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Nonlinear Oscillations across a Point of Resonance for Nonselfadjoint Systems*

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1. INTRODUCTION

We consider here abstract operational equations of the form

$$Ex + \alpha Ax = Nx, \tag{1}$$

where $E\colon \mathfrak{D}(E) \to Y$, $\mathfrak{D}(E) \subset X$, $N\colon X \to Y$, $A\colon X \to Y$ are operators, E linear, not necessarily bounded, N and A continuous, not necessarily linear, X, Y Banach spaces over the reals, α a real parameter.

In any application E may be a linear differential operator in some domain $G \subset E^\nu$, $\nu \geqslant 1$, with linear homogeneous boundary conditions.

We consider the case in which $Ex = 0$ has a nontrivial set $X_0 = \ker E$ of solutions; in other words, the equation $Ex + \lambda x = 0$ has $\lambda = 0$ as an eigenvalue. We assume, however, that X_0 is finite dimensional, thus, $1 \leqslant \dim \ker E < \infty$. On the continuous operator $A\colon X \to Y$ we assume that it maps bounded sets of X into bounded sets of Y.

On the continuous operator $N\colon X \to Y$ we assume that it satisfies certain hypotheses in the large which represent an abstract extension of those of the Landesman and Lazer and analogous theorems. The hypotheses on N guarantee the existence of solutions to the equation at resonance $Ex = Nx$.

In the present paper we prove, in terms of the alternative method and the Schauder fixed point theorem, that the same assumptions on N actually have a stronger implication, Namely, under such assumption, there are numbers $\alpha_0 > 0$, $C > 0$ such that, for every real α with $|\alpha| \leqslant \alpha_0$, the equation $Ex + \alpha Ax = Nx$ has at least a solution $x \in X$ with $\|x\| \leqslant C$ (existence of solutions across a point of resonance). In other words, the parameter α is allowed to go through the point of resonance $\alpha = 0$, and yet uniformly bounded solutions x of (1) can be guaranteed.

This phenomenon has physical significance. For the case of periodic solutions

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of ordinary differential systems with forcing terms of given period $T = 2\pi/\omega$, our statements may imply the existence of uniformly bounded periodic solutions of the same period T (entrainement of frequency). Even in this situation, the results are new since they are proved under sole qualitative hypotheses on N and A.

We discuss problem (1) here under assumptions which do not imply selfadjointness. (For a discussion of the same problems for the sole selfadjoint case see Cesari [6]). Applications of the theorems of the present paper to the ordinary differential equations taken into consideration by Lazer and Leach [20] are made briefly at the end of this paper (Sect. 5). Applications to the partial differential equations taken into consideration by Landesman and Lazer [19], Williams [20], and De Figueiredo [10–12], are made in [5].

An existence theorem at resonance ($\alpha = 0$) for the bounded case and selfadjoint problems ($\| Nx \| \leqslant J_0$, $X = Y$ a Hilbert space) has been proved by the author and Kannan in [8] in terms of the Schauder fixed point theorem, and the same statement has been proved by Kannan and McKenna [18], the latter in connection with his thesis at Michigan, by the alternative method and the Leray–Schauder topological degree argument. The same combination of the alternative method and the Leray–Schauder argument could equivalently be used also in the proof of the theorems of the present paper. We prefer to use here an argument, based on Schauder's fixed point theorem, which is closer to the original arguments of Landesman and Lazer, and of Williams.

2. Notations and Main Assumptions

Let X, Y be Banach spaces over the reals, and let $\| x \|_X$, $\| y \|_Y$ denote the norms in X and Y, respectively.

Let $P: X \to X$, $Q: Y \to Y$ be projection operators (i.e., linear, bounded, and idempotent), with ranges and null spaces

$$\mathfrak{R}(P) = PX = X_0, \qquad \ker P = \mathfrak{R}(I - P) = (I - P)X = X_1,$$
$$\mathfrak{R}(Q) = QY = Y_0, \qquad \ker Q = \mathfrak{R}(I - Q) = (I - Q)Y = Y_1.$$

Let $E: \mathfrak{D}(E) \to Y$ be a linear operator with domain $\mathfrak{D}(E) \subset X$ and let us assume that

$$\ker E = X_0 = PX, \quad \mathfrak{R}(E) = Y_1 = (I - Q)Y, \quad 1 \leqslant m = \dim X_0 < \infty.$$

Then E, as a linear operator from $\mathfrak{D}(E) \cap X_1$ into Y_1 is one–one and onto, so that the partial inverse $H: Y_1 \to \mathfrak{D}(E) \cap X_1$ exists as a linear operator. We assume that H is a bounded linear compact operator, and that the usual axioms of [3] hold:

$$(k_1)\ H(I-Q)E = I - P; \quad (k_2)\ EP = QE; \quad (k_3)\ EH(I-Q) = I - Q.$$

We have depicted here a situation which is rather typical for a large class of differential systems, nonnecessarily selfadjoint, in the alternative method (cf. [3–7, 15]).

Let $A: X \to Y$ be a continuous operator, not necessarily linear, for which we only assume that A is bounded, that is, A maps bounded sets into bounded sets, or equivalently $\| Ax \| \leqslant \omega(\| x \|)$ for all $x \in X$ and some given monotone nondecreasing function $\omega(\zeta) \geqslant 0$, $0 \leqslant \zeta < +\infty$.

Let $N: X \to Y$ be a continuous operator, not necessarily linear, and let us consider the equation

$$Ex + \alpha\, Ax = Nx, \qquad x \in \mathfrak{D}(E). \tag{2}$$

As we know from [3], this equation is equivalent to the system of auxiliary and bifurcation equations

$$x = Px + H(I-Q)[-\alpha\, Ax + Nx], \tag{3}$$

$$Q(Ex + \alpha\, Ax - Nx) = 0. \tag{4}$$

Having assumed $\ker E = X_0$, the bifurcation equation (4) reduces to $Q[-\alpha\, Ax + Nx] = 0$. Also, for $x^* = Px$, the auxiliary equation (3) takes the form $x = x^* + H(I-Q)[-\alpha\, Ax + Nx]$.

We shall now further assume that Y is a space of linear operators on X so that the operation $\langle y, x\rangle$, $Y \times X \to R$ is defined, is linear both in x and y, and we assume that $|\langle y, x\rangle| \leqslant K \| y \|_Y \| x \|_X$ for some constant K and all $x \in X$, $y \in Y$. We can always choose norms in X and in Y, or we can always choose the linear operator $\langle y, x\rangle$, in such a way that $K = 1$.

The following examples are of interest. Here G denotes a bounded domain in any t-space R^ν, $t = (t_1, \dots, t_\nu)$, $\nu \geqslant 1$.

(a) $X = Y = L_2(G)$, $|\langle y, x\rangle| = |\int_G y(t)\, x(t)\, dt| \leqslant \| y \| \| x \|$, with usual norms in L_2.

(b) $X = L_2(G)$ with L_2-norm $\| x \|$, $Y = L_\infty(G)$ with norm $\| y \|_\infty$, and then $|\langle y, x\rangle| = |(\text{meas } G)^{-1/2} \int_G y(t)\, x(t)\, dt| \leqslant \| y \|_\infty \| x \|$.

(c) $X = L_\infty(G)$ with usual norm $\| x \|_\infty$, $Y = L_\infty(G)$ with norm $\| y \|_\infty$, and then again $|\langle y, x\rangle| = |(\text{meas } G)^{-1} \int_G y(t)\, x(t)\, dt| \leqslant \| y \|_\infty \| x \|_\infty$.

