Multiple Zeros for Eigenvalues and the Multiplicity of Traps of a Linear Compartmental System

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ABSTRACT

It is known that a linear compartmental system has a trap if and only if the associated system of differential equations has zero as an eigenvalue. In this paper, we show that if such a system has zero as an eigenvalue of multiplicity m, then the system contains m irreducible traps.

In [1], Fife showed that a linear compartmental system has a trap if and only if the associated system of differential equations has zero as an eigenvalue. The question arises as to what information about the system can be gained from knowing the multiplicity of this eigenvalue. In this note, we will use some results given by Hearon [2] to examine this question.

Let S be a linear compartmental system consisting of n compartments $C_1,\ldots,C_n$, and let q_j be the amount of material in C_j. Let f_{ij} be the fractional exchange coefficient, so that the rate of flow of material from C_j to C_i is $f_{ij}q_j$. Let $f_{0j}q_j$ be the rate of flow of material from C_j to the environment. Setting

$$f_{jj} = -\left(f_{0j} + \sum_{\substack{i=1\\ i\neq j}}^{n} f_{ij} \right), \tag{1}$$

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we observe that $f_{jj}q_j$ is the total amount of material leaving C_j per unit time. Therefore, the rate of change of material in C_j is given by

$$\frac{dq_j}{dt}=f_{jj}q_j+\sum_{\substack{i=1\\ i\neq j}}^{n} f_{ji}q_i, \qquad i=1,\ldots,n. \tag{2}$$

In matrix notation, (2) becomes

$$\dot{q}=Fq, \tag{3}$$

where $q=(q_1,\ldots,q_n)^T$ and $F=(f_{ij})$. We call (f_{ij}) the matrix of the system (relative to $\{C_1,\ldots,C_n\}$).

The system S is called a separable linear system if it can be partitioned into a disjoint union of subsystems

$$S=S_1\,\dot{\cup}\,S_2\,\dot{\cup}\cdots\dot{\cup}\,S_k \tag{4}$$

such that S_i receives no input from $S_{i+1},\ldots,S_k$, $i=1,\ldots,k-1$. Suppose S is separable. Then it is clear that we may renumber the compartments of S so that

$$\begin{aligned} S_1&=\{C_1,\ldots,C_{k_1}\},\\ S_2&=\{C_{k_1+1},\ldots,C_{k_2}\},\\ &\vdots\\ S_k&=\{C_{k_{k-1}},\ldots,C_n\}, \end{aligned} \tag{5}$$

where, relative to this numbering, F assumes block triangular form

$$F=\begin{bmatrix} F_{11} & & & \\ F_{21} & F_{22} & 0 & \\ \vdots & \vdots & \ddots & \\ F_{k1} & F_{k2} & \cdots & F_{kk} \end{bmatrix}. \tag{6}$$

Now let Σ_n be the permutation group on $\{1,2,\ldots,n\}$. Then an arbitrary n by n matrix $A=(a_{ij})$ is, by definition, reducible iff there exists a σ in Σ_n such that

$$(a_{\sigma(i),\sigma(j)})=\begin{bmatrix} A_{11} & 0\\ A_{21} & A_{22}\end{bmatrix}, \tag{7}$$

where A_{11} and A_{22} are square matrices of dimension less than n [2].

Consequently, if $\sigma \in \Sigma_n$, then $(f_{\sigma(i),\sigma(j)})$ is the matrix of S relative to $\{C_{\sigma(1)},\ldots,C_{\sigma(n)}\}$. It follows easily that if F is reducible, then S is separable, and conversely.

Let us consider (4). We recall that a subsystem S_m is a trap iff it has no output to the rest of the system or to the environment. Thus, if S_m is a trap, $F_{im}=0$ for $i=1,\ldots,k$, $i\neq m$, and, according to Fife [1], $\det(F_{mm})=0$.

Now suppose S is not separable and open (in the sense that $f_{0i}\neq 0$ for at least one i). Then F is irreducible and, for at least one i,

$$|f_{ii}| > \sum_{\substack{j=1 \\ i\neq j}}^{n} f_{ji}.$$

In this situation, F is non-singular (see Taussky [3]). On the other hand, if S is closed, then zero is a simple eigenvalue of F (see Hearon [2], p. 45). We note that adding an excretion from any compartment of S to the environment makes S open, but still leaves F irreducible. Thus the system as a trap can have the trap removed by adding an excretion from any compartment.

