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An Asymptotic Estimate Related to Selberg's Sieve

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Suppose $1 \leqslant z_1 \leqslant z_2 \leqslant N$, and let $\Lambda_i(d) = \mu(d) \max(\log(z_i/d), 0)$ for $i = 1, 2$. We show that

$$\sum_{N \leqslant n} \Big(\sum_{d|n} \Lambda_1(d)\Big)\Big(\sum_{e|n} \Lambda_2(e)\Big) = N \log z_1 + O(N).$$

We then use this to improve a result of Barban–Vehov which has applications to zero-density theorems.

Let $\lambda_0(d)$ denote a real-valued arithmetic function with the property that $\lambda_0(1) = 1$, $\lambda_0(d) = 0$ for $d \geqslant z_0$. If all prime divisors of a number n exceed z_0, then $\sum_{d|n} \lambda_0(d) = 1$, so consequently

$$\pi(N) - \pi(z_0) \leqslant \sum_{1 \leqslant n \leqslant N} \Big(\sum_{d|n} \lambda_0(d)\Big)^2.$$

The quadratic form on the right is asymptotically minimized, subject to the constraints on λ_0, by taking

$$\lambda_0(d) = \mu(d) d \Big(\sum_{\substack{q \leqslant z_0 \\ d|q}} \frac{\mu^2(q)}{\varphi(q)}\Big) \Big/ \Big(\sum_{q \leqslant z_0} \frac{\mu^2(q)}{\varphi(q)}\Big). \tag{1}$$

But it is well known [5] that

$$\sum_{q \leqslant Q} \frac{\mu^2(q)}{\varphi(q)} = \log Q + O(1); \tag{2}$$

hence the choice (1) suggests that we might instead take

$$\lambda_0(d) = \mu(d)(\log(z_0/d))/(\log z_0).$$

Indeed, with either of these choices for λ_0, it is not difficult to show that

$$\sum_{M+1}^{M+N} \Big(\sum_{d|n} \lambda_0(d)\Big)^2 = \frac{N}{\log z_0} + O\Big(\frac{N + z_0^2}{\log^2 z_0}\Big),$$

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uniformly in M, and we thus obtain an upper bound for $\pi(N+M)-\pi(M)$. This estimate becomes imprecise if z_0 exceeds $N^{1/2}$, and this seems to be in the nature of things, but when $M=0$ we obtain an asymptotic expansion which is sharp for all values of z_0.

We suppose throughout that $1 \leqslant z_1 \leqslant z_2$, and we let

$$\begin{aligned} \Lambda_i(d) &= \mu(d)\log(z_i/d) && \text{if } d \leqslant z_i, \\ &= 0 && \text{if } d > z_i. \end{aligned} \tag{3}$$

THEOREM. *Let* $1 \leqslant z_1 \leqslant z_2 \leqslant N$. *Then*

$$\sum_{1\leqslant n\leqslant N} \Big(\sum_{d\mid n} \Lambda_1(d)\Big)\Big(\sum_{e\mid n} \Lambda_2(e)\Big) = N\log z_1 + O(N).$$

The supposition $z_2 \leqslant N$ occasions no real loss of generality, since the value of $\sum_{d\mid n} \Lambda_i(d)$ is independent of z_i when $1 < n \leqslant z_i$.

If $z_1 < z_2$, we define

$$\begin{aligned} \lambda_d &= \{\Lambda_2(d) - \Lambda_1(d)\}/\log(z_2/z_1) \\ &= \mu(d) && \text{if } 1 \leqslant d \leqslant z_1, \\ &= \mu(d)(\log(z_2/d)/\log(z_2/z_1)) && \text{if } z_1 < d \leqslant z_2, \\ &= 0 && \text{if } d > z_2. \end{aligned} \tag{4}$$

Barban and Vehov [1] proved

$$\sum_{1\leqslant n\leqslant N} \Big(\sum_{d\mid n} \lambda_d\Big)^2 \ll \frac{N}{\log(z_2/z_1)}. \tag{5}$$

Their proof was sketchy, but Motohashi [6] has supplied the details. We use the above theorem to prove the following

