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Collision Problems of Random Walks in Two-Dimensional Time*

NASROLLAH ETEMADI

The University of Michigan

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The collision problems of two-parameter random walks are studied. That is, some criteria have been established in terms of the characteristic functions of two or more mutually independent random walks in order to determine if they meet infinitly often in certain restricted time sets.

INTRODUCTION

Let $\{X_{ij}: i > 0, i > 0\}$ be a double sequence of independently, identically distributed random variables (i.i.d.) which takes values in the d-dimensional integer lattice E_d. The double sequence $\{S_{mn}: m > 0,\ n > 0\}$ defined by $S_{mn} = \sum_{i=1}^{m} \sum_{i=1}^{n} X_{ij}$ is called the random walk in two-dimensional time generated by X_{11}, or a two-parameter random walk, or simply a random walk when there is no danger of confusion. In this paper we study two different but closely related problems. The first one is the recurrence properties of the random walk when the distribution of X_{11} is symmetric and the second one is the collision problems of these random walks. To be more specific, one wants to know if the associated random walk would return to the origin infinitely often in certain time sets, and also whether two or more mutually independent random walks would meet infinitely often in certain time sets of interest.

In this work after giving some notations and preliminary estimates in Section 1, we give, in Section 2, a necessary and sufficient condition in terms of the characteristic function associated with a symmetric random walk so that it will return to the origin infinitely often when the time set is the positive integer lattice in the plane. In Section 3, we use the result of Section 2 in order to establish some criteria in terms of the characteristic functions associated with two or more mutually independent random walks with the same distribution so that they

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would meet infinitely often. In Sections 4 and 5, we basically follow Sections 2 and 3 when the time set is restricted to two different proper subsets of the positive integer lattice in the plane.

1. Notations and Preliminaries

Let $\{X_{ij}: (i, j) \in I^+ \times I^+\}$ (I^+ = the set of positive integers) be the corresponding double sequence of i.i.d. random variables of a two-parameter random walk $\{S_{mn}: (m, n) \in I^+ \times I^+\}$ defined on the probability space $(\Omega, \mathscr{F}, P)$. Then, adapting the notations and terminologies in Spitzer's book [4], we have

Definition 1.1. The two-parameter random walk $\{S_{mn}: (m, n) \in I^+ \times I^+\}$ is called genuinely d-dimensional (aperiodic, strongly aperiodic) if the associated one-parameter random walk $\{S_{m1}: m \in I^+\}$ is genuinely d-dimensional (aperiodic, strongly aperiodic).

The one-parameter random walk $\{S_{m1}: m \in I^+\}$ may take place on a proper subgroup of E_d. In this case, the subgroup is isomorphic to some E_k, $k \leqslant d$; if $k < d$, then the transformation should be made (see [4, p. 66]) and the problem should be considered in k dimensions. We assume throughout the paper that this reduction has been made, if necessary, and the random walk is aperiodic and genuinely d-dimensional.

For an arbitrary time set A in $I^+ \times I^+$, $\sum_{(i,j)\in A} X_{ij}$ will be denoted by S_A. The following theorem will give us an estimate for $P\{S_A = 0\}$, where A is a finite time set with cardinality $|A|$.

Theorem 1.1. *For a genuinely d-dimensional random walk generated by* X_{11} *there exist constants* $c_1, c_2 > 0$ *such that for every finite time set A in* $I^+ \times I^+$, $P\{S_A = 0\} \leqslant c_1 |A|^{-d/2}$. *Furthermore, if* $EX_{11} = 0$ *and* $E|X_{11}|^2 < \infty$, *then* $P\{S_A = 0\} \sim c_2 |A|^{-d/2}$, *as* $|A| \to \infty$, *provided that the random walk is strongly aperiodic.*

For the proof, note that $P\{S_A = 0\} = P\{S_{|A|1} = 0\}$ and see [3, p. 371] for the first part and [4, p. 72], for the second part of the theorem.

Definition 1.2. Let $N: I^+ \to A$ be a numbering of an infinite time set A. Then

$$\{S_{mn} = 0 \text{ i.o. in } A\} = \bigcap_{p=1}^{\infty} \bigcup_{q=p}^{\infty} \{S_{N(q)} = 0\}, \tag{1.1}$$

where $S_{N(q)} = S_{ij}$ if $N(q) = (i, j)$.

Notice that this definition clearly does not depend on the choice of N.

In order to study the recurrent properties of a random walk, besides the standard Borel–Cantelli lemma, the following is also needed.

THEOREM 1.2 (Generalized Boral–Cantelli Lemma). *Let* $\{E_n : n \in I^+\}$ *be a sequence of events in a probability space* $(\Omega, \mathscr{F}, P)$.

If $\sum_{k=1}^{\infty} P(E_k) = \infty$ *and if for some* $c > 0$

$$\varliminf_{n\to\infty} \frac{\sum_{k=1}^{n} \sum_{l=1}^{n} P(E_k \cap E_l)}{(\sum_{k=1}^{n} P(E_k))^2} \leqslant c, \tag{1.2}$$

then $P\{E_n : i.o.\} \geqslant 1/c$, *where* $\{E_n : i.o.\} = \bigcap_{l=1}^{\infty} \bigcup_{k=l}^{\infty} E_k$.