(d) $X = H^m(G)$ with usual Sobolev norm $\| x \|_m$, $Y = L_2(G)$, and then $|\langle y, x\rangle| = |\int_G y(t)\, x(t)\, dt| \leqslant \| y \| \| x \| \leqslant \| y \| \| x \|_m$.

Note that whenever $X \subset Y$ and $X_0 \subset Y_0$, then for the elements x of the finite-dimensional space X_0 the norms in X and in Y are equivalent, that is, their quotient is bounded above and below (in X_0).

We shall use below the following notations, with X and Y Banach spaces and norms $\| x \|_X$, $\| y \|_Y$. The indication X or Y will be omitted when the meaning is clear. Let $w = (w_1, ..., w_m)$ be an arbitrary basis for the finite-dimensional space $X_0 = \ker E = PX$, $1 \leqslant m = \dim \ker E < \infty$. By $\langle y, w \rangle$ we shall denote the m-vector $\langle y, w_i \rangle$, $i = 1, ..., m$. For $x^* \in X_0$ we have $x^* = \sum_1^m c_i w_i$, or briefly $x^* = cw$, $c = (c_1, ..., c_m) \in R^m$, and there are constants $0 < \gamma' \leqslant \gamma < \infty$, such that $\gamma' \mid c \mid \leqslant \| cw \| \leqslant \gamma \mid c \mid$, where $\mid \ \mid$ is the Euclidean norm in R^m.

We shall now assume that the operation $\langle y, x \rangle$ from $X \times Y$ into the reals has the following property (π). For $y \in Y$ we have $y \in \mathfrak{R}E = Y_1$, that is, $Qy = 0$, if and only if $\langle Qy, x^* \rangle = 0$ for all $x^* \in X_0$, that is, if and only if $\langle Qy, w_i \rangle = 0$, $i = 1, ..., m$, or $\langle Qy, w \rangle = 0$.

System (3), (4) of the auxiliary and bifurcation equations can now be written in the form $x = cw + H(I - Q)[-\alpha Ax + Nx]$, and $\langle Q[-\alpha Ax + Nx], w \rangle = 0$.

Let $k_0 = \| P \|$, $k' = \| I - P \|$, so that $\| Px \| \leqslant k_0 \| x \|$, $\|(I - P)x \| \leqslant k' \| x \|$ for all $x \in X$. Analogously, let $\chi = \| Q \|$, $\chi' = \| I - Q \|$, so that $\| Qy \| \leqslant \chi \| y \|$, $\|(I - Q)y \| \leqslant \chi' \| y \|$ for all $y \in Y$. Also, let $L = \| H \|$, and note that there is a constant $\mu > 0$ such that $\langle y, w \rangle = d$, that is, $\langle y, w_i \rangle = d_i$, $i = 1, ..., m$, $d = (d_1, ..., d_m)$, $y \in Y$, implies $\mid d \mid \leqslant \mu \| y \|$.

Whenever X and Y are Hilbert spaces (as in cases (a) and (b) above), and P and Q are orthogonal projections, then $k_0 = k' = \chi = \chi' = 1$. If X is a Hilbert space and $w = (w_1, ..., w_m)$ is orthonormal in X, then $\gamma = \gamma' = 1$. If $X = Y$ are Hilbert and $(w_1, ..., w_m)$ orthonormal, then $\mu = 1$.

Note that, if X^* denotes the dual of X, then the linear operation $\langle z, x \rangle$, $x \in X$, is defined for all $z \in X^*$, and we may have $Y \subset X^*$.

3. Existence Theorems at Resonance

(a) *The Case of N Bounded*

THEOREM 1 (existence at resonance). *Let X, Y be Banach spaces, let E, H, P, Q satisfy (k_{123}), let $N: X \to Y$ be a continuous operator, let $X_0 = \ker E$ be finite-dimensional, let H be linear, bounded and compact, and $\langle y, x \rangle$ be defined such that $|\langle y, x \rangle| \leqslant \| y \| \| x \|$ and satisfying (π). If (B_0) there is a constant $J_0 > 0$ such that $\| Nx \| \leqslant J_0$ for all $x \in X$; and if (N_0) there is a constant $R_0 \geqslant 0$ such that $\langle Q\, Nx, x^* \rangle \leqslant 0$ [or $\langle Q\, Nx, x^* \rangle \geqslant 0$] for all $x \in X$, $x^* \in X_0$ with $Px = x^*$, $\| x^* \| \geqslant R_0$, $\| x - x^* \| \leqslant L\chi' J_0$, then equation $Ex = Nx$ has at least a solution $x \in \mathfrak{D}(E) \subset X$.*

Proof. Let us assume we always have $\langle Q\, Nx, x^*\rangle \leqslant 0$. We take now positive numbers R_1, R_2, R, S, η satisfying the relations

$$\begin{gathered} R_0 \leqslant R_1 < R_2 < R < S, \quad 0 < \eta < \gamma R_1/2, \quad R_0 \leqslant \gamma' R_1, \\ \gamma R + L\chi' J_0 \leqslant S, \quad R_2 + \mu\chi J_0 \leqslant R, \end{gathered} \tag{5}$$

and we consider the transformation T: $(x, c) \to (\bar{x}, \bar{c})$ defined by

$$\begin{gathered} T\colon \bar{x} = cw + H(I-Q)\, Nx, \quad \bar{c} = c + g(\bar{x}, c), \\ (x, c) \in \mathfrak{C} = [(x, c) \mid x \in X, c \in R^m, \| x \| \leqslant S, \mid c \mid \leqslant R], \end{gathered} \tag{6}$$

where $x^* = cw = \sum_1^m c_i w_i = P\bar{x}$, $\bar{x}^* = \bar{c}w = \sum_1^m \bar{c}_i w_i$, $c, \bar{c} \in R^m$, and $g(\bar{x}, c) = (g_1, \ldots, g_m)$ is explicitely given below. Note that

$$\bar{x}^* = x^* + g(\bar{x}, c)w = x^* + \sum_1^m g_i w_i .$$

Here, for $0 \leqslant \mid c \mid \leqslant R_1$ we take $g(\bar{x}, c) = \langle Q\, N\bar{x}, w\rangle$.

For $R_2 \leqslant \mid c \mid \leqslant R$, we take

$$g(\bar{x}, c) = [\langle Q\, N\bar{x}, x^*\rangle - \eta \| Q\, N\bar{x} \|](2\chi J_0 \gamma \mid c \mid)^{-1} c. \tag{7}$$

For $R_1 \leqslant \mid c \mid \leqslant R_2$ we take

$$\begin{aligned} g(\bar{x}, c) &= \lambda\langle Q\, N\bar{x}, w\rangle + (1-\lambda)[\langle Q\, N\bar{x}, x^*\rangle - \eta \| Q\, N\bar{x} \|](2\chi J_0 \gamma \mid c \mid)^{-1} c, \\ \bar{c} &= c + \lambda\langle Q\, N\bar{x}, w\rangle + (1-\lambda)[\langle Q\, N\bar{x}, x^*\rangle - \eta \| Q\, N\bar{x} \|](2\chi J_0 \gamma \mid c \mid)^{-1} c, \\ \lambda &= (R_2 - R_1)^{-1}(R_2 - \mid c \mid), \qquad 0 \leqslant \lambda \leqslant 1. \end{aligned}$$

For any $(x, c) \in \mathfrak{C}$, we have $P\bar{x} = cw = x^*$, and $\| \bar{x} - x^* \| = \| H(I-Q)\, Nx \| \leqslant L\chi' J_0$. Hence, for $\mid c \mid \geqslant R_1$, and consequently $\| x^* \| = \| cw \| \geqslant \gamma' R_1 \geqslant R_0$, by (N_0) we have $\langle Q\, N\bar{x}, x^*\rangle \leqslant 0$.