COROLLARY. *Suppose* $1 \leqslant z_1 < z_2$. *If* $z_2 \leqslant N$, *then*

$$\sum_{1\leqslant n\leqslant N} \Big(\sum_{d\mid n} \lambda_d\Big)^2 = \frac{N}{\log(z_2/z_1)} + O\Big(\frac{N}{\log^2(z_2/z_1)}\Big).$$

If $z_1 < N < z_2$, *then*

$$\sum_{1\leqslant n\leqslant N} \Big(\sum_{d\mid n} \lambda_d\Big)^2 = \frac{N\log(N/z_1)}{\log^2(z_2/z_1)} + O\Big(\frac{N}{\log^2(z_2/z_1)}\Big).$$

The estimate (5) has been used by Motohashi [7] and Jutila [4] to prove zero-density theorems for L-functions which are sensitive near $\sigma = 1$. In a later paper, we use the corollary to prove a quantitative form of Jutila's result, and thus obtain a new estimate for the constant in Linnik's theorem on the least prime in an arithmetic progression.

We employ the usual notation for the classical arithmetic functions, and in addition we put

$$\sigma_a(n) = \sum_{d \mid n} d^a, \qquad \kappa(n) = n \prod_{p \mid n} \left(1 + \frac{1}{p}\right).$$

Note that without subscripts, $\lambda(n)$ and $\Lambda(n)$ denote the arithmetic functions of Liouville and von Mangoldt, respectively.

2. Preparatory Lemmas

At various points we appeal to weak quantitative forms of the prime number theorem. For convenience, we present them in advance.

For any $A > 0$,

$$\sum_{q \leqslant Q} \mu(q) \ll_A Q(\log 2Q)^{-A}, \tag{6}$$

$$\sum_{q \leqslant Q} \mu(q)\, q^{-1} \ll_A (\log 2Q)^{-A}, \tag{7}$$

$$\sum_{q \leqslant Q} \frac{\mu(q) \log q}{q} = -1 + O_A((\log 2Q)^{-A}), \tag{8}$$

$$\sum_{p \leqslant Q} \log p = Q + O_A(Q(\log 2Q)^{-A}). \tag{9}$$

Lemma 1. *If $a < 0$ then*

$$\sum_{q \leqslant Q} \sigma_a(q) \leqslant \sum_{q \leqslant Q} \sigma_a^2(q) \ll_a Q.$$

Bellman [2] and Halberstam [3] have proved results which are similar but much stronger. Here we give a simple proof which is sufficient for our purposes.

Proof. The first inequality is trivial. For the second inequality, we have

$$\sum_{q \leqslant Q} \sigma_a^2(q) = \sum_{q \leqslant Q} \sum_{d,e \mid q} d^a e^a \leqslant Q \sum_{d,e} d^a e^a [d,e]^{-1}$$

$$\leqslant Q \sum_{d,e} [d,e]^{-1+a} \leqslant Q \sum_{n} d^2(n)\, n^{-1+a} \ll_a Q.$$

LEMMA 2. *For any integer r and any $A > 0$,*

$$\sum_{\substack{n\leqslant Q\\(n,r)=1}} \frac{\mu(n)}{n}\log\left(\frac{Q}{n}\right) = \frac{r}{\varphi(r)} + O_A(\sigma_{-1/2}(r)\log^{-A} 2Q), \tag{10}$$

$$\sum_{\substack{n\leqslant Q\\(n,r)=1}} \frac{\mu(n)}{\kappa(n)}\log\left(\frac{Q}{n}\right) = \frac{\pi^2}{\sigma}\frac{\kappa(r)}{r} + O_A(\sigma_{-1/2}(r)\log^{-A} 2Q), \tag{11}$$

$$\sum_{\substack{n\leqslant Q\\(n,r)=1}} \frac{\mu(n)}{\kappa(n)} \ll_A \sigma_{-1/2}(r)\log^{-A} 2Q. \tag{12}$$

Proof. Let

$$\mathscr{D} = \{d\colon p \mid d \Rightarrow p \mid r\}.$$

We see easily that

$$\begin{aligned}\sum_{\substack{d\in\mathscr{D}\\d\mid n}} \mu(n/d) &= \mu(n) \qquad \text{if} \quad (n,r)=1,\\ &= 0 \qquad \text{if} \quad (n,r)>1.\end{aligned}$$

