JOURNAL OF DIFFERENTIAL EQUATIONS 23, 244–269 (1977)

Mixed Problems for Nonlinear Conservation Laws

TAKAAKI NISHIDA*

Kyoto University, Kyoto, Japan

AND

JOEL SMOLLER*

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48104

Received May 14, 1975; revised September 9, 1975

1. INTRODUCTION

We consider the mixed problem for the system of equations

$$v_t - u_x = 0, \qquad u_t + p(v)_x = 0, \tag{1}$$

where $p(v) = K^2 v^{-\gamma}$, $\gamma = 1 + 2\epsilon$, $\epsilon \geqslant 0$, $K = \text{const} > 0$, in regions (a) $x > 0$, $t > 0$, and in regions (b) $0 < x < 1$, $t > 0$. In both cases we prescribe initial data

$$(v(x, 0), u(x, 0)) = (v_0(x), u_0(x)), \tag{2}$$

where $0 < \mathbf{v} \leqslant v_0(x) \leqslant \bar{v} < +\infty$. In regions (a) we also prescribe boundary data of the form

$$u(0, t) = u_1(t), \qquad t \geqslant 0, \tag{3}$$

while in regions (b) we prescribe boundary data (3) and

$$u(1, t) = u_2(t), \qquad t \geqslant 0. \tag{4}$$

Systems of the type (1) describe one-dimensional motion of an isothermal gas, in Lagrangian coordinates, in the absence of dissipative effects. Here v denotes the specific volume (the reciprocal of the density ρ), and u is the velocity of the gas. The mixed problem (1)–(3) is sometimes called the "piston problem," and the function $u_1(t)$ denotes the velocity of the piston. The problem (1)–(4) can be

* Smoller's research was supported by the Air Force Office of Scientific Research, Contract Number AFOSR-71-2122, and Nishida's research was supported in part by a Sloan Postdoctoral Fellowship at the Courant Institute of Mathematical Sciences.

ISSN 0022–0396

called the "double piston problem" (although we also allow "rigid walls," $u_1 \equiv 0$ and/or $u_2 \equiv 0$).

We always assume that each of the functions v_0, u_0, u_1, and u_2 are bounded and have finite total variation. We set

$$T_1 = TV\{v_0\} + TV\{u_0\} + TV\{u_1\},$$
$$T_2 = T_1 + TV\{u_2\}.$$

We first show that the problem (1)–(3) has a global solution defined for all $t \geqslant 0$ provided that ϵT_1 is sufficiently small. The case where $\epsilon = 0$ is considered in [4], and for the case where the variation of the data is sufficiently small, see [3]. Our result is related to our paper [5], where we considered the pure initial value problem. The piston problem is more complicated, due, mainly, to the reflection of shock waves at the boundary $x = 0$, whereby the strength of the reflected shock is usually greater than the strength of the incoming shock. Our technique is to use Glimm's method [2], however, we must modify his functional in order to take care of the reflections of shock waves on the boundary $x = 0$. Such a procedure requires an estimate of the strength of the reflected shock wave in terms of the strength of the incoming shock, together with the contribution of the boundary data at the point of reflection. This estimate is obtained by showing that if the incoming shock impinges on the boundary $x = 0$ at a point of continuity of u_1, then this reflection can be reduced to a problem of interacting shocks for a free (i.e., initial value) problem.

For the "double piston" problem, (1)–(4), the existence of a solution is much more delicate due to the continued reflection of shock waves across both boundaries $x = 0$, and $x = 1$. We first present an example which shows that the problem cannot have a global (in time) solution without some additional restrictions on the data. We give a fairly careful analysis of this example which points out just where the difficulty lies; namely, it is necessary to put conditions on the data which prevent the pistons from coming together in a finite time ($\rho = +\infty$), and also prevent the vacuum ($\rho = 0$) from appearing. These "physical" conditions are made precise in inequality (14). In order to handle the problem of multiple reflections of shock waves off the boundaries $x = 0$ and $x = 1$, we employ the generalized Riemann invariants introduced by DiPerna in [1]. Measured in terms of these coordinates, the strengths of the reflected shock waves do not increase, modulo contributions from the boundary data. Thus, we can again use Glimm's method, where we now employ a functional analogous to that used in [1] (which, however, is supplemented by additional terms needed to take boundary interactions into account). The desired decrease of our functional is obtained only if ϵT_2 is sufficiently small and the approximate solutions lie in the region in which the generalized Riemann invariants are defined. This requires a short detour; namely we first fix ϵT_2 to be sufficiently small and then take t to be sufficiently small. We thus get a weak solution defined in $0 \leqslant t \leqslant t_0$, $0 \leqslant x \leqslant 1$, which is L^1-continuous in t. It therefore satisfies the above "physical" conditions on the data in this time interval. Thus using this a priori bound, we can take as new "initial data" the functions $(v(x, t_0), u(x, t_0))$. We then proceed to solve the problem locally and repeatedly in regions $nt_0 \leqslant t \leqslant (n+1)t_0$, $0 \leqslant x \leqslant 1$, where $n = 1, 2, \ldots$. This then yields the desired global solution.

2. Preliminaries to the Piston Problem

Solutions of nonlinear hyperbolic systems are usually discontinuous; thus, by a solution of (1)–(3), we mean a pair of bounded measurable functions $(v(x, t), u(x, t))$, which satisfy the two equations

$$\int_{\substack{t\geqslant 0\\ x\geqslant 0}}\int (v\phi_t - u\phi_x)\,dx\,dt + \int_{t=0} v_0\phi\,dx + \int_{x=0} u_1\phi\,dt = 0, \qquad \phi \in C_0^1,$$

$$\int_{\substack{t\geqslant 0\\ x\geqslant 0}}\int (u\psi_t + p(v)\,\psi_x)\,dx\,dt + \int_{t=0} u_0\psi\,dx = 0, \quad \psi \in C_0^1, \ \ \psi(0, t) = 0, \ \ t \geqslant 0.$$

We recall from [5], that a pair of Riemann invariants for (1) may be taken as

$$r = u - K\gamma^{1/2}[(\rho^\epsilon - 1)/\epsilon], \qquad s = u + K\gamma^{1/2}[(\rho^\epsilon - 1)/\epsilon].$$

In these coordinates, we can solve the simplest piston problem, i.e., the analog of Riemann's problem for the mixed problem.

Lemma 2.1. *Consider the system* (1) *with data* $v(x, 0) = v_+$, $u(x, 0) = u_+$, *and* $u(0, t) = u_-$, *where* v_+, u_+, u_- *are constants, and* $v_+ > 0$. *This problem has a piecewise continuous solution in* $x \geqslant 0$, $t \geqslant 0$ *satisfying the estimates*

$$\begin{aligned} r(x, t) &\equiv r(v(x, t), u(x, t)) \geqslant r(v_+, u_+) \equiv r_+, \\ s(x, t) &\equiv s(v(x, t), u(x, t)) \leqslant \max[s(v_+, u_+) \equiv s_+, 2u_- - r_+], \\ \Delta s &\leqslant 2 \max\,[0, u_- - u_+] \equiv 2\Delta u, \end{aligned}$$

where Δs *is the variation of* s *across*[1] S_2 *in the solution.*

Remark. The term $2u_- - r_+$ is new to the mixed problem, and is due to shock waves, reflecting off, or coming out of, the boundary $x = 0$.

Proof. We consider two cases: $u_+ < u_-$, or $u_+ > u_-$. Suppose first that $u_+ < u_-$. The solution to our problem is given by a shock wave of the second kind coming out of the origin (see Fig. 1).

[1] S_i denotes an i shock, and R_i denotes an i rarefaction wave $i = 1, 2$ (see [5]).

Using Fig. 2, we see $r_- \geqslant r_+$, and since $s_0 = 2u_- - r_+$ and $s_+ = 2u_+ - r_+$, we have

$$\begin{aligned} \Delta s &= s_- - s_+ < s_0 - s_+ = 2u_- - r_+ - (2u_+ - r_+) = 2\Delta u, \\ s_- &< s_0 = s_+ + (s_0 - s_+) < (2u_+ - r_+) + 2(u_- - u_+) = 2u_- - r_+ . \end{aligned}$$

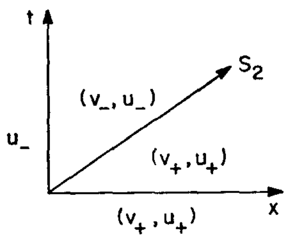


FIGURE 1

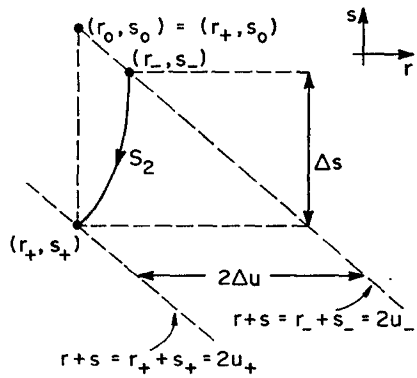


FIGURE 2

If we consider the case where $u_+ > u_-$, we must, of course, exclude the case where $\rho = 0$ (i.e., $v = \infty$) so we must assume here that[2]

$$s_- - r_+ > 2(u_+ - u_-) - (2K\gamma^{1/2}/\epsilon).$$

[2] To see this, we consider the equation $s - r = 2K\gamma^{1/2}[(\rho^\epsilon - 1)/\epsilon]$, together with Fig. 3. Note that $\rho = 0$ corresponds to $s - r = -2K\gamma^{1/2}/\epsilon$, while $\rho > 0$ corresponds to $s - r > -2K\gamma^{1/2}/\epsilon$. Since $s_- - s_+ = 2(u_- - u_+)$, we have $s_- - r_+ = (s_- - s_+) + (s_+ - r_+) > -2K\gamma^{1/2}/\epsilon$.