From (6) we see that we have $\bar{c} = c$, or $\bar{x}^* = x^*$, if and only if $g(\bar{x}, c) = 0$. For $\mid c \mid \leqslant R_1$ we have $g = 0$ if and only if $Q\, N\bar{x} = 0$. For $R_2 \leqslant \mid c \mid \leqslant R$ we have $\langle Q\, N\bar{x}, x^*\rangle - \eta \| Q\, N\bar{x} \| \leqslant -\eta \| Q\, N\bar{x} \|$, $c \neq 0$, and again g, as given by (7), is zero if and only if $Q\, N\bar{x} = 0$. For $R_1 < \mid c \mid < R_2$, we have

$$\begin{aligned} g(\bar{x}, c)c = \sum_{i=1}^m g_i c_i &= \lambda\langle Q\, N\bar{x}, x^*\rangle + (1-\lambda)[\langle Q\, N\bar{x}, x^*\rangle \\ &\quad - \eta \| Q\, N\bar{x} \|](2\chi J_0 \gamma \mid c \mid)^{-1} \mid c \mid^2, \end{aligned}$$

where $\lambda > 0$, $1-\lambda > 0$, $\langle Q\, N\bar{x}, x^*\rangle \leqslant 0$, $\langle Q\, N\bar{x}, x^*\rangle - \eta \| Q\, N\bar{x} \| \leqslant -\eta \| Q\, N\bar{x} \|$, $\mid c \mid^2 > 0$. Thus, $g(\bar{x}, c)c < 0$ for $Q\, N\bar{x} \neq 0$; $g(\bar{x}, c)c = 0$ for $Q\, N\bar{x} = 0$, that is, $g = 0$ if and only if $Q\, N\bar{x} = 0$. Thus, in any case $\bar{c} = c$ if and only if $g = 0$, and $g = 0$ if and only if $Q\,N\bar{x} = 0$. We conclude that $(x, c) \in \mathfrak{C}$ is a fixed point of T if and only if x, $x^* = Px$ satisfy the relations $x = cw + H(I - Q)\,Nx$, $Q\,Nx = 0$, that is, the auxiliary and bifurcation equations for $Ex = Nx$. Thus, $(x, c) \in \mathfrak{C}$ is a fixed point of T if and only if x is a solution of $Ex = Nx$.

Let us prove that T maps $\mathfrak{C}$ into itself. First, for $(x, c) \in \mathfrak{C}$ we have $x^* = cw$, $\| x^* \| = \| cw \| \leqslant \gamma \mid c \mid \leqslant \gamma R$, and

$$\| \bar{x} \| \leqslant \| cw \| + \| H(I - Q)\,Nx \| \leqslant \gamma R + L\chi' J_0 \leqslant S.$$

For $\mid c \mid \leqslant R_1$ we have $\bar{c} = c + g(\bar{x}, c) = c + \langle Q\,N\bar{x}, w\rangle$; hence

$$\mid \bar{c} \mid \leqslant \mid c \mid + |\langle Q\,N\bar{x}, w\rangle| \leqslant \mid c \mid + \mu \| Q\,N\bar{x} \| \leqslant R_1 + \mu\chi J_0 \leqslant R.$$

For $R_2 \leqslant \mid c \mid \leqslant R$ we have

$$\bar{c} = \{1 + [\langle Q\,N\bar{x}, x^*\rangle - \eta \| Q\,N\bar{x} \|](2\chi J_0 \gamma \mid c \mid)^{-1}\}c = \Lambda c,$$

where Λ is the number in braces, $\langle Q\,N\bar{x}, x^*\rangle \leqslant 0$, $|\langle Q\,N\bar{x}, x^*\rangle| \leqslant \chi J_0 \gamma \mid c \mid$, $\| Q\,N\bar{x} \| \leqslant \chi J_0$, $\mid c \mid \geqslant R_2 \geqslant R_1$, $\eta/2\gamma R_1 \leqslant 1/4$, and $1/4 = 1 - 1/2 - 1/4 \leqslant \Lambda \leqslant 1$. Thus $\bar{c}$ is a point on the segment between c and $c/4$ in R^m, and $\mid \bar{c} \mid \leqslant \mid c \mid \leqslant R$.

For $R_1 < c < R_2$ we have $0 < \lambda < 1$,

$$\bar{c} = \lambda c + \lambda\langle Q\,N\bar{x}, w\rangle + (1 - \lambda)\{1 + [\langle Q\,N\bar{x}, x^*\rangle - \eta \| Q\,N\bar{x} \|](2\chi J_0 \gamma \mid c \mid)^{-1}\}c,$$

and

$$\begin{aligned} \mid \bar{c} \mid &\leqslant \lambda \mid c \mid + \lambda\,|\langle Q\,N\bar{x}, w\rangle| + (1 - \lambda)|\,\{\ \}c\,| \\ &\leqslant \lambda \mid c \mid + \mu\chi J_0 + (1 - \lambda) \mid c \mid = \mu\chi J_0 + \mid c \mid \leqslant \mu\chi J_0 + R_2 \leqslant R. \end{aligned}$$

We have proved that $T\colon \mathfrak{C} \to \mathfrak{C}$.

Let us prove that T is compact. For this we consider any (bounded) sequence (x_k, c_k), $k = 1, 2, \ldots$, of points of $\mathfrak{C}$. Then the sequence Nx_k is bounded, actually $\| Nx_k \| \leqslant J_0$, $\| z_k \| = \| H(I - Q)\,Nx_k \| \leqslant L\chi' J_0$, and since H is compact, there is a subsequence, say still $[k]$, so that z_k is convergent in X. Certainly c_k, $g(x_k, c_k) = d_k$ are bounded sequences, $\mid c_k \mid \leqslant R$, $\mid d_k \mid \leqslant R$, both c_k and d_k in R^m, a finite-dimensional space. Thus, we can extract the subsequence, say still $[k]$, so that c_k, d_k are convergent in R^m, and then $\bar{x}_k = c_k w + z_k$, $\bar{c}_k = c_k + d_k$ are convergent in S and R^m, respectively. We have proved that T is compact.

By Schauder's fixed point theorem $T\colon \mathfrak{C} \to \mathfrak{C}$ has at least one fixed point $(x, c) = T(x, c)$ in $\mathfrak{C}$. Theorem 1 is thereby proved.

(b) *The Case of Limited Growth of N*

For the case of limited growth of N, we need consider a suitable monotone nondecreasing function $\phi(\zeta) \geqslant 0$, $0 \leqslant \zeta < +\infty$, and assume that $\| Nx \| \leqslant \phi(\| x \|)$ for all $x \in X$. On $\phi(\zeta)$ we could simply require that $\phi(\zeta)/\zeta \to 0$ as $\zeta \to \infty$. Actually, it is of some advantage to require less on ϕ.

