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OPPONENT PROCESS ADDITIVITY—II. YELLOW/BLUE EQUILIBRIA AND NONLINEAR MODELS[1]

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Abstract—A yellow/blue equilibrium light is one which appears neither yellowish nor bluish (i.e. uniquely red, uniquely green, or achromatic). The spectral locus of the monochromatic greenish equilibrium (around 500 nm) shows little, if any, variation over a luminance range of 2 $\log_{10}$ units. Reddish equilibria are extraspectral, involving mixtures of short- and long-wave light. Their wavelength composition is noninvariant with luminance: a reddish equilibrium light turns bluish-red if luminance is increased with wavelength composition constant.

The additive mixture of the reddish and greenish equilibria is again a yellow/blue equilibrium light.

We conclude that yellow/blue equilibrium can be described as the zeroing of a nonlinear functional, which is, however, approximately linear in the short-wavelength ("blue") and middle-wavelength ("green") cone responses and nonlinear only in the long-wavelength ("red") cone response. The "red" cones contribute to yellowness, but via a compressive function of luminance. This effect works against the direction of the Bezold–Brücke hue shift.

The Jameson–Hurvich yellow/blue chromatic-response function is only approximately correct; the relative values of yellow/blue chromatic response for an equal energy spectrum must vary somewhat with the energy level.

INTRODUCTION

In the first paper of this series (Larimer, Krantz and Cicerone, 1974), we confirmed Grassmann-type additivity laws for the equilibrium colors of the red/green opponent process. We showed that if both *a* (e.g. a yellow light) and *b* (e.g. a blue light) appear neither reddish nor greenish, then any linear combination of these lights is also a red/green equilibrium light. One implication of this finding is that the red/green chromatic-response function (Jameson and Hurvich, 1955), denoted ϕ_1, is linear with light input and the solutions of the equation $\phi_1(a) = 0$ predict the red/green equilibrium colors. As a second consequence of the empirically verified additivity laws, it can be proved (Krantz, 1975b) that ϕ_1 is a linear combination of the three photopigment quantum-catch functions. These psychophysical findings and the strength of their implications, which link the red/green opponent-response function in such a simple way to the underlying pigments, encouraged us to investigate the yellow/blue process.

Jameson and Hurvich (1955) determined a series of yellow/blue equilibrium colors in the course of their measurement of the yellow/blue chromatic-response function. Their measurements are shown in Fig. 1. For yellowish wavelengths (above 500 nm) the open symbols show the relative amounts of a standard blue cancellation light that had to be added so that the mixture of the yellowish and the blue light was in equilibrium, i.e. neither yellowish nor bluish. For bluish wavelengths, cancellation was done with a standard yellow, the results being shown as closed symbols and the sign taken conventionally as negative.

These measurements rely for their interpretation on the assumption that yellow/blue equilibrium lights are *closed* under scalar multiplication (overall intensity

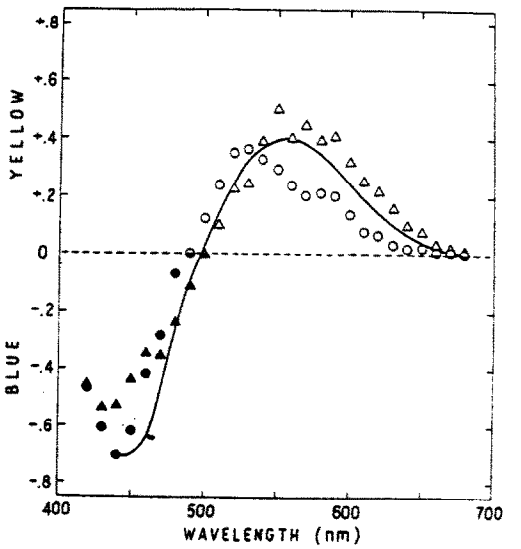

Fig. 1. Opponent-cancellation coefficients for yellowness (open symbols) and blueness (filled symbols) for observers H (circles) and J (triangles), for an equal energy spectrum. Data replotted from Jameson and Hurvich (1955). The solid line is a linear functional for the CIE Standard Observer (Judd, 1951).

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changes) and under addition (additive color mixture). We denote these operations by $*$ and $\ominus$ respectively: that is, if two lights are denoted a,b, then $t*a$ denotes the light formed by changing the radiance of light a by the positive factor t at every wavelength, and $a \oplus b$ denotes the additive mixture of a and b, in which energies are added wavelength by wavelength. Using this notation, and denoting the set of yellow/blue equilibrium lights as A_2, the two basic closure assumptions can be written formally as follows:

(i) if a is in A_2, then $t*a$ is in A_2;

(ii) if a is in A_2, then b is in A_2 if and only if $a \oplus b$ is in A_2.

For a detailed discussion of these properties and their relationships to the Jameson and Hurvich measurements and to other visual phenomena, see Krantz (1975b) and part I of this series (Larimer *et al.*, 1974). We briefly summarize the main points as follows. Assumption (i) is required for the chromatic-response function of Fig. 1 to be independent of luminance in its shape and, indeed, (i) was presupposed in plotting Fig. 1, since the values there are for an equal-energy spectrum, although the measurements were made at constant brightness. Assumption (ii) is required for predicting the chromatic response of mixed lights from their monochromatic components, and in particular is necessary if the shape of the function is to be independent of the choice of cancellation standard. If both (i) and (ii) hold, then the measured chromatic-response function, which we denote ϕ_2, can be shown to be linear with light input, i.e.

$$\phi_2[(s*a) \oplus (t*b)] = s\phi_2(a) + t\phi_2(b). \qquad (1)$$

This also implies that ϕ_2 is a linear combination of the color-matching functions or of any other linear trichromatic coordinates, such as photopigment primaries.