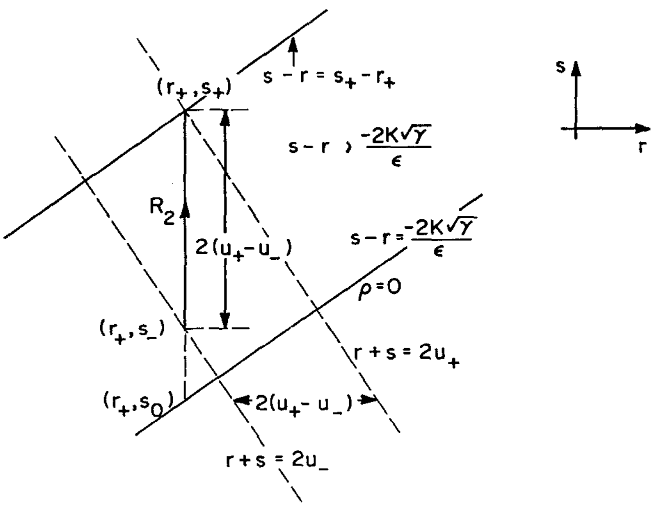


FIGURE 3

In this case, we can find an $s_- < s_+$ such that $s_+ - s_- = 2(u_+ - u_-)$. Hence the solution to our problem is given by an R_2 coming out of the origin (see Fig. 4). Using Fig. 5, we see that $r_- = r_+$, and $s_+ - s_- = 2(u_+ - u_-)$, $s_- = 2u_- - r_+$. This completes the proof of the lemma.

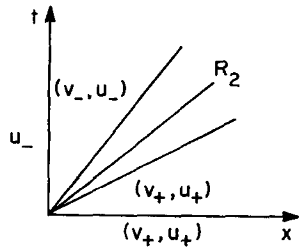


FIGURE 4

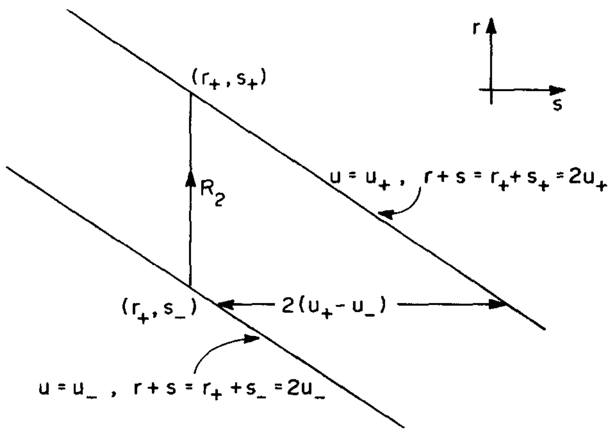


FIGURE 5

3. THE DIFFERENCE SCHEME AND NONLINEAR FUNCTIONALS

We next consider general data u_0, v_0, u_1, all three functions being bounded and of bounded total variation. To handle this general case we shall use a modified form of Glimm's scheme. Thus, let

$$Y = \{(m, n) : m = 1, 3, 5, \ldots; \quad n = 1, 2, 3, \ldots\},$$
$$A = \prod_{(m,n) \in Y} ([(m-1)\,l, (m+1)l] \times \{nh\}),$$

and choose the mesh lengths l, h to satisfy

$$l/h = (1/K\gamma^{1/2})[1 + (\epsilon/2K\gamma^{1/2})(s_0 - r_0)]^{-((1+\epsilon)/\epsilon)},$$

where

$$r_0 = \inf_{x \geqslant 0} r(v_0(x), u_0(x)),$$

and

$$s_0 = \max[\sup_{x \geqslant 0} s(v_0(x), u_0(x)), \quad 2 \sup_{t \geqslant 0} u_1(t) - r_0].$$

Let $\{\alpha_n : n = 1, 2,...\}$ be a random sequence of numbers equidistributed in $(-1, 1)$, and let $a_{m,n} = (ml + \alpha_n l, nh)$, $m = 1, 3,...$, $n = 1, 2,...$; $a_{0,n} = (0, nh - \frac{1}{2}h)$, $n = 1, 2,...$; $a_{m,0} = (ml, 0)$, $m = 1, 3,...$, be the mesh points.

We define the I curve $\mathcal{O}$ to be any spacelike curve joining points $a_{m,0}$ $(m = 1, 3, 5,...)$ and not containing points $((m + 1)\, l, 0)$, $m = 1, 3, 5,...$, which lies in $0 \leqslant t \leqslant h$ if $x > 0$, and which also includes the half-ray $t \geqslant h/2$, $x = 0$, and the straight-line segment joining a_{10} to a_{01} (see Fig. 6).

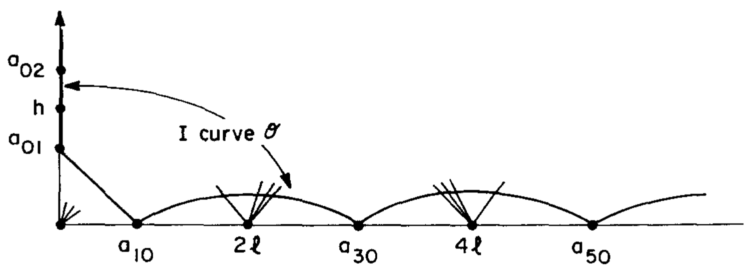


FIGURE 6

In order to define the I curve J, we first let i_m^n- (respectively i_m^n+), $m = 2, 4, 6,...$ be any space-like curve joining $a_{m-1,n}$ and $a_{m+1,n}$ lying in $(n - 1)h < t \leqslant nh$ (respectively $nh \leqslant t < (n + 1)h$), and not passing through the point (ml, nh); i_0^n+ (respectively i_0^n-) is the straight-line segment joining the points a_{1n} and $[0, nh + (h/2)]$, (respectively $[0, nh - (h/2)]$). Then the I curve J is composed of curves $i_m^{n\pm}$, $m = 0, 2, 4,...$ and straight-line segments joining the mesh points $a_{m-1,n}$ and $a_{m+1,n-1}$, (or $a_{m-1,n}$ and $a_{m+1,n+1}$), $m = 2, 4,...$ on which the index m increases to infinity, together with the half-ray $t \geqslant nh + (h/2)$ (or $t \geqslant nh - (h/2)$).

Next, we use Glimm's method [2], to obtain approximate solutions to our problem. Namely, we solve the Riemann problem in the region $nh \leqslant t < (n + 1)h$, $ml \leqslant x \leqslant (m + 2)l$, $m = 1, 3,...$, $n = 0, 1, 2,...$, and we also solve the problem (1)–(3), with constant data (see Lemma 2.1) in the region $nh \leqslant t < (n + 1)h$, $0 \leqslant x \leqslant l$. This is analogous to what is done in [4, 5].

In order to obtain the desired estimates on these approximate solutions, we define certain functionals as follows. For an I curve J, we let

$$L(J) = \sum_J |\alpha_k| + |\beta_l| + |\gamma_j|,$$

where α_k is an S_1 crossing J, β_l is an S_2 crossing J, and

$$|\gamma_j| = 2\Delta u_1 = 2\max\{0, u_1(a_{0,j+1}) - u_1(a_{0,j})\}$$

for all j such that $[0, jh \pm (h/2)] \in J$. Here $|\alpha_k|$ and $|\beta_l|$ denote the strength of the shock waves α_k and β_l, respectively (see [5]). We next let

$$Q(J) = \sum_J \{|\alpha_k||\beta_l| + |\alpha_k||\alpha_l| + \tfrac{1}{2}|\alpha_k|^2 + |\alpha_k||\gamma_j|\} \tag{5}$$

where α_k, β_l, and γ_j are as above. Here the term $|\alpha_k||\beta_l|$ is included only if α_k and β_l are approaching (cf. [2]); the term $|\alpha_k||\alpha_l|$ is included for $k < l$ and the terms $\frac{1}{2}|\alpha_k|^2$ and $|\alpha_k||\gamma_j|$ are included for all k, j. Finally we set

$$F(J) = L(J) + KQ(J), \tag{6}$$

where $K = O(\epsilon)$ will be chosen later. Note that

$$F(\mathscr{O}) \leqslant L(\mathscr{O}) + KL(\mathscr{O})^2 \leqslant 2L(\mathscr{O}), \tag{7}$$

since we may assume that $KL(\mathscr{O}) \leqslant 1$, if ϵ is small.