Hence

$$\sum_{\substack{n\leqslant Q\\(n,r)=1}} \frac{\mu(n)}{n}\log\left(\frac{Q}{n}\right) = \sum_{d\in\mathscr{D}}\frac{1}{d}\sum_{m\leqslant Q/d}\frac{\mu(m)}{m}\log\left(\frac{Q}{md}\right).$$

By (7) and (8), the above is

$$\begin{aligned}&= \sum_{\substack{d\in\mathscr{D}\\d\leqslant Q}} d^{-1}\left\{1 + O_A\left(\log^{-A}\left(\frac{2Q}{d}\right)\right)\right\}\\ &= \sum_{d\in\mathscr{D}} d^{-1} + O\Big(\sum_{\substack{d\in\mathscr{D}\\d>Q}} d^{-1}\Big) + O_A\Big(\sum_{\substack{d\in\mathscr{D}\\d\leqslant Q}} d^{-1}\log^{-A}\left(\frac{2Q}{d}\right)\Big).\end{aligned}$$

Here the first term is $=r/\varphi(r)$. The first error term is

$$\begin{aligned}&\ll Q^{-1/3}\sum_{d\in\mathscr{D}} d^{-2/3}\\ &= Q^{-1/3}\prod_{p\mid r}(1 + (p^{2/3}-1)^{-1})\\ &\ll (\log 2Q)^{-A}\,\sigma_{-1/2}(r).\end{aligned}$$

The last inequality follows from the fact that

$$1 + (p^{2/3}-1)^{-1} \leqslant 1 + p^{-1/2}$$

for all but finitely many p. In the second error term, we consider separately those $d \leqslant Q^{1/2}$ and $Q^{1/2} < d \leqslant Q$. In the first place,

$$\sum_{\substack{d \in \mathscr{D} \\ d \leqslant Q^{1/2}}} \ll (\log 2Q)^{-A} \sum_{d \in \mathscr{D}} d^{-1} \ll (\log 2Q)^{-A}\, \sigma_{-1/2}(r),$$

while on the other hand,

$$\sum_{\substack{d \in \mathscr{D} \\ Q^{1/2} < d \leqslant Q}} \ll Q^{-1/6} \sum_{d \in \mathscr{D}} d^{-2/3} \ll_A (\log 2Q)^{-A}\, \sigma_{-1/2}(r).$$

To prove (11), we use the identity

$$\frac{1}{\kappa(n)} = \frac{1}{n} \sum_{d \mid n} \frac{\mu(d)}{\kappa(d)}.$$

This implies

$$\sum_{\substack{n \leqslant Q \\ (n,r)=1}} \frac{\mu(n)}{\kappa(n)} \log\left(\frac{Q}{n}\right)$$

$$= \sum_{\substack{d \leqslant Q \\ (d,r)=1}} \frac{\mu(d)}{\kappa(d)} \sum_{\substack{n \leqslant Q \\ (n,r)=1, d \mid n}} \frac{\mu(n)}{n} \log\left(\frac{Q}{n}\right)$$

$$= \sum_{\substack{d \leqslant Q \\ (d,r)=1}} \frac{\mu^2(d)}{d\kappa(d)} \sum_{\substack{m \leqslant Q/d \\ (m,dr)=1}} \frac{\mu(m)}{m} \log\left(\frac{Q}{md}\right).$$

By (10), this is

$$= \frac{r}{\varphi(r)} \sum_{\substack{d \leqslant Q \\ (d,r)=1}} \frac{\mu^2(d)}{\kappa(d)\,\varphi(d)} + O_A\left(\sigma_{-1/2}(r) \sum_{d \leqslant Q} \frac{\sigma_{-1/2}(d)}{d^2} \log^{-A}\left(\frac{2Q}{d}\right)\right)$$

$$= \frac{r}{\varphi(r)} \prod_{p \nmid r} (1 - p^{-2})^{-1} + O_A\left(\frac{r}{\varphi(r)} \sum_{d > Q} d^{-2}\right)$$

$$+ O_A\left(\sigma_{-1/2}(r) \sum_{d \leqslant Q} d^{-3/2} \log^{-A}\left(\frac{2Q}{d}\right)\right)$$

$$= \frac{\kappa(r)}{r} \frac{\pi^2}{6} + O_A(\sigma_{-1/2}(r) \log^{-A} 2Q),$$

and this proves (11).