Assumption (i) is closely related to invariance of equilibrium hues with respect to the Bezold–Brücke hue shift, while (ii) is related to invariance with respect to the Abney hue shift, when an equilibrium light is used as a desaturant.

In this paper we study the luminance-invariance and additivity of the equilibria of the yellow/blue opponent process. Figure 1 shows that there is one monochromatic equilibrium light, a green around 500 nm. But there is no single monochromatic equilibrium light in the long-wavelength spectral region—all long-wavelength reds are slightly yellowish. We can adapt our previous method of using a wavelength staircase to determine the monochromatic equilibrium green, but we will have to use mixtures of yellowish-red and bluish light to generate equilibrium reds.

GENERAL DESIGN

One method to study equilibrium red would be to use a procedure similar to the cancellation procedure which generated Fig. 1. A fixed bluish cancellation wavelength would be added to the long-wavelength yellowish red, with variable luminance of the blue component. The equilibrium point would be determined by a luminance staircase. Property (i) would be confirmed for equilibrium reds if this equilibrium luminance were proportional to the luminance of the red component, wavelengths being fixed.

This method would have two disadvantages. First, different response metrics would be used for determining equilibrium red and equilibrium green. Secondly, at low luminance, the reds really do not look very yellowish at all and, consequently, the observer would never see any very yellowish reds. This would interfere with the delicate perceptual learning involved in detecting small amounts of yellow or blue sensation in predominantly reddish lights, and it could bias the results in ways that cannot be foreseen precisely.

For these two reasons, we fixed the red component at 650 nm, and varied the wavelength of the short-wavelength component, holding its luminance approximately constant. From Fig. 1 it can be seen that the blue chromatic response increases steeply below 500 nm; the steepness is even greater at constant luminance, rather than constant radiance as shown in the figure. By using a dim short-wavelength component (relative to the red) we could affect the blue/yellow aspect of the mixture relatively steeply while affecting the redness relatively little. This method, which was introduced in the studies of equilibrium mixture in our previous paper, allows us to use a wavelength rather than a luminance staircase, keeping response metric the same. Moreover, by carrying the short-wavelength component past the equilibrium green point, into the yellow-green, we could guarantee that the mixture looked distinctly yellow. We are thus in a position to test by quite objective means the contention that the long wavelengths appear slightly yellowish, even at low luminance—we confirm this by showing that in reddish equilibrium lights the wavelength of the short-wavelength component is in fact found to be on the bluish side of equilibrium green.

To study mixtures of reddish and greenish A_2 equilibria, we redetermined the wavelength of the short-wavelength component of red in mixed lights with an additional greenish component. For example, we might fix both the 650 nm red component and a yellowish-green component around 510 nm; we then show that in the presence of the 510 nm addend the bluish component is shifted to still shorter (bluer) wavelengths.

As a consequence of this method, we plot only the wavelengths of the equilibrium green and of the short-wave component of equilibrium red. Our mixture experiment looks like a tradeoff between the yellowness and the blueness of the two short-wavelength lights, in the presence of a fixed 650 nm component.

METHODS

Apparatus

A detailed description of the apparatus can be found in Larimer *et al.* (1974). Briefly, the stimulus was a disk, subtending a visual angle of 2·6°, presented in Maxwellian view on a dark background without fixation point. The stimuli consisted of various intensity combinations of three monochromatic beams, superimposed optically by two beam-splitting cubes, as shown in the schematic diagram of Fig. 2.

Observers

The same two males and three females who served in our previous study served as observers in this experiment. Observers PS and TC remained completely naive about the goals of the experiment, and observers CC, DK, and JL were kept naive about their own performance. All observers used their right eyes. Observer PS was unavailable at the time the others were run. Later, she was run in a somewhat reduced design. (Her data nevertheless confirm all our main findings.)

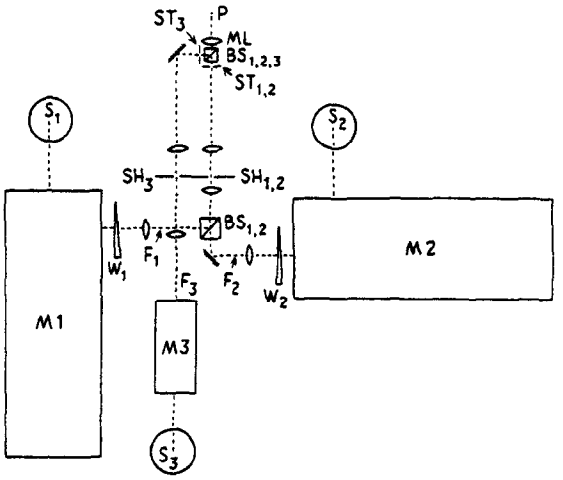

Fig. 2. A schematic diagram of the optical system. One channel combines beams from grating monochromators M1, M2 at beamsplitter $BS_{1,2}$. The second channel has a single monochromatic input from grating monochromator M3. S_1, S_2, S_3 are tungsten sources; W_1, W_2 neutral-density wedges; F_1, F_2, F_3 stray-light and neutral-density filters; $SH_{1,2}$ and SH_3 shutters; $ST_{1,2}$ and ST_3 stops. The channels are combined at beamsplitter $BS_{1,2,3}$ and the Maxwellian view is provided by lens ML. All three exit slits are imaged at P, the observer's pupil. (The horizontal distances are shown nearly to scale but the M3 channel is simplified by omission of a vertical segment of the light path and two auxiliary lenses.)

