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Generalized Appell Connection Sequences

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1. INTRODUCTION

Let $P = \{P_n(x)\}_{n=0}^{\infty}$ be a sequence of polynomials and suppose there exists a function

$$\Phi(t) = \sum_{n=0}^{\infty} \phi_n t^n \qquad (\phi_n \neq 0)$$

such that

$$\sum_{n=0}^{\infty} \phi_n P_n(x)\, t^n = G(t)\, \Phi(xH(t)) \tag{1}$$

where $G(t)$ and $H(t)/t$ have power series expansions with nonzero initial coefficients. We are not concerned here with questions of convergence, the entire paper being in the context of formal power series.

Polynomial sequences generated by relations of the form (1) were first discussed in full generality by Boas and Buck [3] who later [2] studied their expansion properties. Using the terminology of Boas and Buck, we call (1) a generalized Appell representation and refer to P as *a generalized Appell sequence*. We introduce here, moreover, the symbol (Φ) to denote the class of generalized Appell sequences whose representations (1) involve a fixed $\Phi(t)$. When $\Phi(t) = \exp t$, for example, we have the class (exp) of all Sheffer sequences [8]. An important subclass of (Φ) consists of the Brenke sequences [4] in (Φ), occurring when $H(t) = t$ in their representations (1). The Brenke sequences in (exp) are known as Appell sequences [1].

Suppose now that two sequences $P = \{P_n(x)\}_{n=0}^{\infty}$ and $\tilde{P} = \{\tilde{P}_n(x)\}_{n=0}^{\infty}$ are in some given generalized Appell class (Φ). It follows [2, p. 18] that they are simple sequences; that is, $P_n(x)$ and $\tilde{P}_n(x)$ are of degree exactly n for any n. This fact assures us that there exist so-called connection constants $q_{n,k}$ such that

$$P_n(x) = \sum_{k=0}^{n} q_{n,k} \tilde{P}_k(x) \qquad (n = 0, 1, 2, \ldots), \tag{2}$$

and we call the sequence $Q = \{Q_n(x)\}_{n=0}^{\infty}$ of uniquely determined polynomials $Q_n(x) = \sum_{k=0}^{n} q_{n,k} x^k$ *the connection sequence from* P *to* $\tilde{P}$.

The chief aim of this paper is to show that the connection sequence Q is also in the class (Φ) and to furnish a representation for that sequence. This is the content of the theorem in Section 2. In Section 3 we relate our result to a recent paper by Mullin and Rota [6] who derived a method for finding connection sequences within a certain subclass of (exp). Taking the approach of generalized Appell representations, we use our theorem to modify their derivation, which was based exclusively on certain operator methods, in a novel and efficient way.

2. Representations for Connection Sequences

The theorem which follows is a direct generalization of one of Appell's observations [1, p. 123] regarding his sequences.

THEOREM. *Let*

$$P = \{P_n(x)\}_{n=0}^{\infty} \qquad \text{and} \qquad \tilde{P} = \{\tilde{P}_n(x)\}_{n=0}^{\infty}$$

be any two sequences in a given generalized Appell class (Φ) *with representations*

$$\sum_{n=0}^{\infty} \phi_n P_n(x)\, t^n = G(t)\, \Phi(xH(t)) \tag{3}$$

and

$$\sum_{n=0}^{\infty} \phi_n \tilde{P}_n(x)\, t^n = \tilde{G}(t)\, \Phi(x\tilde{H}(t)), \tag{4}$$

respectively. The connection sequence $Q = \{Q_n(x)\}_{n=0}^{\infty}$ *from* P *to* $\tilde{P}$ *is also in* (Φ) *and has the representation*

$$\sum_{n=0}^{\infty} \phi_n Q_n(x)\, t^n = \frac{G(t)}{\tilde{G}(\tilde{H}^{-1}(H(t)))}\, \Phi(x\tilde{H}^{-1}(H(t))). \tag{5}$$

Here $\tilde{H}^{-1}(t)$ is the formal power series inverse of $\tilde{H}(t)$; that is,

$$\tilde{H}(\tilde{H}^{-1}(t)) = \tilde{H}^{-1}(\tilde{H}(t)) = t.$$

Proof. We impose on the class (Φ) a binary operation often used for polynomial sequences. It was used, in fact, by Appell. If

$$P_n(x) = \sum_{k=0}^{n} p_{n,k} x^k \qquad (n = 0, 1, 2,\ldots), \tag{6}$$

we define the product $P\tilde{P} = \{(P\tilde{P})_n (x)\}_{n=0}^{\infty}$ by means of the equations

$$(P\tilde{P})_n (x) = \sum_{k=0}^{n} p_{n,k} \tilde{P}_k(x) \qquad (n = 0, 1, 2,\ldots). \tag{7}$$