To prove (12), we note that for any $c > 1$,

$$(\log c) \sum_{\substack{n \leqslant Q \\ (n,r)=1}} \frac{\mu(n)}{\kappa(n)}$$

$$= \sum_{\substack{n \leqslant Qc \\ (n,r)=1}} \frac{\mu(n)}{\kappa(n)} \log\left(\frac{Qc}{n}\right) - \sum_{\substack{n \leqslant Q \\ (n,r)=1}} \frac{\mu(n)}{\kappa(n)} \log\left(\frac{Q}{n}\right)$$

$$- \sum_{\substack{Q < n \leqslant Qc \\ (n,r)=1}} \frac{\mu(n)}{\kappa(n)} \log\left(\frac{Qc}{n}\right). \tag{14}$$

We take $c = 1 + \log^{-A} 2Q$. By (11) (with A replaced by $2A$) the difference between the first two sums on the right-hand side of (14) is

$$\ll_A \sigma_{-1/2}(r) \log^{-2A} 2Q.$$

The third sum is

$$\ll (\log c) \sum_{Q < n \leqslant Qc} n^{-1} \ll \log^2 c.$$

Since $\log c \ll \log^{-A} 2Q$, this completes the proof of (12).

LEMMA 3. *For any integer r,*

$$\sum_{\substack{n \leqslant Q \\ (n,r)=1}} \mu^2(n) = \frac{6}{\pi^2} \frac{r}{\kappa(r)} Q + O(Q^{1/2}\sigma_{-1/3}(r)).$$

Proof. Let $\mathscr{D}$ be as in (13). We see easily that

$$\sum_{\substack{d \in \mathscr{D} \\ dm=n}} \lambda(d)\, \mu^2(m) = \mu^2(n) \quad \text{if} \quad (n,r) = 1,$$
$$= 0 \quad \text{if} \quad (n,r) > 1.$$

Hence the expression in the lemma is

$$= \sum_{d \in \mathscr{D}} \lambda(d) \sum_{m \leqslant Q/d} \mu^2(m)$$

$$= \frac{6}{\pi^2} Q \sum_{d \in \mathscr{D}} \frac{\lambda(d)}{d} + O\left(Q^{1/2} \sum_{d \in \mathscr{D}} d^{-1/2}\right)$$

$$= \frac{6}{\pi^2} \frac{r}{\kappa(r)} Q + O\left(Q^{1/2} \prod_{p \mid r} (1 + (p^{1/2} - 1)^{-1})\right)$$

$$= \frac{6}{\pi^2} \frac{r}{\kappa(r)} Q + O(Q^{1/2}\sigma_{-1/3}(r)).$$

3. Proof of the Theorem for $z_1 z_2 \leqslant N$

It is clearly immaterial to the result whether the range of summation is $1 < n \leqslant N$ or $1 \leqslant n \leqslant N$; we work with the former range, since it allows a slight simplification in the proof.

If $z_1 z_2 \leqslant N$, then

$$\sum_{1<n\leqslant N} \Big(\sum_{d|n} \Lambda_1(d)\Big)\Big(\sum_{e|n} \Lambda_2(e)\Big)$$

$$= (N-1) \sum_{d,e} \frac{\Lambda_1(d)\,\Lambda_2(e)}{[d,e]} + O\Big(\sum_{d,e} |\Lambda_1(d)|\,|\Lambda_2(e)|\Big). \tag{15}$$

The error term is

$$\ll \sum_{d\leqslant z_1} \log\left(\frac{z_1}{d}\right) \sum_{e\leqslant z_2} \log\left(\frac{z_2}{e}\right) \ll z_1 z_2 \ll N. \tag{16}$$

For the main term, we have

$$\sum_{d,e} \frac{\Lambda_1(d)\,\Lambda_2(e)}{[d,e]}$$

$$= \sum_{d,e} \frac{\Lambda_1(d)\,\Lambda_2(e)}{de} \sum_{r|(d,e)} \varphi(r)$$

$$= \sum_{r\leqslant z_1} \frac{\mu^2(r)\,\varphi(r)}{r^2} \Big(\sum_{\substack{m\leqslant z_1/r\\(m,r)=1}} \frac{\mu(m)}{m} \log\left(\frac{z_1}{mr}\right)\Big)\Big(\sum_{\substack{n\leqslant z_2/r\\(n,r)=1}} \frac{\mu(n)}{n} \log\left(\frac{z_2}{nr}\right)\Big).$$

