. Thus, to prove that $u(T) = v(T) \notin (D_+{}^\rho)^\perp$, we use the following fact:

$$\text{If } v(T) \in (D_+{}^\rho)^\perp \text{ ir follows that } v \text{ is smooth for} \\ t < T - \rho - (1/C_{\min}) \, |x| \,. \tag{3}$$

For odd space dimensions even more is true, that is, v vanishes in the stated region. This is a consequence of the fact that D_+ and D_- are orthogonal for n odd. The proof of (3) for n even is given in the next section. ∎

Results similar to Theorem 1 are also valid for perturbations of the wave equation $(\partial^2/\partial t^2)u - \Delta u = 0$.

2. A Result on $D_\pm$ in Even Dimensions

Our purpose is to prove the result (3) used in Section 1. For the sake of simplicity we consider the Radon transform giving a translation representation for solutions to the wave equation $(\partial^2/\partial t^2)u - \Delta u = 0$ as developed by Lax and Phillips [3, Sect. 1, 2] for even space dimensions. The case of symmetric hyperbolic systems presents no new difficulties. Let us state some useful properties of the Radon transform, which takes an element $f = \binom{\phi}{\psi}$ with finite energy norm, $\|f\|_{H_0}^2 = \int (|\nabla\phi|^2 + \psi^2)$ into $Rf(s, \omega)$.

(i) $[Ru(t)](s, \omega) = [Ru(0)](s + t, \omega)$ if u solves the wave equation.

(ii) R is an isometry of H_0 onto $W_{(n-1)/2}[\mathbb{R}, L^2(S^{n-1})]$.

(iii) In this context, $D_- = \{f \in H_0: K\partial_s^{n-2} Rf(s, \omega) = 0, \text{ for } s > 0\}$ with a similar representation for D_+. Solutions of the wave equation with data in D_- vanish in the backward case $|x| < -t$.

(iv) If $f \in (D_+)^\perp$, then for $s > 0$, $Rf(s, \omega)$ is a polynomial in s of degree less than $(n/2) - 1$,(i.e., for $s > 0$,

$$Rf = \sum_{j=0}^{(n/2)-2} a_j(\omega)\, s^j.$$

We are now ready for the main result of this section.

THEOREM 2. *If* $f = \binom{\phi}{\psi} \in (D_+)^{\perp}$, *then the solution* $u(t, x)$ *to the wave equation* $u_{tt} - \Delta u = 0$, $u(0, x) = \phi$, $u_t(0, x) = \psi$ *is smooth in the backward light cone* $|x| < -t$.

Proof. For any $\epsilon > 0$, we show that u is smooth for $|x| < -t - \epsilon$. Choose $g \in C^{\infty}(\mathbb{R})$ such that $g(s) = 1$ for $s \geqslant \epsilon$ and $g(s) = 0$ for $s \leqslant 0$. Write $f = f_1 + f_2$ where $Rf_1 = gRf$.

If A_0 in the generator of the group $U_0(t)$ of solution operators to the wave equation ($u(t) = U_0(t)f$) then $f_1 \in \bigcap \mathscr{D}(A_0{}^m)$. In fact, in the translation representation provided by R, A_0 corresponds to $\partial/\partial s$, and $\mathscr{D}(A_0{}^m)$ corresponds to $W_{(n-1)/2+m} \cap W_{(n-1)/2}$. But $(\partial^m/\partial s^m)\, Rf_1$ is smooth, supported in $\{s \geqslant 0\}$, and for $s > \epsilon$ is a polynomial of degree less than $(n-1)/2 - m$, so $Rf \in W_{(n-1)/2+m} \cap W_{(n-1)/2}$. Since $f_1 \in \bigcap_m \mathscr{D}(A_0{}^m)$, it follows that $U_0(t) f_1$ is smooth on $\mathbb{R}^{n+1}$. This is a simple consequence of classical smoothness results, and is discussed for example in [6].