We need the constant $R_0 \geqslant 0$ which appears in the condition (N_ϕ) below. Let σ_1, σ_2, σ be arbitrary constants, $0 < \sigma_1 < \sigma_2 < \sigma < \min[1, \gamma^{-1}]$, and let us consider numbers

$$\lambda_0 \geqslant \max[1, (\gamma')^{-1}], \quad \lambda_1 \leqslant \min[(L\chi')^{-1}(1 - \gamma\sigma), (\mu\chi)^{-1}(\sigma - \sigma_2)]$$

The only requirement we need for the monotone function ϕ is that there is a constant S satisfying

$$S \geqslant \sigma_1^{-1} \lambda_0 R_0, \qquad \phi(S)/S \leqslant \lambda_1. \tag{8}$$

Thus, if $\phi(\zeta)/\zeta \to 0$ as $\zeta \to +\infty$, then certainly such a constant S can be determined.

For instance, if $\| Nx \| \leqslant J_0 + J_1 \| x \|^k$ for all $x \in X$ and some constants $J_0 \geqslant 0$, $J_1 > 0$, $0 < k < 1$, then $\phi(\zeta) = J_0 + J_1\zeta^k$, $\phi(\zeta)/\zeta \to 0$ as $\zeta \to +\infty$, and the constant S can be found.

If $\| Nx \| \leqslant J_0 + J_1 \| x \|^k$ for all $x \in X$ and constants $J_0 \geqslant 0$, $J_1 > 0$ and $k \geqslant 1$, then $\phi(\zeta) = J_0 + J_1\zeta^k$ and $\phi(\zeta)/\zeta$ does not approach zero as $\zeta \to +\infty$. However, a constant S satisfying (8) can be found provided J_1 is sufficiently small. Indeed, it is enough we take

$$S \geqslant \max[\sigma_1^{-1} \lambda_0 R_0, 2J_0\lambda_1^{-1}], \qquad J_1 \leqslant 2^{-1} \lambda_1 S^{1-k}.$$

since then $\phi(S)/S = J_0S^{-1} + J_1S^{k-1} \leqslant \lambda_1/2 + \lambda_1/2 = \lambda_1$.

THEOREM 1* (existence at resonance). *Under the same general hypotheses as in Theorem* 1, *let* $\phi(\zeta) \geqslant 0$, $\psi(\zeta) \geqslant 0$, $0 \leqslant \zeta < +\infty$, *be monotone nondecreasing functions. Let us assume that* (B_ϕ) $\| Nx \| \leqslant \phi(\| x \|)$ *for all* $x \in X$; *and that* (N_ϕ) $\langle Q\, Nx, x^* \rangle \leqslant 0$ [*or* $\langle Q\, Nx, x^* \rangle \geqslant 0$] *for all* $x \in X$, $x^* \in X_0$ *with* $Px = x^*$, $\| x^* \| \geqslant R_0$, $\| x - x^* \| \leqslant \psi(\| x \|)$. *Let us assume further that there is a number* $S \geqslant \sigma_1^{-1} \lambda_0 R_0$ *with* $0 < \phi(S)/S \leqslant \lambda_1$, *and*

$$L\chi'\phi(S) \leqslant \psi(k_0^{-1} \gamma'\sigma_1 S). \tag{9}$$

Then, the equation $Ex = Nx$ *has at least a solution* $x \in \mathfrak{D}(E) \subset X$ *with* $\| x \| \leqslant S$.

For instance, if we take ψ so that $L\chi'\phi(\zeta) = \psi(k_0^{-1}\gamma'\sigma_1\zeta)$, then relation (N_ϕ) is required to hold for $\| x - x^* \| \leqslant L\chi'\phi(\lambda_2 \| x \|)$ with $\lambda_2 = (\sigma_1\gamma')^{-1}k_0$, relation (9) is trivially satisfied for all S, and all we require on ϕ is that there is some S satisfying (8). For instance, in the case $\phi(\zeta) = J_0 + J_1\zeta^k$, $0 < k < 1$, this choice of ψ would yield $\psi(\zeta) = L\chi' J_0 + L\chi' J_1((\gamma')^{-1}k_0\sigma_1^{-1}\zeta)^k$.

If we require (N_ϕ) to hold for $\| x - x^* \| \leqslant \psi(\| x^* \|)$, then (9) shall be replaced by

$$L\chi'\phi(S) \leqslant \psi(\gamma'\sigma_1 S). \tag{9$'$}$$

If we choose ψ so that $L\chi'\phi(\zeta) = \psi(\gamma'\sigma_1\zeta)$ for all ζ, then relation (N_ϕ) is required to hold for $\| x - x^* \| \leqslant L\chi'\phi(\lambda_3 \| x \|^*)$ with $\lambda_3 = \sigma_1^{-1}(\gamma')^{-1}$, relation (9)$'$ is trivially satisfied, and all we require on ϕ again is that there is some S satisfying (8).

Proof. By repeating the proof of Theorem 1, we need determine the positive constants R_1, R_2, R, S, η in such a way that

$$R_0 \leqslant R_1 < R_2 < R < S, \quad 0 < \eta < \gamma R_1/2, \quad R_0 \leqslant \gamma' R_1, \tag{10}$$

$$\gamma R + L\chi'\phi(S) \leqslant S, \quad R_2 + \mu\chi\phi(S) \leqslant R, \tag{11}$$

the last two relations being equivalent to

$$\gamma R/S + L\chi'\phi(S)/S \leqslant 1, \quad R_2/S + \mu\chi\phi(S)/S \leqslant R/S.$$

First we take $R_1 = \sigma_1 S$, $R_2 = \sigma_2 S$, $R = \sigma S$ and thus $R_1 < R_2 < R < S$. Now $S \geqslant \sigma_1^{-1} \lambda_0 R_0$ implies $S \geqslant \sigma_1^{-1} R_0$, $S \geqslant \sigma_1^{-1}(\gamma')^{-1} R_0$, $R_1 = \sigma_1 S \geqslant R_0$, $\gamma' R_1 = \gamma'\sigma_1 S \geqslant R_0$. By taking any $0 < \eta < \gamma R_1/2$, we have satisfied relations (10). We have now $R_0 \leqslant R_1 < R_2 < R < S$, and

$$\gamma R/S + L\chi'\phi(S)/S \leqslant \gamma\sigma + L\chi'\lambda_1 \leqslant \gamma\sigma + (1 - \gamma\sigma) = 1,$$
$$R_2/S + \mu\chi\phi(S)/S \leqslant \sigma_2 + \mu\chi\lambda_1 \leqslant \sigma_2 + (\sigma - \sigma_2) = \sigma = R/S.$$

Thus, relations (11) are also satisfied.

Now we can proceed as in the proof of Theorem 1 where we replace everywhere $\phi(S)$ for J_0. Attention should be made to what occurs for $(x, c) \in \mathfrak{C}$ with $R_1 < | c | \leqslant R$. First, $P\bar{x} = cw = x^*$, and from (9) we have

$$\| \bar{x} - x^* \| = \| H(I - Q) Nx \| \leqslant L\chi'\phi(S) \leqslant \psi(k_0^{-1} \gamma'\sigma_1 S).$$

Hence, from $\gamma' | c | \leqslant \| cw \| = \| x^* \| \leqslant \gamma | c |$, $\| x^* \| = \| P\bar{x} \| \leqslant k_0 \| \bar{x} \|$, we see that, for $R_1 \leqslant c \leqslant R$ we have

$$\| \bar{x} - x^* \| \leqslant \psi(k_0^{-1} \gamma'\sigma_1 S) = \psi(k_0^{-1} \gamma' R_1) \leqslant \psi(k_0^{-1} \gamma' | c |)$$
$$\leqslant \psi(k_0^{-1} \| x^* \|) \leqslant \psi(\| \bar{x} \|),$$

and by (N_ϕ) also $\langle Q N\bar{x}, x^* \rangle \leqslant 0$. The remaining of the proof of Theorem 1 remains now unchanged with the sole replacement of $\phi(S)$ for J_0.