Alignment

Alignment for this experiment was more critical than in the red/green equilibrium experiment (where monochromator M3 was not used). Monochromators M1 and M3 were set at 650 nm and M2 at 505 nm. With the beam from M3 occluded the observer adjusted $ST_{1,2}$ for good focus if necessary, and then adjusted the position of his pupil in the plane perpendicular to the optic axis so that the white image of $ST_{1,2}$ had neither red nor green fringes. At this point M1 was occluded and the beams from M2 and M3 were exposed. The observer's task was to adjust ST_3 so that it appeared in focus with $ST_{1,2}$ and was perfectly superimposed on it. Thus, again no fringes appeared. The wavelength of M1 was changed to 505 nm and the superposition of $ST_{1,2}$ and ST_3 was rechecked with the M1, M3 combination. A final check of alignment was sometimes carried out by setting M2 and M3 to 580 nm and M1 to 480 nm and again checking for focus and fringes in various pairings of M1, M2, and M3 as well as all three in concert.

Procedure

The same procedure as that employed in the red/green equilibrium experiment was used. Observers were dark adapted for 10 min, after which the experimental session began. Stimuli were delivered for 1 sec with an intertrial interval of 20 sec of darkness. Wavelengths required for equilibrium were determined by staircase procedures (Larimer *et al.*, 1974), with luminance approximately constant. Staircases for three or more conditions were irregularly interspersed, thereby avoiding anticipatory responses.

Response

The observer's task was to judge whether the stimulus seen on a trial was yellowish or bluish. With the exception of some of the mixture conditions, where the stimulus appeared mostly whitish, the stimuli were generally either overwhelmingly red or green. Thus it was essential that the observer pay attention to very minute traces of yellowness or blueness. As we noted in our study of red/green equilibria, practice plays an important role in these judgments. The judgment of yellowness or blueness of saturated reds was especially difficult, and required much concentration to reduce variability to a minimum.

Filter calibration

Whenever the luminance of a non-monochromatic light is increased or decreased careful attention must be paid to the luminance changes at every point in the spectrum. Our Wratten 96 filters show as much as a 10 per cent density difference in the short to long regions of the visual spectrum. We compensated by wedge adjustments for these slight colorations in the neutral density filters, thereby assuring true scalar multiplication in the nonspectral red and the addend conditions.

EXPERIMENT 1: SCALAR MULTIPLICATION FOR YELLOW/BLUE EQUILIBRIA

Introduction

A direct test of (i), closure under scalar multiplication, was performed by determining the spectral locus of equilibrium green at three luminance levels, and by determining the spectral locus of the short-wavelength component of equilibrium red, also at three luminances. For the latter test, the luminance ratio between the 650 nm and short-wavelength component was fixed at about 3·4 to 1. If closure obtains, then the wavelength loci of both the equilibrium greens and the short-wavelength component of the equilibrium reds should be independent of luminance.

These loci were determined for each of the five observers using the staircase method mentioned previously. On a single day a determination was made at each luminance level. The experiment was repeated on each of 4 days. The exact spectral locus for a given luminance and hue on a single day was determined by linear interpolation to the wavelength that would have generated 50 per cent "too yellow" responses and 50 per cent "too blue" responses.

Results

The four daily determinations at each luminance-hue combination were averaged and an estimate of the standard error was computed using the between-day variability. A wavelength by luminance plot of these loci for the first four observers is given in Fig. 3. The equilibrium greens are plotted to the right on a scale from 485 to 505 nm. The short-wavelength component of the equilibrium red mixture is plotted on the left with a scale range of 465–500 nm. It is important to recall that this is the wavelength of the dim blue that was used to counter the yellowness in the much brighter 650-nm component of the red mixture. The horizontal bars at each point are the 80 per cent confidence intervals for the mean, based on between-day estimates of the standard error.

For linearity to hold the line connecting the determinations at each different luminance should be perpendicular to the wavelength axis. The equilibrium greens do not deviate systematically from that prediction. The equilibrium reds, however, show a small but systematic shift in every subject—at the highest luminance, the short-wave component shifts toward the yellow. (See the yellow/blue cancellation function, Fig. 1.)

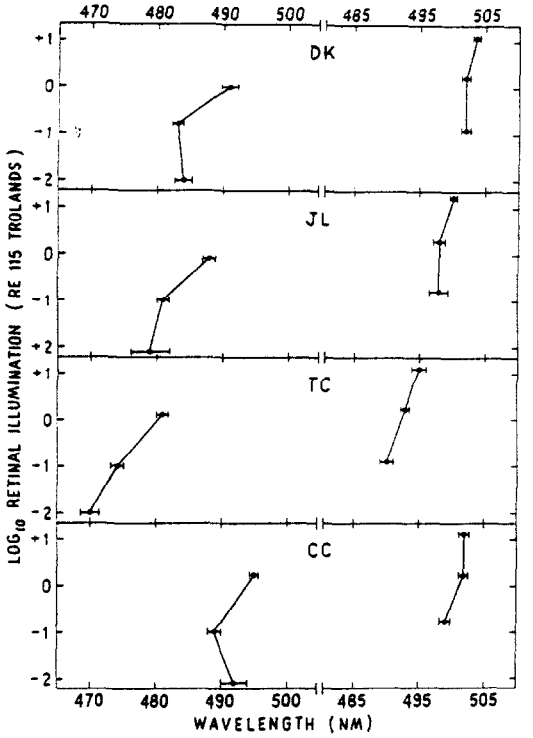

Fig. 3. Mean wavelength of the short-wave (variable) component of equilibrium red (left) and mean wavelength of equilibrium green (right), at various luminance levels, for four observers. Error bars are 80% confidence intervals based on between-session variability.