By (10), the above is

$$= \sum_{r\leqslant z_1} \frac{\mu^2(r)\,\varphi(r)}{r^2} \left\{\frac{r}{\varphi(r)} + O\left(\sigma_{-1/2}(r) \log^{-2}\left(\frac{2z_1}{r}\right)\right)\right\}^2$$

$$= \sum_{r\leqslant z_1} \frac{\mu^2(r)}{\varphi(r)} + O\Big(\sum_{r\leqslant z_1} \frac{\sigma_{-1/2}(r)}{r} \log^{-2}\left(\frac{2z_1}{r}\right)\Big)$$

$$+ O\Big(\sum_{r\leqslant z_1} \frac{\sigma^2{}_{1/2}(r)}{r} \log^{-2}\left(\frac{2z_1}{r}\right)\Big). \tag{17}$$

The first error term is clearly majorized by the second, which is

$$\ll z_1^{-1} \sum_k k^{-2} \cdot 2^k \sum_{2^{-k}z_1 < r \leqslant 2^{-k+1}z_1} \sigma^2_{-1/2}(r),$$

and by Lemma 1, this is

$$\ll \sum_k k^{-2} \ll 1. \tag{18}$$

The theorem in the case $z_1 z_2 \leqslant N$ follows from (2) and (15)–(18).

4. The case $z_1 z_2 > N$

The estimate of the remainder term given in (16) does not allow one to use the proof of the previous section when $N = o(z_1 z_2)$. Indeed, some well-known examples of Selberg suggest that one must use special properties of the integers if one is to extend the theorem to this case.

We will make essential use of the relation

$$\sum_{d \mid n} \mu(d) \log\left(\frac{z}{d}\right) = \Lambda(n) \qquad \text{if} \quad n > 1,$$
$$= \log z \qquad \text{if} \quad n = 1.$$

It follows that for $n > 1$,

$$\sum_{d \mid n} \Lambda_i(d) = \Lambda(n) - \sum_{\substack{d \mid n \\ z_i < d}} \mu(d) \log\left(\frac{z_i}{d}\right)$$
$$= \Lambda(n) + \sum_{\substack{d \mid n \\ d < n/z_i}} \mu\left(\frac{n}{d}\right) \log\left(\frac{n}{dz_i}\right).$$

This implies

$$\sum_{1 < n \leqslant N} \Big(\sum_{d \mid n} \Lambda_1(d)\Big)\Big(\sum_{e \mid n} \Lambda_2(e)\Big)$$
$$= \sum_{1 < n \leqslant N} \Lambda^2(n)$$
$$+ \sum_{1 < n \leqslant N} \Lambda(n) \Big\{ \sum_{\substack{d \mid n \\ d < n/z_1}} \mu\left(\frac{n}{d}\right) \log\left(\frac{n}{dz_1}\right) + \sum_{\substack{e \mid n \\ e < n/z_2}} \mu\left(\frac{n}{e}\right) \log\left(\frac{n}{ez_2}\right) \Big\}$$
$$+ \sum_{1 < n \leqslant N} \Big(\sum_{\substack{d \mid n \\ d < n/z_1}} \mu\left(\frac{n}{d}\right) \log\left(\frac{n}{dz_1}\right)\Big) \Big\{ \sum_{\substack{e \mid n \\ e < n/z_2}} \mu\left(\frac{n}{e}\right) \log\left(\frac{n}{ez_2}\right)\Big\}. \tag{19}$$

By (9) and partial summation, the first sum on the right-hand side of (19) is $= N \log N + O(N)$. In the second sum, we may write $n = p^k$.