If (N_ϕ) holds for $\| x - x^* \| \leqslant \psi(\| x^* \|)$, then from (9)$'$ we have, for $R_1 \leqslant | c | \leqslant R$,

$$\| \bar{x} - x^* \| \leqslant \psi(\gamma'\sigma_1 S) = \psi(\gamma' R_1) \leqslant \psi(\gamma' | c |) \leqslant \psi(\| x^* \|).$$

4. Existence Theorems across a Point of Resonance

THEOREM 2. (existence across a point of resonance). *Under the same general assumptions of Theorem* 1, *and* $A: X \to Y$ *a continuous bounded operator, if* (B_0) *there is a constant* $J_0 > 0$ *such that* $\| Nx \| \leqslant J_0$ *for all* $x \in X$; *and if* (N_ϵ) *there are constants* $R_0 \geqslant 0$, $\epsilon > 0$, $K > L\chi' J_0$ *such that* $\langle Q\, Nx, x^* \rangle \leqslant -\epsilon \| x^* \|$ [*or* $\langle Q\, Nx, x^* \rangle \geqslant \epsilon \| x^* \|$] *for all* $x \in X$, $x^* \in X_0$ *with* $Px = x^*$, $\| x^* \| \geqslant R_0$, $\| x - x^* \| \leqslant K$, *then there are also constants* $\alpha_0 > 0$, $C > 0$ *such that, for every real* α *with* $| \alpha | \leqslant \alpha_0$, *equation* $Ex + \alpha Ax = Nx$ *has at least a solution* $x \in \mathfrak{D}(E) \subset X$ *with* $\| x \| \leqslant C$.

THEOREM 3 (existence across a point of resonance). *Under the same general assumptions of Theorem* 2, *if* (B_k) *there are constants* $J_0 \geqslant 0$, $J_1 > 0$, $0 < k < 1$, *such that* $\| Nx \| \leqslant J_0 + J_1 \| x \|^k$ *for all* $x \in X$; *and if* $(N_{\epsilon k})$ *there are constants* $R_0 \geqslant 0$, $\epsilon > 0$, $K_0 > L\chi' J_0$, $K_1 > L\chi' J_1((\gamma')^{-1} k_0 \gamma_0^{-1})^k$ *such that* $\langle Q\, Nx, x^* \rangle \leqslant -\epsilon \| x^* \|^{1+k}$ [*or always* $\langle Q\, Nx, x^* \rangle \geqslant \epsilon \| x^* \|^{1+k}$] *for all* $x \in X$, $x^* \in X_0$ *with* $Px = x^*$, $\| x^* \| \geqslant R_0$, $\| x - x^* \| \leqslant K_0 + K_1 \| x \|^k$; *then, there are also constants* $\alpha_0 > 0$, $C > 0$ *such that, for every real* α *with* $| \alpha | \leqslant \alpha_0$, *equation* $Ex + \alpha Ax = Nx$ *has at least a solution* $x \in \mathfrak{D}(E) \subset X$ *with* $\| x \| \leqslant C$.

Both Theorems 2 and 3 are actually particular cases of a unique statement which contains also other cases of interest. Thus, by proving only Theorem 4 we give only one proof, instead of two separate and very similar ones.

Let $R_0 \geqslant 0$ denote the constant which will appear in the assumption (N_ϕ) below. Let σ_1, σ_2, σ be arbitrary constants, $0 < \sigma_1 < \sigma_2 < \sigma < \min[1, \gamma^{-1}]$, and let us consider two other positive constants

$$\lambda_0 \geqslant \max[1, (\gamma')^{-1} k_0], \quad \lambda_1 < \min[(L\chi')^{-1}(1 - \gamma\sigma), (\mu\chi)^{-1}(\sigma - \sigma_2)].$$

THEOREM 4 (existence across a point of resonance). *Under the same general assumptions of Theorem* 2, *let* $\phi(\zeta)$, $\phi_1(\zeta)$, $\psi(\zeta) \geqslant 0$, $0 \leqslant \zeta < +\infty$, *be monotone nondecreasing functions, both* ϕ_1 *and* ψ *positive for* $\zeta \geqslant R_0$. *Let us assume that* (B_ϕ) $\| Nx \| \leqslant \phi(\| x \|)$ *for all* $x \in X$; *and that* (N_ϕ) $\langle Q\, Nx, x^* \rangle \leqslant -\phi_1(\| x^* \|)$ [*or* $\langle Q\, Nx, x^* \rangle \geqslant \phi_1(\| x^* \|)$] *for all* $x \in X$, $x^* \in X_0$ *with* $Px = x^*$, $\| x^* \| \geqslant R_0$, $\| x - x^* \| \leqslant \psi(\| x \|)$. *Let us assume further that there is a constant* $S \geqslant \sigma_1^{-1} \lambda_0 R_0$ *with* $0 < \phi(S)/S < \lambda_1$, *and*

$$L\chi'\phi(S) < \psi(k_0^{-1} \gamma' \sigma_1 S). \tag{12}$$

Then, there is $\alpha_0 > 0$ *such that, for every real* $| \alpha | \leqslant \alpha_0$, *the equation* $Ex + \alpha Ax = Nx$ *has at least a solution* $x \in \mathfrak{D}(E) \subset X$ *with* $\| x \| \leqslant S$.

The same occurs even if (N_ϕ) *holds with* $\| x - x^* \| \leqslant \psi(\| x^* \|)$ *and* (12) *is replaced by*

$$L\chi'\phi(S) < \psi(\gamma' \sigma_1 S). \tag{12'}$$

Proof. The proof is similar to the ones for Theorems 1 and 1*. First we need determine constants R_1, R_2, R, S in such a way that

$$R_0 \leqslant R_1 < R_2 < R < S, \qquad R_0 \leqslant \gamma' R_1\,, \; R_0 \leqslant \gamma' k_0^{-1} R_1\,, \tag{13}$$

$$\gamma R + L\chi'\phi(S) < S, \tag{14}$$

$$R_2 + \mu\chi\phi(S) < R, \tag{15}$$

$$L\chi'\phi(S) < \psi(k_0^{-1}\gamma' R_1). \tag{16}$$

We take here $R_1 = \sigma_1 S$, $R_2 = \sigma_2 S$, $R = \sigma S$, and then $R_1 < R_2 < R < S$. Now $S \geqslant \sigma_1^{-1}\lambda_0 R_0$ and $k_0 \geqslant 1$ imply $S \geqslant \sigma_1^{-1} R_0$, $S \geqslant \sigma_1^{-1}(\gamma')^{-1} R_0$, $S \geqslant \sigma_1^{-1}(\gamma')^{-1} k_0 R_0$, and finally $R_1 = \sigma_1 S \geqslant R_0$, $\gamma' R_1 = \gamma'\sigma_1 S \geqslant R_0$, $\gamma' k_0^{-1} R_1 = \gamma' k_0^{-1}\sigma_1 S \geqslant R_0$. Thus relations (13) are satisfied. Since $\phi(S)/S \leqslant \lambda_1$ we have, as in the proof of Theorem 1*,

$$\gamma R/S + L\chi'\phi(S)/S < \gamma\sigma + (1 - \gamma\sigma) = 1,$$
$$R_2/S + \mu\chi\phi(S)/S < \sigma_2 + (\sigma - \sigma_2) = \sigma = R/S,$$

and relations (14), (15) are satisfied. Finally (16) is identical to (12).