Proof. See [4, p. 317].

We also need a notation for the characteristic function of a random walk. For convenience we use Greek letters to denote the elements of R^d. A typical element will be $\theta = (\theta_1, \theta_2, ..., \theta_d)$, where each θ_i is a real number for $i = 1, 2, ..., d$. Now define the characteristic function of the random walk generated by X_{11} by

$$\varphi(\theta) = \sum_{x \in E_d} P\{X_{11} = x\}\, e^{ix\cdot\theta}, \tag{1.3}$$

where $x \cdot \theta = \sum_{i=1}^{d} x_i \theta_i$. To set up a convenient notation for integration, let

$$C = \{\theta \in R^d : |\theta_i| \leqslant \pi \text{ for } i = 1, 2, ..., d\}.$$

Then for complex-valued functions $g(\theta)$ which are Lebesgue measurable on C, the integral over C is denoted by

$$\int g(\theta)\, d\theta = \int_C g(\theta)\, d\theta = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} g(\theta)\, d\theta_1 \cdots d\theta_d. \tag{1.4}$$

Thus $d\theta$ always denotes the volume element. Using this notation, the inversion formula for characteristic functions becomes

$$P\{S_A = x\} = \frac{1}{(2\pi)^d} \int_C e^{-ix\cdot\theta} (\varphi(\theta))^{|A|}\, d\theta, \qquad x \in E_d, \tag{1.5}$$

where A is a finite time set in $I^+ \times I^+$. For ease of reference we state the following as a lemma.

LEMMA 1.1. *For a random walk whose associated characteristic function is a nonnegative real-valued function the following hold true*:

(i) *if* $|A| \leqslant |B|$, *then* $P\{S_B = 0\} \leqslant P\{S_A = 0\}$,

(ii) $P\{S_A = x\} \leqslant P\{S_A = 0\}$, $x \in E_d$,

where A and B are two finite time sets in $I^+ \times I^+$.

Proof. The proof follows immediately from (1.5).

2. Recurrence Properties of Symmetric Random Walks for the Entire Time Set $I^+ \times I^+$

First we show that the event introduced in (1.1) occurs with probability zero or one.

THEOREM 2.1 (0–1 Law). *Let A be an infinite time set. Then*

$$P\{S_{mn} = 0 \text{ i.o. in } A\} = 0 \text{ or } 1.$$

Proof. Define

$$A_m = \bigcup_{\{l:(m,l)\in A\}} \{S_{ml} = 0\}, \qquad A^n = \bigcup_{\{k:(k,n)\in A\}} \{S_{kn} = 0\}.$$

Clearly,

$$\{S_{mn} = 0 \text{ i.o. in } A\} = \{A_m \text{ i.o.}\} \cup \{A^n \text{ i.o.}\}. \tag{2.2}$$

Therefore, it suffices to show that the zero–one law holds for both $\{A_m$ i.o.$\}$ and $\{A^n$ i.o.$\}$. We will only show that the zero-one law holds for the event $\{A^n$ i.o.$\}$, for the other one follows by a similar argument. Define

$$X_n = (X_{1n}, X_{2n}, X_{3n}, \ldots), \qquad n \in I^+. \tag{2.3}$$

Since X_{ij}'s are i.i.d., $\{X_n\}$, $n \in I^+$, is a stationary independent process and by the Hewitt–Savage zero–one law (see [2, Theorem 8.1.4]) each permutable event with respect to $\{X_n\}$, $n \in I^+$, has probability zero or one. Now since $M \in I^+$ in the following can be as large as we want

$$\{A^n : \text{i.o.}\} = \bigcap_{p=M}^{\infty} \bigcup_{q=p}^{\infty} A^q, \tag{2.4}$$

any finite permutation of X_n's would not change the event $\{A^n$ i.o.$\}$ and we are through. Notice that $\{A^n$ i.o.$\}$ could be empty, which in this case means that it has probability zero.

DEFINITION 2.1. The random walk generated by X_{11} is called symmetric if $P\{X_{11} = x\} = P\{X_{11} = -x\}$, $x \in E_d$.

LEMMA 2.1. *Let $s \in (0, 1)$. Then*

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} s^{mn} \sim -\frac{\log(1-s)}{1-s}, \quad \text{as} \quad s \nearrow 1.$$

Proof. For $s \in (0, 1)$, $s^m/(1 - s^m)$ decreases as m increases and we have

$$\frac{s}{1-s} + \int_2^\infty \frac{s^x}{1-s^x}\,dx \leqslant \sum_{m=1}^\infty \frac{s^m}{1-s^m} \leqslant \frac{s}{1-s} + \int_1^\infty \frac{s^x}{1-s^x}\,dx. \quad (2.5)$$

Therefore,

$$\frac{s}{1-s} + \frac{\log(1-s^2)}{\log s} \leqslant \sum_{m=1}^\infty \sum_{n=1}^\infty s^{mn} \leqslant \frac{s}{1-s} + \frac{\log(1-s)}{\log s}. \quad (2.6)$$

Now a simple limit argument shows that

$$\frac{s}{1-s} + \frac{\log(1-s)}{\log s} \sim -\frac{\log(1-s)}{1-s} \sim \frac{s}{1-s} + \frac{\log(1-s^2)}{\log s}, \quad (2.7)$$

as $s \nearrow 1$.