For such n, we find that

$$\sum_{\substack{d|n \\ d<n/z_i}} \mu\left(\frac{n}{d}\right) \log\left(\frac{n}{dz_i}\right) = -\log\left(\frac{p}{z_i}\right) \quad \text{if} \quad z_i < p,$$
$$= 0 \quad \text{if} \quad p \leqslant z_i.$$

Thus the second sum on the right-hand side of (19) is

$$= -\sum_{z_1<p\leqslant N} (\log p)\left(\log\left(\frac{p}{z_1}\right)\right) - \sum_{z_2<p\leqslant N} (\log p)\left(\log\left(\frac{p}{z_2}\right)\right)$$
$$+ O(N^{1/2}\log 2N);$$

here the contribution of those $n = p^k$ with $k > 1$ has been absorbed into the error term. By (9) and partial summation, the above is

$$= N\log(z_1 z_2/N^2) + O(N).$$

Thus to complete the proof of the theorem, it suffices to show that

$$\sum_{1<n\leqslant N}\Big(\sum_{\substack{d|n \\ d<n/z_1}} \mu\left(\frac{n}{d}\right)\log\left(\frac{n}{dz_1}\right)\Big)\Big(\sum_{\substack{e|n \\ e<n/z_2}} \mu\left(\frac{n}{e}\right)\log\left(\frac{n}{ez_2}\right)\Big)$$
$$= N\log\left(\frac{N}{z_2}\right) + O(N). \tag{20}$$

Let $(d, e) = q$, $d = qr$, $e = qt$ (here $(r, t) = 1$), and $n = mqrt$. The left-hand side of (20) is

$$= \sum_{d\leqslant N/z_1}\sum_{e\leqslant N/z_2}\sum_{\substack{dz_1<n\leqslant N \\ ez_2<n\leqslant N \\ [d,e]|n}} \mu\left(\frac{n}{d}\right)\mu\left(\frac{n}{e}\right)\log\left(\frac{n}{dz_1}\right)\log\left(\frac{n}{ez_2}\right)$$
$$= \sum_{q\leqslant N/z_2}\sum_{t\leqslant N/qz_2}\mu(t)\sum_{\substack{r\leqslant N/qz_1 \\ (r,t)=1}}\mu(r)\sum_{\substack{m\leqslant N/qrt \\ m>z_1/t,z_2/r \\ (m,rt)=1}}\mu^2(m)\log\left(\frac{mt}{z_1}\right)\log\left(\frac{mr}{z_2}\right).$$

We apply Lemma 3 and partial summation to the innermost sum. If $w = \max(z_1/t, z_2/r)$ and $y = \min(z_1/t, z_2/r)$, this sum is

$$= \frac{6}{\pi^2}\frac{N}{q\kappa(r)\,\kappa(t)}\left\{\log\left(\frac{N}{eqrz_1}\right)\log\left(\frac{N}{eqtz_2}\right) + 1\right\}$$
$$+ \frac{6}{\pi^2}\frac{wrt}{\kappa(r)\,\kappa(t)}\log\left(\frac{w}{ye^2}\right)$$
$$+ O\left(\left(\frac{N}{qrt}\right)^{1/2}\sigma_{-1/3}(rt)\log\left(\frac{2N}{qtz_2}\right)\log\left(\frac{2N}{qrz_1}\right)\right).$$

(Here and below e represents Napier's constant.)

Thus the expression in (21) may be written as

$$\frac{6}{\pi^2} N \sum_{q\leqslant N/z_2} \frac{1}{q} \sum_{t\leqslant N/qz_2} \frac{\mu(t)}{\kappa(t)} \sum_{\substack{r\leqslant N/qz_1\\(r,t)=1}} \frac{\mu(r)}{\kappa(r)} \left\{\log\left(\frac{N}{erqz_1}\right)\log\left(\frac{N}{etqz_2}\right)+1\right\}$$

$$+\frac{6}{\pi^2} z_1 \sum_{q\leqslant N/z_2} \sum_{r\leqslant N/qz_1} \frac{\mu(r)r}{\kappa(r)} \sum_{\substack{t\leqslant z_1r/z_2\\(t,r)=1}} \frac{\mu(t)}{\kappa(t)} \log\left(\frac{z_1r}{z_2te^2}\right)$$

$$+\frac{6}{\pi^2} z_2 \sum_{q\leqslant N/z_2} \sum_{t\leqslant N/qz_2} \frac{\mu(t)t}{\kappa(t)} \sum_{\substack{r<z_2t/z_1\\(r,t)=1}} \frac{\mu(r)}{\kappa(r)} \log\left(\frac{z_2t}{z_1re^2}\right)$$