Now we can determine $\alpha_0 > 0$ sufficiently small so that the following relations also hold:

$$\gamma R + L\chi'\phi(S) + L\chi'\alpha_0\omega(S) \leqslant S, \tag{17}$$

$$R_2 + \mu\chi\phi(S) + \mu\chi\alpha_0\omega(S) \leqslant R, \tag{18}$$

$$L\chi'\phi(S) + L\chi'\alpha_0\omega(S) \leqslant \psi(k_0^{-1}\gamma' R_1), \tag{19}$$

$$\alpha_0\chi\gamma R\omega(S) < \phi_1(\gamma' R_1). \tag{20}$$

Let $T\colon (x, c) \to (\bar{x}, \bar{c})$, or $\mathfrak{C} \to X \times R^m$, denote the transformation defined by

$$\begin{aligned} &T\colon \bar{x} = cw + H(I - Q)\tilde{N}x, \quad \bar{c} = c + g(\bar{x}, c), \\ &(x, c) \in \mathfrak{C} = [(x, c) \mid x \in X,\, c \in R^m,\, \|x\| \leqslant S,\, |c| \leqslant R], \end{aligned} \tag{21}$$

where $\tilde{N}x = -\alpha Ax + Nx$, where $x^* = cw = \sum_1^m c_i w_i = P\bar{x}$, $\bar{x}^* = \bar{c}w = \sum_1^m \bar{c}_i w_i$, $c, \bar{c} \in R^m$, and $g(\bar{x}, c) = (g_1, \ldots, g_m)$ is explicitely given below. Note that

$$\bar{x}^* = x^* + g(\bar{x}, c)w = x^* + \sum_1^m g_i w_i\,.$$

Here, for $0 \leqslant |c| \leqslant R_1$ we take $g(\bar{x}, c) = \langle Q\tilde{N}\bar{x}, w\rangle$. For $R_2 \leqslant |c| \leqslant R$ we take

$$g(\bar{x}, c) = \langle Q\tilde{N}\bar{x}, x^*\rangle\,\eta c, \qquad \text{with} \quad \eta = (\chi\gamma R(\alpha_0\omega(S) + \phi(S)))^{-1}.$$

For $R_1 \leqslant |c| \leqslant R_2$ we take

$$g(\bar{x}, c) = \lambda\langle Q\tilde{N}\bar{x}, w\rangle + (1-\lambda)\langle Q\tilde{N}\bar{x}, x^*\rangle \eta c,$$
$$\bar{c} = c + \lambda\langle Q\tilde{N}\bar{x}, w\rangle + (1-\lambda)\langle Q\tilde{N}\bar{x}, x^*\rangle \eta c,$$
$$\lambda = (R_2 - R_1)^{-1}(R_2 - |c|), \qquad 0 \leqslant \lambda \leqslant 1.$$

For any $(x, c) \in \mathfrak{C}$, we have $P\bar{x} = cw = x^*$, and

$$\begin{aligned}\|\bar{x} - x^*\| &= \|H(I-Q)\tilde{N}x\| \\ &= \|H(I-Q)[-\alpha Ax + Nx]\| \leqslant L\chi'(\alpha_0\omega(S) + \phi(S)).\end{aligned}$$

Since $\|x^*\| = \|cw\| \leqslant \gamma|c| \leqslant \gamma R$, by (17) we have $\|\bar{x}\| \leqslant \gamma R + L\chi'(\alpha_0\omega(S) + \phi(S)) \leqslant S$. For $R_1 \leqslant |c| \leqslant R$, we have $\|x^*\| = \|cw\| \geqslant \gamma'|c| \geqslant \gamma' R_1 \geqslant R_0$, $\|x^*\| = \|P\bar{x}\| \leqslant k_0\|\bar{x}\|$. Hence, by using (12), (19), we also have

$$\begin{aligned}\|\bar{x} - x^*\| &\leqslant L\chi'\alpha_0\omega(S) + L\chi'\phi(S) \\ &\leqslant \psi(\gamma' k_0^{-1}R_1) \leqslant \psi(\gamma' k_0^{-1}|c|) \\ &\leqslant \psi(k_0^{-1}\|x^*\|) \leqslant \psi(\|\bar{x}\|).\end{aligned}$$

Thus, $\langle QN\bar{x}, x^*\rangle \leqslant -\phi_1(\|x^*\|)$, and by using (20) also

$$\begin{aligned}\langle Q\tilde{N}\bar{x}, x^*\rangle &= \langle Q(-\alpha A\bar{x}), x^*\rangle + \langle QN\bar{x}, x^*\rangle \\ &\leqslant \chi\alpha_0\omega(S)\gamma R - \phi_1(\|x^*\|) \\ &\leqslant \chi\alpha_0\omega(S)\gamma R - \phi_1(\gamma' R_1) < 0;\end{aligned}$$

hence, $\langle Q\tilde{N}\bar{x}, x^*\rangle < 0$ for every $R_1 \leqslant |c| \leqslant R$. From (21) we see that $\bar{c} = c$, $\bar{x}^* = x^*$ if and only if $g(\bar{x}, c) = 0$. For $R_2 \leqslant |c| \leqslant R$ we have $\langle Q\tilde{N}\bar{x}, x^*\rangle < 0$ and hence $g \neq 0$. For $R_1 < |c| < R_2$ we have

$$g(\bar{x}, c)c = \sum_1^m g_i c_i = \lambda\langle Q\tilde{N}\bar{x}, x^*\rangle + (1-\lambda)\langle Q\tilde{N}\bar{x}, x^*\rangle \eta |c|^2,$$

where $\lambda > 0$, $1 - \lambda > 0$, $\langle Q\tilde{N}\bar{x}, x^*\rangle < 0$ and again $g \neq 0$. Thus a fixed point (x, c) for T may occur only for $|c| \leqslant R_1$, and $Q\tilde{N}x = 0$.

Let us prove that T maps $\mathfrak{C}$ into itself. First, for $(x, c) \in \mathfrak{C}$ we have $x^* = cw$, $\|x^*\| = \|cw\| \leqslant \gamma|c| \leqslant \gamma R$, and by using (17) also

$$\|\bar{x}\| \leqslant \|cw\| + \|H(I-Q)\tilde{N}x\| \leqslant \gamma R + L\chi'\alpha_0\omega(S) + L\chi'\phi(S) \leqslant S.$$

For $|\bar{c}| \leqslant R_1$ we have by using (18)

$$|\bar{c}| \leqslant |c| + |\langle Q\tilde{N}\bar{x}, w\rangle| \leqslant R_1 + \mu\chi\alpha_0\omega(S) + \mu\chi\phi(S) \leqslant R.$$

For $R_2 \leqslant |c| \leqslant R$ we have

$$\bar{c} = \{1 + \langle Q\tilde{N}\bar{x}, x^*\rangle\eta\}c = \Lambda c,$$

where Λ denotes the number in braces, $\langle Q\tilde{N}\bar{x}, x^*\rangle < 0$ and

$$\eta\,|\langle Q\tilde{N}\bar{x}, x^*\rangle| \leqslant \eta\chi(\alpha_0\omega(S) + \phi(S))\,\gamma R = 1.$$