It is interesting to compare the F defined by (6) to the associated F of the pure initial-value problem, defined in [5]. The new terms here are $\sum|\gamma_j|$ in $L(J)$ and $\sum|\alpha_k||\alpha_l|$, $\frac{1}{2}\sum|\alpha_k|^2$, $\sum|\alpha_k||\gamma_j|$ in $Q(J)$. These terms are due to reflection of shocks on the boundary $x = 0$; see Fig. 7. That is, each γ_j comes from an S_2

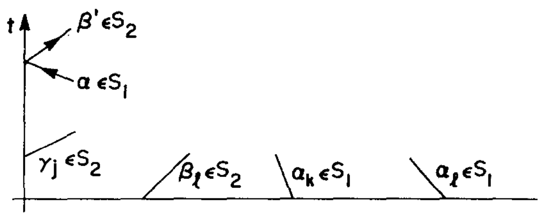


FIGURE 7

when we solve (1)–(3). The term $\frac{1}{2}|\alpha_k|^2$ is needed in order to handle the reflections of shocks on the boundary $x = 0$ ($\alpha \to \beta'$ in Fig. 7). The strength of the S_2 reflected wave, β', is greater than the strength of the incoming S_1; i.e., α. We will show below that $|\beta'| \leqslant |\alpha| + |\gamma| + C\epsilon|\alpha|^2$. The term $|\alpha_k||\alpha_l|$, $k < l$, goes into the term $|\beta_k'||\alpha_l|$ after α_k is reflected at $x = 0$ (that is, we do not get $|\alpha_k'||\alpha_l|$, i.e., a new S_1, as in the Cauchy problem, but we get an S_2). Finally, $|\alpha_k||\gamma_j|$ goes into $|\alpha_k||\beta_l|$ away from the boundary; that is, γ_j becomes an S_2.

The interactions which do not contain reflections of shock waves on the boundary $x = 0$ are quite the same as for the Cauchy problem, and for these, the estimates in [5] are valid.

4. Estimate of the Reflected Shock Waves

In this section we shall consider the reflection of a shock wave of the first kind on the boundary $x = 0$, at time $t = nh$. Thus, we suppose α is an S_1 coming into the boundary $x = 0$; it gets reflected into a shock wave $\beta' \in S_2$, and our task is to estimate $|\beta'|$ in terms of $|\alpha|$ and the boundary data. We first consider the case where the piston velocity u is constant near the point $(0, nh)$.

PROPOSITION 4.1. *Let* $0 \leqslant \epsilon < \frac{1}{2}$, *and consider the reflection* $\alpha \to \beta'$ *on* $x = 0$, *at* $t = nh$, *where* $\alpha \in S_1$, $\beta' \in S_2$. *If u is constant on* $(n-1)h < t < (n+1)h$, $x = 0$, *then*

$$|\beta'| \leqslant |\alpha| + C\epsilon |\alpha|^2, \tag{8}$$

where C is a positive constant independent of α, β', *and* ϵ, *provided that the waves* α *and* β' *are contained in the strip* $\rho \in [\underline{\rho}, \bar{\rho}]$, $u \in \mathbb{R}$ *in the* r, s *plane.*

In order to prove this proposition, we need a lemma. This lemma states roughly, that shocks reflected off a part of the boundary in which u_1 is constant may be considered as coming from a pure Cauchy problem.

LEMMA 4.2. *Given* α, β' *as above, there exists* $\alpha' \in S_1$, $\beta \in S_2$ *with* $|\alpha| = |\beta|$, $|\alpha'| = |\beta'|$ *such that* $\beta + \alpha \to \alpha' + \beta'$.

Proof. Using the remark [5, p. 192] we note that the shock wave curve S_1 starting at any point (r_-, s_-) is symmetric to the inverse shock wave curve S_2' starting at (r_-, s_-), with respect to the line $r + s = r_- + s_-$. Similarly, this symmetry is valid for S_2 and S_1' starting at any point $(\tilde{r}, \tilde{s})$, with respect to the line $r + s = \tilde{r} + \tilde{s}$ (see Fig. 8).

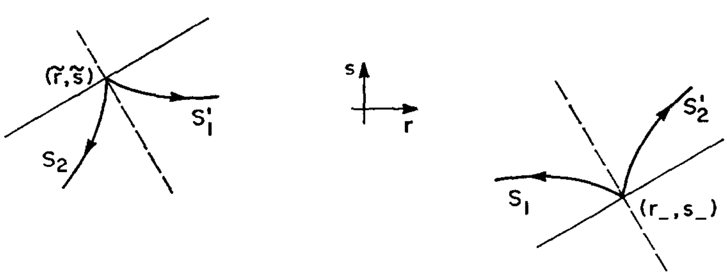


FIGURE 8

Given α, we construct β', β, and α' as follows: First consider the diagram in Fig. 9.

The intersection of the inverse shock curve S_2' starting at (r_+, s_+) with the line $r + s = r_- + s_-$ is denoted by $(\tilde{r}, \tilde{s})$; the shock curve S_2 from $(\tilde{r}, \tilde{s})$ to

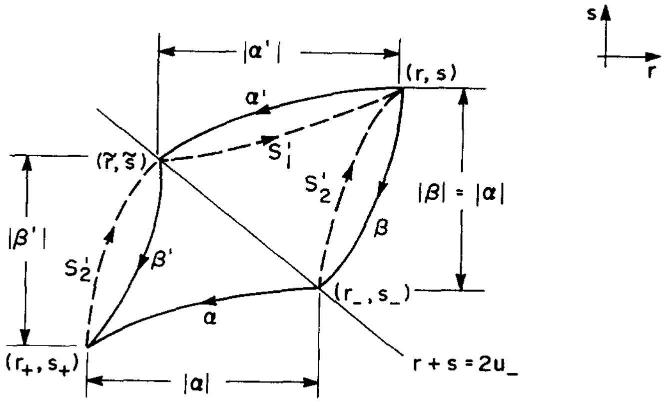


FIGURE 9

(r_+, s_+) is β'; the point on the inverse shock curve S_2' starting at (r_-, s_-) with strength $|\alpha|$ is (r, s), and the shock curve S_2 from (r, s) to (r_-, s_-) is β where $|\beta| = |\alpha|$. If we draw the inverse shock curve S_1' from $(\tilde r, \tilde s)$, it crosses the S_2' starting from (r_-, s_-), at (r, s) because of the symmetry with respect to the line $r + s = r_- + s_- = \tilde r + \tilde s$. Therefore the shock curve S_1 from (r, s) to $(\tilde r\ \tilde s)$ is α' where $|\alpha'| = |\beta'|$. Q.E.D.

Proof of Proposition 4.1. From Fig. 9, Lemma 4.2 and the estimate of [5, Lemma 4(i)(a)], we have

$$|\alpha'| = |\beta'| \leqslant |\alpha| + C\epsilon |\alpha| |\beta| = |\alpha| + C\epsilon |\alpha|^2,$$

as asserted.

PROPOSITION 4.3. *Let* $0 \leqslant \epsilon < \frac{1}{2}$, *and consider the reflection* $\alpha \to \beta'$ *on* $x = 0$ *at* $t = nh$, *where* $\alpha \in S_1$, $\beta' \in S_2$. *Then if* $|\gamma| = 2\max(0, u_1(a_{0,n+1}) - u_1(a_{0,n}))$, *we have*

$$|\beta'| \leqslant |\alpha| + |\gamma| + C\epsilon |\alpha|^2 \tag{9}$$

where C is a positive constant, independent of α, β', $|\gamma|$, *and* ϵ *(see Proposition* 4.1).

Proof. The reflection $\alpha \to \beta'$ is described by the diagram in Fig. 10, where $u_- = u_1(a_{0,n})$, $\bar u = u_1(a_{0,n+1})$.

We let $(r_\pm, s_\pm) = (r(v_\pm, u_\pm), s(v_\pm, u_\pm))$ and $(\bar r, \bar s) = (r(\bar v, \bar u), s(\bar v, \bar u))$. First consider the case when $\bar u < u_-$. If we refer to Fig. 7, we see that the region $r + s < 2\bar u$ lies below the line $r + s = 2u_-$, so that the starting point of the shock β' lies below the line $r + s = 2u_-$. Now we construct β, α' as in Lemma 4.2,

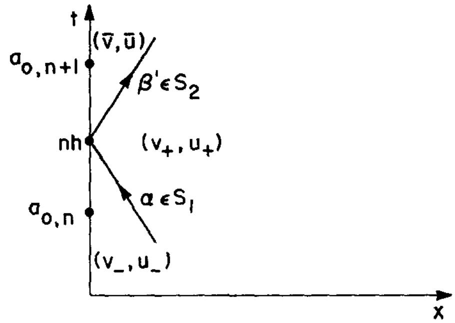


FIGURE 10

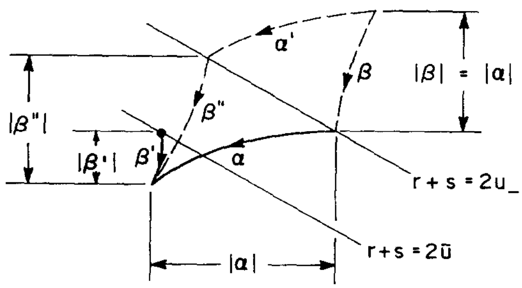


FIGURE 11

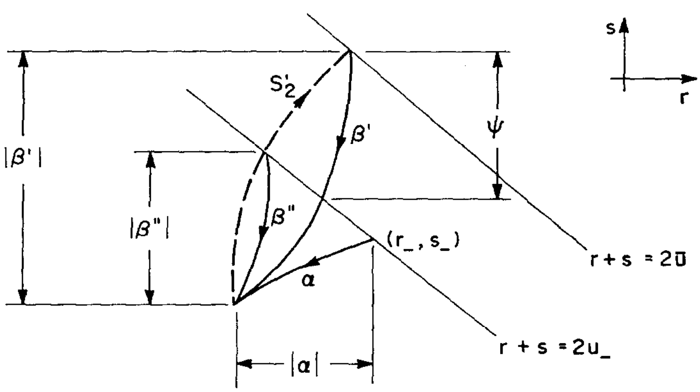


FIGURE 12

and let β'' denote the S_2 as in Fig. 11, ($|\alpha'| = |\beta''|$). Then using Proposition 4.1, we have the estimate,

$$|\beta'| \leqslant |\beta''| \leqslant |\alpha| + C\epsilon |\alpha|^2,$$

so that (9) holds if $\bar{u} \leqslant u_-$. Suppose now that $\bar{u} > u_-$. If we refer to Fig. 12, we see that

$$|\beta'| < |\beta''| + \psi \leqslant |\alpha| + C\epsilon |\alpha|^2 + \psi.$$

But from Lemma 2.1, we have $\psi < 2(\bar{u} - u_-)$ so that (9) holds and the proof is complete.

