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SIMPLE KNOTS WITH UNIQUE SPANNING SURFACES

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§1. INTRODUCTION AND PRELIMINARIES

IT IS easily shown (e.g. cf. [8]) that any two incompressible spanning surfaces for a possibly trivial Neuwirth (i.e. fibered) knot are isotopic, contrasting sharply with the fact that some knots [2, p. 60] have incompressible spanning surfaces of arbitrarily high genus. In answer to a question raised by Alford and Schaufele [1], Whitten [8] has shown that doubles of certain knots have unique isotopy types of minimal (knotted genus one) spanning surfaces. This paper generalizes his construction and extends his results in several directions. We establish the existence of:

(i) a class containing knots of every genus greater than one, each nonfibered with *all of its incompressible spanning surfaces isotopic* and *knotted*;

(ii) a class containing knots of every genus, each nonfibered with *all of its incompressible spanning surfaces isotopic* and *unknotted*; and

(iii) an infinite subclass of class (ii), each member of which is *simple*.

Our proofs are generally independent of Whitten's, shorter, and more geometric.

Our notation and terminology follow [2, 3, 4 and 8]. The following lemma follows directly from Waldhausen [7].

LEMMA 1. *If T is a properly embedded incompressible surface in $S \times I$, where S is an oriented 2-manifold with connected boundary, if $T \cap (S \times \{1\}) = \varnothing$, if $\chi(T) \neq 1$, and if $T \cap ((\text{Bd } S) \times I)$ is connected or empty, then T is parallel, in $S \times I$, to a subset of $S \times \{0\}$.*

We will also have occasion to use the following two technical lemmas, the proofs of which are straightforward.

LEMMA 2. *If the knot space K contains a properly embedded incompressible and boundary incompressible annulus, then K is either a composite knot space or a nontrivial cable knot space, and the boundaries of all such annuli are isotopic in* Bd K.

LEMMA 3. *If K is a nontrivial torus knot space containing the properly embedded incompressible and boundary incompressible annulus A, then any bounded, properly embedded* *incompressible and boundary incompressible surface in K is isotopic either to A or to a fiber in K.*

§2. THE CONSTRUCTION

Let $k' \subset S^3$ be a Neuwirth knot with minimal spanning surface S' containing an unknotted simple closed curve w which is noncontractible in $S^3 - k'$, and let W' be the solid torus $\text{Cl}(S^3 - N(w))$. Assume that the curves $S' \cap \text{Bd } W'$ have winding number $n \geq 0$ in W' and that they separate Bd W' into annuli A' and B'. Let z be any (possibly trivial) tame knot in S^3 with $W = N(z)$ and $Z = \text{Cl}(S^3 - W)$, and let $f\colon W' \to W$ be any homeomorphism. If $f(A') = A$, $f(B') = B$, and $f(k') = k$, then k is a knot in S^3 with knot space $K = \text{Cl}(S^3 - N(k))$, and $S = (f(S' \cap W) \cap K) \cup A$ is a spanning surface in K. Note that $\text{Cl}(K - (S \cup Z))$ has a natural product structure. If z is trivial, let each component of Bd A have winding number $m \geq 0$ in the solid torus Z, so that the g.c.d. $(m, n) = 1$.

LEMMA 4. *The surface S is compressible in K iff $m = 0$ and z is trivial. In particular, if S is compressible, then $n = 1$ and z is trivial.*

Proof. If $m = 0$ and z is trivial, then each component of Bd A bounds a disk in $\text{Cl}(K - S)$, so S is compressible. If S is compressible, then, by the usual argument, there exists a nonsingular disk D such that $D \cap S = \text{Bd } D$ and Bd D is noncontractible in S. Put D in general position with B and assume a minimal number of components in $B \cap D$. The closure D' of at least one component of $D - B$ must be a disk. If D' were contained in W, then D' could be pulled back by f^{-1} to contradict the incompressibility of S'. Hence we may assume that $D' \subset Z$. But since the intersection $B \cap D$ is minimal, this means that D' must be a nonseparating disk in Z, so z is trivial and $m = 0$. Since $(m, n) = 1$, the compressibility of S implies $n = 1$.