The results for the fifth observer, PS, are very similar, except that all of the lowest-luminance reds were judged to be bluish. We therefore did not measure any equilibrium red mixture at the lowest luminance. This difficulty will be discussed below, in connection with a brief review of Westphal's (1909) report.

Discussion

The set A_2 is not closed under scalar multiplication and hence the yellow/blue chromatic response function is not linear. In particular, when the luminance of a monochromatic long-wave stimulus is increased, the cancellation yellowness does not increase proportionally, and so relatively less blue is required to balance the yellow.

Note that the shift in cancellation yellow runs contrary to the Bezold–Brücke shift of perceived yellowness. Increasing the luminance of a 650-nm light yields a striking increase in *apparent* yellowness (the classic Bezold–Brücke shift); whereas in isolation the short-wavelength cancellation light of course appears less blue as its wavelength is increased. This strongly suggests a complex relationship between the cancellation code or chromatic response functions on one hand and the magnitude code or the appearance of yellowness or blueness in a light, on the other (see Krantz, 1975b).

The shift in equilibrium red, in conjunction with the lack of shift in equilibrium green, suggests that the nonlinearity may be due to the role of the long-wavelength pigment whose quantum catch is much larger for an equilibrium red than for an equilibrium green. Several models of this nonlinearity will be discussed below.

EXPERIMENT 2: ADDITION OF YELLOW BLUE EQUILIBRIA

Introduction

A direct test of condition (ii) would be to take the equilibrium red and equilibrium green lights and mix them in some luminance ratio. If the mixture were judged neither yellowish nor bluish, then (ii) would be confirmed. In the introduction to the comparable red/green equilibria addition experiment (Larimer *et al.*, 1974), we gave several reasons for a slight modification of the direct approach. Briefly these reasons are as follows. First, the concept of an equilibrium light is statistical. This means that a light which is equilibrium on one day may not be on the next day. Therefore it is more reasonable to test (ii) by fixing one component (called the *addend*) and then using the staircase method on the other component (called the *variable*). In this experiment a green wavelength plus the fixed 650-nm light were mixed to form the addend, and the variable was always the short-wavelength component of the equilibrium red. We further argued that the prior determination of equilibrium hues would force a temporal separation of the two experiments, with consequent possible criterion variation. A third reason was that the *statistical* nature of an equilibrium green would make the direct approach vulnerable to lack of statistical precision, especially when the green component of the addend is brighter than the red component. Therefore we used wavelengths of the green component on either side of the equilibrium point, and the locus of the short-wavelength variable component was determined as a function of the wavelength of the green component.

These experiments were carried out simultaneously with the determinations of equilibrium reds and greens (Experiment 1). For each observer three green/red luminance ratios (approx 4, 1/2, and 1/6 relative to the 650-nm component), and two values of the red luminance were selected (approx 390 and 30 td for the 650-nm component). For each of the resulting six combinations, we determined the wavelength of the short-wavelength component of the red, as a function of the wavelength of the green. Each data point represents the mean of two determinations made on two different days of the 4-day experiment.

Results

The results of this experiment for the first four observers are presented graphically in Fig. 4, which plots the short-wavelength component of the red mixture, the variable, against the green wavelength component of the addend. The different symbols (□, △, and ○) denote the low, medium, and high green/red luminance ratios. The upper graphs are the plots for the high-luminance red and the lower ones are the plots for the low-luminance red. The solid six-pointed star is a replot of equilibrium wavelengths from Experiment 1: its abscissa is the locus of the short-wave component of red, at the corresponding luminance, and its ordinate is the mean locus of equilibrium green.

Straight lines were fitted by eye to each set of points having a fixed luminance ratio and a fixed red luminance. Property (ii) implies that the intersection of these lines should be at the solid six-pointed star in each plot. This prediction is confirmed by these data. The nonlinearity apparent in Experiment 1 is also