5. Convergence of the Approximating Solutions

We shall now obtain uniform bounds on the total variation of the approximating solutions using Glimm's method. That is, if J_1 and J_2 are two I curves with J_2 an immediate successor to J_1, we shall show that $F(J_2) \leqslant F(J_1)$ provided that $\epsilon F(\mathcal{O})$ is sufficiently small. The interactions which do not involve reflections of shock waves at the boundary $x = 0$ are quite the same as in [5]; we need only consider the case where J_1 and J_2 differ only on $a_{0,n} \leqslant t \leqslant a_{0,n+1}$, $0 \leqslant x \leqslant a_{1,n}$, and $\alpha + \gamma \to \beta'$ (cf. Fig. 13, where $\gamma = \gamma(nh)$). For this interaction, we note that α and γ cross J_1 but not J_2 and β' crosses J_2 but not J_1. We have

$$L(J_2) - L(J_1) = |\beta'| - |\alpha| - |\gamma|,$$

where of course, $\gamma = \gamma(nh)$ is a contribution due only to the boundary data, and

$$Q(J_2) - Q(J_1) = |\beta'| \textstyle\sum |\alpha_l| - |\alpha| \sum |\alpha_l| - \frac{1}{2}|\alpha|^2 - |\alpha||\gamma|$$
$$- |\alpha| \textstyle\sum |\gamma_j| - |\gamma| \sum |\alpha_l|,$$

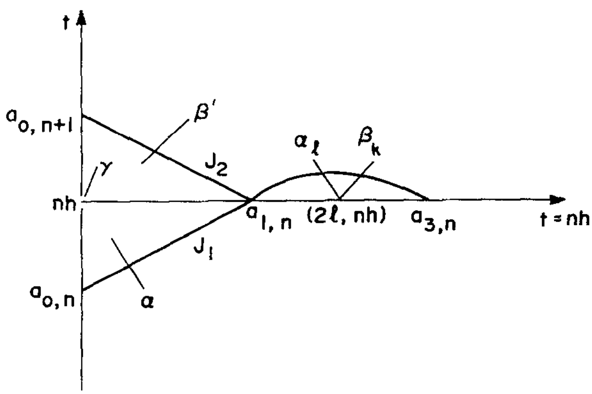


Figure 13

so that using Proposition 4.3, where[3] $K = 4C\epsilon$

$$\begin{aligned} F(J_2) - F(J_1) &= |\beta'| - |\alpha| - |\gamma| - (K/2)|\alpha|^2 \\ &\quad + K\left\{(|\beta'| - |\alpha| - |\gamma|)\sum|\alpha_l| - \left(|\gamma| + \sum|\gamma_j|\right)|\alpha|\right\} \\ &\leqslant C\epsilon|\alpha|^2 - (K/2)|\alpha|^2 + KC\epsilon|\alpha|^2\sum|\alpha_l| \\ &\leqslant |\alpha|^2\{C\epsilon + C\epsilon KF(J_1) - (K/2)\} \\ &\leqslant |\alpha|^2\{C\epsilon(1 + KF(\mathcal{O})) - (K/2)\} \\ &\leqslant 0 \end{aligned}$$

if $KF(\mathcal{O}) \leqslant 1$ and $K = 4C\epsilon$; i.e., $\epsilon F(\mathcal{O}) \leqslant (4C)^{-1}$. We have thus proved.

LEMMA 5.1. *If $0 \leqslant \epsilon < \frac{1}{2}$, and $\epsilon F(\mathcal{O})$ is sufficiently small, then $F(J_2) \leqslant F(J_1)$, where J_i $(i = 1, 2)$, are two I curves, and J_2 is an immediate successor to J_1.*

From this lemma, it follows that $F(J) \leqslant F(\mathcal{O}) \leqslant 2L(\mathcal{O}) \leqslant \text{const} \cdot TV\{v_0, u_0, u_1\}$,[4] for any I curve J. This estimate yields a uniform bound on the total variation of any of our approximating solutions, on each line $t = \text{const} > 0$. We thus have the following theorem.

THEOREM 5.1. *Let the data functions $(\rho_0(x), u_0(x), u_1(t))$ each have bounded total variation, and be bounded; i.e., $|u_0(x)| + |u_1(t)| \leqslant M$, $0 < \rho_- \leqslant \rho_0(x) \leqslant \rho_+ < +\infty$. Then there exists a constant γ_0, $1 < \gamma_0 < 2$ such that for $\gamma \in [1, \gamma_0]$, the mixed problem (1)–(3) has a global (weak) solution which has bounded total variation on each line $t = \text{const} > 0$. γ_0 depends on the total variation of the data.*

We remark here that the extension to $\gamma_0 \geqslant 2$ is considered in Section 9.

6. AN EXAMPLE

Consider the system (1) in the region $0 < x < 1$, $t > 0$, with initial data (2) in $0 \leqslant x \leqslant 1$, and boundary data

$$u(0, t) = u_1(t), \qquad u(1, t) = u_2(t), \qquad t \geqslant 0. \tag{10}$$

Such a problem is not well posed, if one does not provide supplementary conditions on the boundary data. To see this, consider the case where $\epsilon = 0$, $(u_1(t), u_2(t)) \equiv (u_1, u_0)$, where u_1, u_0 are constants with $u_1 - u_0 = 1$. Suppose

[3] This comes from the interactions in $x \geqslant a_{1,n}$, as in [5].

[4] By $TV(f, g)$, we mean Tot Var f + Tot Var g.

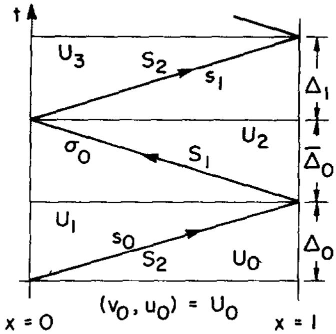


FIGURE 14

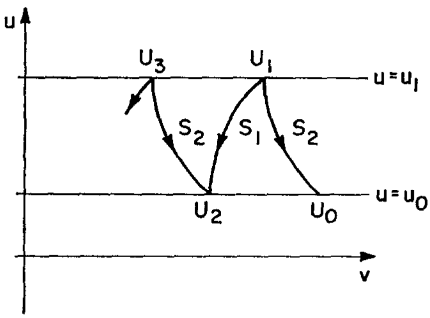


FIGURE 15

further that $(v_0(x), u_0(x)) \equiv (v_0, u_0)$, where v_0 is a constant. Using Lemma 2.1, we see that an S_2 shock wave shoots out of the corner (0, 0), and impinges on the boundary $x = 1$. It is then reflected as an S_1 shock wave which impinges on the boundary $x = 0$; this shock in turn is reflected as an S_2, and so on (see Figs. 14, 15). If we let $U_i = (v_i, u_i)$, $i = 0, 1, 2,\ldots$, then we see that $u_{2n} = u_0$, $u_{2n+1} = u_1$, $n = 0, 1, 2,\ldots$.

We have, with $p(v) = v^{-1}$,

$$-[(v_1 - v_0)(p(v_0) - p(v_1))]^{1/2} = u_0 - u_1 = -[(v_1 - v_2)(p(v_2) - p(v_1))]^{1/2}$$

so that if we square and collect terms, we get $v_1^2 = v_2 v_0$. Similarly, $v_2^2 = v_1 v_3$ so that $v_1/v_3 = v_0/v_2 = K > 1$. Then $v_3 = K^{-1}v_1$, $v_5 = K^{-2}v_1,\ldots$, and in general, $v_{2n+1} = K^{-n}v_1$; likewise $v_{2n} = K^{-n}v_0$. If we let the shock speed s_n be defined by

$$s_n(v_{2n+1} - v_{2n}) = -(u_1 - u_0) = -1,$$

then $s_n K^{-n}(v_1 - v_0) = -1$, $s_n = K^n(v_0 - v_1)^{-1}$. If we let σ_n denote the shock speed of the S_1 shocks, we have $\sigma_n = K^n(v_2 - v_1)^{-1}$. If Δ_n and $\bar{\Delta}_n$ are defined as in Fig. 14, then

$$T = \sum \Delta_n + \bar{\Delta}_n = (v_0 - v_2) \sum_{n=0}^{\infty} K^{-n} = v_0 < \infty.$$

Since $v_n \to 0$ as $n \to \infty$, we see $p(v_n) \to \infty$, as $n \to \infty$. Thus, the pressure becomes infinite after a finite time!

