Brief Communication

*The Communication Problem on Graphs and Digraphs**

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I. Introduction

The communication problem discussed in this article has been given various names in the literature. For example, Baker and Shostak (**1**) call it the "gossip problem" while Hajnal *et al.* (**3**) refer to it as the "telephone disease". This is analogous to the well-known term "the four color disease" [(**4**), p. 126], which arose from the fact that the problem has so many features of an ailment, the main difference being that this one was settled so effortlessly, relatively speaking. In the literature of psychology the "common symbol problem" (**2**) is equivalent to it, although in disguise. Recently, Moon (**5**) studied a probabilistic variation of the communication problem in which the calls are made at random.

II. Background

In brief, the communication problem has as its setting the hypothesis that each of n individuals knows a unique item of information. What is the minimum number of communications between pairs of people needed to exchange all the information? It is assumed that these are two-way contacts such as a telephone conversation and that each individual can phone any other. This amounts to the stipulation that the complete graph K_n represents the communication network. Thus the communication problem is generalized when we take as the given network not K_n but a connected graph G with n points. The lines of G can then be regarded as those two-way calls which do not entail toll charges.

A further modification of the original problem results when the communication acts are all one-way, such as writing a letter. Of course the underlying network must then be a strongly connected digraph D, in order that all the information eventually reaches everyone.

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We now summarize the results obtained to date for K_n, G and D. It was shown in (**1**), (**3**) and (**6**) (and also in an unpublished note by Bumby and Spencer) that the solution to the original communication problem (for K_n, $n \geqslant 4$) is $2n-4$. This means that $2n-4$ calls (two-way contacts) suffice and that no smaller number would do.

We show that for an arbitrary connected graph G, $2n-3$ calls suffice and that if G contains a quadrilateral then just as for a complete graph, $2n-4$ calls do.

Finally, for a given strong digraph D, $2n-2$ is the minimum number of one-way contacts for all the information to be disseminated to everyone.

III. Noncomplete Graphs

By *the solution* of the communication problem for a nontrivial connected graph G is meant the minimum number of calls to acquaint all n individuals with all n items of information. We state now the result of (**1**), (**3**) and (**5**).

Theorem I. The solution for K_n with $n \geqslant 4$ is $2n-4$. (For K_3 and K_2, the solution is $2n-3$.)

When G is not complete, it is logically convenient to consider trees first.

Theorem II. The solution for a tree T with n points is $2n-3$.

Proof: It is known [(**4**), p. 34] that every non-trivial tree T has at least two endpoints, and that T has $n-1$ lines. The following procedure will achieve a solution in $2n-3$ calls. Let each endpoint of T first call its unique neighboring point. Let T' [see (**4**), p. 35] be the tree obtained when the endpoints of T are removed. Repeat in T' the process of "endpoint calling followed by deletion", and so forth for T'', T''', ... Then this ingoing procedure terminates after $n-1$ calls when the center of T is reached, whether T has one or two central points.

The two individuals involved in the last call now know all n items of information. They proceed to disseminate it by calling back those who had called them previously, so that $n-2$ outgoing calls are made. These are the first $n-2$ calls repeated in reverse order. Thus after $2n-3$ calls, everyone has all the information.

If fewer than $2n-3$ calls could suffice, then at least two lines of the tree would be used only once. But the removal of these two lines results in three subtrees, just one of which is "between" the other two. Thus there is no way for the people in one of the "extreme" trees to receive any information from the other extreme tree.

Corollary 1(a). The solution for a connected graph G with n points is either $2n-3$ or $2n-4$.

Proof: For $G = K_n$, the solution is $2n-4$ by Theorem I. Every connected graph has a spanning tree T and by Theorem II the solution for T is $2n-3$. Obviously the solution for G is at most that for T, and the corollary is established.

Theorem III. If a connected graph G contains a quadrilateral C_4, then the solution for G is $2n-4$.

Proof: Clearly, G contains a spanning unicyclic graph H whose cycle is C_4. It is well known that the removal of the lines of C_4 from H results in four trees, each of which may be regarded as rooted at a point of C_4.

We now show that the solution for H is $2n-4$, which implies the theorem. Apply the same process of endpoint calling followed by deletion, as in the proof of Theorem II, to the endpoints of H, forming $H', H'', \ldots, H^{(r)} = C_4$. These $n-4$ calls, one for each line of H not in C_4, convey all the information to the four points of C_4. Now, looking at Fig. 1, four calls are made to exchange

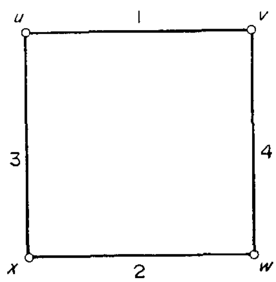

FIG. 1. A quadrilateral and its calling scheme.

all the information in H (and hence in G) among the four people in C_4. These four calls are made in the order that the lines are labeled. A glance at Fig. 1 shows that after the first two calls u and v have "one half" of the information and x and w have "the other half". After the third call u and x know everything, and after the fourth, so do v and w. Finally, repeating the first $n-4$ calls in reverse order disseminates all the information to everybody.

Thus only for graphs not containing C_4 is the question still open. We are so convinced of the next statement that even though it is by definition a conjecture, we shall call it a *True Conjecture*: a connected graph G with n points has solution $2n-4$ if and only if G contains a quadrilateral.