Thus $0 \leqslant \Lambda < 1$, $\bar{c}$ is on the segment from the origin to c, and $|\bar{c}| \leqslant |c|$. For $R_1 < |c| < R_2$ we have $0 < \lambda < 1$,

$$\bar{c} = \lambda c + \lambda\langle Q\tilde{N}\bar{x}, w\rangle + (1-\lambda)\{1 + \langle Q\tilde{N}\bar{x}, x^*\rangle\eta\}c,$$

and by using (18) also

$$\begin{aligned}|\bar{c}| &\leqslant \lambda\,|c| + \mu\chi(\alpha_0\omega(S) + \phi(S)) + (1-\lambda)\,|c| \\ &\leqslant R_2 + \mu\chi(\alpha_0\omega(S) + \phi(S)) \leqslant R.\end{aligned}$$

We have proved that $T\colon \mathfrak{C} \to \mathfrak{C}$ maps $\mathfrak{C}$ into itself. The proof of the compactness of T is the same as for Theorem 1. Since $\mathfrak{C}$ is convex and closed in $X \times R^m$, by Schauder's fixed point theorem we conclude that there is at least one fixed point $(x, c) = T(x, c)$ in $\mathfrak{C}$. Theorem 4 is thereby proved.

It remains to show that the conditions of Theorem 4 can be easily satisfied, and that in particular they are satisfied in the situations of Theorems 2 and 3, and in other relevant cases.

For the sake of simplicity, we shall consider below only the first one of the two alternatives in assumption (N_ϕ).

(a) First, let us prove that if (B_ϕ) and $N_\phi)$ of Theorem 4 hold with ϕ satisfying $\phi(\zeta)/\zeta \to 0$ as $\zeta \to +\infty$, with an arbitrary $\phi_1(\zeta)$ as stated, and any ψ satisfying

$$\psi(\zeta) > L\chi'\phi((\gamma')^{-1}\,k_0\sigma_1^{-1}\zeta), \tag{22}$$

then all conditions of Theorem 4 hold.

Indeed, inequality (22) implies that relation (12) holds for all S. Thus, it is enough to determine S in such a way that $S \geqslant \sigma_1^{-1}\lambda_0 R_0$ and $0 < \phi(S)/S < \lambda_1$.

(b) Let us assume that (B_0) and (N_ϵ) hold, that is, the conditions of Theorem 2. Let us prove that the conditions of Theorem 3 hold.

Here we have

$$\phi(\zeta) = J_0 > 0, \qquad \phi_1(\zeta) = \epsilon\zeta, \qquad \psi(\zeta) = K_0$$

for some constants $\epsilon > 0$ and $K_0 > L\chi' J_0$. Then relation (22) reduces here to $L\chi' J_0 < K_0$, which is satisfied by hypothesis. Since $\phi(\zeta)/\zeta = J_0/\zeta \to 0$ as $\zeta \to +\infty$, we have only to determine $S \geqslant \sigma_1^{-1}\lambda_0 R_0$ satisfying $J_0/S < \lambda_1$.

(c) Let us assume that (B_k) and $(N_{\epsilon k})$ hold, that is, the conditions of Theorem 3. Let us prove that the conditions of Theorem 4 hold.

First we note that $K_1 > L\chi' J_1((\gamma')^{-1}\,k_0\gamma_0)^k$; hence, there is some number

σ_1, $0 < \sigma_1 < \gamma_0 = \min[1, \gamma^{-1}]$, so close to γ_0, so that we also have $K_1 > L\chi' J_1((\gamma')^{-1} k_0 \sigma_1^{-1})^k$. We then take constants σ_2, σ so that $0 < \sigma_1 < \sigma_2 < \sigma < \gamma_0 = \min[1, \gamma^{-1}]$, and we take λ_0, λ_1 accordingly as stated.

Here we have

$$\phi(\zeta) = J_0 + J_1\zeta^k, \quad J_0 \geqslant 0, \quad J_1 > 0, \quad 0 < k < 1,$$
$$\phi_1(\zeta) = \epsilon\zeta^{1+k}, \quad \epsilon > 0,$$
$$\psi(\zeta) = K_0 + K_1\zeta^k, \quad K_0 > L\chi' J_0, \quad K_1 > L\chi' J_1((\gamma')^{-1} k_0 \sigma_1^{-1})^k.$$

Now relation (22) reduces to

$$K_0 + K_1\zeta^k > L\chi'[J_0 + J_1((\gamma')^{-1} k_0 \sigma_1^{-1}\zeta)^k],$$

and this is true for every $\zeta \geqslant 0$ since $K_0 > L\chi' J_0$ and $K_1 > L\chi' J_1((\gamma')^{-1} k_0 \sigma_1^{-1})^k$. Thus, (12) is true for every S. Here $\phi(\zeta)/\zeta \to 0$ as $\zeta \to +\infty$, and all we have to do is to determine $S \geqslant \sigma_1^{-1}\lambda_0 R_0$ satisfying $\phi(S)/S < \lambda_1$.

(d) Let us assume that a relation (B_ϕ) holds with $\phi(\zeta) = J_0 + J_1\zeta^k$ for constants $k \geqslant 1$, $J_0 > 0$ fixed, and J_1 sufficiently small, and that (N_ϕ) holds with $\phi_1(\zeta) = \epsilon\zeta$ and $\psi(\zeta) = K_0 > L\chi' J_0$. Let us prove that for $J_1 > 0$ sufficiently small, all conditions of Theorem 4 hold. Indeed we take S so that $S \geqslant \sigma_1^{-1}\lambda_0 R_0$ and $J_0/S < \lambda_1$. Then we can determine $J_1 > 0$ so small that we also have $\phi(S)/S = J_0/S + J_1 S^{k-1} < \lambda_1$.

Remark. In Theorems 2, 3, 4 the term αAx, (with $A: X \to Y$, $\| Ax \| \leqslant \omega(\| x \|)$, ω monotone nondecreasing), could be replaced by $A_\alpha x$, $A_\alpha : X \to Y$, depending on a vector valued α, (with $\| A_\alpha x \| \leqslant \omega(\alpha, \| x \|)$, $\omega(\alpha, \zeta)$ monotone nondecreasing in ζ, $\omega(\alpha, \zeta) \to 0$ as $\alpha \to 0$ uniformly in any $[0, \zeta]$.