To analyze this example, we write the system in Eulerian coordinates

$$\rho_t + (\rho u)_q = 0, \qquad (\rho u)_t + (\rho u^2 + p(\rho))_q = 0,$$

and for simplicity, we assume $u_0 = 0$, $u_1 = 1$. Here q is the position of the gas particle, and q and x are related by

$$x = \int_0^q \rho(s, t)\, ds > 0.$$

Also, $\partial q/\partial t = u$, $\partial q/\partial x = \rho^{-1}$, $u_x = v_t$. Thus,

$$q(x, t) = q(x, 0) + \int_0^t u(x, t)\, dt,$$

so $q(x, t)$ labels the q position of that point where the amount of gas between that point and 0 is x; the piston corresponds to $x = 0$. The piston path is

$$q(0, t) = q(0, 0) + \int_0^t u(0, t)\, dt = \int_0^t u_1\, dt = t.$$

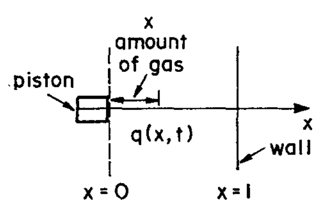


FIGURE 16

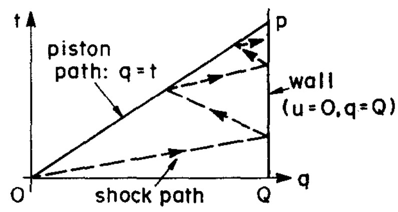


FIGURE 17

If we set $q(1, 0) = \int_0^1 \rho^{-1}(\xi, 0)\, d\xi$, then the wall is given by $q(1, t)$, and we have

$$q(1, t) = q(1, 0) + \int_0^t u(1, t)\, dt \equiv Q + \int_0^j u(1, t)\, dt.$$

Thus, in Eulerian coordinates, the piston moves with velocity 1, and the piston path is given by $q(0, t) = t$; see Fig. 17. Our example corresponds to the triangular region OPQ in Eulerian coordinates; i.e., the piston collides with the wall at a finite time, $t = Q$. This is physically impossible, for at this time, $\rho = +\infty$, $p = +\infty$, so that the force on the piston is infinite.

From this example, we see that the trouble is that the piston collides with the wall. It is thus natural to assume that both boundaries $q(0, t)$ and $q(1, t)$ do not collide in finite time. Hence, it is necessary to impose the following restriction on the boundary data (1): For all $t \geqslant 0$, $q(0, t) < q(1, t)$; i.e.,

$$\int_0^t u_1(s)\, ds < Q + \int_0^t u_2(s)\, ds \tag{11}$$

where Q is defined by

$$Q = \int_0^1 \rho_0^{-1}(\xi)\, d\xi = \int_0^1 v_0(\xi)\, d\xi.$$

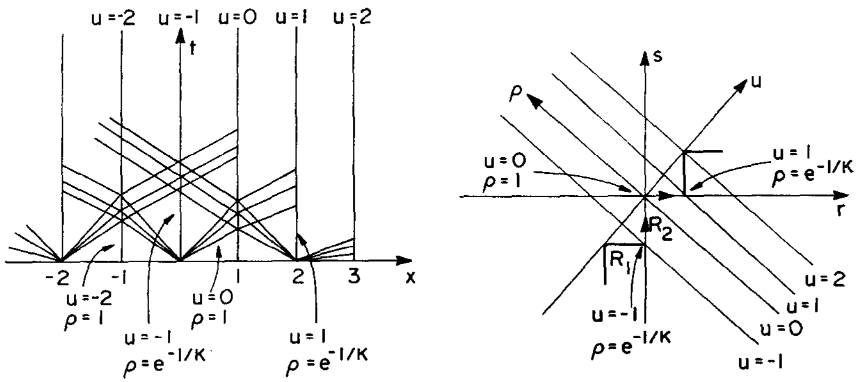
Observe that (11) fails for our example since here $Q = v_0$, and (11) yields

$$t = (u_1 - u_0)t < v_0 .$$

That is, (11) does not hold for $t \geqslant v_0$, and as we have seen earlier, blow-up occurs precisely at time v_0.

Next, we also cannot admit boundary data which allows the pistons to move infinitely far apart from each other, since generally speaking, this will lead to $\rho \to 0$ as $t \to \infty$. For example, suppose that we consider the problem (1)–(4) where we take $p(v) = v^{-1}$, $u(0, t) = -1$, $u(1, t) = 0$, and $(v(x, 0), u(x, 0)) = (1, 0)$. Then from (11) we see that $q(1, t) - q(0, t) = 1 + t \to +\infty$ as $t \to +\infty$. Moreover, an analysis similar to that given above shows that an R_2 rarefaction wave shoots out of the corner (0, 0) and impinges on the boundary $x = 1$. It then gets reflected as an R_1, and so on.[5] Here we see that $v \to +\infty$ as $t \to \infty$,

[5] The solution to this problem can be constructed via the methods in [6]. That is, we can consider this mixed problem as an initial-value problem as follows (where for brevity, we merely sketch the solution).



Also, we may note that for $p(v) = v^{-(1+2\epsilon)}$, $\epsilon > 0$, it can happen that $v = +\infty$ for all $t > 0$ (cf. Lemma 2.1 in case $u_+ > u_-$, $s_- - r_+ < 2(u_+ - u_-) - [2K\gamma^{1/2}/\epsilon]$).

i.e., $\rho \to 0$ as $t \to +\infty$. Thus, in order to avoid ρ coming arbitrarily close to 0, it is necessary to bound $q(1, t) - q(0, t)$ from above; i.e., it is necessary that

$$\int_0^1 v_0(\xi)\, d\xi + \int_0^t [u_2(s) - u_1(s)]\, ds \leqslant \text{const}$$

for all $t > 0$. In the next section we shall see that this condition, together with (11) yields a global existence theorem for the problem (1)–(4).

7. Preliminaries to the Existence Theorem in $0 \leqslant x \leqslant 1$

For the problem in $0 \leqslant x \leqslant 1$, we define a solution of (1), (2), (10) to be a pair of bounded, measurable functions u, v satisfying the pair of equations

$$\int_0^T \int_0^1 (v\phi_t - u\phi_x)\, dx\, dt - \int_{j=T} v\phi\, dx + \int_{t=0} v_0\phi\, dx + \int_{x=1} u_2\phi\, dt - \int_{x=0} u_1\phi\, dt = 0 \tag{12a}$$

for all $T > 0$ and for all $\phi \in C^1$, and

$$\int_{t>0} \int_0^1 (u\psi_t + p(v)\,\psi_x)\, dx\, dt + \int_{t=0} u_0\psi\, dx = 0 \tag{12b}$$

for all $\psi \in C_0^1$, satisfying $\psi(0, t) = \psi(1, t) = 0$ for all $t \geqslant 0$. Here we tacitly assume that $\int_{t=T} v\phi\, dx$ is defined for all $T > 0$; in fact, we shall obtain a (weak) solution which is a continuous function in $t \geqslant 0$ with values in $L^1(0, 1)$.

In order to solve this problem, we shall again use Glimm's method, together with the functional introduced by DiPerna in [1], now supplemented by the boundary terms. We assume that the initial conditions $v_0(x)$, $u_0(x)$ and the boundary conditions $u_1(t)$ and $u_2(t)$ are bounded and have finite total variation; we also assume that $0 < \mathrm{v} \leqslant v_0(x) \leqslant \bar{v} < \infty$.

Let

$$Q(t) = \int_0^1 v_0(x)\, dx + \int_0^t (u_2(s) - u_1(s))\, ds. \tag{13}$$

From the examples in the last section, we see that in order to avoid the vacuum ($\rho = 0$) in the solution and to avoid the collision of the boundaries ($\rho = +\infty$) for all time, it is necessary to assume that there exist constants Q_1, Q_2 such that

$$0 < Q_1 \leqslant Q(t) \leqslant Q_2 < +\infty, \tag{14}$$

for all $t \geqslant 0$.[6] In what follows, we shall actually prove that (14) is also sufficient

[6] Note that (14) holds in the case of "rigid walls"; namely $u_1(t) = u_2(t) = 0$.

for global existence of a solution for the problem (1)–(4), provided that $\epsilon T_2 < \text{const.}$

We note here that for a weak solution of (1)–(4) in $0 \leqslant x \leqslant 1$, $0 \leqslant t \leqslant T$, Eq. (12a) with $\phi \equiv 1$ yields the equality

$$\int_0^1 v(t, x)\, dx = \int_0^1 v_0(x)\, dx + \int_0^t (u_2(s) - u_1(s))\, ds = Q(t) \tag{15}$$

for $0 \leqslant t \leqslant T$. It follows from (14) and the finiteness of the total variation of u_1 and u_2 that the following limits exist:

$$\begin{aligned} v_\infty &= \overline{\lim_{t\to\infty}}\, Q(t) < \infty, \\ u_\infty &= \lim_{t\to+\infty} u_2(t) = \lim_{t\to+\infty} u_1(t). \end{aligned} \tag{16}$$

We let L be the following line segment in the r, s plane:

$$L = \{(r, s)\colon \tfrac{1}{2}Q_1 \leqslant v \leqslant Q_2 + \tfrac{1}{2}Q_1\,,\ u = u_\infty\}, \tag{17}$$

where $r = r(v, u)$, $s = s(v, u)$ are the Riemann invariants.