Theorem III establishes the sufficiency; the necessity still eludes us, however.* By way of heuristic evidence in favor of the conjecture, we now show that no cycle other than a quadrilateral will serve to yield the minimum possible solution, $2n-4$, for a unicyclic graph.

Theorem IV. If G is a unicyclic graph whose cycle is not a quadrilateral, then the solution for G is $2n-3$.

Proof: We know by Corollary 1(a) that the solution for G is either $2n-3$ or $2n-4$.

Suppose a solution of $2n-4$ calls exists. We first observe that every line of G must be used, for if a bridge is omitted, the people on opposite sides cannot learn each other's information. On the other hand, if a line of the cycle is not used, then the calling sequence is actually based on a tree, so $2n-3$ calls are needed by Theorem II. Now since $2n-4$ calls are to be made

* We offer U.S. \$10 (regardless of its fluctuations relative to other currencies) for the first proof or (highly unlikely) disproof thereof.

on n lines and each line is used at least once, we see that at least four lines are used exactly once. We call such a line *singular*.

We now show that all four of these singular calls must lie on the cycle. For if one of them is a bridge, then it separates G into two components, say of order b and $n-b$. Now since this bridge uv is used just once, u and v must each know all the information from his own side at the time of that call. This requires $(b-1)+(n-b-1) = n-2$ prior calls. After the call uv, $n-2$ additional calls are needed to disseminate the information just obtained. Thus $(n-2)+1+(n-2) = 2n-3$ calls have been made in this case. Since this violates the hypothesis that $2n-4$ calls suffice, we know that the four singular calls must all lie on the cycle.

Now the removal of these four lines leaves four components T_i (each a tree) as shown in Fig. 2, in which u_i and v_i may coincide for some choices of i.

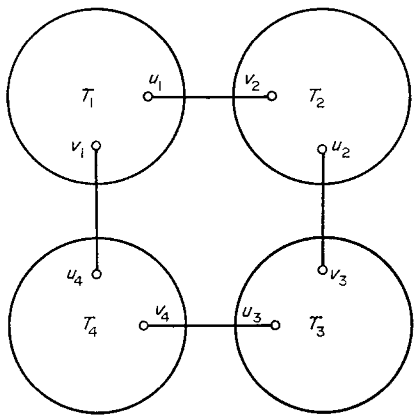

FIG. 2. The four singular lines and the four associated trees.

Of the four singular calls $u_1 v_2$, $u_2 v_3$, $u_3 v_4$ and $u_4 v_1$, the first two made cannot be "consecutive" on the cycle as, for example, $u_1 v_2$ and $u_2 v_3$, for then there is no way for component T_2 to receive any information from T_4. Thus, we may assume without loss of generality that the relative order of these four singular calls is $u_1 v_2$, $u_3 v_4$, $u_4 v_1$ and then $u_2 v_3$. Now in order to transmit all the information among the four components, $n-4$ calls must precede the four singular calls, and $n-4$ calls must follow them, as in the proof of Theorem III. Thus, this accounts for the allotted total of $2n-4$ calls, so calls $u_1 v_2, u_3 v_4, u_4 v_1, u_2 v_3$ must occur in sequence with no other intervening calls.

Remember however, that the cycle is not a quadrilateral. Therefore, while $u_i = v_i$ is possible for some i, there must be a choice of i for which $u_i \neq v_i$. Thus, the two singular calls into this particular component T_k arrive at distinct points, hence T_k cannot be used to relay information between its two neighboring components. However, this is a contradiction because each component must provide one such relay in order to exchange information between the diagonally opposite pairs of components. Consequently, no sequence of $2n-4$ calls exists, and so the minimum solution is in fact $2n-3$ calls.

Thus we see that the True Conjecture is more plausible in the light of Theorem IV in that the presence of a quadrilateral is seen to be important, rather than a cycle of any other length.

IV. One-way Communications

A variation of the original communication problem for graphs is obtained when instead of considering two-way contacts (such as phone calls), only one-way messages (such as letters) are possible. The setting for the resulting problem is of course a digraph D which must be strong (strongly connected) if each item of information is to reach all n people. At first glance one might assume that it would require nearly twice the number of letters as calls since it takes two letters for mutual contact between two people. Thus the theorem which establishes the minimum number of letters comes as a bit of a surprise. In fact, only one more letter is needed for a strong digraph than the number of calls needed for a tree.

Theorem V. The solution for a strong digraph D with n points is $2n-2$.

Proof: By definition, every pair of points of a strong digraph are joined by two directed paths, one in each direction. Thus, at any particular point v, it is possible to find a tree T_{in} spanning D with all its arcs toward v. This is called an *in-tree*; an *out-tree* T_{out} is similarly defined. These two trees can now be used to provide a solution. Starting at the endpoints of T_{in}, one letter is sent for each arc in T_{in} so that after $n-1$ letters, v is fully informed. Now the $n-1$ arcs of T_{out} are used to convey this information from v to everyone else. Thus $2n-2$ letters serve to inform everyone about all n items.

We observe that no smaller number can suffice, for obviously no one can be fully informed after only $n-2$ letters. Each subsequent letter fully informs at most one person, so at least $(n-2)+n = 2n-2$ letters are required. Since we have achieved this bound, the proof is complete.

References

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Note added in proof: We have recently learnt from R. L. Graham that the True Conjecture has been proved by D. Kleitman,