THEOREM 2.2. *Let $\{S_{mn}: m > 0, n > 0\}$ be a symmetric random walk. Then the following conditions are equivalent:*

(i) $\displaystyle\int -\frac{\log(1 - |\varphi(\theta)|}{1 - |\varphi(\theta)|}\,d\theta = \infty,$

(ii) $\displaystyle\sum_{m=1}^\infty \sum_{n=1}^\infty P\{S_{mn} = 0\} = \infty,$

(iii) $P\{S_{mn} = 0 \text{ i.o.}\} = 1,$

where $\varphi(\theta)$ is the characteristic function associated with the random walk.

Proof. To prove this theorem, we assume that $\varphi(\theta) \geqslant 0$, $\theta \in R^d$. Then at the end we will remove this assumption. The equivalence of (i) and (ii) follows from the preceding lemma and the inversion formula (1.5). One implication, namely, (iii) $\Rightarrow$ (ii), follows trivially using the standard Borel–Cantelli lemma. To prove (ii) $\Rightarrow$ (iii), we use the generalized Borel–Cantelli lemma. In order to have the relevant setup, let us number the entire time set $I^+ \times I^+$ by

$$(m, n) \leftrightarrow \begin{cases} (m-1)^2 + n & \text{if } n \leqslant m, \\ n^2 - m + 1 & \text{if } m < n. \end{cases} \quad (2.8)$$

Let $N: I^+ \to I^+ \times I^+$ be the function induced by (2.8), and define $E_i = \{\omega: S_{N(i)}(\omega) = 0\}$. Using the zero–one law for the event $\{E_i \text{ i.o.}\}$, it suffices to show

$$\sum_{i=1}^\infty P(E_i) = \infty, \quad (2.9a)$$

$$\varliminf_{m\to\infty} \frac{\sum_{i=1}^m \sum_{j=1}^m P(E_i \cap E_j)}{(\sum_{i=1}^m P(E_i))^2} < \infty. \quad (2.9b)$$

Part (a) is immediate. To see (2.9b), observe that

$$\varliminf_{m\to\infty} \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} P(E_i \cap E_j)}{(\sum_{i=1}^{m} P(E_i))^2} \leqslant \varliminf_{n\to\infty} \frac{\sum_{i=1}^{n^2}\sum_{j=1}^{n^2} P(E_i \cap E_j)}{(\sum_{i=1}^{n^2} P(E_i))^2}. \quad (2.10)$$

But

$$\begin{aligned}\sum_{i=1}^{n^2}\sum_{j=1}^{n^2} P(E_i \cap E_j) &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} P\{S_{ij}=0,\, S_{kl}=0\},\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} P\{S_{ij}=0\},\\ &\quad + 2\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=i}^{n}\sum_{l=j+1}^{n} P\{S_{ij}=0,\, S_{kl}=0\}\\ &\quad + 2\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{i-1}\sum_{l=j}^{n} P\{S_{ij}=0,\, S_{kl}=0\}\\ &= I_n + 2II_n + 2III_n\,.\end{aligned}$$

For $k \geqslant i$, $l > j$, let $A = \{(p, q)\colon 1 \leqslant p \leqslant k,\ 1 \leqslant q \leqslant l\}\backslash\{(p, q)\colon 1 \leqslant p \leqslant i,\ 1 \leqslant q \leqslant j\}$. Then by Lemma 1.1,

$$\begin{aligned}P\{S_{ij}=0,\, S_{kl}=0\} &= P\{S_{ij}=0,\, S_A=0\},\\ &= P\{S_{ij}=0\}\,P\{S_A=0\}, \qquad (2.12)\\ &\leqslant P\{S_{ij}=0\}\,P\{S_{k(l-j)}=0\}.\end{aligned}$$

Also for $k < i$, $l > j$, let $B = \{(p, q)\colon 1 \leqslant p \leqslant k,\ 1 \leqslant q \leqslant l\}\backslash\{(p, q)\colon 1 \leqslant p \leqslant k,\ 1 \leqslant q \leqslant j\}$. Then

$$\begin{aligned}P\{S_{ij}=0,\, S_{kl}=0\} &= \sum_{x\in E_d} P\{S_{ij}=0,\, S_B=x,\, S_{kj}=-x\},\\ &= \sum_{x\in E_d} P\{S_{ij}=0,\, S_{kj}=-x\}\,P\{S_B=x\},\\ &\leqslant \sum_{x\in E_d} P\{S_{ij}=0,\, S_{kj}=-x\}\,P\{S_{k(l-j)}=0\},\\ &= P\{S_{ij}=0\}\,P\{S_{k(l-j)}=0\}. \qquad (2.13)\end{aligned}$$

Now for sufficiently large n, an easy calculation yields that $II_n \leqslant (I_n)^2$ and $III_n \leqslant (I_n)^2$. Hence, the left-hand side of (2.10) is bounded by 4 and we are through.