We choose the space mesh length $l = 1/2M$, where M is an integer; the time mesh length $h = h(l)$ will be chosen later. We set

$$\tilde{Y} = \{(m, n)\colon m = 1, 3, 5,\ldots, 2M - 1, n = 1, 2, 3,\ldots\},$$

and

$$\tilde{A} = \Pi\{\{(m - 1)\, l, (m + 1)l\} \times \{nh\}\colon (m, n) \in \tilde{Y}\}.$$

Let $\{\alpha_n\}$ be a random sequence in $(-1, 1)$ as in Section 3, and let $a_{m,n}$ be the mesh points, where $a_{m,n} = (ml + \alpha_n l, nh)$, $(m, n) \in \tilde{Y}$, $a_{0,n} = (0, nh - \frac{1}{2}h)$, $n = 1, 2,\ldots$, $a_{2M,n} = (1, nh - \frac{1}{2}h)$, $n = 1, 2,\ldots$, $a_{m,0} = (ml, 0)$, $m = 1, 3,\ldots, 2M - 1$. The I curve $\tilde{\mathcal{O}}$ is any space-like curve joining the points $a_{m,0}$, $m = 1, 3,\ldots, 2M - 1$, and not containing points $((m + 1)l, 0)$, $m = 1, 3,\ldots, 2M - 3$, which lies in $0 \leqslant t \leqslant h$ if $l \leqslant x \leqslant (2M - 1)l$, and which also includes the two half-rays $t \geqslant \frac{1}{2}h$, $x = 0$, and $t \geqslant \frac{1}{2}h$, $x = 1$, and the straight-line segments joining $a_{1,0}$ to $a_{0,1}$ and $a_{2M-1,0}$ to $a_{2M,1}$. The I curves J are defined in an analogous manner; see Section 3 and Fig. 18.

We now recall the main theorem in [1] concerning the transformation from the Riemann invariants r, s to the generalized Riemann invariants $\phi(r)$, $\psi(s)$. Let

$$\begin{aligned} \sigma &= 2(u - u_\infty) = r + s - 2u_\infty\,, \\ \eta &= (2K\gamma^{1/2}/\epsilon)\, \rho^\epsilon = s - r + (2K\gamma^{1/2}/\epsilon) \end{aligned} \tag{18}$$

and

$$W(k, \Theta) = \{(\sigma, \eta): |\,\sigma\,| < k\eta,\ c_1 < \Theta\eta < c_2\}$$

where $1 > k > 0$, $\Theta > 0$, $c_1, c_2 > 0$ (see Fig. 19).

Let us consider the following transformation from (r, s) to $(\phi(r), \psi(s))$:

$$T_\theta : \begin{cases} \sigma' = \exp \theta(\eta + \sigma) - \exp \theta(\eta - \sigma), \\ \eta' = \exp \theta(\eta + \sigma) + \exp \theta(\eta - \sigma) - 2, \end{cases} \tag{19}$$

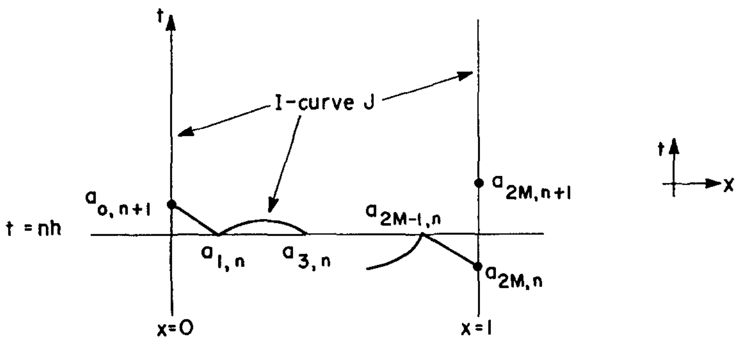


FIGURE 18

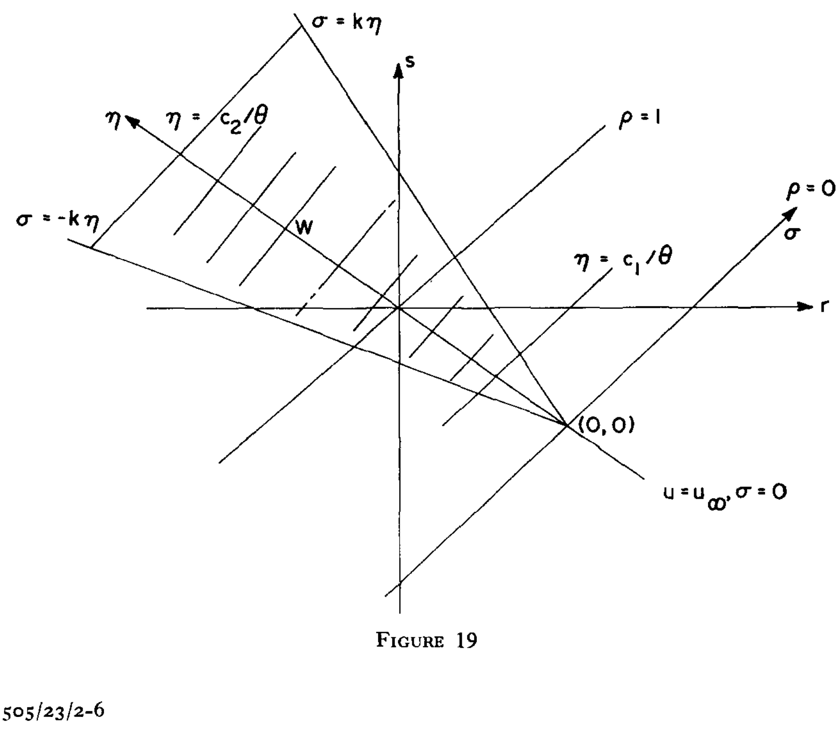


FIGURE 19

where $\sigma' = \phi + \psi$, $\eta' = \psi - \phi$ gives $\phi = \phi(r)$, $\psi = \psi(s)$. For this transformation applied to the system (1) we have the following properties [1]:

(i) $T_\Theta W \supset W' = \{(\sigma', \eta'): |\sigma'| < k\eta', d_1 < \eta' < d_2\}$, $d_1 = d_1(c_1)$, $d_2 = d_2(c_2)$.

(ii) The shock curves S_1 and S_2 in the r, s plane, $s_0 - s = g_1(r_0 - r; \rho_0)$, and $r_0 - r = g_2(s_0 - s; \rho_0)$, are transformed by T_Θ to curves

$$\psi_0 - \psi = G_1(\phi_0 - \phi; \eta_0') \qquad \text{and} \qquad \phi_0 - \phi = G_2(\psi_0 - \psi; \eta_0'),$$

respectively. Let the shock strengths of S_1 and S_2 in terms of ϕ, ψ be $\Delta\phi = \phi_0 - \phi$ and $\Delta\psi = \psi_0 - \psi$, respectively. If we take k sufficiently small, $c_1 = c_1(k)$ and $c_2 = c_2(k)$, then the shock waves in the region W' do not increase in strength (measured in terms of ϕ, ψ) after interacting with each other.

(iii) As $k \to 0$, c_1 and d_1 remain finite, but

$$\lim_{k\to 0} c_2(k) = +\infty, \qquad \text{and} \qquad \lim_{k \to 0} d_2(k) = +\infty.$$

In the next section we shall apply these results to our problem.

8. Solution of Problem (1)–(4)

We now return to our problem. We set

$$\begin{aligned} \eta_\infty &= \eta(\rho_\infty), \qquad \rho_\infty = 1/v_\infty\,, \\ \theta &= [c_1 + c_2(k)]/2\eta_\infty\,. \end{aligned} \tag{20}$$

The line segment L (see (17)) becomes, in terms of σ, η,

$$L = \{(\sigma, \eta): \sigma = 0, 0 < \eta_1 \leqslant \eta \leqslant \eta_2 < +\infty\},$$

where $\eta_1 < \eta_\infty < \eta_2$, and so

$$L' = T_\Theta L = \{(\sigma', \eta'): \sigma' = 0, 0 < \eta_1' \leqslant \eta' \leqslant \eta_2' < +\infty\}.$$

Hence for sufficiently small k, we have, using (iii) and (20), that

$$L' \subset W' = \{(\sigma', \eta'): |\sigma'| < k\eta', d_1 < \eta' < d_2\};$$

that is, $d_1 < \eta_1' < \eta_\infty' < \eta_2' < d_2$, where $(0, \eta_\infty') = T_\theta(0, \eta_\infty)$. Therefore, we may put, for small k

$$S = \tfrac{1}{4}\,\mathrm{dist}\{L', \partial W'\} > 0, \tag{21}$$

where $\partial W'$ is the boundary of W' and, as usual, $\operatorname{dist}\{(\phi_1, \psi_1), (\phi_2, \psi_2)\} = |\phi_1 - \phi_2| + |\psi_1 - \psi_2|$. Hereafter we fix k small as above.

We set

$$\bar{\lambda} = |\lambda(\rho = [c_2(k)/2K\gamma^{1/2}]^{1/\epsilon})|$$

and we take the time mesh length to be

$$h = l/\bar{\lambda},$$

where we note that when ϵ is small, Θ is small and has the same order as ϵ in view of (18) and (20).

In order to define the functional F on J, we set (see Fig. 20)

$$\begin{aligned} \sigma_i(t) &= 2(u_i(t) - u_\infty), \qquad & i = 1, 2, \\ \sigma_i'(t) &= \theta(2 + d_2)\sigma_i(t), \qquad & i = 1, 2. \end{aligned} \tag{22}$$

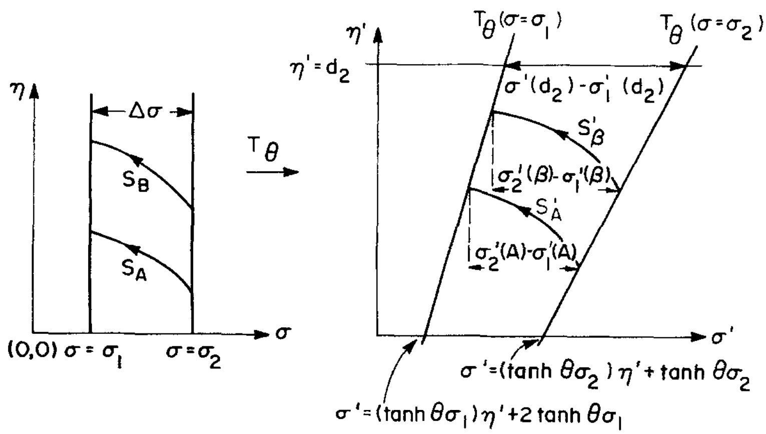


FIGURE 20

We remark that the indices 1 and 2 in the $\sigma - \eta$ plane in Fig. 20 refer to just the two values of σ on $x = 1$, and are independent of those of $\sigma_i(t)$, $i = 1, 2$. Furthermore, the two S_1 shock waves S_A and S_B having the same strength $\Delta\sigma$ are transformed by T_0 into S_A' and S_B', respectively, which have different strength in terms of the σ' coordinate.