5. Nonlinear Oscillations

We consider here the ordinary differential equation

$$x'' + m^2 x + \alpha g(t, x) = p(t) + h(x), \tag{23}$$

where x is a scalar, m is an integer, g, p, h are continuous functions, and g, p are 2π-periodic in t. This is a stronger form of the Lazer and Leach theorem:

5(i) *If* $| h(x)| \leqslant M$ *for all real* x *and some constant* M, *and if there are constants* $c < d$, $C < D$ *such that* $h(x) \leqslant C$ *for* $x \leqslant c$, *and* $h(x) \geqslant D$ *for* $x \geqslant d$, *and* $(A^2 + B^2)^{1/2} < 2(D - C)$, *where* $A = \int_0^{2\pi} p(t) \cos mt\, dt$, $B = \int_0^{2\pi} p(t) \sin mt\, dt$, *then there are constants* $\alpha_0 > 0$, $\Lambda > 0$ *such that, for every* $| \alpha | \leqslant \alpha_0$, *Eq.* (23) *has at least a* 2π*-periodic solution* $x(t)$ $-\infty < t < +\infty$, *with* $| x(t)| \leqslant \Lambda$. *The roles of the inequalities* $h(x) \leqslant C$, $h(x) \geqslant D$ *could be exchanged.*

Proof. We shall write (23) in the form $Ex + \alpha Ax = Nx$, where E is the differential operator $Ex = x'' + m^2x$, with boundary conditions $x(0) = x(2\pi)$, $x'(0) = x'(2\pi)$, and where $Ax = g(t, x(t))$, $Nx = p(t) + h(x(t))$. Let X denote the space of all 2π-periodic functions $x(t)$ which are continuous in $(-\infty, +\infty)$, and absolutely continuous (AC) in $[0, 2\pi]$ with derivative $x' \in L_2[0, 2\pi]$. Thus X is a Sobolev space H^1, a Hilbert space, with usual inner product, and norm $\| x \|_1$, or $\| x \|_X$. Let $\mathfrak{D}(E) \subset X$ denote the set of all functions $x \in X$ which are continuous in $(-\infty, +\infty)$ and AC in $[0, 2\pi]$ together with x', and with $x'' \in L_2[0, 2\pi]$. Let $Y = L_2[0, 2\pi]$ with usual inner product and square norm $\| y \|$, the functions $y \in Y$ extended to $(-\infty, +\infty)$ by 2π-periodicity. Because of the continuity hypotheses on h and g we see that $A\colon X \to Y$, $N\colon X \to Y$, and that A and N are continuous as operators from X into Y. Moreover $E\colon \mathfrak{D}(E) \to Y$. For $x \in Y, y \in Y$ we take $\langle y, x \rangle = \int_0^{2\pi} y(t)\, x(t)\, dt$, so that $|\langle y, x \rangle| \leqslant \| x \| \, \| y \| \leqslant \| x \|_1 \, \| y \|$. Let X_0, Y_0 be the spaces spanned by $\cos mt$, $\sin mt$, and P, Q be the usual projections of X and Y onto X_0, Y_0, and let $X_1 = (I - P)X$, $Y_1 = (I - Q)Y$. Here Q is an orthogonal projection, $Y \to Y$; hence $\| Q \| = \| I - Q \| = 1$ (or $\chi = \chi' = 1$). Every element $y \in Y_1$ has the Fourier representation

$$y(t) = (1/2)a_0 + \sum_{k \geqslant 1, k \neq m} (a_k \cos kt + b_k \sin kt),$$

and we define $H\colon Y_1 \to \mathfrak{D}(E) \cap X_1$ by taking

$$Hy = (1/2m^2)a_0 + \sum_{k \geqslant 1, k \neq m} (m^2 - k^2)^{-1} (a_k \cos kt + b_k \sin kt).$$

Thus, H is a bounded map from Y_1 into H^2, hence, a compact map from Y_1 into X_1. Let $L = \| H \|$. We should note here that in X the two norms are equivalent

$$\| x \|_1 = \| x \| + \| x' \|, \qquad \| x \|_1' = \| x' \| + \operatorname*{Sup}_t | x(t) |.$$

In the finite-dimensional subspace X_0 the norms $\| x \|_1$, $\| x \|_1'$ and $\| x \|$ are of course equivalent.

Because of the boundedness of $h\colon R^1 \to R^1$ we see that $\| Nx \| \leqslant J_0$ for some constant J_0 and all $x \in X$. Moreover, if we define the constant μ by taking

$$2\mu = \pi^{-1/2}[2(D - C) - (A^2 + B^2)^{1/2}],$$

then N has the relevant property: $\langle Nx, x^* \rangle \leqslant -\mu \| x^* \|$ (square norm), for all $x \in X$, $x^* = Px$, with $\| x^* \| \geqslant R_1$ and $\| x - x^* \|_X \leqslant K_0$ for suitable constants R_1 and K_0. Namely, if we take an arbitrary constant $K_0 > L\chi' J_0$, then $\| x - x^* \|_X \leqslant K_0$ implies $| x(t) - x^*(t) | \leqslant L_0$, for some constant L_0 which depends on K_0 but not on x, or x^*. Then, an argument similar to the one of Lazer's and Leach's proof shows that we can determine R_1 so that the relations above hold (see, e.g., [5] for details). Statement 5(i) is now a corollary of Theorem 2.

Remark. In statement 5(i) we could as well assume only that $p \in L_2[0, 2\pi]$. We could also assume that $h(t, x)$ is a function $R^2 \to R^1$ which is 2π-periodic in t, continuous in x for a.a. t, measurable in t for every k, satisfying $h(t, s) \leqslant C$ for all t and $s \leqslant c$; $h(t, s) \geqslant D$ for all t and $s \geqslant d$, and also satisfying

$$| h(t, s)| \leqslant H(t), \quad | h(t, s_1) - h(t, s_2)| \leqslant \sigma(\eta, \zeta) H_1(t),$$

for all t, s, s_1, s_2 real with $| s_1 - s_2 | \leqslant \eta$, $| s_1 |, | s_2 | \leqslant \zeta$, where $H, H_1 \in L_2[0, 2\pi]$, and $\sigma(\eta, \zeta) \to 0$ as $\eta \to 0$ uniformly for ζ in any $[0, \zeta_0]$. Analogously, we could assume that $g(t, x, x')$ is a function $R^3 \to R^1$ which is 2π-periodic in t, continuous in (x, x') for a.a. t, measurable in t for every (x, x'), satisfying

$$| g(t, s, u)| \leqslant G_1(t) + \phi(| s |) G_2(t) + | u | G_3(t),$$
$$| g(t, s_1, u_1) - g(t, s_2, u_2)| \leqslant \sigma(\eta, \zeta) G_2(t) + | u_1 - u_2 | G_3(t),$$

for all t, s_1, s_2, u_1, u_2 real with $| s_1 - s_2 | \leqslant \eta$, $| s_1 |, | s_2 | \leqslant \zeta$, and σ as above, and where $G_1, G_2, G_3 \in L_2[0, 2\pi]$. Indeed, under these hypotheses, the operators $Ax = g(t, x(t), x'(t))$ and $Nx = p(t) + h(t, x(t))$ map X into Y, and are continuous as maps from X into Y.

For instance the equations

(a) $x'' + x + x^2 \sin t = \cos t + 2 \arctan x$, (with $A = B = \pi$, $D = 3/2$, $C = -3/2$);

(b) $x'' + x + \beta x' + \alpha x = \cos t + 2 \arctan x + e^{-x^2}$, $| \alpha |, | \beta |$ small;

(c) $x'' + x + \alpha\varphi(t)x^2 + \beta\varphi(t)x' = \cos t + 2 \arctan(x + \sin t) + e^{-x^2}$, $| \alpha |$, $| \beta |$ small, $\varphi(t) = t^{-1/3}$ for $0 < t \leqslant 2\pi$, $\varphi(t + 2\pi) = \varphi(t)$;

all satisfy the conditions above and have therefore 2π-periodic solutions.

Statement 5(i) will be extended elsewhere to arbitrary differential operators $Ex = x^{(n)} + a_1(t)x^{(n-1)} + \cdots + a_n(t)x$, $a_1, \ldots, a_n$ continuous and 2π-periodic, for which the homogeneous equation $Ex = 0$ possesses nontrivial 2π-periodic solutions.

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