Finally, to remove the positivity assumption on $\varphi(\theta)$, let

$$Y_{mn} = S_{m(2n)} - S_{m(2n-2)} - S_{(m-1)(2n)} + S_{(m-1)(2n-2)}\,, \qquad (2.14)$$

where $S_{mn} = 0$ if either $m = 0$ or $n = 0$. Clearly,

$$Y_{11} = X_{11} + X_{12} \qquad \text{and} \qquad \eta(\theta) = \varphi^2(\theta) \qquad \theta \in R^d, \tag{2.15}$$

where $\eta(\theta)$ is the characteristic function of Y_{11}. Since $\eta(\theta) \geqslant 0$ by our previous argument, (i), (ii), and (iii) are equivalent for the new random walk

$$S^*_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} Y_{ij} = S_{m(2n)}, \tag{2.16}$$

But $\{S^*_{mn} = 0 \text{ i.o.}\} \subset \{S_{mn} = 0 \text{ i.o.}\}$. Thus, if $P\{S^*_{mn} = 0 \text{ i.o.}\} = 1$, then $P\{S_{mn} = 0 \text{ i.o.}\} = 1$. Now observe that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) hold true in general. Therefore, it is sufficient to show that

$$\int -\frac{\log(1 - |\varphi(\theta)|)}{1 - |\varphi(\theta)|}\, d\theta = \infty \Rightarrow \int -\frac{\log(1 - \varphi^2(\theta))}{1 - \varphi^2(\theta)}\, d\theta = \infty, \tag{2.17}$$

whose validity follows from the fact that

$$\frac{\log(1 - s^2)}{1 - s^2} \sim \frac{\log(1 - s)}{2(1 - s)}, \tag{2.18}$$

as $s \nearrow 1$.

3. Collision Problems of Random Walks for the Entire Time Set $I^+ \times I^+$

Definition 3.1. Let $\{X^\nu_{ij}: (i, j, \nu) \in I^+ \times I^+ \times I^+\}$ be a sequence of d-dimensional integer-valued i.i.d. random variables. Then the random walks

$$S^\nu_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} X^\nu_{ij}, \qquad \nu \in I^+ \tag{3.1}$$

are said to be mutually independent with the same distribution.

Theorem 3.1. *Let $\{S^\nu_{mn}: (m, n) \in I^+ \times I^+\}$, $\nu = 1, 2$, be two mutually independent random walks with the same distribution. Then the following conditions are equivalent.*

(i) $\displaystyle\int -\frac{\log(1 - |\varphi(\theta)|)}{1 - |\varphi(\theta)|}\, d\theta = \infty,$

(ii) $\displaystyle\sum_{m,n} \sum_{x \in E_d} P^2\{S^1_{mn} = x\} = \infty,$

(iii) $P\{S^1_{mn} = S^2_{mn} \text{ i.o.}\} = 1,$

where $\varphi(\theta)$ is the common characteristic function associated with the random walks.

Proof. Define a new random walk

$$S_{mn} = S_{mn}^1 - S_{mn}^2 . \tag{3.2}$$

Since the characteristic function of $X_{11}^1 - X_{11}^2$ is $|\varphi(\theta)|^2$, Theorem 2.2 and (2.18) together with the following simple observation will justify our assertion.

$$P\{S_{mn} = 0\} = P\{S_{mn}^1 = S_{mn}^2\} = \sum_{x \in E_d} P\{S_{mn}^1 = x,\, S_{mn}^2 = x\}$$

$$= \sum_{x \in E_d} P^2\{{}_{mn}^{1} = x\}. \tag{3.3}$$

THEOREM 3.2. *Let $\{S_{mn}^\nu : (m, n) \in I^+ \times I^+\}$, $\nu = 1, 2, 3, 4$, be four mutually independent random walks with the same distribution. Then the following first four conditions are equivalent and condition* (v) *implies* (iv).

(i) $\displaystyle\int -\frac{\log(1 - |\varphi(\theta)\,\varphi(\delta)|)}{1 - |\varphi(\theta)\,\varphi(\delta)|}\, d\theta\, d\delta = \infty,$

(ii) $\displaystyle\sum_{m,n} \Big(\sum_x P^2\{S_{mn}^1 = x\}\Big)^2 = \infty,$

(iii) $P\{(S_{mn}^1, S_{mn}^2) = (S_{mn}^3, S_{mn}^4) \text{ i.o.}\} = 1,$

(iv) $\sum \sum P^3\{S_{mn}^1 = x\} = \infty,$

(v) $P\{S_{mn}^1 = S_{mn}^2 = S_{mn}^3 \text{ i.o.}\} = 1.$

Remark 3.1. Although the implication (iv) $\Rightarrow$ (v) seems quite plausible, we have not been able to prove it yet.