Now the class $\mathscr{S}$ of all simple polynomial sequences forms a group under this same binary operation [7, pp. 15–17], and we begin our proof by showing that the subclass (Φ) is actually a subgroup of $\mathscr{S}$. The identity element is, of course, the sequence $I = \{x^n\}_{n=0}^{\infty}$ whose representation is

$$\sum_{n=0}^{\infty} \phi_n x^n t^n = \Phi(xt).$$

In order to show closure within (Φ), we exhibit a representation for the product $P\tilde{P}$ of the arbitrary sequences P and $\tilde{P}$ as follows. We note from (6) that

$$\sum_{n=0}^{\infty} \phi_n P_n(x)\, t^n = \sum_{k=0}^{\infty} \left\{ \sum_{n=0}^{\infty} \phi_{n+k} p_{n+k,k} t^{n+k} \right\} x^k. \tag{8}$$

Also, since

$$\Phi(t) = \sum_{k=0}^{\infty} \phi_k t^k,$$

$$G(t)\, \Phi(xH(t)) = \sum_{k=0}^{\infty} \{G(t)\, \phi_k (H(t))^k\}\, x^k. \tag{9}$$

Now the right sides of (8) and (9) are equal because of (3), and so

$$\sum_{n=0}^{\infty} \phi_{n+k} p_{n+k,k} t^{n+k} = G(t)\, \phi_k (H(t))^k \qquad (k = 0, 1, 2,\ldots). \tag{10}$$

Therefore,

$$\sum_{n=0}^{\infty} \phi_n \sum_{k=0}^{n} p_{n,k}\tilde{P}_k(x)\, t^n = \sum_{k=0}^{\infty} \left\{ \sum_{n=0}^{\infty} \phi_{n+k} p_{n+k,k} t^{n+k} \right\} \tilde{P}_k(x)$$

$$= \sum_{k=0}^{\infty} \{G(t)\, \phi_k (H(t))^k\}\, \tilde{P}_k(x)$$

$$= G(t) \sum_{k=0}^{\infty} \phi_k \tilde{P}_k(x)\, (H(t))^k.$$

Finally, in view of (7) and (4), this result can be written in the form

$$\sum_{n=0}^{\infty} \phi_n (P\tilde{P})_n\, (x)\, t^n = G(t)\, \tilde{G}(H(t))\, \Phi(x\tilde{H}(H(t))) \tag{11}$$

which is the desired representation.

In our demonstration that (Φ) is a subgroup of $\mathscr{S}$ it remains to show that the inverse P^{-1} of the arbitrary sequence P is in (Φ). While this follows from an earlier note of ours [5], we observe here that it is immediate upon writing $\tilde{G}(t) = 1/G(H^{-1}(t))$ and $\tilde{H}(t) = H^{-1}(t)$ in (11). For in that case the right side of (11) reduces to $\Phi(xt)$, the function generating the identity sequence I. Hence the inverse sequence $P^{-1} = \{P_n^{-1}(x)\}_{n=0}^{\infty}$ is in (Φ) with representation

$$\sum_{n=0}^{\infty} \phi_n P_n^{-1}(x)\, t^n = \frac{1}{G(H^{-1}(t))}\, \Phi(xH^{-1}(t)). \tag{12}$$

Regarding the sequences P and $\tilde{P}$ as group elements, consider now the element $Q = P\tilde{P}^{-1}$. Since $P = Q\tilde{P}$, it follows from definition (7) of product that $P_n(x) = \sum_{k=0}^{n} q_{n,k}\tilde{P}_k(x)$ where the $q_{n,k}$ are the coefficients in the polynomials $Q_n(x)$ of the sequence Q. Also, since $Q_n(x) = (P\tilde{P}^{-1})_n\,(x)$, we have representation (5) by referring to (12) and then formula (11) for the representation of a product. The proof of the theorem is now complete.