$$+O\Bigg(N^{1/2} \sum_{q\leqslant N/z_2} q^{-1/2} \sum_{t\leqslant N/qz_2} \frac{\sigma_{-1/3}(t)\log(2N/tqz_2)}{t^{1/2}}$$

$$\times \sum_{r\leqslant N/qz_1} \frac{\sigma_{-1/3}(r)\log(2N/rqz_1)}{r^{1/2}}\Bigg)$$

$$= N\Sigma_A + z_1\Sigma_B + z_2\Sigma_C + O(N^{1/2}\Sigma_D), \tag{22}$$

say. Using (11) and (12) for the innermost sum in $\sum_A$, we have

$$\Sigma_A = \sum_{q\leqslant N/z_2} q^{-1} \sum_{t\leqslant N/qz_2} \mu(t)\, t^{-1} \log\left(\frac{N}{eqtz_2}\right)$$

$$+ O\Big(\sum_{q\leqslant N/z_2} q^{-1} \log^{-4}\left(\frac{2N}{qz_2}\right) \sum_{t\leqslant N/qz_2} \sigma_{-1/2}(t)\, t^{-1} \log\left(\frac{2N}{tqz_2}\right)\Big).$$

Using (7), (8), Lemma 1, and partial summation, we obtain

$$\Sigma_A = \sum_{q\leqslant N/z_2} q^{-1} + O\Big(\sum_{q\leqslant N/z_2} q^{-1} \log^{-2}\left(\frac{2N}{qz_2}\right)\Big) = \log\left(\frac{N}{z_2}\right) + O(1). \tag{23}$$

To estimate Σ_B, we note that (11) and (12) imply

$$\Sigma_B \ll \sum_{q\leqslant N/z_2} \Big| \sum_{z_2/z_1\leqslant r\leqslant N/qz_1} \mu(r) \Big|$$

$$+ \sum_{q\leqslant N/z_2} \sum_{z_2/z_1\leqslant r\leqslant N/qz_1} \sigma_{-1/2}(r) \log^{-2}\left(\frac{2z_1r}{z_2}\right).$$

For the first sum we use (6); for the second sum we use Lemma 1 and partial summation. We get

$$\Sigma_B \ll \left(\frac{z_2}{z_1}\right) \sum_{q\leqslant N/z_2} 1 + \left(\frac{N}{z_1}\right) \sum_{q\leqslant N/z_2} q^{-1} \log^{-2}\left(\frac{2N}{qz_2}\right) \ll \frac{N}{z_1}. \tag{24}$$

A similar procedure shows that

$$\Sigma_C \ll N/z_2 \tag{25}$$

To estimate Σ_D, we first note that Lemma 1 implies

$$\sum_{n\leqslant Q} n^{-1/2}\sigma_{-1/3}(n)\log\left(\frac{2Q}{n}\right) \ll Q^{-1/2}\sum_k k2^{k/2} \sum_{Q2^{-k}<n\leqslant Q2^{-k+1}} \sigma_{-1/3}(n)$$
$$\ll Q^{1/2}\sum_k k2^{-k/2} \ll Q^{1/2}.$$

It follows that

$$\Sigma_D \ll N^{1/2}(N/z_1z_2)^{1/2}\sum_q q^{-3/2} \ll N^{1/2}, \tag{26}$$

since $N \leqslant z_1z_2$.

To prove (20), we combine (21) through (26). This completes the proof of the theorem.

5. Proof of the Corollary

By (4),

$$\log^2\left(\frac{z_2}{z_1}\right)\sum_{1<n\leqslant N}\left(\sum_{d\mid n}\lambda_d\right)^2 = \sum_{1<n\leqslant N}\left(\sum_{\substack{d\mid n\\ d\leqslant z_1}}\Lambda_1(d)\right)^2$$
$$+\sum_{1<n\leqslant N}\left(\sum_{\substack{e\mid n\\ e\leqslant z_2}}\Lambda_2(e)\right)^2$$
$$-2\sum_{1<n\leqslant N}\left(\sum_{\substack{d\mid n\\ d\leqslant z_1}}\Lambda_1(d)\right)\left(\sum_{\substack{e\mid n\\ e\leqslant z_2}}\Lambda_2(e)\right).$$

The right-hand side of the above is unchanged if $\min(N, z_2)$ is substituted for z_2. This substitution, together with three applications of the theorem, proves the corollary.

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