Proof. Define two new random walks

$$\bar{S}_{mn}^1 = (S_{mn}^1, S_{mn}^2), \qquad \bar{S}_{mn}^2 = (S_{mn}^3, S_{mn}^4). \tag{3.4}$$

Then the equivalence of (i), (ii), and (iii) follows immediately from theorem 3.1, for the characteristic function associated with $\bar{S}_{mn}^1$ is $\varphi(\theta)\,\varphi(\delta)$ and

$$\sum_{x \in E_d \times E_d} P^2\{\bar{S}_{mn}^1 = x\} = \Big(\sum_{x \in E_d} P^2\{S_{mn}^1 = x\}\Big)^2. \tag{3.5}$$

Now consider another random walk

$$S_{mn} = (S_{mn}^1 - S_{mn}^2,\, S_{mn}^2 - S_{mn}^3). \tag{3.6}$$

Since

$$P\{S_{mn} = 0\} = P\{S_{mn}^1 = S_{mn}^2 = S_{mn}^3\} = \sum_{x \in E_d} P^3\{S_{mn}^1 = x\} \tag{3.7}$$

the standard Borel–Cantelli lemma gives us the implication (v) $\Rightarrow$ (iv). To show (iv) $\Rightarrow$ (i), observe that the characteristic function associated with S_{mn} is $\bar{\varphi}(\theta)\,\varphi(\theta-\delta)\,\varphi(\delta)$. Therefore, using the inverseion formula we have

$$\sum_{m,n} P\{S_{mn}=0\} \leqslant \frac{1}{(2\pi)^{2d}} \int \sum_{m,n} |\,\varphi(\theta)\,\varphi(\delta)|^{mn}\, d\theta\, d\delta, \tag{3.8}$$

which gives us the desired result if we use Lemma 2.1.

To conclude the proof it suffices to show that (ii) $\Rightarrow$ (iv). This can be done easily by applying the Schwarz inequality, as follows:

$$\begin{aligned}\Big(\sum_x P^2\{S^1_{mn}=x\}\Big)^2 &= \Big(\sum_x P\{S^1_{mn}=x\}\,P\{S^1_{mn}=x\}\Big)^2\\ &\leqslant \Big(\sum_x P^2\{S^1_{mn}=x\}\,P\{S^1_{mn}=x\}\Big)\Big(\sum_x P\{S^1_{mn}=x\}\Big)\\ &= \sum_x P^3\{S^1_{mn}=x\}. \qquad \text{Q.E.D.}\end{aligned} \tag{3.9}$$

Remark 3.2. For analogous problems with one-parameter time, see [1].

4. Recurrence Properties of Random Walks in Certain Restricted Time Sets

We assume, throughout this section, that the random walk is "nice," in the sense that it is strongly aperiodic with $EX_{11}=0$ and $E\,|\,X_{11}\,|^2<\infty$, where $|\ \ |$ is the usual d-dimensional norm. We will take a nondecreasing path going to infinity in the first quadrant, say $f(x)$, and then we will "watch" the random walk on this path to see if it returns to zero infinitely often. There are two natural time sets associated with this path that we will consider. One is

$$A_1=\{(m,[f(m)]): m\in I^+\}, \tag{4.1}$$

and for the other one we go along the path arcwise one step at a time and we take the time to be the "closest" integer lattice point to our position. To be more precise, consider the class

$$\begin{aligned}\mathscr{D}=\{f\colon [0,\infty)\to[0,\infty)\colon f \text{ is differentiable with } f^1\geqslant 0 \text{ and}\\ f(x)\geqslant 1 \text{ for some } x\geqslant 0\},\end{aligned} \tag{4.2}$$

and let $(x_f(s), y_f(s))$ (or simply $(x(s), y(s))$) be the parametric representation of $f(x)$, where s is the arc length measured from $(0, f(0))$. The second natural time set is

$$A_2=\{([x(m)],[y(m)]): m\in I^+\}. \tag{4.3}$$

Remark 4.1. Since the time sets contain only integer lattice points for any nondecreasing function, say $f(x)$, it is easy to see that it is always possible to replace $f(x)$ by a piecewise smooth function without changing the time sets. Therefore, we will see that the result of this section and the following section would not alter if we had nondecreasing functions in $\mathscr{D}$ with $f(x) \geqslant 0$ for some $x \geqslant 0$. The reason we have the differentiability assumption and also the last condition on the functions in $\mathscr{D}$ is simply to avoid technical difficulties and non-interesting time sets. Also, we do not consider the case when $f(x)$ has a vertical or a horizontal asymptote, for in this case we are dealing with one-parameter random walk and the results are well known.

Remark 4.2. Let A be a time set. Then, using Theorem 1.1, without any moment assumptions, we obtain $P\{S_{mn} = 0 \text{ i.o. in } A\} = 0$. Therefore, from now on, although the results are true for any dimension, we will concentrate on either one- or two-dimensional random walks.

Before proceeding further, we need the following lemma.

LEMMA 4.1. *Let $\{E_n : n \in I^+\}$ be a sequence of events in a probability space $(\Omega, \mathscr{F}, P)$.*

If $\sum_{k=1}^{\infty} P(E_k) = \infty$ and if for some $a > 0$, $b \geqslant 1$ and $M \in I^+ P(E_m \cap E_n) \leqslant aP(E_m)\, P(E_{[(n-m)/b]})$, $n - m > M$, then $P\{E_n \text{ i.o.}\} > 0$.