Now suppose S is separable, and zero is a simple eigenvalue of F. Then by an easy argument (see Hearon [2]), S can be expressed as a disjoint union of two subsystems

$$S = T_1 \dot{\cup} T_2, \tag{8}$$

where T_2 is a trap and, as a system, is not separable. The matrix of this system assumes the form

$$F = \begin{bmatrix} G_{11} & 0 \\ G_{21} & G_{22} \end{bmatrix}, \tag{9}$$

where G_{11} is non-singular and G_{22} has zero as a simple eigenvalue and is irreducible. As above, adding an excretion from any compartment in T_2 will remove T_2 as a trap.

We are now ready to give

THEOREM

Suppose S is a linear compartmental system, and suppose zero is an eigenvalue of multiplicity m of the system matrix.

1. *If $m=1$, then S is separable iff the system matrix is reducible.*
2. *If $m>1$, the system S is separable.*

3. *There exists a partitioning of S into a disjoint union of subsystems*

$$S = S_1 \dot{\cup} S_2 \dot{\cup} \cdots \dot{\cup} S_k$$

such that S_i *receives no input from* $S_{i+1}, \ldots, S_k$, $i = 1, \ldots, k-m$, *and* $S_{k-m+1}, \ldots, S_k$ *are traps.*

4. *Relative to this partitioning (renumbering the compartments of S if necessary), the system matrix is given by*

$$F = \begin{bmatrix} F_{11} & & & & & \\ F_{21} & F_{22} & & & 0 & \\ \vdots & \vdots & \ddots & & & \\ F_{k-m+1,1} & F_{k-m+1,2} & \cdots & F_{k-m+1,k-m+1} & & \\ \vdots & \vdots & & & \ddots & \\ F_{k1} & F_{k2} & \cdots & 0 & \cdots & F_{kk} \end{bmatrix},$$

where, for $i = k-m+1, \ldots, k$, *zero is a simple eigenvalue of* F_{ii}.

5. *The addition of an excretion from any compartment of* S_i, $i = k-m+1, \ldots, k$ *removes* S_i *as a trap.*

Proof. The proof is by induction on m, the case when $m = 1$ being proved in the remarks preceding the statement of the theorem. Moreover, if $m > 1$, it is clear that S is separable.

Now let us assume that the theorem is valid for all systems whose system matrices have zero as an eigenvalue of multiplicity m, and suppose S is an n compartmental system whose system matrix has zero as an eigenvalue of multiplicity $m+1$. Then S is separable, whence

$$S = T_1 \dot{\cup} T_2, \tag{10}$$

where T_1 receives no input from T_2. Renumbering the compartments of S if necessary, the system matrix F assumes the form

$$F = \begin{bmatrix} G_{11} & 0 \\ G_{21} & G_{22} \end{bmatrix}, \tag{11}$$

where $G_{11} = (g_{ij})$, $i,j = 1, \ldots, p$; $G_{22} = (g_{ij})$, $i,j = p+1, \ldots, n$; and $G_{21} = (g_{ij})$,

$i=p+1,\ldots,n,\ j=1,\ldots,p$. Notice that

$$g_{jj}=\begin{cases}-\left(g_{0j}+\sum_{\substack{i=1\\ i\neq j}}^{n} g_{ij}\right), & j=1,\ldots,p\\ -\left(g_{0j}+\sum_{\substack{i=p+1\\ i\neq j}}^{n} g_{ij}\right), & j=p+1,\ldots,n.\end{cases} \tag{12}$$

We now make two critical observations. First, let us write

$$g_{jj}=-\left[\left(g_{0j}+\sum_{i=p+1}^{n} g_{ij}\right)+\sum_{\substack{i=1\\ i\neq j}}^{p} g_{ij}\right]$$

$$=g'_{0j}+\sum_{\substack{i=1\\ i\neq j}}^{p} g_{ij}, \qquad j=1,\ldots,p.$$

If we do this, we see that we may regard G_{11} as the system matrix of T_1, considering T_1 as a separate system. Notice how we have lumped excretions into T_2 as "environmental" to accomplish this. It is easy to see how G_{22} may be regarded as the system matrix of T_2, considering T_2 as a separate system.