LEMMA 5. *Assume S is incompressible. Then the knot space K is fibered iff z is trivial and $m = 1$.*

Proof. If K is fibered, then $\text{Cl}(K - S)$ is a product, and by Lemma 1, the annulus B is parallel to A. Hence Z is a solid torus, z is trivial, and $m = 1$. If Z is a solid torus and $m = 1$, then $\text{Cl}(K - S)$ is a product and K is fibered.

From this point onward we will be concerned with showing that, under certain circumstances, two incompressible spanning surfaces in K are isotopic in K. It then follows by Whitten's argument [8] that this isotopy may be extended to all of S^3, leaving k fixed.

THEOREM 1. *Assume S is incompressible and that Z contains no properly embedded incompressible and boundary incompressible annuli whose boundary components are isotopic, in* Bd Z, *to components of* Bd A. *Then any minimal spanning surface $T \subset K$ is isotopic, in K, to S.*

Proof. We may clearly assume that initially, and at the end of each isotopy arising in the proof, the surface T satisfies the condition $(\text{Bd } T) \cap (\text{Bd } S) = \varnothing$. Our first goal is to show that we can move T by an isotopy until $T \cap Z = \varnothing$. To do this we put T in general position with S, assume that $S \cap T$ is minimal, and consider two cases.

Case 1. $S \cap T \neq \varnothing$. Since T cannot carry any homology of K, there is some component of $T - S$ whose closure t meets S only from the side opposite B. Put t in general position with B and assume $t \cap B$ is minimal, so that $t \cap B$ must consist of points in (Bd t) $\cap$ (Bd B) and simple closed curves in (Int t) $\cap$ (Int B). If there were no such simple closed curve, then, by Lemma 1, t would be parallel to a subset of S and we could reduce $S \cap T$, a contradiction. Hence at least one component of $T \cap$ Bd Z is parallel, in Bd Z, to a component of Bd A. However, since T is incompressible, it follows that we can remove all components of $T \cap$ Bd Z which are contractible in Bd Z, leaving us with each component of $T \cap$ Bd Z isotopic, in Bd Z, to a component of Bd A. Since both A and B are incompressible in Z, no component of $T \cap Z$ is a disk. If $\chi(T \cap Z)$ were negative, then we could replace the components of $T \cap Z$ by boundary parallel annuli in Z and obtain a spanning surface of genus lower than that of T, another contradiction. Hence $\chi(T \cap Z) = 0$, each component of $T \cap Z$ is an annulus, and our hypotheses assure us that T may be moved by an isotopy until $T \cap Z = \varnothing$.

Case 2. $S \cap T = \varnothing$. Put T in general position with B, maintaining $S \cap T = \varnothing$. If $T \cap B = \varnothing$, then S is isotopic to T by Lemma 1. If $T \cap B \neq \varnothing$, we will have $T \cap$ Bd B = $\varnothing = T \cap$ Bd A and we can proceed as in Case 1. (Note that the final isotopy there might destroy the condition $S \cap T = \varnothing$.)

Now assume $S \cap T$ is again minimal, this time subject to $T \cap Z = \varnothing$. If $S \cap T \neq \varnothing$, then there exists, as before, a component of $T - S$ whose closure t meets S only from the side opposite B. This time, however, we have $T \cap B = \varnothing$, so by Lemma 1, t is parallel to a subset of S, and this subset contains A, because if not, we could reduce $S \cap T$. Without loss of generality we can assume t is an outermost such component in Cl($K - (S \cup Z)$). Similarly, there exists an outermost component of $T - S$ whose closure u meets $(S - A) \cup B$ only from the side opposite A and which is parallel to a subset of $(S - A) \cup B$ containing B. But this means that $t \cup u$ separates K, and since $t \cup u \subset T$, we have a contradiction. Hence $S \cap T = \varnothing = T \cap Z$, so $T \cap (S \cup Z) = \varnothing$, and by Lemma 1, T is isotopic to S.