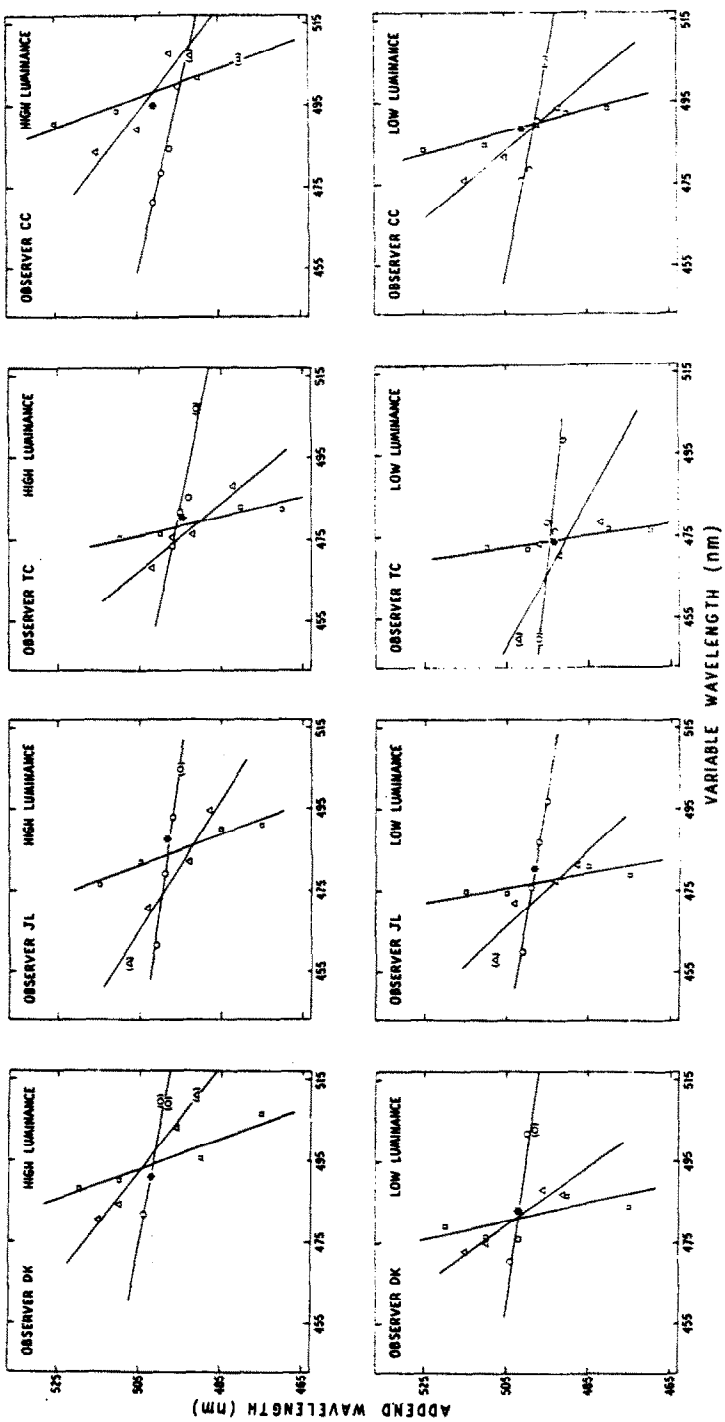

Fig. 4. Wavelength loci of addend-plus-variable combinations which, mixed with 650-nm light, are neither yellowish nor bluish. Upper graphs: 650 nm at 390 td; lower graphs: 650 nm at 30 td. The luminance ratio of the 650 nm component to the (blue) variable component was fixed at approx 3·4 to 1. The luminance ratios of the (green) addend to the 650-nm light were: □—1/6; △—1/2; ○—4. The solid star has ordinate equal to the equilibrium green locus (Fig. 3, right, averaged over luminance) and abscissa equal to the short-wave component of equilibrium red (Fig. 3, left, for the corresponding luminance of 650 nm). Straight lines were fit by eye to each set equilibrium loci with fixed addend/variable ratio and fixed luminance level.

apparent in these results and is seen in the fact that the lines for the high and low luminance reds intersect near the corresponding six-pointed stars. That is, corresponding lines are generally shifted to the right in the high-luminance graphs.

The data for the fifth observer, PS, are virtually identical, except that the design was only 2 × 2, with the lowest green/red ratio omitted.

Discussion

The closure under the addition, property (ii), is confirmed in these data within luminance levels. But a marked nonlinearity occurs between luminance levels. A more detailed discussion of these results will follow after the introduction of several potential nonlinear models of this process.

INDIVIDUAL DIFFERENCES IN YELLOW/BLUE EQUILIBRIA

Previous work on this subject has comprised two kinds of studies: determinations of yellow/blue equilibria and studies of hue shifts, especially the Bezold–Brücke effect. Some of the main studies, and their logical interrelations, were reviewed in our previous paper on red/green equilibria. We shall mention only some further points that are of interest in connection with the yellow/blue system.

Our main points agree with some findings of Purdy (1931). He found that equilibrium green is invariant with luminance and coincides with an invariant wavelength for the Bezold–Brücke shift. He found that equilibrium red is extraspectral and noninvariant with luminance. (He reported that there is a red-plus-blue mixture which is hue-invariant with luminance increase, but that its hue is bluish red, not equilibrium red.)

Westphal (1909) found that "fundamental green" shifted to slightly shorter wavelengths as luminance increased. His findings for reds are more discrepant from ours. For three of his observers, long-wavelength reds were judged yellow at high luminance, but scarcely so at low luminance. For three others, a monochromatic "fundamental red" was located in the long wavelengths at all luminance levels. These same observers tended to show their equilibrium green wavelengths shifted to unusually long wavelengths—a finding that has recently been a focus of debate (Rubin, 1961; Hurvich, Jameson and Cohen, 1968). He classified these latter observers as "yellow-indolent."

In considering the problem posed by the discrepant results for the yellow/blue equilibria, three possible explanations must be considered: adaptation effects, response biases, and individual differences in the organization of the yellow/blue process. Hurvich *et al.* (1968) showed clearly that there are large individual differences in the effect of "minor" adapting effects on the equilibrium green locus. We have taken pains to keep the observers neutral-adapted, but this could have been a factor in Westphal's results. Response bias is a serious problem. Some naive observers, presented with a series of long-wavelength reds, and asked to judge them as "yellowish" or "bluish" may comply with the task by judging the ones that are not obviously orange as "bluish." This is particularly the case at low luminance, where all observers agree that the salience of any "yellowness" is reduced or nearly absent.

As stated under General Design, our methods avoid some of this difficulty, since we can vary the stimulus by adding an extremely yellowish or extremely bluish short-wavelength component to the standard red. The observer can have no idea of his performance—specifically, no idea of whether the short-wave component on any trial is less than or greater than the equilibrium green in wavelength. We in fact noted that one of our observers (DK), in early pilot data, sometimes required short-wave components at wavelengths equal to or longer than equilibrium green. With more experience (but with no feedback) perceptual learning seems to take place: the observer becomes skilled at identifying small traces of yellowness or blueness. In later pilot data, and in the main experiment reported above, the four main observers judged mixtures of 650 nm with the dimmer green or slightly bluish/green component as yellowish; the equilibrium values of the short-wave component were 5–25 nm below the equilibrium green wavelength. At the same time, it should be noted that the lowest luminance reds were subjectively very difficult and the results were correspondingly more variable.