Proof. For n large enough we have

$$
\begin{aligned}
&\sum_{k=1}^{n} \sum_{l=1}^{n} P(E_k \cap E_l) \\
&\quad = \sum_{k=1}^{n} P(E_k) + 2 \sum_{k=1}^{n} \sum_{l=k+1}^{n} P(E_k \cap E_l), \\
&\quad \leqslant (2M+1) \sum_{k=1}^{n} P(E_k) + 2a \sum_{k=1}^{n-M-1} \sum_{l=k+M+1}^{n} P(E_{[(l-k)/b]})\, P(E_k), \\
&\quad \leqslant (2M+1) \sum_{k=1}^{n} P(E_k) + 2a([b]+1) \sum_{k=1}^{n} \sum_{l=1}^{n} P(E_k)\, P(E_l), \\
&\quad \leqslant c \left(\sum_{k=1}^{n} P(E_k) \right)^2,
\end{aligned}
\tag{4.4}
$$

where $c = 2M + 1 + 2a([b] + 1)$. Now use the generalized Borel-Cantelli lemma (Theorem 1.2).

Remark 4.3. Throughout the remaining part of this section the lower limit of integration could be any fixed positive number.

THEOREM 4.1. *Let $f(x)$ be in $\mathscr{D}$. Then*

$$\int^{\infty} \frac{dx}{(xf(x))^{d/2}} \begin{matrix} < \infty \\ = \infty \end{matrix} \quad \textit{iff} \quad \{S_{m[f(m)]} = 0 \textit{ i.o.}\} = \begin{matrix} 0 \\ 1 \end{matrix}.$$

Proof. It suffices to show

$$P\{S_{m[f(m)]} = 0 \text{ i.o.}\} = \begin{matrix} 0 \\ 1 \end{matrix} \qquad \text{iff} \quad \sum_{m=1}^{\infty} P\{S_{m[f(m)]} = 0\} \begin{matrix} < \infty \\ = \infty \end{matrix},$$

for Theorem 1.1 easily gives us the integral test.

Let $E_i = \{S_{i[f(i)]} = 0\}$, $i \in I^+$. Then using Theorem 1.1 for $n > M$, we obtain

$$\begin{aligned} P(E_m \cap E_n) &\leqslant cP(E_m) \cdot \frac{1}{(n[f(n)] - m[f(m)])^{d/2}}, \\ &\leqslant cP(E_m) \frac{1}{((n-m)[f(n)])^{d/2}} \leqslant cP(E_m) \frac{1}{((n-m)[f(n-m)])^{d/2}}, \\ &\leqslant cP(E_m)\, P(E_{n-m}). \end{aligned} \tag{4.5}$$

Here c is a positive constant and may change in each step in the computation and we adapt such a constant in the future estimation without further notifications. To finish up the proof, use the preceding lemma and the zero–one law. Q.E.D.

The following corollary is an immediate consequence of this theorem.

COROLLARY 4.1. *Let $g(x)$ be in $\mathscr{D}$ such that $P\{S_{m[g(m)]} = 0 \textit{ i.o.}\} = 1$ $(=0)$. Then for every $f(x)$ in $\mathscr{D}$ with $f(x) \leqslant g(x)$ $f(x) \geqslant g(x))$ for large x we have $P\{S_{n[f(m)]} = 0 \textit{ i.o.}\} = 1$ $(=0)$.*

EXAMPLE 4.1. Let $d = 1$; then it is easy to see that one can let $g(x) = ax(\log x)^2$ $(=ax(\log x)^{2+\epsilon})$ in Corollary 4.1 with $a, \epsilon > 0$.

EXAMPLE 4.2. Let $d = 2$; then it is easy to see that one can let $g(x) = a \log x$ $(=a(\log x)^{1+\epsilon})$ in Corollary 4.1 with $a, \epsilon > 0$.

THEOREM 4.2. *Let $(x(s), y(s))$ be the parametric representation of $f(x)$ in $\mathscr{D}$. Then*

$$\int^{\infty} \frac{ds}{(x(s)\, y(s))^{d/2}} \begin{matrix} < \infty \\ = \infty \end{matrix} \quad \textit{iff} \quad P\{S_{[x(m)][y(m)]} = 0 \textit{ i.o.}\} = \begin{matrix} 0 \\ 1 \end{matrix}.$$

Proof. Let $E_i = \{S_{[x(i)][y(i)]} = 0\}$, $i \in I^+$; then, using Theorem 1.1, it is easy to see that for $n > m$

$$P(E_m \cap E_n) \leqslant cP(E_m) \cdot \frac{1}{([x(n)][y(n)] - [x(m)][y(m)])^{d/2}}. \tag{4.6}$$

Now let $l = n - m$; then we have

$$[x(n)][y(n)] - [x(m)][y(m)] = [x(m+l)][y(m+l)] - [x(m)][y(m)]$$
$$\geqslant \begin{cases} ([x(m+l)] - [x(m)])[y(m+l)] \\ ([y(m+l)] - [y(m)])[x(m+l)] \end{cases}$$
$$\geqslant \begin{cases} ([x(m+l)] - [x(m)])[y(l)] \\ ([y(m+l)] - [y(m)][x(l)]. \end{cases} \tag{4.7}$$