We note in passing that, no matter what the class (Φ) is, the sequences P and $\tilde{P}$ in the theorem can always be connected by a *Sheffer* sequence in the following sense.

$$P_n(x) = \sum_{k=0}^{n} \frac{k!}{n!} \frac{\phi_k}{\phi_n}\, c_{n,k} \tilde{P}_k(x) \qquad (n = 0, 1, 2, \ldots) \tag{13}$$

where the $c_{n,k}$ are the coefficients in the polynomials of the sequence $C = \{C_n(x)\}_{n=0}^{\infty}$ whose representation is

$$\sum_{n=0}^{\infty} C_n(x) \frac{t^n}{n!} = \frac{G(t)}{\tilde{G}(\tilde{H}^{-1}(H(t)))} \exp(x\tilde{H}^{-1}(H(t))). \tag{14}$$

This follows from the general fact that when (3) is a representation for the sequence of polynomials (6), then

$$\sum_{n=0}^{\infty} \psi_n \Pi_n(x)\, t^n = G(t)\, \Psi(xH(t))$$

is a representation for the related sequence

$$\Pi_n(x) = \sum_{k=0}^{n} \frac{\phi_n}{\psi_n} \frac{\psi_k}{\phi_k} p_{n,k} x^k \qquad (n = 0, 1, 2,\ldots),$$

where $\Psi(t) = \sum_{n=0}^{\infty} \psi_n t^n$ is any power series with nonzero coefficients. To see that this is so, we need only refer to (10) and write

$$\begin{aligned} \sum_{n=0}^{\infty} \psi_n \Pi_n(x)\, t^n &= \sum_{k=0}^{\infty} \left\{ \sum_{n=0}^{\infty} \phi_{n+k} p_{n+k,k} t^{n+k} \right\} \frac{\psi_k}{\phi_k}\, x^k \\ &= \sum_{k=0}^{\infty} \{G(t)\, \phi_k (H(t))^k\} \frac{\psi_k}{\phi_k}\, x^k \\ &= G(t)\, \Psi(xH(t)). \end{aligned}$$

Evidently then,

$$c_{n,k} = \frac{n!}{k!} \frac{\phi_n}{\phi_k} q_{n,k}$$

where $c_{n,k}$ are the coefficients in the polynomials of (14) and $q_{n,k}$ are the coefficients in the polynomials of (5). Since $Q = \{Q_n(x)\}_{n=0}^{\infty}$ is the connection sequence from P to $\tilde{P}$, (13) now follows.

3. Sequences of Binomial Type

A simple polynomial sequence $B = \{B_n(x)\}_{n=0}^{\infty}$ is said to be a sequence of binomial type if

$$B_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(x)\, B_k(y) \qquad (n = 0, 1, 2,\ldots).$$

In the paper mentioned in Section 1 Mullin and Rota established a one to one correspondence between such sequences and differential operators of the form $J = \sum_{k=1}^{\infty} j_k d^k/dx^k$ where the j_k are constants and $j_1 \neq 0$. They showed that

$$\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \exp(xH(t)) \tag{15}$$

where $H(t)$ is the formal power series inverse of the function $J(t) = \sum_{k=1}^{\infty} j_k t^k$. Mullin and Rota thus recognized sequences of binomial type to be certain sequences in (exp) which were studied earlier, along with their operators, by Sheffer [8]. We use Sheffer's notation here and follow him in saying that the function $J(t)$ generates the operator J.

Mullin and Rota's stated goal [6, p. 169] was to provide a general method for finding the connection sequence from one sequence of binomial type to another. The method they discovered is based on the fact that the operator J corresponding to a sequence $B = \{B_n(x)\}_{n=0}^{\infty}$ of binomial type can be used to write the $B_n(x)$ explicitly by means of certain formulas due to Steffensen [9]. Given any two sequences B and $\tilde{B}$ of binomial type along with the corresponding operators, they found the operator for the connection sequence Q from B to $\tilde{B}$, that sequence turning out to be also of binomial type. The new operator then yields the polynomials of Q via Steffensen's formulas. The main result [6, pp. 201–202] in the paper by Mullin and Rota can be stated as follows and is presented here as a corollary to our theorem, once (15) has been established.

COROLLARY (Mullin–Rota). *Let B and $\tilde{B}$ be any two sequences of binomial type corresponding to operators generated by $J(t)$ and $\tilde{J}(t)$, respectively. The connection sequence Q from B to $\tilde{B}$ is a sequence of binomial type corresponding to the operator generated by $J(\tilde{J}^{-1}(t))$, where $\tilde{J}^{-1}(t)$ is the formal power series inverse of $\tilde{J}(t)$.*

Proof. In view of (15), it follows from our theorem that the connection sequence $Q = \{Q_n(x)\}_{n=0}^{\infty}$ is of binomial type and has the representation

$$\sum_{n=0}^{\infty} Q_n(x) \frac{t^n}{n!} = \exp(x\tilde{H}^{-1}(H(t)))$$

where $H(t)$ and $\tilde{H}(t)$ are the inverses of the given functions $J(t)$ and $\tilde{J}(t)$, respectively. Consequently, the operator for Q is generated by the inverse of $\tilde{H}^{-1}(H(t)) = \tilde{J}(J^{-1}(t))$, that inverse being $J(\tilde{J}^{-1}(t))$. This is the desired result.

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