A fifth observer, PS, showed the above difficulty in pilot data not only at the lowest, but also at the middle luminance level. When we tested her (without her knowledge) on 650 nm alone, she judged it as yellowish at high luminance, but bluish at intermediate and low luminance. After several such pilot sessions she abruptly switched (again, we emphasize there was no feedback). Now she judged the intermediate luminance 650-nm stimulus also as yellowish, and gave mixed responses at the lowest luminance. It then became possible to obtain results similar to those for the other observers, at all conditions except for the lowest luminance red equilibrium. During the formal 4-day experiment, we simply discontinued staircases for this condition whenever the wavelength required exceeded about 505 nm; and so we never did obtain a reliable equilibrium point at this luminance.

Obviously PS is somewhat like Westphal's "yellow-indolent" observers, in the red, but not in the green. It is clear that a part of her early response pattern was due to insufficient practice with very slightly orangish reds. To what degree this stemmed from inattention, and to what degree from genuine individual differences in yellow/blue coding, we do not know.

NONLINEAR MODELS FOR YELLOW/BLUE EQUILIBRIA

All five observers showed shifts to somewhat shorter wavelengths of the short-wave component of equilibrium red, at low luminances. We have noted that this effect runs counter to the perceived yellowness of 650-nm light; even though the low-luminance red light *looks* relatively less yellowish, its *cancellation* yellow is relatively higher. The fact that the same phenomenon appears as well in the mixture conditions (bottom part of Fig. 4 is shifted to the left) rules out the possibility that these results could be attributed to response-criterion shift. We are therefore led to consider possible nonlinear functions ϕ_2 such that the equation $\phi_2 = 0$ characterizes the yellow/blue equilibria.

Let us first consider functions of the form

$$\phi_2 = \sum_{j=1}^{3} \alpha_{2j}\rho_j^{n_j}. \quad (2)$$

Here ρ_1, ρ_2, ρ_3 denote the relative quantum-catch functions of the short-wavelength, middle-wavelength, and long-wavelength cones, respectively; α_{2j} is the weight by which ρ_j enters into ϕ_2; and n_j is a positive exponent, presumably ≤ 1. Linearity holds if and only if $n_j = 1$, for $j = 1$, 2 and 3. More precisely, if ϕ_2 satisfies equation (2), and if A_2 is characterized by $\phi_2 = 0$, then A_2 satisfies the empirical conditions (i) and (ii) if and only if $n_1 = n_2 = n_3 = 1$ [i.e. if and only if ϕ_2 satisfies equation (1)]. (Note that condition (i) alone is satisfied if all the n_j are equal, even if they are different from 1.)

The pattern of our observed results is qualitatively consistent with the hypothesis that $n_1 = n_2 = 1$ and $n_3 < 1$, for we can reason as follows. Making n_3 less than n_1 and n_2 means that the 650-nm light, which mainly affects ρ_3, will contribute relatively less than the short-wave component to ϕ_2, as luminance increases. Making $n_1 = n_2$ means that a 500-nm light (near the greenish equilibrium point) will have relatively constant effects on ρ_1 and ρ_2, as luminance changes, resulting in near invariance of that equilibrium point. And making $n_1 = 1$ allows additivity of the effects of the short-wavelength component of equilibrium red and of equilibrium green on ρ_1. (This additivity is shown by the upper and lower parts of Fig. 4, taken separately.) We therefore suggest, as one possible nonlinear model, the equation

$$\phi_2 = -\rho_1 + \alpha_{22}\rho_2 + \alpha_{23}\rho_3^{n_3}, \quad (3)$$

where we have normalized by setting $\alpha_{21} = -1$. Positive values of ϕ_2 correspond to yellowish appearances; negative ones correspond to bluish appearances; and $\phi_2 = 0$ characterizes A_2.

In order to test equation (3) quantitatively, we need to know (or assume) the functions ρ_1, ρ_2, and ρ_3. We assumed for this purpose that ρ_1, ρ_2, and ρ_3 correspond respectively to the Vos and Walraven (1970) B, G, and R primaries. [These primaries were derived on the assumption that tritanopic, deuteranopic, and protanopic color matches are respectively based on the normal (G,R), (B,R), and (B,G) pigment pairs; see Krantz, 1975a.]

The three parameters of equation (3) are α_{22}, α_{23}, and n_3. These had to be estimated (separately for each observer) to optimize the fit to the 30 experimental measurements per observer (six points in Fig. 3 and 24 points in Fig. 4). Given trial values for the three parameters, it is straightforward to find the wavelength λ of the variable light that yields $\phi_2 = 0$ for any experimental condition. For example, in the determination of equilibrium green (right half of Fig. 3) only one monochromator is used; for wavelength λ and relative radiance t the value of ϕ_2 is just

$$-t\rho_1(\lambda) + \alpha_{22}t\rho_2(\lambda) + \alpha_{23}[t\rho_3(\lambda)]^{n_3}.$$

Since for any one staircase, our calibration specifies $t = t(\lambda)$ as a function of λ (approximately constant, but not exactly), this entire expression is a function of λ. For λ far below 500 nm, the ρ_1 term will dominate and the expression is negative (blueness); for λ high, the ρ_2 and ρ_3 terms dominate and the expression is positive (yellowness); and a simple search routine estimates λ such that the expression is 0.