On the other hand, since $Rf_2 = 0$ for $s > \epsilon$, it follows that $U_0(t) f_2 = 0$ for $|x| < -t - \epsilon$, by (iii). Putting these facts together

$$u(t) = U_0(t) f_1 + U_0(t) f_2$$

is smooth for $|x| < -t - \epsilon$. ∎

3. COMPACTLY SUPPORTED DECAYING STATES

Here we consider solutions to an equation

$$Lu = \frac{\partial^2}{\partial t^2} u - Au + b(x) \frac{\partial}{\partial t} u = 0,$$

where A is a strongly elliptic operator of second order, $A(x, D_x) = \Delta$ for $|x| > r$, and $b(x) \in C_0^{\infty}(|x| \leqslant r)$. There may be an obstacle K in the ball $B_r = \{|x| < r\}$, on which boundary conditions are imposed to make a well posed mixed problem on $(0, \infty) \times \mathbb{R}^n \backslash K$. We call L a compactly supported perturbation of the wave equation.

We make the following unique continuation hypothesis:

> (UCP) *If* u *solves the mixed problem above and there is a compact set* Λ *such that* $u(x, t) = 0$ *provided* $x \notin \Lambda$ *and* $t \geqslant 0$, *then* $u \equiv 0$.

In [6] the authors showed that (UCP) always holds if the mixed problem is well posed for negative as well as positive time. In particular this holds if $K = \varnothing$ and the only perturbation comes from variable coefficients.

THEOREM 3. *For a compactly supported perturbation of the wave equation* $(\partial^2/\partial t^2)u - \Delta u = 0$, *possibly involving obstacles, there can be no compactly supported Cauchy data representing decaying states provided that the unique continuation principle* (UCP) *holds.*

Note that no hypothesis about trapped rays is made here. A similar result is true for symmetric hyperbolic systems.

Proof. Suppose $u \neq 0$ is a solution to the mixed problem with finite energy and $u(0)$, $u_t(0)$ of compact support, say in the ball B_r. By UCP we can define $T < \infty$ by

$$T = \inf\{t \geqslant 0 \colon \| u(t)\|_{L^2(\mathbb{R}^n \setminus B_{2r})} > 0\}.$$

Let H_0 be the space of data on $\mathbb{R}^n$ with finite energy and choose $f \in H_0$ to be an extension of $(u(T), u_t(T))$ over K. To describe Rf we need the following Paley–Wiener type theorem:

$$\begin{array}{l} \text{Given that } f \in H_0 \text{ has compact support,} \\ f = 0 \qquad \text{for } |x| > r \qquad \text{if and only if} \\ \qquad\qquad Rf(s, \omega) = 0 \qquad \text{for } s > r. \end{array} \tag{4}$$

In particular, $Rf = 0$ for $s > 2r$.

Suppose the space dimension, n, is odd. If $\mathscr{R}f = \partial_s^{(n-1)/2} Rf$, the map $f \mapsto \mathscr{R}f$ defines a translation representation for solutions to $(\partial^2/\partial t^2)u - \Delta u = 0$, which is unitary from H_0 to $L^2(\mathbb{R} \times S^{n-1})$.

Now D_+^r consists of those q with $\mathscr{R}q$ supported in $[r, \infty)$. We would like to know that the orthogonal projection of f on D_+^r is nonzero. This will show that the energy of $u(t)$ cannot decay to zero by the same argument as in Section 1. Thus, we must show that $\mathscr{R}f$ is not supported in $[-2r, r]$. If it were, this would imply $\mathscr{R}f(\tau)$ supported in $[-2r + \tau, r + \tau]$ where $f(t)$ is the solution to the free space wave equation with Cauchy data f. By the Paley–Wiener theorem, it follows that $f(\tau)$ is supported in B_{2r}, for τ small and positive. However, $f(\tau) = u(T + \tau)$ for $|x| \geqslant 2r$ and τ small positive, which implies

$$u(T + \tau) = 0, \qquad |x| \geqslant 2r, \quad \tau \text{ small positive},$$

contradicting the definition of T. The theorem is proved if n is odd.