But clearly for any path $x(s) + y(s) \geqslant s$. Hence, either $x(s) \geqslant s/2$ or $y(s) \geqslant s/2$. Therefore, either

$$x(m+l) - x(m) \geqslant l/2 \qquad \text{or} \qquad y(m+l) - y(m) \geqslant l/2. \tag{4.8}$$

Consequently, for $l \geqslant 4$, either

$$[x(m+l)] - [x(m)] \geqslant l/2 - 1 \geqslant l/4 \geqslant [x[l/4]]$$
or
$$[y(m+l)] - [y(m)] \geqslant [y[l/4]]. \tag{4.9}$$

Utilizing this in (4.7) and then (4.6) for $n - m \geqslant 4$, we obtain

$$P(E_m \cap E_n) \leqslant cP(E_m) \frac{1}{([x[(n-m)/4]][y[(n-m)/4]])^{d/2}}$$
$$\leqslant cP(E_m)\, P(E_{[(n-m)/4]}). \qquad \text{Q.E.D.} \tag{4.10}$$

To have a better picture of what is happening, we prove the following theorem.

THEOREM 4.4. *Let $(x(s).\, y(s))$ be the parametric representation of $f(x)$ in $\mathscr{D}$. Then*

(i) *For $d \geqslant 3$,*

$$P\{S_{[x(m)][y(m)]} = 0 \ i.o.\} = 0.$$

(ii) *For $d = 2$,*

$$\int \frac{ds}{x(s)\, y(s)} \begin{matrix} < \infty \\ = \infty \end{matrix} \quad \text{iff} \quad P\{S_{[x(m)][y(m)]} = 0 \text{ i.o.}\} \begin{matrix} = 0 \\ = 1 \end{matrix}.$$

(iii) *For $d = 1$,*

$$P\{S_{[x(m)][y(m)]} = 0 \ i.o.\} = 1.$$

Proof. In light of Remark 4.2 and the preceding theorem, only part (iii) needs justification. Now assume for large x, $f(x) \leqslant x$. Then,

$$\int^{\infty} \frac{ds}{(x(s)\,y(s))^{1/2}} \geqslant \int^{\infty} \frac{dx}{(xf(x))^{1/2}} \geqslant \int^{\infty} \frac{dx}{x} = \infty. \tag{4.11}$$

By symmetry, if for large x, $f(x) \geqslant x$, then we still have the result. Therefore, the only case left to be verified is the case when $f(x)$ oscillates around $y = x$. But since $P\{S_{[x(m)][y(m)]} = 0\} = P\{S_{[y(m)][x(m)]} = 0\}$, the sum

$$\sum_{m=1}^{\infty} P\{S_{[x(m)][y(m)]} = 0\}$$

does not change if we replace $f(x)$ by a new function whose graph coincides with the one of $f(x)$ when the graph of $f(x)$ is below $y = x$ and with the mirror image of the graph of $f(x)$ with respect to $y = x$ when the graph of $f(x)$ is above $y = x$. Therefore, without loss of generality (see Remark 4.1), the result holds true in this case, too. Q.E.D.

COROLLARY 4.2. *Let $d = 2$ and $f(x)$ be in $\mathscr{D}$ with $(x(s), y(s))$ as its parametric representation such that for large x, $f(x) \leqslant a \log x$ or $f(x) \geqslant ae^{x}$ $(a(\log x)^{1+\beta} \leqslant f(x) \leqslant be^{x^{\alpha}}$, $\alpha \in (0, 1)$, β, a, $b > 0)$, $a > 0$, then $P\{S_{[x(m)][y(m)]} = 0$ i.o.$\} = 1$ $(=0)$.*

Proof. The proof follows easily from the integral test.

5. COLLISION PROBLEMS OF RANDOM WALKS IN CERTAIN RESTRICTED TIME SETS

In this section we will study the collision problems of the random walk in the time sets A_1 and A_2 introduced in (4.1) and (4.3). The case when they are strongly aperiodic and mutually independent with common distributions and finite second moments can be worked out completely by simply using the results of Section 4. In order to study them in terms of their characteristic functions, we first need the following lemma.

LEMMA 5.1. *Let $s \in (0, 1)$ and $f(x) \in \mathscr{D}$ with its associated time sets A_1 and A_2. Then,*

$$\sum_{(m,n)\in A_i} s^{mn} \leqslant 3 + 4 \sum_{(m,n)\in A_i} s^{2mn}, \qquad i = 1, 2.$$

Proof. For A_1 we have

$$\sum_{(m,n)\in A_1} s^{mn} = \sum_{m=1}^{\infty} s^{m[f(m)]} = \sum_{m=1}^{\infty} s^{2m[f(2m)]} + \sum_{m=0}^{\infty} s^{(2m+1)[f(2m+1)]}$$

$$\leqslant 2\sum_{m=1}^{\infty} s^{2m[f(m)]} + s^{[f(1)]} \leqslant 1 + 2\sum_{m=1}^{\infty} s^{2m[f(m)]}$$

$$\leqslant 1 + 2\sum_{(m,n)\in A_1} s^{2mn}. \tag{5.1}$$

For A_2 observe that

$$x(s) \geqslant s/2 \quad \text{or} \quad y(s) \geqslant s/2. \tag{5.2}$$

Therefore,

$$[x(4m)] \geqslant 2m \geqslant 2[x(m)] \quad \text{or} \quad [y(4m)] \geqslant 2m \geqslant 2[y(m)]. \tag{5.3}$$