In the determination of equilibrium red (left half of Fig. 3) two monochromators are involved and the expression for ϕ_2 becomes

$$-t\rho_1(\lambda) - u\rho_1(650) + \alpha_{22}t\rho_2(\lambda) + \alpha_{22}u\rho_2(650) + \alpha_{23}[t\rho_3(\lambda) + u\rho_3(650)]^{n_3}.$$

Here, λ and t are the wavelength and relative radiance of the short-wavelength component and u is the relative radiance of the 650-nm component. Since u and $\rho_i(650)$ ($i = 1, 2, 3$) are known constants and $t(\lambda)$ is a known (approximately constant) function of λ, we can again compute a predicted λ for equilibrium.

Finally, in the addition conditions (Fig. 4) we also include terms involving the radiance and the ρ_i values of the green addend, but again, the estimation of a predicted equilibrium λ is straightforward.

Goodness of fit requires that the 30 predicted λ's be close to the 30 experimental measurements. A variety of goodness-of-fit criteria is possible, since some of the 30 measurements are less reliable than others and some of the predictions are much more sensitive than others to the detailed shapes of the Vos/Walraven ρ_i curves. We ended up using one formal criterion, the simple, unweighted sum of squared deviations between predicted and observed values of λ, and one informal criterion, "qualitative" goodness of fit.

After extensive computations with this model (using a computer program written by the authors for this purpose) we obtained some fairly low sums of squared deviations. The results are shown in Table 1. The second column of the table shows the root mean square deviation of observed from predicted λ, for all 30 conditions. The third column shows the same quantity when 0–4 very poorly fit data are omitted. The fourth column shows $1/\sqrt{2}$ times the pooled estimate of the standard deviation of repeated measurements of a variable wavelength by our staircase measurement. The tabled quantity, $\sigma/\sqrt{2}$, is thus exactly the esti-

Table 1. Fit of equation (3)

Observer	$\lvert\lambda_{pred} - \lambda_{obs}\rvert$ (root mean sq., nm)		Estimated $\sigma/\sqrt{2}$ (nm)	Shift of equilibrium green		Parameter values		
	all data	best subset		pred.	obs.	α_{22}	α_{23}	n_3
DK	6.1	3.4 (26 pts.)	2.0	7.4	0.6	.0052	.0348	.7
JL	7.5	7.5 (30 pts.)	3.5	6.6	1.2	.0085	.0466	.7
TC	6.4	4.4 (27 pts.)	1.6	4.7	4.7	.0093	.0363	.8
CC	5.8	5.3 (29 pts.)	1.8	1.6	3.0	.0110	.0084	.7

Table 2. Fit of equation (4)

Observer	$\lvert\lambda_{pred} - \lambda_{obs}\rvert$ (root mean sq., nm) all data	best subset	Estimated $\sigma/\sqrt{2}$ (nm)	Shift of equilibrium green pred.	obs.	Parameter values α_{22}	α_{23}	n_3
DK	5.8	3.4 (27 pts.)	2.0	3.6	0.6	.0081	.252	.4
JL	6.2	6.2 (30 pts.)	3.5	1.9	1.2	.0105	.126	.6
TC	6.9	4.0 (27 pts.)	1.6	0.8	4.7	.0136	.0976	.7
CC	5.7	5.0 (29 pts.)	1.8	0.3	3.0	.0114	.0140	.6

mated standard error of λ_{obs}, for the 24 addition conditions (two replications each), and is $\sqrt{2}$ times the standard error of λ_{obs} for the six conditions of Fig. 3 (four replications each). Obviously the deviations from the model's predictions are too large to be accounted for by sampling error alone. However, in view of the rather heavy dependence of the fit on the Vos/Walraven primaries, the deviations may be tolerable. The last three columns of the table show the values of the three parameters, α_{22}, α_{23}, and n_3, for which the fit was obtained.

However, equation (3) is qualitatively wrong for observers JL and DK, because it predicts a nonexistent shift of the green equilibrium wavelength. This is shown in the fifth and sixth columns of the table, where the difference between the equilibrium green wavelengths at the highest and lowest luminance is given for the model (predicted) and for the actual data observed.

Above, we reasoned that equation (3) would not predict an appreciable shift in equilibrium green, because the equation is linear in ρ_1 and ρ_2. However the parameter estimates in columns 7 and 8 show that DK and JL have a large contribution of ρ_3, compared with ρ_2; and since the sensitivity of ρ_3 is actually a bit higher around 500 nm than at 650 nm, this led to a large predicted nonlinearity around 500 nm. Thus under the assumption that the Vos–Walraven primaries are correct, the model is incorrect, at least for these observers. Observer CC seems to have a very small contribution of ρ_3 compared with ρ_2, and the model therefore behaves qualitatively correctly for her data. Observer TC has parameters more like those of JL and DK, but her data actually show the predicted shift in equilibrium green, and so again, the predictions are satisfactory.

The fact that ρ_3 is as sensitive around equilibrium green as it is in the far red makes it puzzling how to account for the data of DK and JL, who show a nonlinearity involving the red but not the green. We noted that the Vos/Walraven curves for ρ_2 and ρ_3 were actually nearly equal around 500 nm, and this suggested that a model that had a nonlinear term involving $\rho_3 - \rho_2$ might be better behaved in the green, while being virtually identical to equation (3) in the red, where ρ_2 has negligible sensitivity. So we tried the following:

$$\phi_2 = -\rho_1 + \alpha_{22}\rho_2 \pm \alpha_{23}|\rho_3 - \rho_2|^{n_3}. \qquad (4)$$

We took the $\pm$ sign as $+$ if $\rho_3 - \rho_2 > 0$ [so the model coincides with equation (3) in the red] but as $-$ if $\rho_3 - \rho_2 < 0$ (this decision makes little difference, because $|\rho_3 - \rho_2|$ is always small when $\rho_3 - \rho_2 < 0$).