COROLLARY 1.1. *The knot k has a unique isotopy type of minimal spanning surface if one of the following hold:*

(a) $n > 1$;

(b) $n = 1$ *and z is a nontrivial, noncable knot*; *or*

(c) $n = 0$ *and z is prime.*

Proof. If Z contains a properly embedded incompressible and boundary incompressible annulus with boundary components isotopic, in Bd Z, to components of Bd A, then by Lemma 2, z is either a nontrivial cable knot or a composite knot. In the first case, $n = 1$; and in the second, $n = 0$.

Note from Whitten's figure [8] that the curve K_2 may be moved by an isotopy in $S^3 - K_1$ until it lies in the surface spanning K_1. Hence his construction is a special case of ours, with $n = 1$, and his main theorem follows from Corollary 1.1b.

THEOREM 2. *Assume S is incompressible and that Z contains no properly embedded incompressible and boundary incompressible bounded surface whose boundary components are* *isotopic, in* Bd Z, *to components of* Bd A. *Then any incompressible spanning surface* $T \subset K$ *is isotopic to* S.

Proof. Proceed exactly as in the proof of Theorem 1 to the point where each component of $T \cap$ Bd Z is isotopic, in Bd Z, to a component of Bd A. Our hypotheses then guarantee that we can eliminate all of $T \cap Z$, and the proof again proceeds exactly as that of Theorem 1.

COROLLARY 2.1. *The knot* k *has a unique isotopy type of incompressible spanning surface if either of the following hold:*

(a) $n \neq 1$ *and* z *is a (possibly trivial) torus knot; or*

(b) z *is trivial and* $m > 0$.

Proof. If z is nontrivial, the hypotheses (a) combine with Lemmas 3 and 4 to satisfy the the hypotheses of Theorem 2. If z is trivial and $n \neq 1$ or $m > 0$, then Lemma 4 assures us of satisfying the hypotheses of Theorem 2. Note that Corollary 2.1a generalizes our result in [3].

§3. EXAMPLES

Whitten's construction involved a nonseparating w, with $n = 1$ twist, in a trefoil knot surface, and in [3] we exhibited a nonseparating w, with $n = 0$ twists, in a surface spanning a genus two Neuwirth knot. Thus given integers $0 \leq n \leq g$ where $g > 0$ and $n \neq 0$ if $g = 1$, we can take an appropriate direct sum of these knots to obtain a Neuwirth knot of genus g whose fiber contains a nonseparating w with n twists.

If we apply our construction in any situation where $n \neq 1$ and z is a nontrivial torus knot, then Corollary 2.1a appiles. Moreover, since the incompressible spanning surfaces all may be moved by an isotopy until they miss Z, they are all knotted and class (i) is established.

If we apply our construction in any situation where $n = 1$, w is nonseparating, and z is trivial, the map f may be chosen so that $m > 1$. Hence Corollary 2.1b applies and the resulting knots are nonfibered by Lemma 5. In this case, $\text{Cl}(K - S)$ is homeomorphic to $\text{Cl}(S^3 - S')$ with a solid torus sewn along w', one of the two copies of w. Since $\text{Cl}(S^3 - S')$ has a product structure and w doesn't separate S', w' represents a primitive element in the free fundamental group of $\text{Cl}(S^3 - S')$. Thus the fundamental group of $\text{Cl}(K - S)$, being free with a root adjoined along a primitive element, is also free, so S is unknotted and class (ii) is established.

If we apply our construction to Whitten's link, with z trivial, the resulting knots are the ρ-twist knots (see [5, pp. 226f] for example) with $m = |\rho|$. (Compare the figure in [8] with Fig. 4.1 in [5, p. 227].) Hence, if $|\rho| > 1$, the ρ-twist knot is in class (ii). Since Schubert [6, Satz 4, p. 242] has shown the twist knots to be simple, class (iii) is established.

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