The second observation involves what happens when we apply row and column operations to F. Specifically, if we consider Σ_p as a subgroup of Σ_n, then for any $\sigma\in\Sigma_p\subset\Sigma_n$, $(g_{\sigma(i),\sigma(j)})$ is still block triangular. The effect of putting G_{11} into block triangular form by this method simply rearranges the columns of G_{21}. Similarly if Σ^*_{n-p} is the permutation group on $\{p+1,\ldots,n\}$ and is identified in the natural way as a subgroup of Σ_n, then the effect of putting G_{22} into block triangular form by this method is simply to alter the rows of G_{21}.

From these two observations, the proof is immediate. Indeed, without loss of generality, we may assume $\det(G_{11})=\det(G_{22})=0$. By the induction hypothesis T_1 and T_2 can be arranged according to the statement of the theorem, i.e.

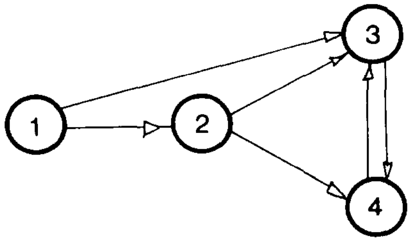
$$T_1=T_{11}\dot{\cup}\cdots\dot{\cup}T_{1r}\dot{\cup}\cdots\dot{\cup}T_{1s},$$

$$T_2=T_{21}\dot{\cup}\cdots\dot{\cup}T_{2u}\dot{\cup}\cdots\dot{\cup}T_{2v},$$

where $T_{1r+1},\ldots,T_{1s},\ T_{2u+1},\ldots,T_{2v}$ are the $m+1$ traps. Since traps have no excretions, the final arrangement is obvious. (Notice that this is true of the whole matrix, since the column entries under a trap are zeros). Q.E.D.

We wish to consider two examples to illustrate what the theorem will and won't do.

Example 1. Consider a four compartment system $S=\{C_1,C_2,C_3,C_4\}$ whose connectivity diagram is given by



The system matrix according to this numbering is given by

$$F=\begin{bmatrix} f_{11} & 0 & 0 & 0 \\ f_{21} & f_{22} & 0 & 0 \\ f_{31} & f_{32} & f_{33} & f_{34} \\ 0 & f_{42} & f_{43} & f_{44} \end{bmatrix}.$$

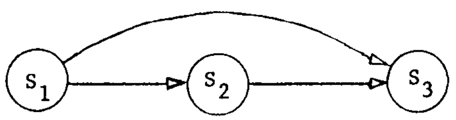
According to the theorem, the block triangular form of F is

$$F=\begin{bmatrix} F_{11} & 0 & 0 \\ F_{21} & F_{22} & 0 \\ F_{31} & F_{32} & F_{33} \end{bmatrix},$$

where $F_{11}=(f_{11})$, $F_{22}=(f_{22})$, and $F_{33}=\begin{bmatrix} f_{33} & f_{34} \\ f_{43} & f_{44} \end{bmatrix}$, and zero is a simple eigenvalue of F_{33}. Notice this decomposition is relative to

$$S=S_1\dot{\cup}S_2\dot{\cup}S_3,$$

where $S_1=\{C_1\}$, $S_2=\{C_2\}$, and $S_3=\{C_3,C_4\}$. Now S_3 is a trap. But $S_2 \dot{\cup} S_3$ is also a subsystem which is a trap, and the whole system, being closed, is a trap. Thus how many traps are there in this system? In light of this, what the theorem tells us directly is the number of irreducible subsystems (subsystems which, when regarded as systems, are not separable) which are traps. Indeed, this is the multiplicity of zero as an eigenvalue. In this example, S_3 is such a system. What the theorem tells us indirectly is how these traps will nest with other subsystems to form more complex traps. From the form for F given in the theorem, we may draw a connectivity diagram for the S_i, for S_i receives from S_j iff $F_{ij} \neq 0$. In this way, we see how complex subsystems which are traps in themselves may be formed. In this particular example,

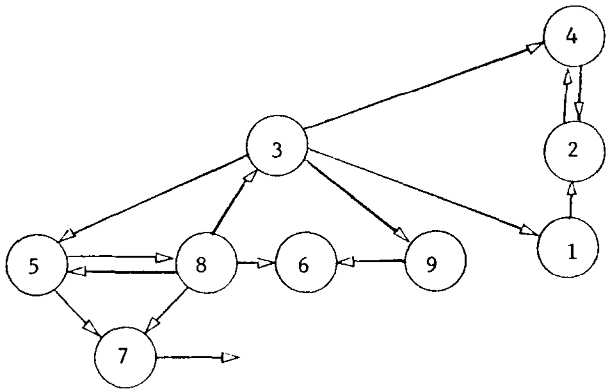


Notice that even though S_3 is a trap, $S_1 \dot{\cup} S_3$ is not a trap.