Another way to look at equation (4) is to think of two intermediate opponent processes: a ρ_3, ρ_2 cancellation, yielding $\rho_3 - \rho_2$, and a ρ_2, ρ_1 cancellation, yielding $\alpha_{22}\rho_2 - \rho_1$. The yellow/blue mechanism sums them, after first applying a compressive transformation to $\rho_3 - \rho_2$.

Table 2 shows results for equation (4), analogous to those in Table 1 for equation (3). The overall fits are about the same as in Table 1—perhaps slightly better, for column 3, where a few badly fitting points are dropped in both tables. The qualitative fit to equilibrium green is much better for JL and DK, but worse for TC whose actual shift is no longer predicted. Note that for DK, the nonlinearity in $\rho_3 - \rho_2$ is really marked: $n_3 = 0{\cdot}4$ and $\alpha_{23} = 0{\cdot}252$. By contrast, the n_3 values for equation (3) (Table 1) are very similar among observers.

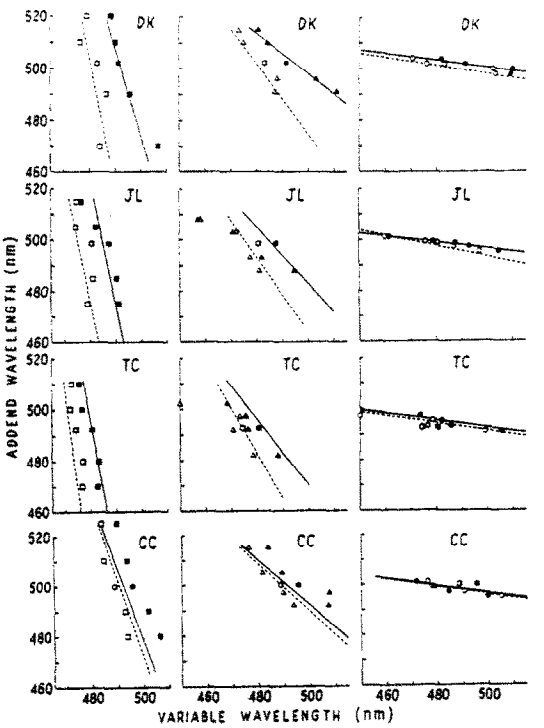

Fig. 5. The data of Fig. 4 (loci of addend plus variable combinations which equilibrate 650 nm on the yellow/blue code) are replotted, together with theoretical curves based on the three parameter nonlinear model of equation (4). Solid symbols and lines: higher luminance (390 td at 650 nm). Open symbols, dashed lines: lower luminance (30 td at 650 nm). The four lefthand graphs have luminance ratio (green addend/650 nm) of 1/6; center, 1/2; right, 4·0.

Another way of looking at the nonlinear results, and their fit by equation (4), is shown in Fig. 5. Here the data of Fig. 4 are replotted, except that high- and low-luminance results are compared on the same axes, and different addend/variable luminance ratios are separated for clarity. (Thus, Fig. 4 emphasizes additivity within luminance levels, by showing that at each luminance level, the three loci are nearly concurrent, near the predicted point based on equilibrium red and equilibrium green measurements; while Fig. 5 emphasizes the difference between the two luminance levels, implying nonadditivity.)

The straight lines in Fig. 5 are not drawn through the data points; rather, they are a very close fit to the *predicted* points based on equation (4). For observer DK (top three graphs) it seems that deviations from equation (4) are not systematic: the three-parameter fit to 26 data points (the two stars are the same in each graph) is excellent. Observer CC (bottom three graphs) is fit worst by the model. The reason is clear from the data: for the highest addend/variable luminance ratio (right-hand graph) the luminance-level effect is washed out, or even reversed (the filled circles tend to be to the left of the open ones, rather than to the right as in the other 11 graphs). For the lowest and the middle addend/variable ratios, on the other hand, the luminance-level effect is around 10 nm, similar to that found in other observers. The resulting best fit underpredicts the luminance-level effects at these lower addend/variable ratios. (Note the very small value of α_{23} in Tables 1 and 2 for CC.) Most likely the power-function form of nonlinearity will not do for CC's data; a better fit might be attained, e.g. by a function that is more nearly linear in ρ_3, or in $\rho_3 - \rho_2$, when ρ_1 and ρ_2 are large relative to ρ_3.

The fit for observers JL and TC is qualitatively better than for CC. The largest deviations could only be rectified by a theory that predicted shallower curves for high-luminance mixtures; we don't know what sort of nonlinearity this would entail, nor whether it would be a real improvement.

A figure like Fig. 5 could be constructed for equation (3), but it would differ relatively little. The quantitative fit of equation (4) is slightly better overall, but its qualitative fit to the mixture data is little different from that of equation (3). For the green equilibrium point, equation (3) is superior for two observers and equation (4) for the other two.

To summarize, nonlinearity appears most strikingly when the 650-nm light is mixed with short-wavelength lights, and for that reason we hypothesized that the complication resides chiefly in the contribution of the long-wave (ρ_3) pigment to yellowness. (As intensity increases, this contribution becomes relatively easier to cancel.) The moderately good fit of quantitative models, to the entire data set, using standard Vos/Walraven primaries, supports this general hypothesis, though it is possible that a model involving a nonlinear contribution of $\rho_3 - \rho_2$ (long minus middle) to yellowness fits as well or better. We feel that further refinement of the nonlinear model is probably not warranted on the basis of these data. We hope to return to this problem in the future by bringing to bear other data that may dictate the nature of the nonlinear interaction in the yellow/blue code.

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