α^3 CONTRIBUTIONS TO THE ANOMALOUS MAGNETIC MOMENT OF AN ELECTRON IN THE MASS-OPERATOR FORMALISM ☆

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We evaluate numerically the α^3 contributions to $(q-2)/2$ of an electron due to graphs without electron loop(s) by the mass operator formalism Our result is 0 74(6) $(\alpha/\pi)^3$

The anomalous magnetic moment of an electron has been calculated to α^3 by several groups of authors In view of its significant implication on the validity of QED, it is desirable that a completely different approach should be used for an independent evaluation

Specifically, we use the mass operator formulation [1, 2] to estimate numerically the contribution to $(g-2)/2$ due to graphs without electron loops. These constitute a gauge invariant set. The advantage of using the mass operator is made evident by the fact that sets of graphs which have profuse cancellations otherwise are grouped together at the beginning. These sets are shown in fig. 1.

In the following, we will briefly describe our calculation. Details will be published elsewhere [3]

(a) Let M be the mass operator of an electron in an external e.m. field. (For our purpose, we need to introduce a constant magnetic field only) The Dirac's equation is

$$(-\not\pi + m_0 + M)|\psi\rangle = 0, \tag{1}$$

where $\not\pi = p - eA$ Note that p is a differential operator. M includes all radiative corrections to the motion of the electron. In particular, the term linear in constant field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is what we intend to extract, i.e.,

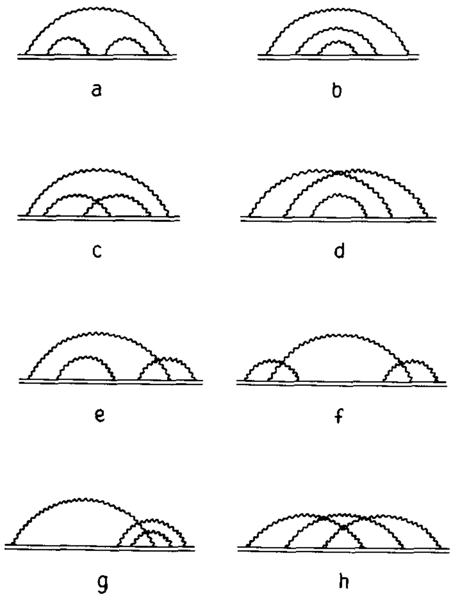

Fig 1 Graphs whose contributions to $(g-2)/2$ are considered in this work. Double lines denote electrons in an external constant magnetic field, and wavy lines are for photons

$$(g-2)/2 = -2m \frac{\partial}{\partial\langle F\rangle} \langle\overline{\psi_0}|M|\psi_0\rangle\Big|_{A=F=0}, \tag{2}$$

where

$$(m - \not\pi)|\psi_0\rangle = 0, \tag{3}$$

and

$$F = \tfrac{1}{4}\, i\, [\gamma^\mu, \gamma^\nu]\, eF_{\mu\nu}. \tag{4}$$

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(b) Let $M_F^{(i)}$ be the portion of the renormalized mass operator of ith order in e and linear in F; then

$$\langle M_F^{(2)}\rangle = \langle \widetilde{M}_F^{(2)}\rangle \tag{5}$$

$$\langle M_F^{(4)}\rangle = \langle \widetilde{M}_F^{(4)}\rangle = \langle \widetilde{M}_F^{(4)}\rangle - \widetilde{B}^{(2)}\langle \widetilde{M}_F^{(2)}\rangle \tag{6}$$

and

$$\langle M_F^{(6)}\rangle = \langle \widetilde{M}_F^{(6)}\rangle - \widetilde{B}^{(2)}\langle \widetilde{M}_F^{(4)}\rangle$$

$$- (\widetilde{B}^{(4)} - \widetilde{B}^{(2)}\widetilde{B}^{(2)})\langle M_F^{(2)}\rangle \tag{7}$$

where

$$(1+\widetilde{B})^{-1} = (1+\widetilde{B}^{(2)}+\widetilde{B}^{(4)}+\ldots)^{-1} \tag{8}$$

is the electron wave function renormalization and the symbol tilde denotes that mass renormalization has been performed and therefore the physical mass of an electron is to be used in its free propagator

(c) A typical integral we encounter is of the form

$$\int \mathrm{d}^4 l_1 \mathrm{d}^4 l_2 \mathrm{d}^4 l_3 \tag{9}$$

$$\times (\ .\ \ \gamma_{\mu_i} \widetilde{G}(\pi_j)\ldots \widetilde{G}(\pi_{j'})\gamma_{\mu_{i'}}\ldots) g^{\mu_i\mu_{i'}}(k_q),$$

where $\widetilde{G}$'s and g's are the free electron and photon Green functions respectively, l's are the loop integration variables, and k's are the photon momenta The π's are non-commuting, and we must either extract the field dependence explicitly or, because of the constant external field assumption, symmetrize the integrand before the l-integrations can be carried out [2]

(d) The algebra to project out the $(g-1)/2$ contribution is done in several ways: (1) by REDUCE II [4], (2) by a FORTRAN based vertex graph program developed by Levine [5], and (3) by multiplication of explicit representation of the γ matrices. They all agree with each other.