We now define the functional F on J:

$$F(J) = \sum_{S_1 \in J \cap \{0<x<1\}} \Delta\phi + \sum_{S_2 \in J \cap \{0<x<1\}} \Delta\psi + \sum_{J \cap \{x=0\}} \Delta\sigma_1' + \sum_{J \cap \{x=1\}} \Delta\sigma_2', \tag{23}$$

where

$$\begin{aligned} \Delta\sigma_1' &= \max\{0, \sigma_1'(nh + \tfrac{1}{2}h) - \sigma_1'(nh - \tfrac{1}{2}h)\} \\ \Delta\sigma_2' &= \max\{0, \sigma_2'(nh - \tfrac{1}{2}h) - \sigma_2'(nh + \tfrac{1}{2}h)\}. \end{aligned} \tag{24}$$

LEMMA 8.1. *Let the image under T_Θ of the approximate solution $\{v_l, u_l\}$ on J be denoted by $\{\phi_l = \phi(v_l, u_l), \psi_l = \psi(v_l, u_l)\}$. If $\{\phi_l, \psi_l\} \in W'$, then*

$$F(J_2) \leqslant F(J_1) \tag{25}$$

for J_2 being an immediate successor of J_1. Also for $J = \tilde{\mathcal{O}}$, we have

$$\begin{aligned} F(\tilde{\mathcal{O}}) &\leqslant TV\{\phi(x, 0), \psi(x, 0), \sigma_1'(t), \sigma_2'(t)\} \\ &\leqslant \Theta M \cdot TV\{v_0(x), u_0(x), u_1(t), u_2(t)\}, \end{aligned} \tag{26}$$

where M is a constant depending only on W'.

Proof. The estimate (25) comes from (ii) and the argument in [1, 4]. Here we must always keep in mind that the boundary conditions on $x = 0$ and $x = 1$ are given by $u = u_i(t)$, $i = 1, 2$, and we can solve the simples piston problem, as in Lemma 2.1 at $x = 0, 1$ and $t = nh$ with data $u_i(t)$. But for the solution we have the estimates

$$\Delta\psi \leqslant \Delta\sigma_1' \qquad \text{near} \qquad t = nh, \qquad x = 0 \qquad \text{for} \quad S_2,$$

and

$$\Delta\phi \leqslant \Delta\sigma_2' \qquad \text{near} \qquad t = nh, \qquad x = 1 \qquad \text{for} \quad S_1,$$

as in Lemma 2.1. Since the solution is contained in W', the strength of the shock coming off the boundary is dominated by $\Delta\sigma_i'$ (cf. Fig. 20). For example, if we consider an S_1 coming off $x = 1$, we have, from Fig. 21,

$$\begin{aligned} \Delta\phi = \phi_2 - \phi_1 &= \tfrac{1}{2}\{\sigma_2'(A) - \eta_2'(A) - (\sigma_1'(A) - \eta_1'(A))\} \\ &= \tfrac{1}{2}\{\sigma_2'(A) - \sigma_1'(A) + \eta_1'(A) - \eta_2'(A)\} \\ &\leqslant \tfrac{1}{2}\{\sigma_2'(A) - \sigma_1'(A) + \sigma_2'(A) - \sigma_1'(A)\} = \sigma_2'(A) - \sigma_1'(A) \\ &\leqslant \sigma_2'(d_2) - \sigma_1'(d_2) = (\tanh\theta\sigma_2)\, d_2 + 2\tanh\theta\sigma_2 \\ &\quad - [(\tanh\theta\sigma_1)\, d_2 + 2\tanh\theta\sigma_1] \\ &\leqslant \theta(2 + d_2)(\sigma_2 - \sigma_1) \equiv \theta(2 + d_2)\,\Delta\sigma_2 \equiv \Delta\sigma_2'. \end{aligned}$$

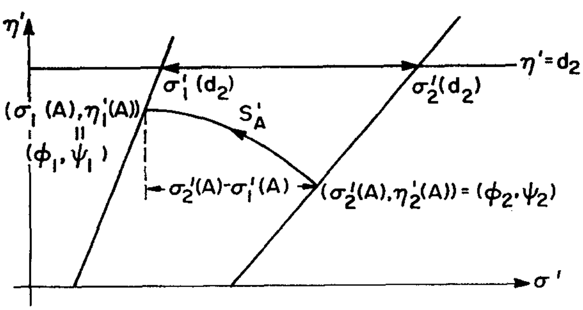


FIGURE 21

(Here our notation $\Delta\sigma_2$ and $\Delta\sigma_2'$ is consistent with that of (22) and (24) since we are considering an S_1 shock coming out of $x = 1$.)

Now as soon as we know that our approximate solution $\{\phi_l, \psi_l\}$ on J lies in W', we can estimate the total variation of $\{\phi_l, \psi_l\}$ on J by $F(J)$ as follows. Since $\lim_{t\to+\infty} \sigma_l'(t) = 0$, for $\sigma_l' = \sigma'(v_l, u_l)$, $\eta_l' = \eta'(v_l, u_l)$, we have

$$\begin{aligned} \underset{J}{TV}\{\sigma_l'\} &= 2(\text{decreasing variation of } \sigma_l' \text{ on } J) \\ &\leqslant 2\left\{\sum_{J\cap\{x=0\}} \Delta\sigma_1' + 2\sum_{J\cap\{0<x<1\}} (\Delta\phi + \Delta\psi) + \sum_{J\cap\{x=1\}} \Delta\sigma_2'\right\} \\ &\leqslant 4F(J), \end{aligned}$$

and

$$\underset{J}{TV}\{\eta_l'\} \leqslant \underset{J}{TV}\{\sigma_l'\} \leqslant 4F(J).$$

Hence from Lemma 8.1, we have

$$\underset{J}{TV}\{\phi_l, \psi_l\} = \underset{J}{TV}\{\sigma_l', \eta_l'\} \leqslant 8F(J) \leqslant 8F(\tilde{\mathscr{O}}). \tag{27}$$

It follows, then, that we must find conditions both on the initial and boundary data and on the time interval, so that the approximate solutions belong to W' for that time interval. This is provided by the following lemma.

Lemma 8.2. *Suppose that initially $\{\phi_l, \psi_l\}_{\tilde{\mathscr{O}}} \subset W'$ and that*

$$F(\tilde{\mathscr{O}}) \leqslant \tfrac{1}{4}S. \tag{28}$$

If in addition the approximate solution satisfies

$$\tfrac{1}{2}Q_1 \leqslant Q_l(t) = \int_0^1 v_l(x, t)\, dx \leqslant \tfrac{1}{2}Q_1 + Q_2, \tag{29}$$

then the approximate solutions $\{\phi_l, \psi_l\}$ are contained in W'.

Proof. It follows from the definition of L (see (17)) and the assumption (29) that $(Q_l(t), u_\infty)$ lies on L, and that

$$T_\Theta(Q_l(t), u_\infty) \in L'. \tag{30}$$

We set

$$\underset{J}{\operatorname{dist}}\{\{\phi_l, \psi_l\}, L'\} \equiv \sup_{(x,t)\in J\cap\{0<x<1\}} \operatorname{dist}\{\{\phi_l(x, t), \psi_l(x, t)\}, L'\}.$$

Using (30) and the fact that $\lim_{t\to+\infty} \sigma_i'(t) = 0$, we have

$$\begin{aligned}\operatorname*{dist}_J \{\{\phi_l, \psi_l\}, L'\} &= \operatorname*{dist}_J \{\{\sigma_l', \eta_l'\}, L'\} \\ &\leqslant TV\{\sigma_l', \eta_l'\} \\ &\leqslant 8F(J) \leqslant 8F(\tilde{\mathscr{O}}) \leqslant 2S.\end{aligned}$$

Here we have also used (27), (28), and the hypothesis $\{\phi_l, \psi_l\}_{\tilde{\mathscr{O}}} \subset W'$. Therefore by (21) we conclude that $\{\phi_l, \psi_l\}_J \subset W'$. This completes the proof.

Now in view of (26), we can choose Θ_0 so small that for any Θ, $0 < \Theta \leqslant \Theta_0$, we have (28). The fact that $\{\phi_l, \psi_l\}_{\tilde{\mathscr{O}}} \subset W'$ follows from (14) at $t = 0$, and the finiteness of the total variation of $v_0(x)$, $u_0(x)$, $u_1(t)$, $u_2(t)$, together with (16), (20), and (21), for $0 < \Theta \leqslant \Theta_0$; i.e., for $0 < \epsilon \leqslant \epsilon_0$. Hereafter, we only consider ϵ in $(0, \epsilon_0]$.