Example 2. Consider a nine compartment system $\{C_1,\ldots,C_9\}$ whose matrix is given:

$$(f_{ij})=\begin{bmatrix} f_{11} & 0 & f_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ f_{21} & f_{22} & 0 & f_{24} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f_{33} & 0 & 0 & 0 & 0 & f_{38} & 0 \\ 0 & f_{42} & f_{43} & f_{44} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f_{53} & 0 & f_{55} & 0 & 0 & f_{58} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & f_{68} & f_{69} \\ 0 & 0 & 0 & 0 & f_{75} & 0 & f_{77} & f_{78} & 0 \\ 0 & 0 & 0 & 0 & f_{85} & 0 & 0 & f_{88} & 0 \\ 0 & 0 & f_{93} & 0 & 0 & 0 & 0 & 0 & f_{99} \end{bmatrix},$$

where $f_{0i}=0$ except for $i=7$. From the matrix, we can read off the connectivity diagram:



It is obvious from the diagram that S is separable. Indeed, if

$$S_1 = \{C_3, C_5, C_6, C_7, C_8, C_9\}$$

and

$$S_2 = \{C_1, C_2, C_4\},$$

then $S = S_1 \dot{\cup} S_2$, where S_1 receives no input from S_2. If we were to relabel the compartments at this stage, the system matrix would be in the form

$$\begin{bmatrix} G_{11} & 0 \\ G_{21} & G_{22} \end{bmatrix}.$$

Notice this is an example of the first stage in the induction in the proof of the theorem. However, both S_1 and S_2 can be broken down. Indeed, let

$$\begin{aligned} &S_{11} = \{C_3, C_5, C_8\}, && S_{21} = \{C_1\}, \\ &S_{12} = \{C_7\}, && S_{22} = \{C_2, C_4\}, \\ &S_{13} = \{C_9\}, && \\ &S_{14} = \{C_6\}, && \end{aligned}$$

Then $S_1 = S_{11} \dot{\cup} S_{12} \dot{\cup} S_{13} \dot{\cup} S_{14}$ and $S_2 = S_{21} \dot{\cup} S_{22}$, where this partitioning has the properties listed in the theorem. Now let us relabel the compartments in S. The permutation for such a renumbering is

$$\sigma=\begin{pmatrix} 3 & 5 & 8 & 7 & 9 & 6 & 1 & 2 & 4 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix}$$

$$=(1\ \ 7\ \ 4\ \ 9\ \ 5\ \ 2\ \ 8\ \ 3)\in\Sigma_9.$$

Thus C_3 becomes C_1, C_5 becomes C_2, etc. The matrix of the system relative to the numbering may be found directly by using σ, i.e., we simply examine $(f_{\sigma(i),\sigma(j)})$, and place the entries in their proper position. The matrix is

$$\left[\begin{array}{cccccc|ccc} f_{11} & 0 & f_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ f_{21} & f_{22} & f_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & f_{32} & f_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & f_{42} & f_{43} & f_{44} & 0 & 0 & 0 & 0 & 0 \\ f_{51} & 0 & 0 & 0 & f_{55} & 0 & 0 & 0 & 0 \\ 0 & 0 & f_{63} & 0 & f_{65} & 0 & 0 & 0 & 0 \\ \hline f_{71} & 0 & 0 & 0 & 0 & 0 & f_{77} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & f_{87} & f_{88} & f_{89} \\ f_{91} & 0 & 0 & 0 & 0 & 0 & 0 & f_{98} & f_{99} \end{array}\right].$$

We have indicated how the system matrix can be broken down for analysis. Notice that the two (simple) traps are S_{14} and S_{22}. Moreover, to put the matrix in the form stated in the theorem can be accomplished by $\rho=(6\ \ 7)\in\Sigma_9$. We can now analyze this system as we did in the previous example.

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