Now if the space dimension n is even, the Paley–Wiener type theorem (4) is still valid, but the free space outgoing translation representation given by $\mathscr{R}f = J_+ * Rf$, where $\operatorname{supp} J_+ = (-\infty, 0]$. We still have $f \in (D_+^r)^\perp$ if and only if $\mathscr{R}f(s, \omega) = 0$ for $s > r$. Taking $f = (u(T), u_t(T))$ as above, so Rf has support in $[2r, 2r]$ we see by Titchmarsh's convolution theorem that $\operatorname{supp} \mathscr{R}f \subset (-\infty, r]$ if and only if $\operatorname{supp} Rf \subset (-\infty, r]$, which is impossible by the argument given above for the odd n case. The proof is complete. ∎

4. Some Examples

The first example we give is an equation of the form

$$u_{tt} + au_t - \Delta u = 0, \tag{5}$$

where $a \in C_0^\infty(\mathbb{R}^n)$, $a \geqslant 0$. The $a(\partial/\partial t)$ term represents a friction, and is responsible for energy loss. The function $u = e^{-t\lambda} v(x)$ is a solution of (5), provided v satisfies the equation

$$(\Delta + \lambda a(x))v = \lambda^2 v.$$

As is well known, there exist positive $b \in C_0^\infty$ such that $\Delta + b(x)$ has a positive eigenvalue, μ say. Taking $\lambda = \mu^{1/2}$ and $a(x) = \mu^{-1/2} b(x)$, we have an example of a decaying state. The results of Sections 1 and 3 apply to this example.

On the other hand, there is a very simple example of a system in one space dimension for which the conclusion of Theorems 1 and 3 are false. Consider $(\partial/\partial t)u = (\partial/\partial x)u$ for $x \in \mathbb{R}\backslash[-1, 1]$, with boundary condition $u(-1) = 0$. It is easy to see that if $u(0)$, smooth or not, is supported in $[1, 2]$, then $u(t) \equiv 0$ for $t \geqslant 1$.

This is not a fluke confined to one space dimension, we will next show how to jack the space dimension up. Consider for example the system $(\partial/\partial t)U - GU = 0$, where

$$G = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial}{\partial r} + \frac{1}{r} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial \theta}, \tag{6}$$

on $\mathbb{R}^2\backslash\{r < 1\}$. We impose a boundary condition on $U = \binom{u}{v}$, namely, $u = 0$ for $r = 1$. This again has nonsmooth, compactly supported states $U(0)$ that get annihilated in finite time. For example, let $v(0)$ be spherically symmetric and supported in $1 \leqslant r \leqslant 2$, $u(0) = 0$. Then $U(t) = 0$ for $t \geqslant 1$. This example does not have the property that the coefficients are constant outside some compact set (in x, y coordinates), and there are topological restrictions to so altering the first-order system (6), in such a manner that there be no zero propagation speed. We can overcome this by considering the following second-order system, derived from (6):

$$\begin{aligned} u_{tt} - \tilde{\Delta} u + [X, Y] \cdot v &= 0, \\ v_{tt} - \tilde{\Delta} v - [X, Y] \cdot u &= 0, \end{aligned} \tag{7}$$

where $X = \partial/\partial r$, $Y = (1/r)(\partial/\partial\theta)$, $\tilde{\Delta} f = X^2 f + Y^2 f$. The boundary conditions are

$$\begin{aligned} u &= 0, \qquad \text{for} \quad |x| = 1; \\ v_t - v_r &= 0, \qquad \text{for} \quad |x| = 1. \end{aligned}$$

Solutions to (6) also solve this system. It is easy to alter the coefficients of (7), outside $r = 2$, in a smooth manner so that outside $r = 3$ the operators become $(\partial^2/\partial t^2) - \Delta$, the usual wave operator. Clearly, this system has no trapped rays.

This last example is unusual insofar as singularities of solutions to the mixed problem get eaten up at the boundary. It would be interesting to classify boundary conditions for which this can happen. It is also tempting to speculate that, other than trapped rays, this is the only obstruction to Theorem 1 and 3 working in the case of obstacles. Examples of disappearing solutions for the scalar wave equation in an exterior domain have recently been found by A. Majda.

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