Consequently, using the monotonicity of $x(s)$ and $y(s)$,

$$[x(4m)][y(4m)] \geqslant 2[x(m)][y(m)]. \tag{5.4}$$

Hence,

$$\sum_{(m,n)\in A_2} s^{mn} = \sum_{m=1}^{\infty} s^{[x(m)][y(m)]} \leqslant \sum_{n=0}^{3}\sum_{m=1}^{\infty} s^{[x(4m+n)][y(4m+n)]} + 3$$

$$\leqslant 3 + 4\sum_{m=1}^{\infty} s^{[x(4m)][y(4m)]} \leqslant 3 + 4\sum_{m=1}^{\infty} s^{2[x(m)][y(m)]}$$

$$\leqslant 3 + 4\sum_{(m,n)\in A_2} s^{2m}. \qquad \text{Q.E.D.} \tag{5.5}$$

THEOREM 5.1. *Let $\{S_{mn}: (m,n)\in I^+ \times I^+\}$ be a random walk such that its associated characteristic function $\varphi(\theta)$ is real and positive. Then, fixing $i \in \{1,2\}$, the following conditions are equivalent*:

(i) $\int \sum_{(m,n)\in A_i} (\varphi(\theta))^{mn}\, d\theta = \infty$

(ii) $\sum_{(m,n)\in A_i} P\{S_{mn} = 0\} = \infty$

(iii) $P\{S_{mn} = 0 \text{ } i.o. \text{ } in \text{ } A_i\} = 1.$

Proof. The equivalence of (i) and (ii) is immediate. To see why (ii) and (iii) are equivalent, follow the proof of Theorems 4.1 and 4.2 for $i = 1$ and $i = 2$, respectively, and use Lemma 1.1. Q.E.D.

THEOREM 5.2. *Let* $\{S_{mn}^{\nu}: (m, n) \in I^+ \times I^+\}$, $\nu = 1, 2$, *be two mutually independent random walks with the same distribution. Then, fixing* $i \in \{1, 2\}$, *the following conditions are equivalent*

$$\text{(i)} \quad \int \sum_{(m,n)\in A_i} |\varphi(\theta)|^{mn}\, d\theta = \infty$$

$$\text{(ii)} \quad \sum_{(m,n)\in A_i} \sum_{x\in E_d} P^2\{S_{mn}^1 = x\} = \infty$$

$$\text{(iii)} \quad P\{S_{mn}^1 = S_{mn}^2 \textit{ i.o. in } A_i\} = 1,$$

where $\varphi(\theta)$ *is the common characteristic function associated with the random walks.*

Proof. Observe that by Lemma 4.1 for $i \in \{1, 2\}$,

$$\int \sum_{(m,n)\in A_i} |\varphi(\theta)|^{2mn}\, d\theta = \infty \quad \text{iff} \quad \int \sum_{(m,n)\in A_i} |\varphi(\theta)|^{mn}\, d\theta = \infty. \tag{5.6}$$

Now the proof follows easily by using Theorem 5.1 and an argument similar to the one in Theorem 3.1. Q.E.D.

THEOREM 5.3. *Let* $\{S_{mn}^{\nu}: (m, n) \in I^+ \times I^+\}$, $\nu = 1, 2, 3, 4$, *be four mutually independent random walks with the same distribution and the common characteristic function* $\varphi(\theta)$. *Then, fixing* $i \in \{1, 2\}$, *the first four conditions are equivalent and* (v) $\Rightarrow$ (iv).

$$\text{(i)} \quad \int \sum_{(m,n)\in A_i} |\varphi(\theta)\, \varphi(\delta)|^{mn}\, d\theta\, d\delta = \infty$$

$$\text{(ii)} \quad \sum_{(m,n)\in A_i} \Big(\sum_{x\in E_d} P^2\{S_{mn}^1 = x\}\Big)^2 = \infty$$

$$\text{(iii)} \quad P\{(S_{mn}^1, S_{mn}^2) = (S_{mn}^3, S_{mn}^4) \text{ i.o. in } A_i\} = 1$$

$$\text{(iv)} \quad \sum_{(m,n)\in A_i} \sum_{x\in E_D} P^3\{S_{mn}^1 = x\} = \infty$$

$$\text{(v)} \quad P\{S_{mn}^1 = S_{mn}^2 = S_{mn}^3 \textit{ i.o. in } A_i\} = 1.$$

Proof. Observe that by Lemma 4.1 for $i \in \{1, 2\}$.

$$\int \sum_{(m,n)\in A_i} |\varphi(\theta)\, \varphi(\delta)|^{2mn}\, d\theta\, d\delta = \infty \quad \text{iff} \quad \int \sum_{(m,n)\in A_i} |\varphi(\theta)\, \varphi(\delta)|^{mn}\, d\theta\, d\delta = \infty. \tag{5.7}$$

Now the proof follows easily by using Theorem 5.1 and an argument analogous to the one in Theorem 3.2. Q.E.D.

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