LEMMA 8.3. *Let $0 < \epsilon \leqslant \epsilon_0$; then the mean velocity $Q_l(t)$ of the approximate solution $\{v_l(t, x), u_l(t, x)\}$ satisfies the inequality (29) for $0 \leqslant t \leqslant t_0$, where t_0 depends only on W' and ϵ_0. Moreover, the following estimate holds for the approximate solution on $0 \leqslant t \leqslant t_0$:*

$$\operatorname*{TV}_{0<x<1} \{r_l, s_l\} \leqslant C_0 TV\{v_0(x), u_0(x), u_1(t), v_1(t)\}, \tag{31}$$

where C_0 is a constant depending only on W' and ϵ_0.

Proof. When the approximate solutions $\{\phi_l, \psi_l\}$ are contained in W', we can estimate the total variation of the approximate solutions $\{r_l, s_l\}$ as follows, using (25), (26), and (27):

$$\begin{aligned}\theta \cdot \operatorname*{TV}_J\{r_l, s_l\} &\leqslant C\operatorname*{TV}_J\{\phi_l, \psi_l\} \leqslant 8CF(J) \\ &\leqslant 8CF(\tilde{\mathscr{O}}) \leqslant 8C\theta M \cdot TV\{v_0, u_0, u_1, u_2\}.\end{aligned}$$

This then yields (31). Next, for $n = 1, 2, \ldots$ we have

$$\begin{aligned}&\int_0^1 |v_l(x, nh+0) - v_l(x, nh-0)|\, dx \\ &= \sum_{m=1,3,\ldots}^{2M-1} \int_{(m-1)l}^{(m+1)l} |v_l(a_{m,n}) - v_l(x, nh-0)|\, dx \\ &\leqslant \sum_{m=1}^{2M-1} TV\{v_l(t, x)\colon (m-1)\, l < x < (m+1)l\} \cdot 2l \\ &= 2l \operatorname*{TV}_{0\leqslant x\leqslant 1} \{v_l(t, x)\} \\ &\leqslant Cl \operatorname*{TV}_{0\leqslant x\leqslant 1} \{r_l, s_l\} \\ &\leqslant C_0 C\ell TV\{v_0, u_0, u_1, u_2\}.\end{aligned}$$

Since $\{v_l, u_l\}$ is a weak solution in the strip $(n-1)h < t < nh$, it follows from (15) that

$$\int_0^1 v_l(x, nh - 0)\, dx = \int_0^1 v_l(x, (n-1)\, h + 0)\, dx + \int_{(n-1)h}^{nh} \{u_{2,l}(s) - u_{1,l}(s)\}\, ds.$$

Therefore,

$$\left| \int_0 v_l(x, t)\, dx - \int_0^1 v_{0,l}(x)\, dx - \int_0^t [u_{2,l}(s) - u_{1,l}(s)]\, ds \right| \leqslant C\bar{\lambda} t \leqslant \tfrac{1}{4} Q_1,$$

for $t \leqslant t_0 \equiv Q_1(4\bar{\lambda}C)^{-1}$. Furthermore, there exists an $l_0 > 0$, such that

$$\left| \int_0^1 v_l(x, t)\, dx - \int_0^1 v_0(x)\, dx - \int_0^t [u_2(s) - u_1(s)]\, ds \right| \leqslant \tfrac{1}{2} Q_1$$

for all $l \leqslant l_0$, because

$$\lim_{l \to 0} \left\{ \int_0^1 v_{0,l}(x)\, dx + \int_0^t [u_{2,l}(s) - u_{1,l}(s)]\, ds \right\}$$
$$= \int_0^1 v_0(x)\, dx + \int_0^j [u_2(s) - u_1(s)]\, ds.$$

This completes the proof of the lemma.

From Lemmas 8.1, 8.2, and 8.3, we conclude that there exists an $\epsilon_0 > 0$ such that for any ϵ, $0 < \epsilon < \epsilon_0$, the approximate solution $\{\phi_l, \psi_l\} \subset W'$ and so $TV\{r_l, s_l\} \leqslant C$ for $0 \leqslant t \leqslant t_0$. Therefore Glimm's argument for the convergence of the approximate solutions yields a weak solution to the problem (1), (2), (10) in the region $0 \leqslant t \leqslant t_0$, $0 \leqslant x \leqslant 1$. Since the limit is also L^1 continuous in t, the equality (15) holds for $0 \leqslant t \leqslant t_0$ and so we also have

$$0 < Q_1 \leqslant Q(t) \leqslant Q_2 < +\infty, \qquad 0 \leqslant t \leqslant t_0 \tag{32}$$

for our weak solution. We now can proceed to construct the weak solution on $t_0 \leqslant t \leqslant 2t_0$, and so on. In fact, since we can assume that there exist integers $n = n(l)$ such that $t_0 = nh$ for $l = 2^{-j}$, $j = 1, 2,\ldots$, the new "initial" data $v(t_0, x)$, $u(t_0, x)$ can be approximated by a subsequence of the piecewise constant functions $v_l(t_0 + 0, x)$, $u_l(t_0 + 0, x)$, which are given by

$$\{v_l(t_0 + 0, x), u_l(t_0 + 0, x)\} = \{v_l(a_{m,n}), u_l(a_{m,n})\}$$

in $(m-1)l < x < (m+1)l$, $m = 1, 3,\ldots, 2M - 1$. The fact that the approximate solutions $\{\phi_l, \psi_l\}$ are contained in W' in $0 \leqslant t \leqslant t_0$ gives us

$$F(J_2) \equiv F(J\colon \{v_l(t_0 + 0, x), u_l(t_0 + 0, x)\})$$
$$\leqslant F(J\colon \{v_l(t_0 - 0, x), u_l(t_0 - 0, x)\}) \equiv F(J_1) \leqslant F(\tilde{\mathcal{O}}).$$

That is, $F(J)$ can be considered as an a priori bound, which together with the successive use of (32), enables us to extend the weak solution for all $t \geqslant 0$. We thus have our main theorem.

THEOREM 8.4. *Let the initial conditions $v_0(x)$, $u_0(x)$, and the boundary conditions $u_1(t)$, $u_2(t)$ be bounded and have finite total variation. Also assume $0 < \mathrm{v} \leqslant v_0(x)$, for $\mathrm{v} = \text{const.}$, and let* (14) *hold. Then there exists a $\gamma_0 > 0$, depending only on the variation of the data, such that for any $\gamma \in (1, \gamma_0]$, the mixed problem* (1)–(4) *has a weak solution defined for all time. The solution has* (*uniformly*) *bounded total variation on each line $t = \text{const.} > 0$.*

9. THE EXTENSION TO $\epsilon \geqslant \frac{1}{2}$

In the previous sections, our results were only valid for $\gamma = 1 + 2\epsilon \leqslant 2$. That is, the piston problem is solvable if $\epsilon T_1 < C_1 = \text{constant}$, and $0 \leqslant \epsilon < \frac{1}{2}$. The reason that we had to take $\epsilon < \frac{1}{2}$ is due to the fact that such a restriction was needed for the estimates of [5] to be valid. We shall now show how to remove this restriction. To see this, we note that $\epsilon < \frac{1}{2}$ was used only on [5, p. 188] in order to prove the two inequalities

$$0 \leqslant (Y-1)(Y+1)^{-1} \leqslant 1 \qquad \text{and} \qquad (Y+1)^{-2} \leqslant Y^{-2} \leqslant \alpha^{-1}. \quad (33)$$

Here Y is defined by

$$Y = [\gamma\alpha^\gamma(\alpha - 1)/\alpha^\gamma - 1]^{1/2}, \qquad \alpha > 1, \quad \gamma = 1 + 2\epsilon, \quad \epsilon \geqslant 0.$$

Now it is a fairly straightforward calculation to check that $Y \geqslant 1$, and thus the first set of inequalities (33) is valid. On the other hand,

$$(Y+1)^{-2} \leqslant Y^{-2} = \frac{\alpha^\gamma - 1}{\gamma\alpha^\gamma(\alpha - 1)} \leqslant \frac{\gamma\alpha^{\gamma-1}(\alpha - 1)}{\gamma(\alpha - 1)\,\alpha^\gamma} = \alpha^{-1},$$

so that the second set of inequalities in (33) is also valid. It follows from this that the piston problem (1)–(3) is solvable provided that $\epsilon T_1 < C_1$, where $\epsilon \geqslant 0$. The fact that the double piston problem is solvable provided that (14) holds and $\epsilon T_2 < C_1$, $\epsilon > 0$, follows from [1].

We must still check the double piston problem when $\epsilon = 0$. But for $\epsilon = 0$, the usual Riemann invariants and our arguments in Sections 7 and 8 yield the existence theorem. In fact, for this case, as was pointed out in [4], the interaction of shocks of the opposite family do not increase in strength in terms of the classical Riemann invariants.

REFERENCES

1. R. DIPERNA, Existence in the large for quasilinear hyperbolic conservation laws, *Arch. Rat. Mech. Anal.* **52** (1973), 244–257.
2. J. GLIMM, Solutions in the large for nonlinear hyperbolic systems, *Comm. Pure Appl. Math.* **18** (1965), 697–715.
3. J. KASSIN, Ph.D. thesis, New York University, 1965.
4. T. NISHIDA, Global solutions for an initial boundary value problem of a quasilinear hyperbolic system, *Proc. Japan Acad.* **44** (1968), 642–648.
5. T. NISHIDA AND J. A. SMOLLER, Solutions in the large for some nonlinear hyperbolic conservation laws, *Comm. Pure Appl. Math.* **26** (1973), 183–200.
6. M. YAMAGUTI AND T. NISHIDA, On some global solution for the quasi-linear hyperbolic equations, *Funkcial. Ekvac.* **2** (1968), 51–57.