(e) After introducing the α-parameters and carrying out the integrations, an integral is converted into the form

$$\int \frac{\mathrm{d}^8\alpha}{\Delta^2}\,\delta\Big(1-\sum_{n=1}^{8}\alpha_i\Big)\Big(\frac{n_1}{\alpha_0}+\frac{n_2}{\alpha_0^2}+\frac{2!n_3}{\alpha_0^3}\Big), \tag{10}$$

where Δ is the characteristic determinant of the graph under consideration, and α_0, in the circuit analogue, is the sum of the resistances of the electron line minus the effective resistance of the graph. The integral, as it stands, is in general both ultraviolet and infrared divergent It is an important observation that the ultraviolet region lies in the vanishing of Δ, whereas the infrared behavior is controlled by α_0 Thus, we perform our renormalization in the following fashion.

(i) When a graph has self-energy insertions(s), we scale the subgraph(s) and subtract an appropriate limiting form of the integrand such that at the point(s) where the scale parameter(s) vanish the subgraph singularity is removed without introducing new infinities We call this the $\beta \triangleq 0$ subtraction This procedure is equivalent to an intermediate renormalization [6].

(ii) When a graph has vertex insertion(s), we subtract away the ultraviolet divergent part(s) only by infrared convergent vertex renormalization

(iii) We use the infrared divergent part of the vertex renormalization (which was not used in (ii)) to subtract away the infrared not removed by the $\beta \triangleq 0$ prescription of (i)

All integrals after this treatment are then finite.

(f) What remains to be done is to numerically integrate the convergent integrals. If one is to make a scale transformation to get rid of the $\delta(1-\Sigma_i)$ of eq (10), then the volume near the surface will be weighted preferentially To avoid this, we use the identity

$$\int\Big(\prod_{i=1}^{n}\mathrm{d}\alpha_i\Big)\,\delta\Big(1-\sum_{j=1}^{n}\alpha_j\Big)f(\alpha) \tag{11}$$

$$=\sum_{j=1}^{n}\int\Big(\prod_{i=1}^{n}\mathrm{d}\alpha_i\Big)\,\delta(1-\alpha_j)f(\alpha),$$

if $f(\lambda\alpha)=\lambda^{-n}f(\alpha)$, which our integrands satisfy In other words, we project the triangular surface of integration in the multi-dimensional α space onto the sides of a unit cube

We also know that the divergences (before subtraction) are isolated singularities at the end points ($\alpha_i = 0$). We therefore make an exponential transformation $\alpha_i = \exp(-x_i)$, so that after subtraction the integrals converge exponentially as $x_i \to \infty$

(g) We use the importance sampling integration program of Sheppy and Dufner to perform the numerical estimate This is done on a PDP-10 computer and typically we make two million samples per integral to reach an accuracy of two decimal places.

(h) Results of this calculation are shown in table 1. We have calculated coefficients of the $\ln\Lambda$ and $\ln^2\Lambda$

Tabie 1

Contributions due to individual graphs of fig 1a-h These have been 'renormalized' by the method discussed in item (e) of the text

Graph	Contribution
a	$-1\ 368(23) \times (\alpha/\pi)^3$
b	$-1\ 412(15)$
c	$+0.952(18)$
d	$+0\ 749(10)$
e × 2	$+0\ 508(23)$
f	$+0\ 735(18)$
g × 2	$+2\ 311(30)$
h	$-1\ 738(14)$
Total	$0\ 737(56)$

(cutoff) terms for each graph and verified that they all sum to zero. Their cancellation is another check on the consistency of our numerical work. We have also explicitly demonstrated the cancellation of all infrared divergent integrals.

Added to the contributions due to the graphs with electron loops ($3.3(0\ 5) \times 10^{-9}$) [8–12] the theoretical value of the 6th order anomalous magnetic moment for an electron is (1 σ limits)

$$a^{\text{6th order}}_{\text{theoretical}} \equiv \left(\frac{g-2}{2}\right)^{\text{6th order}}_{\text{theoretical}} = 12.9(0\ 8) \times 10^{-9} \quad (12)$$

which is to be compared with

$$a_{\text{exp}} - (a^{\text{2nd}} + a^{\text{4th}})_{\text{theoretical}} = 20.0(4.1) \times 10^{-9}, \quad (13)$$

where the 4.1 ppm is attributed to the experimental uncertainty in the anomaly (3.5 ppm) [13] and the fine structure constant (2.2 ppm) [14] determined by the AC Josephson effect. The agreement is therfore fair.

(i) There have been two other recent attempts to estimate the 6th order moment due to the same sets of graphs we consider here. The results are (1 σ limits)

$$0.88(3)\ (\alpha/\pi)^3 \text{ (Levine-Wright [15])}, \quad (14)$$

and

$$1.02(2)\ (\alpha/\pi)^3 \text{ (Cvitanovic-Kinoshita [16])}, \quad (15)$$

to be compared with

$$0.74(6)\ (\alpha/\pi)^3 \text{ (this calculation)}. \quad (16)$$

All the errors quoted are either statistical estimates based on random sampling or educated guesses Biased effects due to nonlinearity of the integrands cannot be accurately assessed Thus, it is unclear whether the discrepancies are genuine or only systematic

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